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Multiple-high-order pole solutions for the NLS equation with quartic terms

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ABSTRACT

The aim of this article is to investigate the multiple-high-order pole solutions to the focusing NLS equation with quartic terms (QNLS) under the non-vanishing boundary conditions (NVBC) via the Riemann–Hilbert(RH) method. The determinant formula of multiple-high-order pole soliton solutions for NVBC is given. Further the double $1nd$-order, mixed $2nd$- and $1nd$-order pole solutions are obtained.

1. Introduction

The NLS equation can well describe the wave evolution of deep water and optical fibers [1,2]. For some multiplicity of nonlinear phenomena in plasma waves, Bose–Einstein condensates and other physical phenomena, the NLS equation is also a generic model [3,4]. Nonetheless, several phenomena cannot be described by low-order dispersion NLS equation. The NLS equation with high-order dispersion is particularly important for describing the more complex phenomena.

In this article, we investigate the multiple-high-order pole soliton solution of focusing QNLS equation [5]

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + \gamma(q_{xxxx} + 6|q|^4 q + 4q\bar{q}_x q_x + 2q^2\bar{q}_{xx} + 6\bar{q}q_x^2 + 8|q|^2 q_{xx}) = 0. \tag{1.1}$$

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Soliton solutions as one of the most important consequence of integrable systems, which can be constructed by many different methods. Recently, the Riemann–Hilbert(RH) method as another form of inverse scattering transform(IST) method is favorite to study the soliton solutions with VBC [6,7], as well as NVBC [8–14]. The N simple poles soliton solution and one N-order pole soliton solution of QNLS equation for VBC had studied in [15]. In the case of NVBC, the multiple-high-order pole solution just investigated the pure one soliton in [15].

The major contributions of this article are to study the QNLS equation with NVBC via RH method. We present the multiple-high-order pole soliton solutions and construct the formula for soliton solution to be a determinant form. For reflection-less and the scatter data $a(k)$ with multiple-high-order pole, the Taylor series on $a(k)$ and sectionally meromorphic matrix $M(x,t,z)$ are fully utilized. The advantage of this method is that the residue conditions of high-order poles are not the necessary conditions.

This work is organized as follows. In Section 2, we investigate IST method of the QNLS equation with NVBC. And we show the analyticities, symmetries and asymptotic of Jost solution and scatter matrix. In Section 3, the formulas of single-high-order pole and multiple-high-order pole solutions are shown. We detailed given the double $1nd$-order, mixed $2nd$- and $1nd$-order pole solutions.

2. Inverse scattering transform for NVBC

The QNLS equation admits the Lax pair

$$\psi_x = \mathfrak{h}\psi = (i\lambda\sigma_3 + iQ)\psi, \qquad \psi_t = \mathfrak{l}\psi = (-8i\gamma\sigma_3\lambda^4 - 8i\gamma Q\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)\psi, \tag{2.1}$$

where

$$\begin{aligned} &A_2 = i(1+4\gamma Q^2)\sigma_3 - 4\gamma\sigma_3 Q_x, \quad A_1 = 2\gamma[Q_x, Q]\sigma_3 + i(Q + 4\gamma Q^3 + 2\gamma Q_{xx}),\\ &A_0 = -3i\gamma\sigma_3 Q^4 + i\gamma\sigma_3 Q_x^2 - i\gamma\sigma_3(QQ_{xx} + Q_{xx}Q) - \frac{i}{2}\sigma_3 Q^2 + \frac{1}{2}\sigma_3 Q_x + 6\gamma\sigma_3 Q^2 Q_x + \gamma\sigma_3 Q_{xxx},\\ &Q = \begin{pmatrix} 0 & \overline{q}(x,t) \\ q(x,t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Considering the NVBC $\lim_{|x|\to\infty} q(x,t) = q_\pm = q_0 e^{i\theta_\pm}$ where $|q_\pm| = q_0$. Naturally, $\mathfrak{h}_\pm = \lim_{x\to\pm\infty}\mathfrak{h}$, $\mathfrak{l}_\pm = \lim_{x\to\pm\infty}\mathfrak{l}$, and $Q_\pm = \lim_{x\to\pm\infty} Q$. The eigenvalues of $\mathfrak{h}_\pm$ and $\mathfrak{l}_\pm$ are $\pm i\sqrt{\lambda^2+q_0^2}$ and $\pm i\sqrt{\lambda^2+q_0^2}(\lambda - 8\gamma\lambda^3 + 4\gamma\lambda q_0^3)$, respectively. Since the eigenvalues are doubly branched, one can transform the λ-plane to z-plane, where $k = \frac{1}{2}(z + \frac{q_0^2}{z})$ and $\lambda = \frac{1}{2}(z - \frac{q_0^2}{z})$.

Now, we introduce the Jost solutions as the simultaneous solutions of Lax pair such that

$$\varPsi_\pm(x,t,z) \sim \varOmega_\pm(z) e^{i\theta(x,t,z)\sigma_3}, \quad z\in\varSigma, \quad x\to\pm\infty, \tag{2.3}$$

where $\theta(x,t,z) = kx + k(\lambda - 8\gamma\lambda^3 + 4\gamma\lambda q_0^3)t$ and $\varSigma = \mathbb{R}\cup\mathcal{C}_0$. $\mathcal{C}_0$ is the circle of radius q_0 in the z-plane. For factoring the asymptotic exponential oscillations, we define a modified Jost solutions

$$\mu_\pm(x,t,z) = \varPsi_\pm(x,t,z) e^{-i\theta(x,t,z)\sigma_3}. \tag{2.4}$$

The modified Jost solutions $\mu_\pm(x,t,z)$ satisfy the asymptotic properties $\lim_{|x|\to\infty}\mu_\pm(x,t,z) = \varOmega_\pm(z)$. And the modified Jost solution admits the Volterra integral equations

$$\mu_\pm(x,t,z) = \varOmega_\pm(z) + \int_{\pm\infty}^{x} \varOmega_\pm(z) e^{ik(x-y)\hat{\sigma}_3} \varOmega_\pm^{-1} \triangle Q_\pm \mu_\pm(y,t,z)\mathrm{d}y, \tag{2.5}$$

where $\triangle Q_\pm = iQ - iQ_\pm$. And one can obtain the analyticities of $\mu_{\pm,j}(x,t,z), (j=1,2)$: $\mu_{+,1}(x,t,z)$ and $\mu_{-,2}(x,t,z)$ analytic in $D_+ = \{z\in\mathbb{C} \parallel (|z|^2 - q_0^2)\mathrm{Im}z > 0\}$, $\mu_{-,1}(x,t,z)$ and $\mu_{+,2}(x,t,z)$ analytic in $D_- = \{z\in\mathbb{C} \parallel (|z|^2 - q_0^2)\mathrm{Im}z < 0\}$. The subscripts '1' and '2' identify the columns of matrix.

Since $\Psi_+(x,t,z)$ and $\Psi_-(x,t,z)$ are the fundamental solutions of the spectral problem (2.1), so there exists a scatter matrix $S(z)=(s_{ij})_{2\times 2}$ which satisfies the linear relationship

$$\mu_+(x,t,z)=\mu_-(x,t,z)e^{i\theta\sigma_3}S(z)e^{-i\theta\sigma_3},\quad z\in\Sigma\setminus\{\pm iq_0\},\tag{2.6}$$

Moreover, (2.6) imply that $\det S(z)=1$. Eq. (2.6) implies that $s_{11}(z)$ is analytic in D_+ and $s_{22}(z)$ is analytic in D_-. However, $s_{12}(z)$ and $s_{21}(z)$ are just continuous on Σ.

For the QNLS equation with NVBC, there exist two kinds of symmetries for the Jost solution $\Psi_\pm(z)$ and scattering matrix $S(z)$ in z-plane:

(1) The symmetries of up-half and low-half of z-plane ($z\leftrightsquigarrow\overline{z}$)

$$\Psi_\pm(z)=-\sigma\overline{\Psi_\pm(\overline{z})}\sigma,\quad \Psi_{\pm,j}=(-1)^{j-1}\sigma\overline{\Psi_{\pm,(3-j)}(\overline{z})},\quad S(z)=-\sigma\overline{S(\overline{z})}\sigma.\tag{2.7}$$

(2) The symmetries of outside and inside of $\mathcal{C}_0$ ($z\leftrightsquigarrow-\frac{q_0^2}{z}$)

$$\Psi_\pm(z)=-\frac{1}{z}\Psi_\pm\big(-\frac{q_0^2}{z}\big)\sigma_3Q_\pm,\quad \Psi_{\pm,j}(z)=(-1)^{j-1}\frac{1}{z}q_\pm\Psi_{\pm,(3-j)}\big(-\frac{q_0^2}{z}\big),\quad S(z)=Q_-^{-1}\sigma_3S\big(-\frac{q_0^2}{z}\big)\sigma_3Q_+.\tag{2.8}$$

According to the above symmetries, the scattering matrix $S(z)$ can be rewritten as a new version $S(z)=\begin{pmatrix} a(z) & -\overline{b(\overline{z})}\\ b(z) & \overline{a(\overline{z})}\end{pmatrix}$.

Next, we consider the asymptotic of Jost solution and scattering matrix. For convenience, we introduce the following notations $A_d=\begin{pmatrix} a_{11} & 0\\ 0 & a_{22}\end{pmatrix}$ and $A_o=\begin{pmatrix} 0 & a_{12}\\ a_{21} & 0\end{pmatrix}$. Consider the asymptotic expansion of $\mu_\pm(x,t,z)$

$$\mu_\pm(x,t,z)=\sum_{n=0}^{\infty}\mu_\pm^{(n)}(x,t,z),\tag{2.9}$$

where

$$\mu_\pm^{(0)}(x,t,z)=\Omega_\pm(z)=I-\frac{1}{z}\sigma_3Q_\pm,\quad \mu_\pm^{(n+1)}(x,t,z)=\int_{-\infty}^{x}\Omega_\pm(z)e^{ik(x-y)\hat{\sigma}_3}\Omega_\pm^{-1}(z)\triangle Q_\pm(y,t)\mu_\pm^{(n)}(y,t,z)\mathrm{d}y.$$

So

$$\begin{aligned}\mu_{\pm,d}^{(n+1)}&=\int_{-\infty}^{x}\left(\Omega_{\pm,o}^{-1}\triangle Q_\pm\mu_{\pm,d}^{(n)}+\Omega_{\pm,d}^{-1}\triangle Q_\pm\mu_{\pm,o}^{(n)}\right)\mathrm{d}y\\&\quad+\int_{-\infty}^{x}\Omega_{\pm,o}e^{ik(x-y)\sigma_3}\left(\Omega_{\pm,d}^{-1}\triangle Q_\pm\mu_{\pm,d}^{(n)}+\Omega_{\pm,o}^{-1}\triangle Q_\pm\mu_{\pm,o}^{(n)}\right)e^{-ik(x-y)\sigma_3}\mathrm{d}y,\\ \mu_{\pm,o}^{(n+1)}&=\int_{-\infty}^{x}\left(\Omega_{\pm,o}\Omega_{\pm,o}^{-1}\triangle Q_\pm\mu_{\pm,d}^{(n)}+\Omega_{\pm,o}\Omega_{\pm,d}^{-1}\triangle Q_\pm\mu_{\pm,o}^{(n)}\right)\mathrm{d}y\\&\quad+\int_{-\infty}^{x}e^{ik(x-y)\sigma_3}\left(\Omega_{\pm,d}^{-1}\triangle Q_\pm\mu_{\pm,d}^{(n)}+\Omega_{\pm,o}^{-1}\triangle Q_\pm\mu_{\pm,o}^{(n)}\right)e^{-ik(x-y)\sigma_3}\mathrm{d}y.\end{aligned}$$

If $z\to\pm\infty$, we note the fact that $k=\frac{1}{2}(z+\frac{u_0^2}{z})$ and $\frac{1}{\omega}=1+\frac{u_0^2}{z^2}+\cdots$. And taking the integration by part for the last two terms of $\mu_{\pm,d}^{(1)}$ and $\mu_{\pm,o}^{(1)}$, one can derive the following results

$$\begin{aligned}\mu_{\pm,d}^{(1)}&=O(\frac{\mu_{\pm,d}^{(0)}}{z})+O(\mu_{\pm,o}^{(0)})+O(\frac{\mu_{\pm,d}^{(0)}}{z^2})+O(\frac{\mu_{\pm,o}^{(0)}}{z^3})=O(\frac{1}{z}),\\ \mu_{\pm,o}^{(1)}&=O(\frac{\mu_{\pm,d}^{(0)}}{z^2})+O(\frac{\mu_{\pm,o}^{(0)}}{z})+O(\frac{\mu_{\pm,d}^{(0)}}{z})+O(\frac{\mu_{\pm,o}^{(0)}}{z^2})=O(\frac{1}{z}).\end{aligned}$$

Iterating the same method, one can obtain the asymptotic

$$\mu_{\pm,d}^{(2n)}=O(\frac{1}{z^n}),\qquad \mu_{\pm,o}^{(2n)}=O(\frac{1}{z^{n+1}}),\qquad \mu_{\pm,d}^{(2n+1)}=O(\frac{1}{z^{n+1}}),\qquad \mu_{\pm,o}^{(2n+1)}=O(\frac{1}{z^{n+1}}).$$

For $z \to 0$, one can obtain the asymptotic in the similar way

$$\mu_{\pm,d}^{(2n)} = O(z^n), \qquad \mu_{\pm,o}^{(2n)} = O(z^{n-1}), \qquad \mu_{\pm,d}^{(2n+1)} = O(z^n), \qquad \mu_{\pm,o}^{(2n+1)} = O(z^n).$$

So we obtain the following asymptotic by some calculations for Eq. (2.9)

$$\mu_\pm(x,t,z) = I + O(\frac{1}{z}), \quad z \to \pm\infty, \qquad \mu_\pm(x,t,z) = -\frac{1}{z}\sigma_3 Q_\pm + O(1), \quad z \to 0. \tag{2.10}$$

And one can derive the solution of QNLS equation $q(x,t) = \lim_{z\to\infty} z(\mu_\pm)_{21}$. By substituting the asymptotic of $\mu_\pm$ into (2.6), one can obtain the asymptotic behaviors of the scattering matrix $S(z)$

$$S(z) = I + O(\frac{1}{z}), \quad z \to \pm\infty, \qquad S(z) = \frac{q_+}{q_-} I + O(z), \quad z \to 0. \tag{2.11}$$

3. The Riemann-Hilbert problem

Based on the analyticity and asymptotic of eigenfunctions $\mu_\pm$ and $S(z)$, the solutions can be derived by RH problem. Firstly, we define a sectionally meromorphic matrix $M(x,t,z)$:

$$M^+(x,t,z)|_{z\in D_+} = \begin{pmatrix} \frac{\mu_{+,1}(x,t,z)}{a(z)} & \mu_{-,2}(x,t,z) \end{pmatrix}, \quad M^-(x,t,z)|_{z\in D_-} = \begin{pmatrix} \mu_{-,1}(x,t,z) & \frac{\mu_{+,2}(x,t,z)}{\overline{a(\overline{z})}} \end{pmatrix}.$$

Proposition 3.1. *The sectionally meromorphic matrix $M(x,t,z)$ satisfies the following RH problem:*

- *Analyticity: $M(x,t,z)$ is meromorphic in $D_+ \cup D_-$.*
- *Jump condition: $M^+(x,t,z) = M^-(x,t,z)\begin{pmatrix} 1-\rho\tilde{\rho} & -e^{2i\theta}\tilde{\rho} \\ e^{-2i\theta}\rho & 1 \end{pmatrix}, \quad z \in \Sigma.$*
- *Asymptotic behavior: $M(x,t,z) = I + O(\frac{1}{z}),\ z \to \infty, \quad M(x,t,z) = -\frac{1}{z}\sigma_3 Q_- + O(1),\ z \to 0.$*

According to the asymptotic of $\mu_\pm$ and $S(z)$, the solution of QNLS can be rewritten as $q(x,t) = \lim_{z\to\infty} z(M(x,t,z))_{21}$. For $a(z)$ with N high order zeros, not only the residue conditions are useful but the coefficients of negative power should be considered. However, these coefficients are not straightforward derived. Moreover, if z_n is the N-order zero point of $a(z)$, so is $-\frac{q_0^2}{\overline{z}_n}$. This situation can be equivalent to the zero points of $a(z)$ always paired in D_+.

We firstly consider single-high-order pole solutions. We assume $a(z)$ with one N-order zero $z_0 \in \{D_+ \cap \mathrm{Im} z_0 > 0\}$. According to the symmetries of $S(z)$, one can derive $-\frac{q_0^2}{\overline{z}_0}$ also is the zero point of $a(z)$. For convenience, we make the following notations $\{\nu_1 \doteq z_0, \nu_2 \doteq -\frac{q_0^2}{\overline{z}_0}, \overline{\nu}_1 \doteq \overline{z}_0, \overline{\nu}_2 \doteq -\frac{q_0^2}{z_0}\}$, where $\nu_1, \nu_2 \in D_+$, and $\overline{\nu}_1, \overline{\nu}_2 \in D_-$. So $a(z)$ can be expanded as the Taylor series

$$a(z) = a_0(z)\prod_{j=1}^{2}(z-\nu_j)^N, \quad j = 1,2,$$

and $a_0(z) \neq 0$ for all $z \in D_+$. Reflection coefficients $\rho(z)$ and $\overline{\rho(\overline{z})}$ with the Laurent expansion are

$$\rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{N}\frac{\rho_{j,m_j}}{(z-\nu_j)^{m_j}}, \qquad \overline{\rho(\overline{z})} = \overline{\rho_{j,0}(\overline{z})} + \sum_{m_j=1}^{N}\frac{\overline{\rho}_{j,m_j}}{(z-\overline{\nu}_j)^{m_j}}, \quad j = 1,2,$$

where $\rho_{j,m_j} = \lim_{z\to\nu_j}\frac{1}{(N-m_j)!}\frac{\partial^{N-m_j}}{\partial z^{N-m_j}}[(z-\nu_j)^N\rho(z)]$, $(m_j = 1,2,\ldots,N)$, $\rho_{j,0}(z)$ is analytic for all $z \in D_+$. According to the definition of $M(x,t,z)$, one can obtain that $z = \nu_j$ are the N-order poles of

$M_1(x,t,z)$ and $z=\overline{\nu}_j$ are the N-order poles of $M_2(x,t,z)$. $M_2(x,t,z)$ analytic as $z=\nu_j$ and $M_1(x,t,z)$ analytic as $z=\overline{\nu}_j$. So we have the following expand

$$M_{21}(z)=\frac{q_-}{z}+\sum_{j=1}^{2}\sum_{p=1}^{N}\frac{G_{j,p}(x,t)}{(z-\nu_j)^p},\quad M_{22}(z)=1+\sum_{j=1}^{2}\sum_{p=1}^{N}\frac{F_{j,p}(x,t)}{(z-\overline{\nu}_j)^p}. \tag{3.1}$$

$G_{j,p}(x,t)$ and $F_{j,p}(x,t)$ are undetermined. According to the analyticity one can get the Taylor expansion

$$e^{-2i\theta(z)}=\sum_{s_j=0}^{+\infty}f_{j,s_j}(x,t)(z-\nu_j)^{s_j},\quad M_{21}(z)=\sum_{s_j=0}^{+\infty}\zeta_{j,s_j}(x,t)(z-\overline{\nu}_j)^{s_j},\quad M_{22}(z)=\sum_{s_j=0}^{+\infty}\mu_{j,s_j}(x,t)(z-\nu_j)^{s_j},$$

where

$$f_{j,s_j}(x,t)=\lim_{z\to\nu_j}\frac{1}{s_j!}\frac{\partial^{s_j}}{\partial z^{s_j}}e^{-2i\theta(z)},\quad \mu_{j,s_j}(x,t)=\lim_{z\to\nu_j}\frac{1}{s_j!}\frac{\partial^{s_j}}{\partial z^{s_j}}M_{22}(z),\quad \zeta_{j,s_j}(x,t)=\lim_{z\to\overline{\nu}_j}\frac{1}{s_j!}\frac{\partial^{s_j}}{\partial z^{s_j}}M_{21}(z).$$

The Taylor expand of $e^{2i\theta(z)}$ also can be obtained. By considering Eq. (2.6) and the definition of $M(x,t,z)$, comparing the corresponding coefficients of $(z-\nu_j)^p$ and $(z-\overline{\nu}_j)^{-p}$, one can derive $G_{j,p}(x,t)$ and $F_{j,p}(x,t)$ as the form

$$F_{j,p}(x,t)=-\sum_{m_j=p}^{N}\sum_{s_j=0}^{m_j-p}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)\zeta_{j,s_j}(x,t),$$
$$G_{j,p}(x,t)=\sum_{m_j=p}^{N}\sum_{s_j=0}^{m_j-p}\rho_{j,m_j}f_{j,m_j-p-s_j}(x,t)\mu_{j,s_j}(x,t),$$

where $p=1,2,\ldots,N$. For $N=1$, $F_{j,p}(x,t)$ and $G_{j,p}(x,t)$ degenerate into the residue conditions. In addition, $\mu_{j,s_j}(x,t)$ and $\zeta_{j,s_j}(x,t)$ can be expressed as $F_{j,p}(x,t)$ and $G_{j,p}(x,t)$ via direct calculation

$$\zeta_{j,s_j}(x,t)=\frac{(-1)^{s_j}q_-}{\overline{\nu}_j^{s_j+1}}+\sum_{p=1}^{N}\begin{pmatrix}p+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}G_{j,p}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+p}},\quad s_j=0,1,2,\ldots$$
$$\mu_{j,s_j}(x,t)=\begin{cases}1+\displaystyle\sum_{p=1}^{N}\frac{F_{j,p}(x,t)}{(\nu_j-\overline{\nu}_l)^p}, & s_j=0,\\ \displaystyle\sum_{p=1}^{N}\begin{pmatrix}p+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}F_{j,p}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+p}}, & s_j=1,2,\ldots.\end{cases}$$

Then one can obtain the following system

$$\begin{aligned}F_{j,p}(x,t)=&-\sum_{m_j=p}^{N}\sum_{s_j=0}^{m_j-p}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)q_-}{\overline{\nu}_j^{s_j+1}}\\&-\sum_{m_j=p}^{N}\sum_{s_j=0}^{m_j-p}\sum_{q=1}^{N}\begin{pmatrix}q+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)G_{j,q}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+q}},\\G_{j,p}(x,t)=&\sum_{m_j=p}^{N}\rho_{j,m_j}f_{j,m_j-p}(x,t)\\&+\sum_{m_j=p}^{N}\sum_{s_j=0}^{m_j-p}\sum_{q=1}^{N}\begin{pmatrix}q+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}\rho_{j,m_j}f_{j,m_j-p-s_j}(x,t)F_{j,q}(x,t)}{(\nu_j-\overline{\nu}_l)^{s_j+q}}.\end{aligned}\tag{3.3}$$

For convenience, we introduce the following symbols $(s, p = 1, 2, \ldots, N)$:

$$|F\rangle = \begin{pmatrix} F_1 & F_2 & \cdots & F_N \end{pmatrix}^T, \quad |G\rangle = \begin{pmatrix} G_1 & G_2 & \cdots & G_N \end{pmatrix}^T, \quad |\beta\rangle = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}^T,$$
$$\beta_j = \begin{pmatrix} \beta_{j,1} & \beta_{j,2} & \cdots & \beta_{j,N} \end{pmatrix}, \quad |\eta\rangle = \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix}^T, \quad \eta_j = \begin{pmatrix} \eta_{j,1} & \eta_{j,2} & \cdots & \eta_{j,N} \end{pmatrix},$$
$$\beta_{j,l} = -\sum_{m_j=l}^{N} \sum_{s_j=0}^{m_j-l} \frac{(-1)^{s_j} \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-l-s_j}(x,t) q_-}{\overline{\nu}_j^{s_j+1}}, \quad \eta_{j,l}(x,t) = \sum_{m_j=l}^{N} \rho_{j,m_j} f_{j,m_j-l}(x,t),$$
$$[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{N\times N} = \left(-\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \begin{pmatrix} q+s_j-1 \\ s_j \end{pmatrix} \frac{(-1)^{s_j} \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-p-s_j}(x,t)}{(\overline{\nu}_j - \nu_l)^{s_j+q}} \right)_{N\times N}.$$

and $\chi = \begin{pmatrix} [\varpi_{11}] & [\varpi_{12}], \\ [\varpi_{21}] & [\varpi_{22}] \end{pmatrix}$. So Eqs. (3.3) can be rewritten as

$$|F\rangle = |\beta\rangle + \chi |G\rangle, \qquad |G\rangle = |\eta\rangle - \overline{\chi} |F\rangle. \tag{3.5}$$

$|F\rangle$ and $|G\rangle$ are solved as

$$|F\rangle = \chi (I + \overline{\chi}\chi)^{-1} |\eta\rangle + (I - \chi (I + \overline{\chi}\chi)^{-1} \overline{\chi}) |\beta\rangle, \qquad |G\rangle = (I + \overline{\chi}\chi)^{-1} (|\eta\rangle - \overline{\chi}|\beta\rangle), \tag{3.6}$$

So the expansions of $M_{21}(x,t,\lambda)$ can be given as

$$M_{21}(x,t,z) = \frac{q_-}{z} + \sum_{j=1}^{2} \sum_{s=1}^{N} \frac{G_{j,s}(x,t)}{(z-\nu_j)^s} = \frac{q_-}{z} + \frac{\det\left[\mathrm{I} + \overline{\chi}\chi + (|\eta\rangle - \overline{\chi}|\beta\rangle)\langle \overline{Y(\overline{\lambda})}|\right]}{\det\left[\mathrm{I} + \overline{\chi}\chi\right]} - 1. \tag{3.7}$$

where

$$\langle Y(z)| = \begin{pmatrix} \frac{1}{z-\overline{\nu}_1} & \frac{1}{(z-\overline{\nu}_1)^2} & \cdots & \frac{1}{(z-\overline{\nu}_1)^N} & \frac{1}{z-\overline{\nu}_2} & \frac{1}{(z-\overline{\nu}_2)^2} & \cdots & \frac{1}{(z-\overline{\nu}_2)^N} \end{pmatrix}.$$

One can derive the solutions of equation QNLS with one N-order pole

$$q(x,t) = q_- + \left(\frac{\det\left(\mathrm{I} + \overline{\chi}\chi + (|\eta\rangle - \overline{\chi}|\beta\rangle)\langle Y_0|\right)}{\det\left(\mathrm{I} + \overline{\chi}\chi\right)} - 1 \right), \tag{3.8}$$

where

$$|\eta\rangle = \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix}^T, \quad |\beta\rangle = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}^T, \quad \langle Y_0| = \begin{pmatrix} Y_1^0 & Y_2^0 \end{pmatrix}, \quad \eta_j = \begin{pmatrix} \eta_{j,1} & \eta_{j,2} & \cdots & \eta_{j,N} \end{pmatrix},$$
$$\eta_{j,l} = \sum_{m_j=l}^{N} \rho_{j,m_j} f_{j,m_j-l}(x,t), \quad \beta_j = \begin{pmatrix} \beta_{j,1} & \beta_{j,2} & \cdots & \beta_{j,N} \end{pmatrix}, \quad Y_j^0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}_{1\times N},$$
$$\beta_{j,l} = -\sum_{m_j=l}^{N} \sum_{s_j=0}^{m_j-l} \frac{(-1)^{s_j} \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-l-s_j}(x,t) q_-}{\overline{\nu}_j^{s_j+1}}, \quad \chi = \begin{pmatrix} [\varpi_{11}] & [\varpi_{12}] \\ [\varpi_{21}] & [\varpi_{22}] \end{pmatrix},$$

and $[\varpi_{j,l}] (j,l = 1,2)$ are $N \times N$ matrix

$$[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{N\times N} = \left(-\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \begin{pmatrix} q+s_j-1 \\ s_j \end{pmatrix} \frac{(-1)^{s_j} \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-p-s_j}(x,t)}{(\overline{\nu}_j - \nu_l)^{s_j+q}} \right)_{N\times N}.$$

Next we consider the multiple-high-order pole solutions. $a(z)$ with N high-order zeros $z_1, z_2, \ldots, z_N$, and the powers are $n_1, n_2, \ldots, n_N$, respectively. Then $-\frac{q_0^2}{\overline{z}_1}, -\frac{q_0^2}{\overline{z}_2}, \ldots, -\frac{q_0^2}{\overline{z}_N}$ also are the zeros of $a(z)$. And the powers are $n_1, n_2, \ldots, n_N$, too. We also make the following notations $\{\nu_j \doteq z_i, \nu_j|_{j=N+i} \doteq -\frac{q_0^2}{\overline{z}_i}, \overline{\nu}_j \doteq \overline{z}_i, \overline{\nu}_j|_{j=N+i} \doteq -\frac{q_0^2}{z_i}\}_{i=1}^{N}$. So $a(z)$ can be expanded as

$$a(z) = a_0(z)(z-\nu_1)^{n_1} \cdots (z-\nu_N)^{n_N} (z-\nu_{N+1})^{n_1} \cdots (z-\nu_{2N})^{n_N},$$

Similar to the one high-order pole, $\rho(z)$ can be expanded as the Laurent series

$$\rho(z)=\rho_{j,0}(z)+\sum_{m_j=1}^{n_j}\frac{\rho_{j,m_j}}{(z-\nu_j)^{m_j}},\qquad \overline{\rho(\overline{z})}=\overline{\rho_{j,0}(\overline{z})}+\sum_{m_j=1}^{n_j}\frac{\overline{\rho}_{j,m_j}}{(z-\overline{\nu}_j)^{m_j}},$$

where $\rho_{j,m_j}=\lim_{z\to\nu_j}\frac{1}{(n_j-m_j)!}\frac{\partial^{n_j-m_j}}{\partial(z-\nu_j)^{n_j-m_j}}\left[(z-\nu_j)^{n_j}\rho(z)\right]$, and $\rho_{j,0}(z)$ is analytic for all $z\in D_+$ and $j=1,\ldots,2N$. So, in the similar method, one can derive the soliton solution formula with multiple-high-order poles:

Theorem 3.1. *For the NVBC, if $a(z)$ with multiple-high-order zeros, then the soliton solutions of Eq. (1.1) are*

$$q(x,t)=q_-+\left(\frac{\det\left(\mathrm{I}+\overline{\chi}\chi+(|\eta\rangle-\overline{\chi}|\beta\rangle)\langle \mathrm{Y}_0|\right)}{\det\left(\mathrm{I}+\overline{\chi}\chi\right)}-1\right),\tag{3.10}$$

where

$$|\eta\rangle=\begin{pmatrix}\eta_1&\eta_2&\cdots&\eta_{2N}\end{pmatrix}^T,\quad |\beta\rangle=\begin{pmatrix}\beta_1&\beta_2&\cdots&\beta_{2N}\end{pmatrix}^T,\quad \langle \mathrm{Y}_0|=\begin{pmatrix}Y_1^0&Y_2^0&\cdots&Y_{2N}^0\end{pmatrix},$$
$$Y_j^0=\begin{pmatrix}1&0&\cdots&0\end{pmatrix}_{1\times n_j},\quad \eta_j=\begin{pmatrix}\eta_{j,1}&\eta_{j,2}&\cdots&\eta_{j,n_j}\end{pmatrix},\quad \beta_j=\begin{pmatrix}\beta_{j,1}&\beta_{j,2}&\cdots&\beta_{j,n_j}\end{pmatrix},$$
$$\eta_{j,l}=\sum_{m_j=l}^{n_j}\rho_{j,m_j}f_{j,m_j-l}(x,t),\quad \beta_{j,l}=-\sum_{m_j=l}^{n_j}\sum_{s_j=0}^{m_j-l}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-l-s_j}(x,t)q_-}{\overline{\nu}_j^{s_j+1}},$$
$$\chi=\begin{pmatrix}[\varpi_{11}]&[\varpi_{12}]&\cdots&[\varpi_{1(2N)}]\\ [\varpi_{21}]&[\varpi_{22}]&\cdots&[\varpi_{2(2N)}]\\ \vdots&\vdots&\cdots&\vdots\\ [\varpi_{(2N)1}]&[\varpi_{(2N)2}]&\cdots&[\varpi_{(2N)(2N)}]\end{pmatrix},$$

and $[\varpi_{j,l}](j,l=1,2,\ldots,2N)$ are $n_j\times n_l$ matrix

$$[\varpi_{j,l}]=([\varpi_{j,l}]_{p,q})_{n_j\times n_l}=\left(-\sum_{m_j=p}^{n_j}\sum_{s_j=0}^{m_j-p}\begin{pmatrix}q+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+q}}\right)_{n_j\times n_l}.$$

Double 1nd − order pole solution

Let ν_1 and ν_2 be the 1-order zeros point of $a(z)$, so is ν_3 and ν_4. The reflection coefficient $\rho(z)$ can be expanded as Laurent series $\rho(z)=\rho_{j,0}(z)+\frac{\rho_{j,1}}{z-\nu_j}$. Now χ in (3.10) is defined as

$$\chi=\begin{pmatrix}[\varpi_{11}]&[\varpi_{12}]&[\varpi_{13}]&[\varpi_{14}]\\ [\varpi_{21}]&[\varpi_{22}]&[\varpi_{23}]&[\varpi_{24}]\\ [\varpi_{31}]&[\varpi_{32}]&[\varpi_{33}]&[\varpi_{34}]\\ [\varpi_{41}]&[\varpi_{42}]&[\varpi_{43}]&[\varpi_{44}]\end{pmatrix}_{4\times4},$$

where $n_1=n_2=n_2=n_4=1,\quad j,l=1,2,3,4$

$$|\eta\rangle=\begin{pmatrix}\eta_1&\eta_2&\eta_3&\eta_4\end{pmatrix}^T,\quad |\beta\rangle=\begin{pmatrix}\beta_1&\beta_2&\beta_3&\beta_4\end{pmatrix}^T,\quad \langle \mathrm{Y}_0|=\begin{pmatrix}1&1&1&1\end{pmatrix},$$
$$\beta_{j,l}=-\sum_{m_j=l}^{n_j}\sum_{s_j=0}^{m_j-l}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-l-s_j}(x,t)q_-}{\overline{\nu}_j^{s_j+1}},\quad \eta_{j,l}=\sum_{m_j=l}^{n_j}\rho_{j,m_j}f_{j,m_j-l}(x,t).$$
$$[\varpi_{j,l}]=[\varpi_{j,l}]_{p,q}=-\sum_{m_j=p}^{n_j}\sum_{s_j=0}^{m_j-p}\begin{pmatrix}q+s_j-1\\ s_j\end{pmatrix}\frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+q}},$$

Mixed 2nd − and 1nd − order pole solution

Let ν_1 and ν_3 be 2-order zero point of $a(z)$, ν_2 and ν_4 be 1-order zeros of $a(z)$. The reflection coefficient $\rho(z)$ can be expanded as Laurent series $\rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(z-\nu_j)^{m_j}}$. Now χ in (3.10) is defined as

$$\chi = \begin{pmatrix} [\varpi_{11}] & [\varpi_{12}] & [\varpi_{13}] & [\varpi_{14}] \\ [\varpi_{21}] & [\varpi_{22}] & [\varpi_{23}] & [\varpi_{24}] \\ [\varpi_{31}] & [\varpi_{32}] & [\varpi_{33}] & [\varpi_{34}] \\ [\varpi_{41}] & [\varpi_{42}] & [\varpi_{43}] & [\varpi_{44}] \end{pmatrix}_{6\times 6},$$

where $n_1 = n_3 = 2, \quad n_2 = n_4 = 1, \quad j, l = 1, 2, 3, 4$

$$|\eta\rangle = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{pmatrix}^T, \quad |\beta\rangle = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}^T, \quad \langle Y_0| = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$\beta_{j,l} = -\sum_{m_j=l}^{n_j} \sum_{s_j=0}^{m_j-l} \frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-l-s_j}(x,t)q_-}{\overline{\nu}_j^{s_j+1}}, \quad \eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t),$$

$$[\varpi_{j,l}] = [\varpi_{j,l}]_{p,q} = -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \begin{pmatrix} q+s_j-1 \\ s_j \end{pmatrix} \frac{(-1)^{s_j}\overline{\rho}_{j,m_j}\overline{f}_{j,m_j-p-s_j}(x,t)}{(\overline{\nu}_j-\nu_l)^{s_j+q}}.$$

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