

RIEMANN-HILBERT APPROACH AND DOUBLE-POLE SOLUTIONS FOR THE THIRD-ORDER FLOW EQUATION OF THE KAUP-NEWELL SYSTEM WITH ZERO/NONZERO BOUNDARY CONDITIONS

JUNCAI PU AND YONG CHEN*

ABSTRACT. The Kaup-Newell (KN) system is a significant integrable system, which reveals many physical phenomena and natural laws. Especially, since the solutions of higher-order KN systems are more complex and various than that of second-order flow KN systems which contain the derivative nonlinear Schrödinger equation (DNLS), Chen-Lee-Liu equation, Gerdjikov-Ivanov equation and so on, the research on it will become more and more important. In this paper, we investigate the third-order flow equation of Kaup-Newell system (TOFKN) with zero boundary conditions (ZBCs) and non-zero boundary conditions (NZBCs) by using the Riemann-Hilbert (RH) approach based on the inverse scattering transformation. Starting from Lax pairs of the TOFKN equation, the direct scattering problem constructs the analyticity, symmetries, asymptotic behaviors and discrete spectral properties for the Jost solutions, modified Jost solutions and scattering matrix. The inverse scattering problems are solved by establishing the matrix RH problem, and the reconstruction formula of potential, trace formula and theta condition are also derived correspondingly. The reflectionless potentials with double poles are exhibited explicitly by means of determinants. Specifically, vivid plots and dynamics analysis which contain double-pole soliton solutions for the ZBCs and double-pole dark-bright soliton solution, breather-breather solutions and breather-breather-dark-bright interaction solutions for the NZBCs, are demonstrated in details. Furthermore, compared with the most classical second-order flow KN system which also be called the DNLS equation, the third-order dispersion and quintic nonlinear term of the KN system change the trajectory and velocity of solutions.

1. INTRODUCTION

For decades, nonlinear integrable systems, which are used to describe complex natural phenomena in the real world, have always been the research hotspots in the field of mathematics and physics [1,2]. Recently, since abundant solutions for nonlinear integrable systems are one of the most significant characteristics to reveal complex natural phenomena, the study of the integrability and exact solutions for nonlinear integrable systems have been paid more and more attention in optical fiber, fluid dynamics, plasma physics, machine learning and others fields [3–7]. General speaking, it is difficult to find the exact solutions of complex nonlinear integrable systems, with the development of soliton theory in recent decades, some effective methods for solving nonlinear integrable systems have been established, such as inverse scattering transformation [8], Hirota bilinear method [9], Darboux transformation [10], Bäcklund transformation [11], symmetry reduction [12] and deep learning method [7].

Among these methods mentioned above, inverse scattering transformation (IST) is one of the most important and basic theories for solving nonlinear integrable systems. In 1967, Gardner, Greene, Kruskal and Miura (GGKM) discovered the classical IST to solve the initial value problem for the Korteweg-de Vries (KdV) equation with lax pairs [8]. After that, many researchers try to extend this method to other nonlinear integrable systems which possess so-called lax pairs [1]. Zakharov and Shabat studied the IST of nonlinear Schrödinger (NLS) equation in 1972 [13]. Later, Ablowitz, Kaup, Newell and Segur (AKNS) proposed a new class of integrable systems, called AKNS systems, and established a general framework for their ISTs in 1973 [14,15]. Subsequently, many classical nonlinear integrable systems are proved to be solvable by IST [16–19]. However, it is found that the solving process of the classical IST for nonlinear integrable systems with second order spectral problems which was based on the Gel'fand-Levitan-Marchenko integral equations is complicated and tedious, and it is difficult to solve the original IST for the higher order spectral problems of nonlinear integrable systems without the Gel'fand-Levitan-Marchenko theory [20]. Later on, a Riemann-Hilbert (RH) approach was developed which streamlines and simplifies the IST considerably by Zakharov and his collaborators [21]. Since 1980s, the RH approach has been applied to nonlinear integrable systems as a more general method than the classical IST. In 1993, Deift and Zhou presented a new and general steepest descent approach to analyzing the asymptotics of oscillatory RH problems, and such problems will arise in the process of evaluating the long-time behavior of nonlinear wave equations solvable by the RH method [22]. Recently, the RH approach has become a powerful tool for constructing IST to obtain abundant solutions of nonlinear integrable systems and dealing with the long time asymptotic behavior of solutions by analysing RH problem. The multisoliton solutions of some important nonlinear integrable systems can be derived by solving a particular RH problem under the reflectionless cases [23–29]. On the other hand, the RH approach has also been utilized to study the initial boundary value problems and the long-time asymptotic behavior for many nonlinear integrable systems [22,30–33].

The Kaup-Newell (KN) systems, which are more complex than AKNS system, are of great significance in mathematical physics [34]. The derivative nonlinear Schrödinger equation (DNLS) is the most classical KN system, which was first derived from Alfvén wave propagation in plasma by Mio et al. in 1976, and well described the propagation

of small amplitude nonlinear Alfvén wave in low plasma [36]. In Ref. [34], Kaup and Newell obtained the one-soliton solution and the infinity of conservation laws for the DNLS via using inverse scattering technique. Furthermore, other important KN systems have been widely studied by means of IST and RH approach, such as the Chen-Lee-Liu equation [37], the Kundu-type equation [38] and the Gerdjikov-Ivanov equation [39]. However, as far as we know, these IST works on these nonlinear integrable systems with ZBCs/NZBCs mainly focus on the case that all discrete spectra are simple. A new research topic is to obtain explicit double-pole solutions for nonlinear integrable systems with ZBCs/NZBCs by utilizing RH method based on IST, and there are few related works on this aspect at present [40–42]. It is worth mentioning that Pichler presented IST for the focusing nonlinear Schrödinger equation with non-zero boundary conditions at infinity and double zeros of the analytic scattering coefficients [43]. In addition, the IST and double-pole solutions for some important nonlinear integrable systems have been investigated [40, 42]. Recently, Zhang and Yan proposed a rigorous theory of the ISTs for the DNLS equation with both ZBCs and NZBCs at infinity and double zeros of analytical scattering coefficients, provided a significant foundation for the IST for KN systems [44]. In this paper, we consider the third-order flow equation of the KN system (TOFKN equation) which contains the third-order dispersion and quintic nonlinear terms. In 1999, Imai firstly proposed the coupled TOFKN equation [45] and pointed out that it is an integrable system, the coupled TOFKN equation as shown below

$$\begin{aligned} q_t &= \frac{a_6}{4\varepsilon^2} \left(q_{xxx} - \frac{3(qrq_x)_x}{\varepsilon} + \frac{3(q^3r^2)_x}{2\varepsilon^2} \right), \\ r_t &= \frac{a_6}{4\varepsilon^2} \left(r_{xxx} + \frac{3(qrr_x)_x}{\varepsilon} + \frac{3(q^2r^3)_x}{2\varepsilon^2} \right), \end{aligned}$$

where $q = q(x, t)$, $r = r(x, t)$.

By imposing the condition $r(x, t) = -q(x, t)^*$ (superscript “*” denotes complex conjugation) and taking appropriate parameters $\varepsilon = i$, $a_6 = 4$ to the coupled TOFKN equation, the coupled TOFKN equation is reduced and obtained the general form of the TOFKN equation:

$$q_t + q_{xxx} - 3i(|q|^2 q_x)_x - \frac{3}{2}(|q|^4 q)_x = 0, \quad (1.1)$$

where $|q|^2 = q(x, t)q^*(x, t)$. To the best of our knowledge, there are few studies on Eq. (1.1), and the inverse scattering transformation for equation (1.1) has not been investigated by utilizing RH method. Lin et al. studied this equation and derived different types of solutions which contain solitons, positons, breathers and rogue waves by using Darboux transformation and extended Darboux transformation for the KN systems [46]. The TOFKN equation is completely integrable and associated with the following modified Zakharov-Shabat eigenvalue problem (Lax pairs) [34]:

$$\Psi_x = X\Psi, \quad (1.2)$$

$$\Psi_t = T\Psi, \quad (1.3)$$

where

$$X = X(x, t, \lambda) = \lambda(i\lambda\sigma_3 + Q),$$

$$T = T(x, t, \lambda) = \left(4\lambda^4 + \frac{3}{2}|q|^4 - 2\lambda^2|q|^2 \right) X + (-2i\lambda^3 + 3i\lambda|q|^2) \sigma_3 Q_x - \lambda^2(-qq_x^* + q_x q^*) \sigma_3 - \lambda Q_{xx},$$

it is easy to check that Eq. (1.1) as the integrability condition (or zero-curvature condition) $\Psi_{xt} = \Psi_{tx}$ (or $X_t - T_x + [X, T] = 0$) of system Eqs. (1.2)-(1.3), $\Psi = \Psi(\lambda; x, t)$ is a 2×2 matrix-valued eigenfunction, $\lambda \in \mathbb{C}$, the potential matrix $Q = Q(x, t)$ is written as

$$Q = \begin{pmatrix} 0 & q(x, t) \\ -q^*(x, t) & 0 \end{pmatrix}, \quad (1.4)$$

the σ_3 is one of the Pauli's spin matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.5)$$

This paper is organized as follows. In Section 2, the IST for the TOFKN equation (1.1) with ZBC at infinity is introduced and solved for the double zeros of analytically scattering coefficients by means of the matrix RH problem. The determinant form of explicit N-double-pole solutions have been presented, and the relative plots at $N = 1, 2$ have been given out in detail. Furthermore, compared with the DNLS in the case of $N = 1$, and the correlation analyses are illustrated in detail. In Section 3, the IST for the TOFKN equation (1.1) with NZBC at infinity is introduced and solved for the double zeros of analytically scattering coefficients. The IST for NZBC at infinity is more complicated than the case of ZBCs since more symmetries and multivalued functions, thus we map the original spectral parameter λ into single-valued parameter z by introducing an appropriate two-sheeted Riemann surface. As a result, we present an explicit expression for the N-double-pole solutions for the case of NZBCs by using determinants, and exhibit vivid

plots of three different types of double-pole solutions in detail. Moreover, some comparison made between the double-pole dark-bright soliton solution of the TOFKN equation and the DNLS equation, and the correlation analyses are explained detailly. Summary and conclusion are given out in last section.

2. THE IST WITH ZBCS AND DOUBLE POLES SOLUTION

In this section, we will seek the N-double-pole solution $q(x, t)$ for the TOFKN equation (1.1) with ZBCs at infinity as follow

$$q(x, t) \sim 0, \text{ as } x \rightarrow \pm\infty. \quad (2.1)$$

In the following subsection, we will present the IST which contain the direct scattering and the inverse problem for Eq. (1.1) with ZBCs by RH method respectively.

2.1. The Direct Scattering with ZBCs.

2.1.1. Jost Solution, Analyticity and Continuity.

Considering the asymptotic scattering spectrum problem ($x \rightarrow \infty$) of the modified Zakharov-Shabat eigenvalue problem

$$\Psi_x = X_0 \Psi, \quad (2.2)$$

$$\Psi_t = T_0 \Psi, \quad (2.3)$$

where $X_0 = i\lambda^2\sigma_3$ and $T_0 = 4\lambda^4X_0 = 4i\lambda^6\sigma_3$, one can obtain the fundamental matrix solution $\Psi^{\text{bg}}(\lambda; x, t)$ of Eqs. (2.2)-(2.3)

$$\Psi^{\text{bg}}(\lambda; x, t) = e^{i\theta(\lambda; x, t)\sigma_3}, \quad \theta(\lambda; x, t) = \lambda^2(x + 4\lambda^4t).$$

Let $\Sigma := \mathbb{R} \cup i\mathbb{R}$. Then, the Jost solutions $\psi_{\pm}(\lambda; x, t)$ can be derived as below

$$\psi_{\pm}(\lambda; x, t) \sim e^{i\theta(\lambda; x, t)\sigma_3}, \quad \lambda \in \Sigma, \text{ as } x \rightarrow \pm\infty. \quad (2.4)$$

In order to obtain the modified Jost solution $\mu_{\pm}(\lambda; x, t)$, we consider a transformation of the form

$$\mu_{\pm}(\lambda; x, t) = \psi_{\pm}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3}, \quad (2.5)$$

it is evident that

$$\mu_{\pm}(\lambda; x, t) \sim I, \text{ as } x \rightarrow \pm\infty,$$

where I is 2×2 identity matrix. According to the modified Zakharov-Shabat eigenvalue problem Eqs. (1.2)-(1.3) and transformation Eq. (2.5), one can obtain $\mu_{\pm}(\lambda; x, t)$ satisfy the following equivalent Lax pair:

$$\mu_{\pm, x}(\lambda; x, t) + i\lambda^2[\mu_{\pm}(\lambda; x, t), \sigma_3] = \lambda Q(x, t)\mu_{\pm}(\lambda; x, t), \quad (2.6)$$

$$\mu_{\pm, t}(\lambda; x, t) + 4i\lambda^6[\mu_{\pm}(\lambda; x, t), \sigma_3] = [T(\lambda; x, t) - T_0]\mu_{\pm}(\lambda; x, t), \quad (2.7)$$

where the Lie bracket $[L_1, L_2] = L_1L_2 - L_2L_1$. Eqs. (2.6) and (2.7) can be written in full derivative form

$$d(e^{-i\theta(\lambda; x, t)\widehat{\sigma_3}}\mu_{\pm}(\lambda; x, t)) = e^{-i\theta(\lambda; x, t)\widehat{\sigma_3}}\{\lambda Q(x, t)dx + [T(\lambda; x, t) - T_0]dt\}\mu_{\pm}(\lambda; x, t), \quad (2.8)$$

and $\mu_{\pm}(\lambda; x, t)$ satisfy the Volterra integral equations

$$\mu_{\pm}(\lambda; x, t) = I + \int_{\pm\infty}^x e^{i\lambda^2(x-y)\widehat{\sigma_3}}(\lambda Q(y, t)\mu_{\pm}(\lambda; y, t))dy, \quad (2.9)$$

where $e^{\lambda\widehat{\sigma_3}}E := e^{\lambda\sigma_3}Ee^{-\lambda\sigma_3}$ with E being a 2×2 matrix. Let $D^{\pm} := \{\lambda \in \mathbb{C} | \pm \text{Re}(\lambda)\text{Im}(\lambda) > 0\}$, as shown in Fig. 1.

Furthermore, one has the following proposition.

Proposition 1. Suppose that $q(x, t) \in L^1(\mathbb{R})$ and $\psi_{\pm i}(\lambda; x, t)$ ($\mu_{\pm i}(\lambda; x, t)$) represent the i th column of $\psi_{\pm}(\lambda; x, t)$ ($\mu_{\pm}(\lambda; x, t)$). Then, the Jost solutions $\psi_{\pm}(\lambda; x, t)$ (modified Jost solutions $\mu_{\pm}(\lambda; x, t)$) possess the properties:

- Eq. (1.2) has the unique Jost solutions $\psi_{\pm}(\lambda; x, t)$ (modified Jost solutions $\mu_{\pm}(\lambda; x, t)$) satisfying Eq. (2.4) (combination of Eqs. (2.4)-(2.5)) on Σ .
- The column vectors $\psi_{+1}(\lambda; x, t)$ ($\mu_{+1}(\lambda; x, t)$) and $\psi_{-2}(\lambda; x, t)$ ($\mu_{-2}(\lambda; x, t)$) can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$.
- The column vectors $\psi_{+2}(\lambda; x, t)$ ($\mu_{+2}(\lambda; x, t)$) and $\psi_{-1}(\lambda; x, t)$ ($\mu_{-1}(\lambda; x, t)$) can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$.

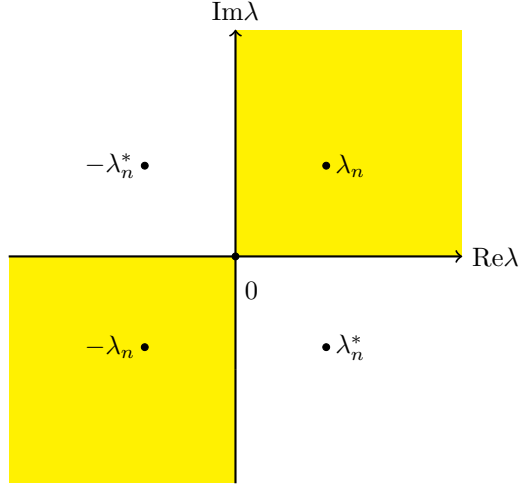


FIGURE 1. (Color online) Distribution of the discrete spectrum and jumping curves for the Riemann-Hilbert problem on complex λ -plane. Region $D^+ = \{\lambda \in \mathbb{C} | \text{Re}\lambda \text{Im}\lambda > 0\}$ (yellow region), region $D^- = \{\lambda \in \mathbb{C} | \text{Re}\lambda \text{Im}\lambda < 0\}$ (white region).

Proof. We define modified Jost solutions $\mu_{\pm} = (\mu_{\pm 1}, \mu_{\pm 2}) = \begin{pmatrix} \mu_{\pm 11} & \mu_{\pm 21} \\ \mu_{\pm 12} & \mu_{\pm 22} \end{pmatrix}$, in which $\mu_{\pm 1}$ and $\mu_{\pm 2}$ is the first and second columns of μ_{\pm} , respectively. Then, taking μ_- as an example, Eq. (2.9) can be rewritten as

$$\begin{pmatrix} \mu_{-11} & \mu_{-21} \\ \mu_{-12} & \mu_{-22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \int_{\pm\infty}^x \begin{pmatrix} q\mu_{-12} & q\mu_{-22}e^{2i\lambda^2(x-y)} \\ -q^*\mu_{-11}e^{-2i\lambda^2(x-y)} & -q^*\mu_{-21} \end{pmatrix} dy,$$

note that $\mu_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \int_{\pm\infty}^x \begin{pmatrix} q\mu_{-12} \\ -q^*\mu_{-11}e^{-2i\lambda^2(x-y)} \end{pmatrix} dy$, where $e^{-2i\lambda^2(x-y)} = e^{-2i(x-y)\text{Re}(\lambda^2)}e^{2(x-y)\text{Im}(\lambda^2)} = e^{-2i(x-y)\text{Re}(\lambda^2)}e^{4(x-y)\text{Re}(\lambda)\text{Im}(\lambda)}$. Since $x - y > 0$ and $\text{Re}(\lambda)\text{Im}(\lambda) < 0$, it indicate that the first column of μ_- is analytically extended to D^- . In addition, since $\text{Im}(\lambda^2) = 0$ when $\lambda \in \Sigma$, it demonstrate that the first column of μ_- is continuously extended to $D^- \cup \Sigma$. In the same way, we also obtain the analyticity and continuity of μ_{-2}, μ_{+1} and μ_{+2} . Similarly, the analyticity and continuity for the Jost solutions $\psi_{\pm}(\lambda; x, t)$ can be simply shown from ones of $\mu_{\pm}(\lambda; x, t)$ and the relation (2.5). \square

Proposition 2. *The Jost solutions $\psi_{\pm}(\lambda; x, t)$ satisfy both parts of the modified Zakharov-Shabat eigenvalue problem (1.2)-(1.3) simultaneously.*

Proof. The Liouville's formula leads to

$$\det(\psi_{\pm}(\lambda; x, t)) = \lim_{x \rightarrow \pm\infty} \det(\psi_{\pm}(\lambda; x, t)) = \lim_{x \rightarrow \pm\infty} \det(\mu_{\pm}(\lambda; x, t)) = 1,$$

that show $\psi_{\pm}(\lambda; x, t)$ are the fundamental matrix solutions on Σ . According to the zero-curvature condition $X_t - T_x + [X, T] = 0$, one can obtain that $\psi_{\pm,t}(\lambda; x, t) - T\psi_{\pm}(\lambda; x, t)$ also solve the x -part (1.2), that is

$$\begin{aligned} (\psi_{\pm,t}(\lambda; x, t) - T\psi_{\pm}(\lambda; x, t))_x &= X(\psi_{\pm,t}(\lambda; x, t) - T\psi_{\pm}(\lambda; x, t)), \\ \psi_{\pm,t,x}(\lambda; x, t) - T_x\psi_{\pm}(\lambda; x, t) - T\psi_{\pm,x}(\lambda; x, t) &= X\psi_{\pm,t}(\lambda; x, t) - XT\psi_{\pm}(\lambda; x, t), \\ (X\psi_{\pm}(\lambda; x, t))_t - T_x\psi_{\pm}(\lambda; x, t) - TX\psi_{\pm}(\lambda; x, t) &= X\psi_{\pm,t}(\lambda; x, t) - XT\psi_{\pm}(\lambda; x, t), \\ X_t\psi_{\pm}(\lambda; x, t) + X\psi_{\pm,t}(\lambda; x, t) - T_x\psi_{\pm}(\lambda; x, t) - TX\psi_{\pm}(\lambda; x, t) &= X\psi_{\pm,t}(\lambda; x, t) - XT\psi_{\pm}(\lambda; x, t), \\ (X_t - T_x + XT - TX)\psi_{\pm}(\lambda; x, t) &= 0. \end{aligned}$$

Thus, there exist the two matrices $U_{\pm}(\lambda; t)$ such that

$$\psi_{\pm,t}(\lambda; x, t) - T\psi_{\pm}(\lambda; x, t) = \psi_{\pm}(\lambda; x, t)U_{\pm}(\lambda; t), \text{ as } \lambda \in \Sigma,$$

multiplying both sides by $e^{-i\theta(\lambda; x, t)\sigma_3}$, we have

$$\begin{aligned} \psi_{\pm,t}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3} - T\psi_{\pm}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3} &= \psi_{\pm}(\lambda; x, t)U_{\pm}(\lambda; t)e^{-i\theta(\lambda; x, t)\sigma_3}, \text{ as } \lambda \in \Sigma, \\ (\psi_{\pm}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3})_t + T_0\psi_{\pm}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3} - T\psi_{\pm}(\lambda; x, t)e^{-i\theta(\lambda; x, t)\sigma_3} &= \psi_{\pm}(\lambda; x, t)U_{\pm}(\lambda; t)e^{-i\theta(\lambda; x, t)\sigma_3}, \text{ as } \lambda \in \Sigma, \\ (\mu_{\pm}(\lambda; x, t))_t + T_0\mu_{\pm}(\lambda; x, t) - T\mu_{\pm}(\lambda; x, t) &= \psi_{\pm}(\lambda; x, t)U_{\pm}(\lambda; t)e^{-i\theta(\lambda; x, t)\sigma_3}, \text{ as } \lambda \in \Sigma, \end{aligned}$$

and letting $x \rightarrow \pm\infty$, one can find $\mu_{\pm}(\lambda; x, t) \sim I, T \sim T_0$, and $U_{\pm}(\lambda; t) = 0$, that is, $\psi_{\pm}(\lambda; x, t)$ also solve the t -part (1.3).

□

2.1.2. Scattering Matrix and Reflection Coefficients.

Since the Jost solutions $\psi_{\pm}(\lambda; x, t)$ solve the both parts of the modified Zakharov-Shabat eigenvalue problem (1.2)-(1.3). Therefore, there exist a constant scattering matrix $S(\lambda) = (s_{ij}(\lambda))_{2 \times 2}$ independent of x and t to satisfy the linear relation between $\psi_+(\lambda; x, t)$ and $\psi_-(\lambda; x, t)$, where $s_{ij}(\lambda)$ are called the scattering coefficients, we have

$$\psi_+(\lambda; x, t) = \psi_-(\lambda; x, t)S(\lambda), \quad \lambda \in \Sigma, \quad (2.10)$$

let $\psi_+(\lambda; x, t) = (\psi_{+1}, \psi_{+2}) = \begin{pmatrix} \psi_{+11} & \psi_{+12} \\ \psi_{+21} & \psi_{+22} \end{pmatrix}$, $\psi_-(\lambda; x, t) = (\psi_{-1}, \psi_{-2}) = \begin{pmatrix} \psi_{-11} & \psi_{-12} \\ \psi_{-21} & \psi_{-22} \end{pmatrix}$, substituting into Eq. (2.10) and using Cramer's rule, one can obtain

$$\begin{aligned} s_{11}(\lambda) &= \frac{\det \begin{pmatrix} \psi_{+11} & \psi_{-12} \\ \psi_{+12} & \psi_{-22} \end{pmatrix}}{\det(\psi_-(\lambda; x, t))} = \det(\psi_{+1}(\lambda; x, t), \psi_{-2}(\lambda; x, t)), \\ s_{12}(\lambda) &= \frac{\det \begin{pmatrix} \psi_{+12} & \psi_{-12} \\ \psi_{+22} & \psi_{-22} \end{pmatrix}}{\det(\psi_-(\lambda; x, t))} = \det(\psi_{+2}(\lambda; x, t), \psi_{-2}(\lambda; x, t)), \\ s_{21}(\lambda) &= \frac{\det \begin{pmatrix} \psi_{-11} & \psi_{+11} \\ \psi_{-21} & \psi_{+21} \end{pmatrix}}{\det(\psi_-(\lambda; x, t))} = \det(\psi_{-1}(\lambda; x, t), \psi_{+1}(\lambda; x, t)), \\ s_{22}(\lambda) &= \frac{\det \begin{pmatrix} \psi_{-11} & \psi_{+12} \\ \psi_{-21} & \psi_{+22} \end{pmatrix}}{\det(\psi_-(\lambda; x, t))} = \det(\psi_{-1}(\lambda; x, t), \psi_{+2}(\lambda; x, t)), \end{aligned} \quad (2.11)$$

where $\det(\psi_-(\lambda; x, t)) = 1$.

Proposition 3. Suppose that $q(x, t) \in L^1(\mathbb{R})$. Then, $s_{11}(\lambda)$ can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$, while $s_{22}(\lambda)$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$. Moreover, both $s_{12}(\lambda)$ and $s_{21}(\lambda)$ are continuous in Σ .

Proof. From proposition (1), we know that $\psi_{+1}(\lambda; x, t)$ and $\psi_{-2}(\lambda; x, t)$ can be extended analytically to D^+ and continuously extended to $D^+ \cup \Sigma$. Since $s_{11}(\lambda) = \det(\psi_{+1}(\lambda; x, t), \psi_{-2}(\lambda; x, t))$, one can obtain $s_{11}(\lambda)$ can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$. Similarly, we can prove $s_{22}(\lambda)$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$. Due to $\psi_{\pm 1}(\lambda; x, t)$ and $\psi_{\pm 2}(\lambda; x, t)$ are continuous in Σ , both $s_{12}(\lambda)$ and $s_{21}(\lambda)$ are continuous in Σ . □

Note that it cannot be ruled out that $s_{11}(\lambda)$ and $s_{22}(\lambda)$ may exist zeros along Σ . In order to study the RH problem in the inverse process, we focus on the potential without spectral singularity. In general, the reflection coefficients $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ are defined by

$$\rho(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)}, \quad \tilde{\rho}(\lambda) = \frac{s_{12}(\lambda)}{s_{22}(\lambda)}. \quad (2.12)$$

2.1.3. Symmetry Properties.

Proposition 4. $X(\lambda; x, t)$, $T(\lambda; x, t)$, Jost solutions, modified Jost solutions, scattering matrix and reflection coefficients have two kinds of symmetry reductions as follow:

- The first symmetry reduction

$$\begin{aligned} X(\lambda; x, t) &= \sigma_2 X(\lambda^*; x, t)^* \sigma_2, \quad T(\lambda; x, t) = \sigma_2 T(\lambda^*; x, t)^* \sigma_2, \\ \psi_{\pm}(\lambda; x, t) &= \sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2, \quad \mu_{\pm}(\lambda; x, t) = \sigma_2 \mu_{\pm}(\lambda^*; x, t)^* \sigma_2, \\ S(\lambda) &= \sigma_2 S(\lambda^*)^* \sigma_2, \quad \rho(\lambda) = -\tilde{\rho}(\lambda^*)^*. \end{aligned} \quad (2.13)$$

- The second symmetry reduction

$$\begin{aligned} X(\lambda; x, t) &= \sigma_1 X(-\lambda^*; x, t)^* \sigma_1, \quad T(\lambda; x, t) = \sigma_1 T(-\lambda^*; x, t)^* \sigma_1, \\ \psi_{\pm}(\lambda; x, t) &= \sigma_1 \psi_{\pm}(-\lambda^*; x, t)^* \sigma_1, \quad \mu_{\pm}(\lambda; x, t) = \sigma_1 \mu_{\pm}(-\lambda^*; x, t)^* \sigma_1, \\ S(\lambda) &= \sigma_1 S(-\lambda^*)^* \sigma_1, \quad \rho(\lambda) = \tilde{\rho}(-\lambda^*)^*. \end{aligned} \quad (2.14)$$

Proof. First, let's prove the first symmetry reduction. Since we know the specific forms of the matrices X and T , we can obtain the symmetry reduction of X and T by direct calculation. In order to obtain the symmetry reduction for Jost solutions, in fact, we just need to demonstrate that $\sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2$ is also a solution of Eq. (1.2) and admits the same asymptotic behavior as the Jost solutions $\psi_{\pm}(\lambda; x, t)$.

Considering

$$\psi_{\pm, x}(\lambda; x, t) = X(\lambda; x, t) \psi_{\pm}(\lambda; x, t), \quad (2.15)$$

replacing λ with λ^* , and taking conjugate both sides simultaneously, we have $\psi_{\pm, x}(\lambda^*; x, t)^* = X(\lambda^*; x, t)^* \psi_{\pm}(\lambda^*; x, t)^*$, substituting $X(\lambda^*; x, t)^* = \sigma_2^{-1} X(\lambda; x, t) \sigma_2^{-1}$ into previous formula, then both sides of the obtained formula multiplying matrix σ_2 on left and right at the same time, one can obtain

$$\begin{aligned} \sigma_2 \psi_{\pm, x}(\lambda^*; x, t)^* \sigma_2 &= \sigma_2 \sigma_2^{-1} X(\lambda; x, t) \sigma_2^{-1} \psi_{\pm}(\lambda^*; x, t)^* \sigma_2, \\ \Rightarrow \sigma_2 \psi_{\pm, x}(\lambda^*; x, t)^* \sigma_2 &= X(\lambda; x, t) \sigma_2^{-1} (\sigma_2^{-1} \sigma_2) \psi_{\pm}(\lambda^*; x, t)^* \sigma_2, \\ \Rightarrow \sigma_2 \psi_{\pm, x}(\lambda^*; x, t)^* \sigma_2 &= X(\lambda; x, t) \sigma_2^{-1} \sigma_2^{-1} (\sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2), \end{aligned}$$

where $\sigma_2^{-1} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, so $\sigma_2^{-1} \sigma_2^{-1} = I$, we have

$$\sigma_2 \psi_{\pm, x}(\lambda^*; x, t)^* \sigma_2 = X(\lambda; x, t) (\sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2). \quad (2.16)$$

By comparing Eqs. (2.15) and (2.16), we know that $\sigma_2 \psi_{\pm, x}(\lambda^*; x, t)^* \sigma_2$ is also a solution of Eq. (2.15), and $\psi_{\pm}(\lambda; x, t) = \sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2$ because they have the same asymptotic behavior

$$\psi_{\pm}(\lambda; x, t), \quad \sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2 \sim I, \text{ as } x \rightarrow \pm\infty.$$

According to Eq. (2.5) and $\theta(\lambda^*)^* = \theta(\lambda)$, we can obtain $\psi_{\pm}(\lambda; x, t) = \mu_{\pm}(\lambda; x, t) e^{i\theta(\lambda; x, t)\sigma_3}$ and $\psi_{\pm}(\lambda^*; x, t)^* = \mu_{\pm}(\lambda^*; x, t)^* e^{-i\theta(\lambda; x, t)\sigma_3}$. Substituting into $\psi_{\pm}(\lambda; x, t) = \sigma_2 \psi_{\pm}(\lambda^*; x, t)^* \sigma_2$, one can derive $\mu_{\pm}(\lambda; x, t) e^{i\theta(\lambda; x, t)\sigma_3} = \sigma_2 \mu_{\pm}(\lambda^*; x, t)^* e^{-i\theta(\lambda; x, t)\sigma_3} \sigma_2 \Rightarrow \mu_{\pm}(\lambda; x, t) = \sigma_2 \mu_{\pm}(\lambda^*; x, t)^* e^{-i\theta(\lambda; x, t)\sigma_3} \sigma_2 e^{-i\theta(\lambda; x, t)\sigma_3} = \sigma_2 \mu_{\pm}(\lambda^*; x, t)^* \sigma_2$.

Next, we demonstrate the symmetry reduction of the scattering matrix. We substitute $\lambda = \lambda^*$ into Eq. (2.10) and conjugate both sides of the equation. We have $\psi_+(\lambda^*; x, t)^* = \psi_-(\lambda^*; x, t)^* S(\lambda^*)^*$, From the symmetry reduction of Jost solutions, one can obtain $\sigma_2^{-1} \psi_+(\lambda; x, t) \sigma_2^{-1} = \sigma_2^{-1} \psi_-(\lambda; x, t) \sigma_2^{-1} (\sigma_2^{-1} \sigma_2) S(\lambda^*)^* \Rightarrow \psi_+(\lambda; x, t) = \psi_-(\lambda; x, t) \sigma_2^{-1} \sigma_2^{-1} (\sigma_2 S(\lambda^*)^* \sigma_2)$, then

$$\psi_+(\lambda; x, t) = \psi_-(\lambda; x, t) (\sigma_2 S(\lambda^*)^* \sigma_2). \quad (2.17)$$

By comparing Eqs. (2.10) and (2.17), we can obtain $S(\lambda) = \sigma_2 S(\lambda^*)^* \sigma_2$. According to the symmetry reduction of scattering matrix, we have

$$\begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} s_{11}(\lambda^*)^* & s_{12}(\lambda^*)^* \\ s_{21}(\lambda^*)^* & s_{22}(\lambda^*)^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} s_{22}(\lambda^*)^* & -s_{21}(\lambda^*)^* \\ -s_{12}(\lambda^*)^* & s_{11}(\lambda^*)^* \end{pmatrix},$$

we then obtain these symmetric relations

$$s_{11}(\lambda) = s_{22}(\lambda^*)^*, \quad s_{12}(\lambda) = -s_{21}(\lambda^*)^*, \quad s_{21}(\lambda) = -s_{12}(\lambda^*)^*, \quad s_{22}(\lambda) = s_{11}(\lambda^*)^*, \quad (2.18)$$

Since $\rho(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)} = \frac{-s_{12}(\lambda^*)^*}{s_{22}(\lambda^*)^*} = -\tilde{\rho}(\lambda^*)^*$, the symmetry reduction of reflection coefficient is $\rho(\lambda) = -\tilde{\rho}(\lambda^*)^*$.

Similarly, by repeating the above process, we can prove the second symmetry reduction. This completes the proof. \square

2.1.4. Asymptotic Behaviors.

In order to propose and solve the matrix RH problem for the inverse problem in the following part, the asymptotic behavior of the modified Jost solution and the scattering matrix must be determined as $\lambda \rightarrow \infty$. The asymptotic behaviors of the modified Jost solution can be derived by employing the usual Wentzel-Kramers-Brillouin (WKB) expansion.

Proposition 5. *The asymptotic behaviors of the modified Jost solutions are as follows:*

$$\mu_{\pm}(\lambda; x, t) = e^{i\nu_{\pm}(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \rightarrow \infty, \quad (2.19)$$

where functions $\nu_{\pm}(x, t)$ read as

$$\nu_{\pm}(x, t) = \frac{1}{2} \int_{\pm\infty}^x |q(y, t)|^2 dy. \quad (2.20)$$

Proof. By utilizing the WKB expansion, we substitute $\mu_{\pm}(\lambda; x, t) = \sum_{i=0}^n \frac{\mu_{\pm}^{[i]}(x, t)}{\lambda^i} + O\left(\frac{1}{\lambda}\right)$ (as $\lambda \rightarrow \infty$) into equivalent Lax pair (2.6), and compare the coefficients of different powers of λ , we have

$$O(\lambda^2) : i\mu_{\pm}^{[0]}(x, t)\sigma_3 - i\sigma_3\mu_{\pm}^{[0]}(x, t) = 0, \Rightarrow \left(\mu_{\pm}^{[0]}(x, t)\right)^{\text{off}} = 0, \quad (2.21)$$

where $\left(\mu_{\pm}^{[0]}(x, t)\right)^{\text{off}}$ represent the off-diagonal parts of $\mu_{\pm}^{[0]}(x, t)$.

$$O(\lambda) : i\mu_{\pm}^{[1]}(x, t)\sigma_3 - i\sigma_3\mu_{\pm}^{[1]}(x, t) = Q(x, t)\mu_{\pm}^{[0]}(x, t), \Rightarrow \left(\mu_{\pm}^{[1]}(x, t)\right)^{\text{off}} = \frac{i}{2}\sigma_3 Q(x, t)\mu_{\pm}^{[0]}(x, t), \quad (2.22)$$

$$O(1) : \mu_{\pm,x}^{[0]}(x, t) + i\mu_{\pm}^{[2]}(x, t)\sigma_3 - i\sigma_3\mu_{\pm}^{[2]}(x, t) = Q(x, t)\mu_{\pm}^{[1]}(x, t), \Rightarrow \mu_{\pm}^{[0]}(x, t) = C^{\text{diag}} e^{i\nu_{\pm}(x, t)\sigma_3}, \quad (2.23)$$

where $\nu_{\pm}(x, t) = \frac{1}{2} \int_{\pm\infty}^x |q(y, t)|^2 dy$, $\mu_{\pm}(\lambda; x, t) = C^{\text{diag}} e^{i\nu_{\pm}(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right)$, as $\lambda \rightarrow \infty$. When $x \rightarrow \pm\infty$, we have $\nu_{\pm}(x, t) \rightarrow 0$, $\mu_{\pm}(\lambda; x, t) \rightarrow I$, one can obtain $C^{\text{diag}} = I$, and deduce the asymptotic behavior as $\lambda \rightarrow \infty$. \square

Proposition 6. *The asymptotic behavior for the scattering matrix is as follows:*

$$S(\lambda) = e^{-\nu\sigma_3} + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \rightarrow \infty, \quad (2.24)$$

where the constant ν reads as

$$\nu = \frac{1}{2} \int_{-\infty}^{+\infty} |q(y, t)|^2 dy. \quad (2.25)$$

Proof. From the relationship among scattering matrix, Jost solution and modified Jost solution, we have

$$\mu_+(\lambda; x, t) e^{i\theta(\lambda; x, t)\sigma_3} = \mu_-(\lambda; x, t) e^{i\theta(\lambda; x, t)\sigma_3} S(\lambda).$$

According to proposition (5), when $\lambda \rightarrow \infty$, $\mu_{\pm}(\lambda; x, t) = e^{i\nu_{\pm}(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right)$, we have

$$\left[e^{i\nu_+(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right) \right] e^{i\theta(\lambda; x, t)\sigma_3} = \left[e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right) \right] e^{i\theta(\lambda; x, t)\sigma_3} S(\lambda),$$

then both sides of the equation pre-multiply $e^{-i[\nu_-(x, t) + \theta(\lambda; x, t)]\sigma_3}$ at the same time, one can obtain

$$S(\lambda) = e^{-i[\nu_-(x, t) - \nu_+(x, t)]\sigma_3} + o\left(\frac{1}{\lambda}\right) = e^{-i\nu\sigma_3} + O\left(\frac{1}{\lambda}\right),$$

where $\nu = \nu_-(x, t) - \nu_+(x, t) = \frac{1}{2} \int_{-\infty}^{+\infty} |q(y, t)|^2 dy$.

On the other hand, substituting the WKB expansion $\mu_{\pm}(\lambda; x, t) = \sum_{i=0}^n \frac{\mu_{\pm}^{[i]}(x, t)}{\lambda^i} + O\left(\frac{1}{\lambda}\right)$ (as $\lambda \rightarrow \infty$) into the time part of equivalent lax pairs (2.7) and matching the $O(\lambda^i)$ ($i = 6, 5, 4, 3, 2, 1, 0$) in order, combining the $O(1) : \mu_{\pm,t}^{[0]}(x, t) + 4i[\mu_{\pm}^{[6]}(x, t), \sigma_3] = (T - T_0)\mu_{\pm}^{[0]}(x, t)$ and $\left(\mu_{\pm}^{[0]}(x, t)\right)^{\text{off}} = 0$, one can yield $\mu_{\pm,t}^{[0]}(x, t) = 0$, $\nu_{\pm,t}(x, t) = 0$ and $\nu_t = 0$ as $x \rightarrow \pm\infty$. That is, ν does not depend on the variable t . Furthermore, for the Zakharov-Shabat spectral problem of the NLS equation, $I_1 = \int_{-\infty}^{\infty} |u(x, t)|^2 dx$ is conservations of mass (also called power), and one has $I_{1,t} = 0$ [48]. Here, one can derive $I_1 = 2\nu = \int_{-\infty}^{+\infty} |q(x, t)|^2 dx$ (here x and y in (2.25) are equivalent) is the conservations of mass for the modified Zakharov-Shabat spectral problem of the TOFKN equation, and we also obtain $\nu_t = 0$ by using the $I_{1,t} = 0$. Next we give a simple proof of pure algebra for $I_{1,t} = 0$.

According to the Eq.(1.1), we have

$$\begin{aligned} q_t &= \frac{9}{2}|q|^4 q_x + 3q^2|q|^2 q_x^* + 3i|q|^2 q_{xx} + 3iq_x^2 q^* + 3iqq_x^* q_x - q_{xxx}, \\ q_t^* &= \frac{9}{2}|q|^4 q_x^* + 3(q^*)^2|q|^2 q_x - 3i|q|^2 q_{xx}^* - 3i(q_x^*)^2 q - 3iq^* q_x q_x^* - q_{xxx}^*, \end{aligned}$$

from $I_1 = \int_{-\infty}^{+\infty} |q(x, t)|^2 dx$, we have

$$\begin{aligned} I_{1,t} &= \int_{-\infty}^{+\infty} [q(x, t)q^*(x, t)]_t dx \\ &= \int_{-\infty}^{+\infty} \left\{ \frac{5}{2}[q^3(q^*)^3]_x + 3i[|q|^2(q_x q^* - q q_x^*)]_x - [(q_{xx} q^* + q q_{xx}^*)_x - (q_x q_x^*)_x] \right\} dx \\ &= \frac{5}{2}q^3(q^*)^3 \Big|_{-\infty}^{+\infty} + 3i|q|^2(q_x q^* - q q_x^*) \Big|_{-\infty}^{+\infty} - (q_{xx} q^* + q q_{xx}^*) \Big|_{-\infty}^{+\infty} + q_x q_x^* \Big|_{-\infty}^{+\infty}, \end{aligned}$$

Since $q(x, t) = 0$ as $x \rightarrow \pm\infty$, one can derive $I_{1,t} = 0$ and $\nu_t = 0$. Therefore, ν is a constant. Proof complete. \square

2.1.5. Discrete Spectrum with Double Zeros.

The discrete spectrum of the studied scattering problem is the set of all values $\lambda \in \mathbb{C} \setminus \Sigma$ which satisfy the eigenfunctions exist in $L^2(\mathbb{R})$. As was proposed by Biondini and Kovačič [47], there are exactly the values of λ in D^+ such that $s_{11}(\lambda) = 0$ and those values in D^- such that $s_{22}(\lambda) = 0$. Differing from the previous results with simple poles [29, 34, 35], we here suppose that $s_{11}(\lambda)$ has N double zeros in $\Lambda_0 = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > 0\}$ denoted by λ_n , $n = 1, 2, \dots, N$, that is, $s_{11}(\lambda_n) = s'_{11}(\lambda_n) = 0$, and $s''_{11} \neq 0$. It follows from the symmetry reduce of the scattering matrix that

$$\begin{cases} s_{11}(\pm\lambda_n) = s_{22}(\pm\lambda_n^*) = 0, \\ s'_{11}(\pm\lambda_n) = s'_{22}(\pm\lambda_n^*) = 0. \end{cases} \quad (2.26)$$

Therefore, the corresponding discrete spectrum is defined by the set

$$\Lambda = \{\pm\lambda_n, \pm\lambda_n^*\}_{n=1}^N, \quad (2.27)$$

whose distributions are shown in Fig. 1. When a given $\lambda_0 \in \Lambda \cap D^+$, one can obtain that $\psi_{+1}(\lambda_0; x, t)$ and $\psi_{-2}(\lambda_0; x, t)$ are linearly dependent by combining Eq. (2.11) and $s_{11}(\lambda_0) = 0$. Similarly, when a given $\lambda_0 \in \Lambda \cap D^-$, one can obtain that $\psi_{+2}(\lambda_0; x, t)$ and $\psi_{-1}(\lambda_0; x, t)$ are linearly dependent by combining Eq. (2.11) and $s_{22}(\lambda_0) = 0$. For convenience, we introduce the norming constant $b[\lambda_0]$ such that

$$\begin{aligned} \psi_{+1}(\lambda_0; x, t) &= b[\lambda_0]\psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ \psi_{+2}(\lambda_0; x, t) &= b[\lambda_0]\psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned} \quad (2.28)$$

For a given $\lambda_0 \in \Lambda \cap D^+$, according to $s_{11}(\lambda_0) = \det(\psi_{+1}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t))$ in Eq. (2.11), and taking derivative respect to λ (here $\lambda = \lambda_0$) on both sides of previous equation, we have

$$\begin{aligned} s'_{11}(\lambda_0) &= \det(\psi'_{+1}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t)) + \det(\psi_{+1}(\lambda_0; x, t), \psi'_{-2}(\lambda_0; x, t)) \\ &= \det(\psi'_{+1}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t)) + \det(b[\lambda_0]\psi_{-2}(\lambda_0; x, t), \psi'_{-2}(\lambda_0; x, t)) \\ &= \det(\psi'_{+1}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t)) - \det(b[\lambda_0]\psi'_{-2}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t)) \\ &= \det(\psi'_{+1}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-2}(\lambda_0; x, t), \psi_{-2}(\lambda_0; x, t)), \end{aligned}$$

note that $s'_{11}(\lambda_0) = 0$, $\psi'_{+1}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-2}(\lambda_0; x, t)$ and $\psi_{-2}(\lambda_0; x, t)$ are linearly dependent. Similarly, for a given $\lambda_0 \in \Lambda \cap D^-$, one can obtain that $\psi'_{+2}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-1}(\lambda_0; x, t)$ and $\psi_{-1}(\lambda_0; x, t)$ are linearly dependent by combining Eq. (2.11) and $s'_{22}(\lambda_0) = 0$. For convenience, we define another norming constant $d[\lambda_0]$ such that

$$\begin{aligned} \psi'_{+1}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-2}(\lambda_0; x, t) &= d[\lambda_0]\psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ \psi'_{+2}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-1}(\lambda_0; x, t) &= d[\lambda_0]\psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned} \quad (2.29)$$

On the other hand, we notice that $\psi_{+1}(\lambda; x, t)$ and $s_{11}(\lambda)$ are analytic on D^+ , Suppose λ_0 is the double zeros of s_{11} . Let $\psi_{+1}(\lambda; x, t)$ and $s_{11}(\lambda)$ carry out Taylor expansion at $\lambda = \lambda_0$, we have

$$\begin{aligned} \frac{\psi_{+1}(\lambda; x, t)}{s_{11}(\lambda)} &= \frac{\psi_{+1}(\lambda_0; x, t) + \psi'_{+1}(\lambda_0; x, t)(\lambda - \lambda_0) + \frac{1}{2!}\psi''_{+1}(\lambda_0; x, t)(\lambda - \lambda_0)^2 + \dots}{s_{11}(\lambda_0) + s'_{11}(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2!}s''_{11}(\lambda_0)(\lambda - \lambda_0)^2 + \frac{1}{3!}s'''_{11}(\lambda_0)(\lambda - \lambda_0)^3 + \dots} \\ &= \frac{\psi_{+1}(\lambda_0; x, t) + \psi'_{+1}(\lambda_0; x, t)(\lambda - \lambda_0) + \frac{1}{2!}\psi''_{+1}(\lambda_0; x, t)(\lambda - \lambda_0)^2 + \dots}{\frac{1}{2!}s''_{11}(\lambda_0)(\lambda - \lambda_0)^2 + \frac{1}{3!}s'''_{11}(\lambda_0)(\lambda - \lambda_0)^3 + \dots} \\ &= \frac{2\psi_{+1}(\lambda_0; x, t)}{s''_{11}(\lambda_0)}(\lambda - \lambda_0)^{-2} + \left(\frac{2\psi'_{+1}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} - \frac{2\psi_{+1}(\lambda_0; x, t)s'''_{11}(\lambda_0)}{3s''_{11}{}^2(\lambda_0)} \right)(\lambda - \lambda_0)^{-1} + \dots, \end{aligned}$$

Then, one has the compact form

$$P_{-2} \left[\frac{\psi_{+1}(\lambda; x, t)}{s_{11}(\lambda)} \right]_{\lambda=\lambda_0} = \frac{2\psi_{+1}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} = \frac{2b[\lambda_0]\psi_{-2}(\lambda_0; x, t)}{s''_{11}(\lambda_0)}, \text{ as } \lambda_0 \in \Lambda \cap D^+,$$

where $P_{-2}[f(\lambda; x, t)]$ denotes the coefficient of $O((\lambda - \lambda_0)^{-2})$ term in the Laurent series expansion of $f(\lambda; x, t)$ at $\lambda = \lambda_0$. and

$$\begin{aligned} \text{Res}_{\lambda=\lambda_0} \left[\frac{\psi_{+1}(\lambda; x, t)}{s_{11}(\lambda)} \right] &= \frac{2\psi'_{+1}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} - \frac{2\psi_{+1}(\lambda_0; x, t)s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} = \frac{2(b[\lambda_0]\psi'_{-2}(\lambda_0; x, t) + d[\lambda_0]\psi_{-2}(\lambda_0; x, t))}{s''_{11}(\lambda_0)} \\ &\quad - \frac{2b[\lambda_0]\psi_{-2}(\lambda_0; x, t)s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} = \frac{2b[\lambda_0]\psi'_{-2}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} + \frac{2d[\lambda_0]\psi_{-2}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} - \frac{2b[\lambda_0]\psi_{-2}(\lambda_0; x, t)s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} \\ &= \frac{2b[\lambda_0]\psi'_{-2}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} + \frac{2b[\lambda_0]}{s''_{11}(\lambda_0)} \cdot \frac{d[\lambda_0]}{b[\lambda_0]} \psi_{-2}(\lambda_0; x, t) - \frac{2b[\lambda_0]}{s''_{11}(\lambda_0)} \cdot \frac{s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} \psi_{-2}(\lambda_0; x, t) \\ &= \frac{2b[\lambda_0]\psi'_{-2}(\lambda_0; x, t)}{s''_{11}(\lambda_0)} + \left[\frac{2b[\lambda_0]}{s''_{11}(\lambda_0)} \left(\frac{d[\lambda_0]}{b[\lambda_0]} - \frac{s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} \right) \right] \psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \end{aligned}$$

where $\text{Res}_{\lambda=\lambda_0}[f(\lambda; x, t)]$ denotes the coefficient of $O((\lambda - \lambda_0)^{-1})$ term in the Laurent series expansion of $f(\lambda; x, t)$ at $\lambda = \lambda_0$.

Similarly, for the case of $\psi_{+2}(\lambda; x, t)$ and $s_{22}(\lambda)$ are analytic on D^- , we repeat the above process and obtain

$$\begin{aligned} P_{-2}_{\lambda=\lambda_0} \left[\frac{\psi_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right] &= \frac{2\psi_{+2}(\lambda_0; x, t)}{s''_{22}(\lambda_0)} = \frac{2b[\lambda_0]\psi_{-1}(\lambda_0; x, t)}{s''_{22}(\lambda_0)}, \text{ as } \lambda_0 \in \Lambda \cap D^-, \\ \text{Res}_{\lambda=\lambda_0} \left[\frac{\psi_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right] &= \frac{2b[\lambda_0]\psi'_{-1}(\lambda_0; x, t)}{s''_{22}(\lambda_0)} + \left[\frac{2b[\lambda_0]}{s''_{22}(\lambda_0)} \left(\frac{d[\lambda_0]}{b[\lambda_0]} - \frac{s'''_{22}(\lambda_0)}{3s''_{22}(\lambda_0)} \right) \right] \psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned}$$

Moreover, let

$$A[\lambda_0] = \begin{cases} \frac{2b[\lambda_0]}{s''_{11}(\lambda_0)}, & \text{as } \lambda_0 \in \Lambda \cap D^+, \\ \frac{2b[\lambda_0]}{s''_{22}(\lambda_0)}, & \text{as } \lambda_0 \in \Lambda \cap D^-, \end{cases} \quad B[\lambda_0] = \begin{cases} \frac{d[\lambda_0]}{b[\lambda_0]} - \frac{s'''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)}, & \text{as } \lambda_0 \in \Lambda \cap D^+, \\ \frac{d[\lambda_0]}{b[\lambda_0]} - \frac{s'''_{22}(\lambda_0)}{3s''_{22}(\lambda_0)}, & \text{as } \lambda_0 \in \Lambda \cap D^-. \end{cases} \quad (2.30)$$

Then, we have

$$\begin{aligned} P_{-2}_{\lambda=\lambda_0} \left[\frac{\psi_{+1}(\lambda; x, t)}{s_{11}(\lambda)} \right] &= A[\lambda_0]\psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ P_{-2}_{\lambda=\lambda_0} \left[\frac{\psi_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right] &= A[\lambda_0]\psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-, \\ \text{Res}_{\lambda=\lambda_0} \left[\frac{\psi_{+1}(\lambda; x, t)}{s_{11}(\lambda)} \right] &= A[\lambda_0][\psi'_{-2}(\lambda_0; x, t) + B[\lambda_0]\psi_{-2}(\lambda_0; x, t)], \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ \text{Res}_{\lambda=\lambda_0} \left[\frac{\psi_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right] &= A[\lambda_0][\psi'_{-1}(\lambda_0; x, t) + B[\lambda_0]\psi_{-1}(\lambda_0; x, t)], \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned} \quad (2.31)$$

Proposition 7. For $\lambda_0 \in \Lambda$, the two symmetry relations for $A[\lambda_0]$ and $B[\lambda_0]$ can be deduced as follows:

- The first symmetry relation $A[\lambda_0] = -A[\lambda_0^*]^*$, $B[\lambda_0] = B[\lambda_0^*]^*$.
- The second symmetry relation $A[\lambda_0] = A[-\lambda_0^*]^*$, $B[\lambda_0] = -B[-\lambda_0^*]^*$.

Proof. Considering the first symmetry relation, for a given $\lambda_0 \in \Lambda$, $s_{11}(\lambda_0) = s_{22}(\lambda_0) = 0$, hence substituting $S(\lambda_0) = \begin{pmatrix} 0 & s_{12}(\lambda_0) \\ s_{21}(\lambda_0) & 0 \end{pmatrix}$ into $\psi_+(\lambda_0; x, t) = \psi_-(\lambda_0; x, t)S(\lambda_0)$, one can obtain

$$\begin{aligned} \psi_{+1}(\lambda_0; x, t) &= s_{21}(\lambda_0)\psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ \psi_{+2}(\lambda_0; x, t) &= s_{12}(\lambda_0)\psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned}$$

Therefore, when $\lambda_0 \in \Lambda \cap D^+$ ($\lambda_0 \in \Lambda \cap D^-$), one has obtain $b[\lambda_0] = s_{21}(\lambda_0)$ ($b[\lambda_0] = s_{12}(\lambda_0)$). According to Eq. (2.18), we have $b[\lambda_0^*]^* = s_{12}(\lambda_0^*)^* = -s_{21}(\lambda_0) = -b[\lambda_0]$ (where $\lambda_0^* \in \Lambda \cap D^-$), and $A[\lambda_0] = \frac{2b[\lambda_0]}{s''_{11}(\lambda_0)} = \frac{-2b[\lambda_0^*]^*}{s''_{22}(\lambda_0^*)^*} = -A[\lambda_0^*]^*$. In addition, for a given $\lambda_0 \in \Lambda \cap D^+$ ($\lambda_0 \in \Lambda \cap D^-$), by taking derivative respect to λ (here $\lambda = \lambda_0$) both sides of $\psi_{+1}(\lambda_0; x, t) = b[\lambda_0]\psi_{-2}(\lambda_0; x, t)$ ($\psi_{+2}(\lambda_0; x, t) = b[\lambda_0]\psi_{-1}(\lambda_0; x, t)$), we have

$$\begin{aligned} \psi'_{+1}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-2}(\lambda_0; x, t) &= b'[\lambda_0]\psi_{-2}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ \psi'_{+2}(\lambda_0; x, t) - b[\lambda_0]\psi'_{-1}(\lambda_0; x, t) &= b'[\lambda_0]\psi_{-1}(\lambda_0; x, t), \text{ as } \lambda_0 \in \Lambda \cap D^-. \end{aligned}$$

Combining with Eq. (2.29), one can obtain

$$\begin{aligned} b[\lambda_0] = s_{21}(\lambda_0) &\Rightarrow d[\lambda_0] = b'[\lambda_0] = s'_{21}(\lambda_0), \text{ as } \lambda_0 \in \Lambda \cap D^+, \\ b[\lambda_0] = s_{12}(\lambda_0) &\Rightarrow d[\lambda_0] = b'[\lambda_0] = s'_{12}(\lambda_0), \text{ as } \lambda_0 \in \Lambda \cap D^-, \end{aligned}$$

then

$$B[\lambda_0] = \frac{d[\lambda_0]}{b[\lambda_0]} - \frac{s''_{11}(\lambda_0)}{3s''_{11}(\lambda_0)} = \frac{s'_{21}(\lambda_0)}{s_{21}(\lambda_0)} - \frac{s''_{22}(\lambda_0)^*}{3s''_{22}(\lambda_0)^*} = \frac{-s'_{12}(\lambda_0)^*}{-s_{12}(\lambda_0)^*} - \frac{s''_{22}(\lambda_0)^*}{3s''_{22}(\lambda_0)^*} = \frac{-d[\lambda_0]^*}{-b[\lambda_0]^*} - \frac{s''_{22}(\lambda_0)^*}{3s''_{22}(\lambda_0)^*} = B[\lambda_0^*].$$

Similarly, by repeating the above process, we can prove the second symmetry relation. \square

2.2. Inverse Problem with ZBCs and Double Poles.

In the following parts, we will propose an inverse problem with ZBCs and solve it to obtain accurate double poles solutions for the TOFKN equation (1.1).

2.2.1. The Matrix Riemann-Hilbert Problem.

In general, according to the relation (2.10) of two Jost solutions $\psi_{\pm}(\lambda; x, t)$, we will investigate the inverse problem with ZBCs by establishing a RH problem. In order to pose and solve the RH problem conveniently, we define

$$\zeta_n = \begin{cases} \lambda_n, & n = 1, 2, \dots, N, \\ -\lambda_{n-N}, & n = N+1, N+2, \dots, 2N. \end{cases} \quad (2.32)$$

Then, a matrix RH problem is proposed as follows.

Proposition 8. *Define the sectionally meromorphic matrices*

$$M(\lambda; x, t) = \begin{cases} M^+(\lambda; x, t) = \left(\frac{\mu_{+1}(\lambda; x, t)}{s_{11}(\lambda)}, \mu_{-2}(\lambda; x, t) \right), & \text{as } \lambda \in D^+, \\ M^-(\lambda; x, t) = \left(\mu_{-1}(\lambda; x, t), \frac{\mu_{+2}(\lambda; x, t)}{s_{11}(\lambda)} \right), & \text{as } \lambda \in D^-, \end{cases} \quad (2.33)$$

where $\lim_{\lambda' \rightarrow \lambda} M(\lambda'; x, t) = M^{\pm}(\lambda; x, t)$. Then, the multiplicative matrix Riemann-Hilbert problem is given below:

- *Analyticity:* $M(\lambda; x, t)$ is analytic in $D^+ \cup D^- \setminus \Lambda$ and has the double poles in Λ , whose principal parts of the Laurent series at each double pole ζ_n or ζ_n^* , are determined as

$$\begin{aligned} \text{Res}_{\lambda=\zeta_n} M(\lambda; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \{ \mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)] \mu_{-2}(\zeta_n; x, t) \}, 0 \right), \\ \text{P}_{-2} M(\lambda; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \mu_{-2}(\zeta_n; x, t), 0 \right), \\ \text{Res}_{\lambda=\zeta_n^*} M(\lambda; x, t) &= \left(0, A[\zeta_n^*] e^{2i\theta(\zeta_n^*; x, t)} \{ \mu'_{-1}(\zeta_n^*; x, t) + [B[\zeta_n^*] + 2i\theta'(\zeta_n^*; x, t)] \mu_{-1}(\zeta_n^*; x, t) \} \right), \\ \text{P}_{-2} M(\lambda; x, t) &= \left(0, A[\zeta_n^*] e^{2i\theta(\zeta_n^*; x, t)} \mu_{-1}(\zeta_n^*; x, t) \right). \end{aligned} \quad (2.34)$$

- *Jump condition:*

$$M^-(\lambda; x, t) = M^+(\lambda; x, t)[I - J(\lambda; x, t)], \text{ as } \lambda \in \Sigma, \quad (2.35)$$

where

$$J(\lambda; x, t) = e^{i\theta(\lambda; x, t)\widehat{\sigma}_3} \begin{pmatrix} 0 & -\tilde{\rho}(\lambda) \\ \rho(\lambda) & \rho(\lambda)\tilde{\rho}(\lambda) \end{pmatrix}. \quad (2.36)$$

- *Asymptotic behavior:*

$$M(\lambda; x, t) = e^{i\nu_{-(x,t)\sigma_3}} + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \rightarrow \infty. \quad (2.37)$$

Proof. For the analyticity of $M(\lambda; x, t)$, It follows from Eqs. (2.5) and (2.31) that for each double poles $\zeta_n \in D^+$ or $\zeta_n^* \in D^-$. Now, we consider $\zeta_n \in D^+$, and obtain

$$\begin{aligned} \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\psi_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} e^{i\theta(\zeta_n; x, t)} \right] = A[\zeta_n] [\psi'_{-2}(\zeta_n; x, t) + B[\zeta_n] \psi_{-2}(\zeta_n; x, t)] \\ &= A[\zeta_n] \{ [\mu_{-2}(\zeta_n; x, t) e^{-i\theta(\zeta_n; x, t)}]' + B[\zeta_n] \mu_{-2}(\zeta_n; x, t) e^{-i\theta(\zeta_n; x, t)} \}, \Rightarrow \\ \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \{ \mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)] \mu_{-2}(\zeta_n; x, t) \}, \Rightarrow \\ \operatorname{Res}_{\lambda=\zeta_n} M(\lambda; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \{ \mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)] \mu_{-2}(\zeta_n; x, t) \}, 0 \right), \end{aligned}$$

where the “’” denotes the partial derivative with respect to λ (here $\lambda = \zeta_n$), and

$$\begin{aligned} P_{-2} \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\psi_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= P_{-2} \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} e^{i\theta(\zeta_n; x, t)} \right] = A[\zeta_n] \mu_{-2}(\zeta_n; x, t) e^{-i\theta(\zeta_n; x, t)}, \Rightarrow \\ P_{-2} \operatorname{Res}_{\lambda=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \mu_{-2}(\zeta_n; x, t), \Rightarrow \\ P_{-2} M(\lambda; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \mu_{-2}(\zeta_n; x, t), 0 \right). \end{aligned}$$

Similarly, we also can obtain the analyticity for $\zeta_n^* \in D^-$. It follows from Eqs. (2.5) and (2.10) that

$$\begin{cases} \mu_{+1}(\lambda; x, t) = \mu_{-1}(\lambda; x, t) s_{11}(\lambda) + \mu_{-2}(\lambda; x, t) e^{-2i\theta(\lambda; x, t)} s_{21}(\lambda), \\ \mu_{+2}(\lambda; x, t) = \mu_{-1}(\lambda; x, t) e^{2i\theta(\lambda; x, t)} s_{12}(\lambda) + \mu_{-2}(\lambda; x, t) s_{22}(\lambda), \end{cases}$$

by combining Eqs. (2.12) and (2.33), one can obtain

$$\begin{aligned} M^+(\lambda; x, t) &= \left(\frac{\mu_{+1}(\lambda; x, t)}{s_{11}(\lambda)}, \mu_{-2}(\lambda; x, t) \right) = \left(\mu_{-1}(\lambda; x, t) + \mu_{-2}(\lambda; x, t) e^{-2i\theta(\lambda; x, t)} \rho(\lambda), \mu_{-2}(\lambda; x, t) \right), \\ M^-(\lambda; x, t) &= \left(\mu_{-1}(\lambda; x, t), \frac{\mu_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right) = \left(\mu_{-1}(\lambda; x, t), \mu_{-1}(\lambda; x, t) e^{2i\theta(\lambda; x, t)} \tilde{\rho}(\lambda) + \mu_{-2}(\lambda; x, t) \right), \end{aligned}$$

and

$$\begin{aligned} \left(\mu_{-1}(\lambda; x, t), \frac{\mu_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right) &= \left(\frac{\mu_{+1}(\lambda; x, t)}{s_{11}(\lambda)}, \mu_{-2}(\lambda; x, t) \right) \begin{pmatrix} 1 & e^{2i\theta(\lambda; x, t)} \tilde{\rho}(\lambda) \\ -e^{-2i\theta(\lambda; x, t)} \rho(\lambda) & 1 - \rho(\lambda) \tilde{\rho}(\lambda) \end{pmatrix}, \\ M^-(\lambda; x, t) &= M^+(\lambda; x, t) (I - J(\lambda; x, t)), \end{aligned}$$

where $J(\lambda; x, t)$ is given by Eq. (2.36). The asymptotic behaviors of the modified Jost solutions $\mu_{\pm}(\lambda; x, t)$ and scattering matrix $S(\lambda)$ given in Propositions (5) and (6) can easily lead to that of $M(\lambda; x, t)$. Specifically,

$$\begin{aligned} M^+(\lambda; x, t) &= \left(\frac{\mu_{+1}(\lambda; x, t)}{s_{11}(\lambda)}, \mu_{-2}(\lambda; x, t) \right) = \begin{pmatrix} \frac{e^{i\nu_+(x, t)}}{e^{-i\nu_-}} & 0 \\ 0 & e^{-i\nu_-(x, t)} \end{pmatrix} + O\left(\frac{1}{\lambda}\right) = e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \in D^+, \\ M^-(\lambda; x, t) &= \left(\mu_{-1}(\lambda; x, t), \frac{\mu_{+2}(\lambda; x, t)}{s_{22}(\lambda)} \right) = \begin{pmatrix} e^{i\nu_-(x, t)} & 0 \\ 0 & \frac{e^{-i\nu_+(x, t)}}{e^{i\nu_-}} \end{pmatrix} + O\left(\frac{1}{\lambda}\right) = e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \in D^-. \end{aligned}$$

Therefore, we obtain the asymptotic behavior (2.37) of $M(\lambda; x, t)$ when $\lambda \rightarrow \infty$. This completes the proof. \square

By subtracting out the asymptotic values as $\lambda \rightarrow \infty$ and the singularity contributions, one can regularize the RH problem as a normative form. Then, applying the Plemelj's formula, the solutions of the corresponding matrix RH problem can be established by an integral equation.

Proposition 9. *The solution of the above-mentioned matrix Riemann-Hilbert problem can be expressed as*

$$\begin{aligned} M(\lambda; x, t) &= e^{i\nu_-(x, t)\sigma_3} + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(\xi; x, t) J(\xi; x, t)}{\xi - \lambda} d\xi + \\ &\quad \sum_{n=1}^{2N} \left(C_n(\lambda) \left[\mu'_{-2}(\zeta_n; x, t) + \left(D_n + \frac{1}{\lambda - \zeta_n} \right) \mu_{-2}(\zeta_n; x, t) \right], \right. \\ &\quad \left. \tilde{C}_n(\lambda) \left[\mu'_{-1}(\zeta_n^*; x, t) + \left(\tilde{D}_n + \frac{1}{\lambda - \zeta_n^*} \right) \mu_{-1}(\zeta_n^*; x, t) \right] \right), \end{aligned} \quad (2.38)$$

where $\lambda \in \mathbb{C} \setminus \Sigma$, \int_{Σ} is an integral along the oriented contour exhibited in Fig. 1, and

$$\begin{aligned} C_n(\lambda) &= \frac{A[\zeta_n]}{\lambda - \zeta_n} e^{-2i\theta(\zeta_n; x, t)}, \quad \tilde{C}_n(\lambda) = \frac{A[\zeta_n^*]}{\lambda - \zeta_n^*} e^{2i\theta(\zeta_n^*; x, t)}, \\ D_n &= B[\zeta_n] - 2i\theta'(\zeta_n; x, t), \quad \tilde{D}_n = B[\zeta_n^*] + 2i\theta'(\zeta_n^*; x, t), \end{aligned} \quad (2.39)$$

μ_{-k} and μ'_{-k} ($k = 1, 2$) satisfy

$$\begin{aligned} \mu_{-1}(\zeta_n^*; x, t) &= e^{i\nu_{-}(x, t)\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{s=1}^{2N} C_s(\zeta_n^*) \left[\mu'_{-2}(\zeta_s; x, t) + \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} \right) \mu_{-2}(\zeta_s; x, t) \right] + \\ &\quad \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t)J(\xi; x, t))_1}{\xi - \zeta_n^*} d\xi, \\ \mu_{-2}(\zeta_s; x, t) &= e^{i\nu_{-}(x, t)\sigma_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{2N} \tilde{C}_j(\zeta_s) \left[\mu'_{-1}(\zeta_j^*; x, t) + \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} \right) \mu_{-1}(\zeta_j^*; x, t) \right] + \\ &\quad \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t)J(\xi; x, t))_2}{\xi - \zeta_s} d\xi, \\ \mu'_{-1}(\zeta_n^*; x, t) &= - \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*)}{\zeta_n^* - \zeta_s} \left[\mu'_{-2}(\zeta_s; x, t) + \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \mu_{-2}(\zeta_s; x, t) \right] + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t)J(\xi; x, t))_1}{(\xi - \zeta_n^*)^2} d\xi, \\ \mu'_{-2}(\zeta_s; x, t) &= - \sum_{j=1}^{2N} \frac{\tilde{C}_j(\zeta_s)}{\zeta_s - \zeta_j^*} \left[\mu'_{-1}(\zeta_j^*; x, t) + \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) \mu_{-1}(\zeta_j^*; x, t) \right] + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t)J(\xi; x, t))_2}{(\xi - \zeta_s)^2} d\xi, \end{aligned} \quad (2.40)$$

where $(M^+(\xi; x, t)J(\xi; x, t))_j$ ($j = 1, 2$) represent the j th column of matrix $M^+(\xi; x, t)J(\xi; x, t)$.

Proof. In order to regularize the RH problem, one has to subtract out the asymptotic values as $\lambda \rightarrow \infty$ which exhibited in Eq. (2.37) and the singularity contributions. Then, the jump condition (2.35) becomes

$$\begin{aligned} M^-(\lambda; x, t) - e^{-i\nu_{-}(x, t)\sigma_3} - \sum_{n=1}^{2N} \left[\frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n} + \frac{\text{Res } M(\lambda; x, t)}{\lambda - \zeta_n} + \frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n^*} + \frac{\text{Res } M(\lambda; x, t)}{\lambda - \zeta_n^*} \right] &= M^+(\lambda; x, t) \\ - e^{i\nu_{-}(x, t)\sigma_3} - \sum_{n=1}^{2N} \left[\frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n} + \frac{\text{Res } M(\lambda; x, t)}{\lambda - \zeta_n} + \frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n^*} + \frac{\text{Res } M(\lambda; x, t)}{\lambda - \zeta_n^*} \right] &- M^+(\lambda; x, t)J(\lambda; x, t), \end{aligned} \quad (2.41)$$

where $\frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n}$, $\frac{\text{Res } M(\lambda; x, t)}{\lambda = \zeta_n}$, $\frac{P_{-2} M(\lambda; x, t)}{\lambda = \zeta_n^*}$, $\frac{\text{Res } M(\lambda; x, t)}{\lambda = \zeta_n^*}$ have given in Eq. (2.34). By using Plemelj's formula, one can obtain the solution (2.38) with formula (2.39) of the matrix RH problem. By combining Eqs. (2.33) and (2.38), $\mu_{-1}(\zeta_n^*; x, t)$ is the first column element of the solution (2.38) as the double-pole $\lambda = \zeta_n^* \in D^-$, $\mu_{-2}(\zeta_s; x, t)$ is the second column element of the solution (2.38) as the double-pole $\lambda = \zeta_s \in D^+$. The specific expressions of $\mu_{-1}(\zeta_n^*; x, t)$, $\mu_{-2}(\zeta_s; x, t)$ and their first derivative to λ are provided in Eq. (2.40). \square

2.2.2. Reconstruction Formula of the Potential.

From the solution (2.38) of the matrix RH problem, we have

$$M(\lambda; x, t) = e^{i\nu_{-}(x, t)\sigma_3} + \frac{M^{[1]}(x, t)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \text{ as } \lambda \rightarrow \infty, \quad (2.42)$$

where

$$\begin{aligned} M^{[1]}(x, t) &= - \frac{1}{2\pi i} \int_{\Sigma} M^+(\xi; x, t)J(\xi; x, t)d\xi + \sum_{n=1}^{2N} \left\{ A[\zeta_n]e^{-2i\theta(\zeta_n; x, t)} [\mu'_{-2}(\zeta_n; x, t) + D_n\mu_{-2}(\zeta_n; x, t)], \right. \\ &\quad \left. A[\zeta_n^*]e^{2i\theta(\zeta_n^*; x, t)} [\mu'_{-1}(\zeta_n^*; x, t) + \tilde{D}_n\mu_{-1}(\zeta_n^*; x, t)] \right\}. \end{aligned} \quad (2.43)$$

Substituting Eq. (2.42) into Eq. (2.6) and matching $O(\lambda)$ term, we have

$$O(\lambda) : i[M^{[1]}(x, t)\sigma_3 - \sigma_3 M^{[1]}(x, t)] = Q(x, t)e^{i\nu_{-}(x, t)\sigma_3},$$

then, by expanding the above equation, one can find the reconstruction formula of the double-pole solution (potential) for the TOFKN equation (1.1) with ZBCs as follows

$$q(x, t) = -2ie^{i\nu_-(x, t)} M_{12}^{[1]}(x, t), \quad (2.44)$$

where $M_{12}^{[1]}(x, t)$ represents the first row and second column element of the matrix $M^{[1]}(x, t)$, and

$$M_{12}^{[1]}(x, t) = -\frac{1}{2\pi i} \int_{\Sigma} (M^+(\xi; x, t) J(\xi; x, t))_{12} d\xi + \sum_{n=1}^{2N} \{A[\zeta_n^*] e^{2i\theta(\zeta_n^*; x, t)} [\mu'_{-11}(\zeta_n^*; x, t) + \tilde{D}_n \mu_{-11}(\zeta_n^*; x, t)]\}, \quad (2.45)$$

where $\mu_{-11}(\zeta_n^*; x, t), \mu'_{-11}(\zeta_n^*; x, t)$ represents the first row element of the column vector $\mu_{-1}(\zeta_n^*; x, t), \mu'_{-1}(\zeta_n^*; x, t)$ respectively. When taking row vector $\alpha = (\alpha^{(1)}, \alpha^{(2)})$ and column vector $\gamma = (\gamma^{(1)}, \gamma^{(2)})^T$, where

$$\begin{aligned} \alpha^{(1)} &= \left(A[\zeta_n^*] e^{2i\theta(\zeta_n^*; x, t)} \tilde{D}_n \right)_{1 \times 2N}, & \alpha^{(2)} &= \left(A[\zeta_n^*] e^{2i\theta(\zeta_n^*; x, t)} \right)_{1 \times 2N}, \\ \gamma^{(1)} &= (\mu_{-11}(\zeta_n^*; x, t))_{1 \times 2N}, & \gamma^{(2)} &= (\mu'_{-11}(\zeta_n^*; x, t))_{1 \times 2N}, \end{aligned} \quad (2.46)$$

we can obtain a more concise reconstruction formulation of the double poles solution (potential) for the TOFKN equation (1.1) with ZBCs and as follows

$$q(x, t) = -2ie^{i\nu_-(x, t)} \left(\alpha \gamma - \frac{1}{2\pi i} \int_{\Sigma} (M^+(\xi; x, t) J(\xi; x, t))_{12} d\xi \right). \quad (2.47)$$

2.2.3. Trace Formulae.

The so-called trace formulae are the scattering coefficients $s_{11}(\lambda)$ and $s_{22}(\lambda)$ are formulated in terms of the discrete spectrum Λ and reflection coefficients $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$. We know that $s_{11}(\lambda), s_{22}(\lambda)$ are analytic on D^+, D^- , respectively. The discrete spectral points ζ_n 's are the double zeros of $s_{11}(\lambda)$, while ζ_n^* 's are the double zeros of $s_{22}(\lambda)$. Define the functions $\beta^{\pm}(\lambda)$ as follows:

$$\begin{aligned} \beta^+(\lambda) &= s_{11}(\lambda) \prod_{n=1}^{2N} \left(\frac{\lambda - \zeta_n^*}{\lambda - \zeta_n} \right)^2 e^{i\nu}, \\ \beta^-(\lambda) &= s_{22}(\lambda) \prod_{n=1}^{2N} \left(\frac{\lambda - \zeta_n}{\lambda - \zeta_n^*} \right)^2 e^{-i\nu}. \end{aligned} \quad (2.48)$$

Then, $\beta^+(\lambda)$ and $\beta^-(\lambda)$ are analytic and have no zero in D^+ and D^- , respectively. Furthermore, we have the relation $\beta^+(\lambda)\beta^-(\lambda) = s_{11}(\lambda)s_{22}(\lambda)$ and the asymptotic behaviors $\beta^{\pm}(\lambda) \rightarrow 1$, as $\lambda \rightarrow \infty$.

According to $\det(S(\lambda)) = s_{11}s_{22} - s_{21}s_{12} = 1$, we can derive

$$\frac{1}{s_{11}s_{22}} = 1 - \frac{s_{21}s_{12}}{s_{11}s_{22}} = 1 - \rho(\lambda)\tilde{\rho}(\lambda),$$

by taking logarithm on both sides of the above equation at the same time, that is

$$-\log(s_{11}s_{22}) = \log[1 - \rho(\lambda)\tilde{\rho}(\lambda)] \Rightarrow \log[\beta^+(\lambda)\beta^-(\lambda)] = -\log[1 - \rho(\lambda)\tilde{\rho}(\lambda)],$$

and employing the Plemelj' formula such that we have

$$\log \beta^{\pm}(\lambda) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(\lambda)\tilde{\rho}(\lambda)]}{\xi - \lambda} d\xi, \quad \lambda \in D^{\pm}. \quad (2.49)$$

Then, substituting Eq. (2.49) into Eq. (2.48), we can obtain the trace formulae

$$\begin{aligned} s_{11}(\lambda) &= \exp \left(-\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(\lambda)\tilde{\rho}(\lambda)]}{\xi - \lambda} d\xi \right) \prod_{n=1}^{2N} \left(\frac{\lambda - \zeta_n}{\lambda - \zeta_n^*} \right)^2 e^{-i\nu}, \\ s_{22}(\lambda) &= \exp \left(\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(\lambda)\tilde{\rho}(\lambda)]}{\xi - \lambda} d\xi \right) \prod_{n=1}^{2N} \left(\frac{\lambda - \zeta_n^*}{\lambda - \zeta_n} \right)^2 e^{i\nu}. \end{aligned} \quad (2.50)$$

2.2.4. Reflectionless Potential: Double-Pole Solitons.

Now, we consider a special kind of reflectionless potential $q(x, t)$ with the reflection coefficients $\rho(\lambda) = \tilde{\rho}(\lambda) = 0$. From the Volterra integral equation (2.9), one obtains $\psi_{\pm}(0; x, t) = \mu_{\pm}(0; x, t) = I$, and $s_{11}(0) = 1$. Combining the trace formula, one obtains that there exists an integer $i \in \mathbb{Z}$ such that $\nu = 8 \sum_{n=1}^N \arg(\zeta_n) + 2\pi i$.

Then Eqs. (2.40) and (2.47) with $J(\lambda; x, t) = 0_{2 \times 2}$ become

$$\begin{aligned}\mu_{-1}(\zeta_n^*, x, t) &= e^{i\nu_-(x, t)\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{s=1}^{2N} C_s(\zeta_n^*) \left[\mu'_{-2}(\zeta_s; x, t) + \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} \right) \mu_{-2}(\zeta_s; x, t) \right], \\ \mu_{-2}(\zeta_s; x, t) &= e^{i\nu_-(x, t)\sigma_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{2N} \tilde{C}_j(\zeta_s) \left[\mu'_{-1}(\zeta_j^*; x, t) + \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} \right) \mu_{-1}(\zeta_j^*; x, t) \right], \\ \mu'_{-1}(\zeta_n^*, x, t) &= - \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*)}{\zeta_n^* - \zeta_s} \left[\mu'_{-2}(\zeta_s; x, t) + \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \mu_{-2}(\zeta_s; x, t) \right], \\ \mu'_{-2}(\zeta_s; x, t) &= - \sum_{j=1}^{2N} \frac{\tilde{C}_j(\zeta_s)}{\zeta_s - \zeta_j^*} \left[\mu'_{-1}(\zeta_j^*; x, t) + \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) \mu_{-1}(\zeta_j^*; x, t) \right],\end{aligned}\tag{2.51}$$

and

$$q(x, t) = -2ie^{i\nu_-(x, t)}\alpha\gamma.\tag{2.52}$$

Theorem 10. *The explicit expression for the double-pole solution of the TOFKN equation (1.1) with ZBCs is given by determinant formula*

$$q(x, t) = 2i \frac{\det(I - G)}{\det R} \left(\frac{\det \tilde{R}}{\det(I - \tilde{G})} \right)^2,\tag{2.53}$$

where

$$\begin{aligned}R &= \begin{pmatrix} 0 & \alpha \\ \tau & I - G \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 0 & \alpha \\ \tau & I - \tilde{G} \end{pmatrix}, \\ \tau &= \begin{pmatrix} \tau^{(1)} \\ \tau^{(2)} \end{pmatrix}, \quad \tau^{(1)} = (1)_{2N \times 1}, \quad \tau^{(2)} = (0)_{2N \times 1},\end{aligned}\tag{2.54}$$

and the $4N \times 4N$ partitioned matrix $G = \begin{pmatrix} G^{(1,1)} & G^{(1,2)} \\ G^{(2,1)} & G^{(2,2)} \end{pmatrix}$ with $G^{(i,j)} = (g_{n,j}^{(i,j)})_{2N \times 2N}$ ($i, j = 1, 2$) is given by

$$\begin{aligned}g_{n,j}^{(1,1)} &= \sum_{s=1}^{2N} C_s(\zeta_n^*) \tilde{C}_j(\zeta_s) \left[-\frac{1}{\zeta_s - \zeta_j^*} \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) + \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} \right) \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} \right) \right], \\ g_{n,j}^{(1,2)} &= \sum_{s=1}^{2N} C_s(\zeta_n^*) \tilde{C}_j(\zeta_s) \left[-\frac{1}{\zeta_s - \zeta_j^*} + \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} \right) \right], \\ g_{n,j}^{(2,1)} &= \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_n^* - \zeta_s} \left[\frac{1}{\zeta_s - \zeta_j^*} \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) - \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} \right) \right], \\ g_{n,j}^{(2,2)} &= \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_n^* - \zeta_s} \left[\frac{1}{\zeta_s - \zeta_j^*} - \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \right],\end{aligned}$$

and the $4N \times 4N$ partitioned matrix $\tilde{G} = \begin{pmatrix} \tilde{G}^{(1,1)} & \tilde{G}^{(1,2)} \\ \tilde{G}^{(2,1)} & \tilde{G}^{(2,2)} \end{pmatrix}$ with $\tilde{G}^{(i,j)} = (\tilde{g}_{n,j}^{(i,j)})_{2N \times 2N}$ ($i, j = 1, 2$) is given by

$$\begin{aligned}\tilde{g}_{n,j}^{(1,1)} &= \zeta_n^* \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_s} \left[-\frac{1}{\zeta_s - \zeta_j^*} \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) + \frac{\zeta_s}{\zeta_j^*} \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} - \frac{1}{\zeta_s} \right) \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} - \frac{1}{\zeta_j^*} \right) \right], \\ \tilde{g}_{n,j}^{(1,2)} &= \zeta_n^* \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_s} \left[-\frac{1}{\zeta_s - \zeta_j^*} + \frac{\zeta_s}{\zeta_j^*} \left(D_s + \frac{1}{\zeta_n^* - \zeta_s} - \frac{1}{\zeta_s} \right) \right], \\ \tilde{g}_{n,j}^{(2,1)} &= \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_n^* - \zeta_s} \left[\frac{1}{\zeta_s - \zeta_j^*} \left(\tilde{D}_j + \frac{2}{\zeta_s - \zeta_j^*} \right) - \frac{\zeta_s}{\zeta_j^*} \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \left(\tilde{D}_j + \frac{1}{\zeta_s - \zeta_j^*} - \frac{1}{\zeta_j^*} \right) \right], \\ \tilde{g}_{n,j}^{(2,2)} &= \sum_{s=1}^{2N} \frac{C_s(\zeta_n^*) \tilde{C}_j(\zeta_s)}{\zeta_n^* - \zeta_s} \left[\frac{1}{\zeta_s - \zeta_j^*} - \frac{\zeta_s}{\zeta_j^*} \left(D_s + \frac{2}{\zeta_n^* - \zeta_s} \right) \right].\end{aligned}$$

Proof. From the Eqs. (2.46), (2.51) and (2.52), the reflectionless potential is derived by determinant formula:

$$q(x, t) = 2i \frac{\det(R)}{\det(I - G)} e^{2i\nu_-(x, t)}.\tag{2.55}$$

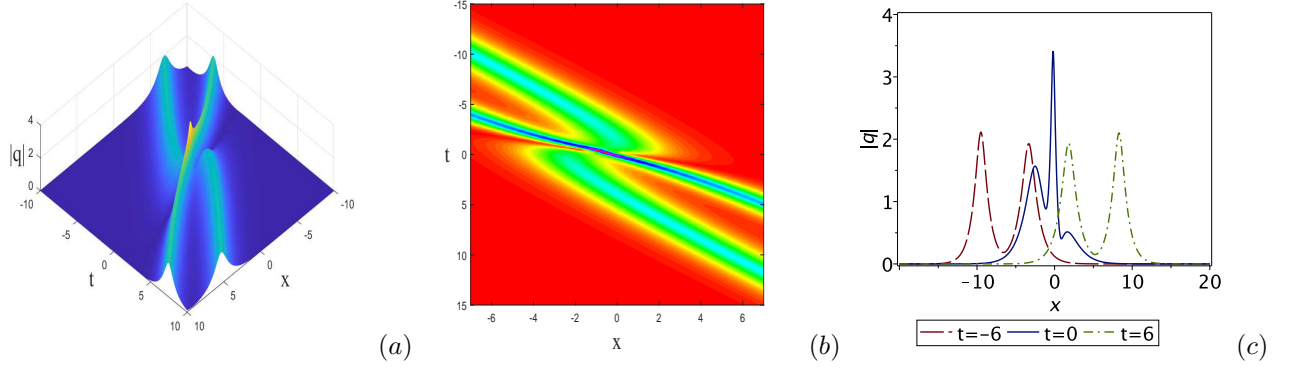


FIGURE 2. (Color online) The 1-double-pole soliton solution of TOFKN equation (1.1) with ZBCs and $N = 1, \zeta_1 = \frac{1}{2} + \frac{1}{2}i, A[\zeta_1] = 1, B[\zeta_1] = 1$. (a) The three-dimensional plot; (b) The density plot; (c) The sectional drawings at $t = -6$ (dashed line), $t = 0$ (solid line), and $t = 6$ (dash-dot line).

However, this formula (2.55) is implicit due to unknown function $\nu_-(x, t)$ is included. We need to deduce an exact form for the reflectionless potential. From the trace formulae (2.50) and Volterra integral equation (2.9) as $x \rightarrow -\infty$, we get

$$M(\lambda; x, t) = I + \lambda \sum_{n=1}^{2N} \left[\frac{P_{-2}(M(\lambda; x, t)/\lambda)}{(\lambda - \zeta_n)^2} + \frac{\text{Res}(M(\lambda; x, t)/\lambda)}{\lambda - \zeta_n} + \frac{P_{-2}(M(\lambda; x, t)/\lambda)}{(\lambda - \zeta_n^*)^2} + \frac{\text{Res}(M(\lambda; x, t)/\lambda)}{\lambda - \zeta_n^*} \right], \quad (2.56)$$

which can yield the γ given by Eq. (2.46) explicitly. Then, substituting γ into the formula of the potential, one yields

$$q(x, t) = 2i \frac{\det(\tilde{R})}{\det(I - \tilde{G})} e^{i\nu_-(x, t)}, \quad (2.57)$$

then, by combining Eq. (2.56) with Eq. (2.57), we can obtain the determinant formula (2.53). Complete the proof. \square

For example, we obtain the N -double-pole solutions of the TOFKN equation (1.1) with ZBCs via Eq. (2.53):

- When taking parameters $N = 1, \zeta_1 = \frac{1}{2} + \frac{1}{2}i, A[\zeta_1] = 1, B[\zeta_1] = 1$, we can obtain the explicit 1-double-pole solution and give out relevant plots in Fig. 2. Figs. 2 (a) and (b) exhibit the three-dimensional and density diagrams for the exact 1-double-pole soliton solution of the TOFKN equation with ZBCs, which is equivalent to the elastic collisions of two bright-bright solitons. Fig. 2 (c) displays the distinct profiles of the exact 1-double-pole soliton solution at $t = \pm 6, 0$.

Compared with the classical second-order flow KN system which also be called the DNLS equation in Ref. [44], the comparison made between the density diagrams of the 1-double-pole soliton solutions for the TOFKN equation and the DNLS equation shows that the trajectories of solutions are different obviously, it also means the introduction of third-order dispersion and quintic nonlinear term of KN systems can affect the trajectories of solutions. By comparing with Ref. [44], we find that the wave heights of the 1-double-pole solution at $x = 0$ and $t = 0$ are consistent, and the wave heights are all 1168/157. Therefore, we make a guess that the energy at $t = 0$ of the central peak for the 1-double-pole solution of the TOFKN equation is the same with that corresponding to the DNLS equation by selecting the same parameters. In other words, third-order dispersion and quintic nonlinear term of KN systems have little effect on the maximum amplitude of the solutions.

Remark 11. The parameter selection of the 1-double-pole soliton solution shown in Fig. 2 is consistent with that in reference [44].

- When taking parameters $N = 2, \zeta_1 = \frac{1}{2} + \frac{1}{2}i, A[\zeta_1] = B[\zeta_1] = 1, \zeta_2 = \frac{1}{3} + \frac{1}{2}i, A[\zeta_2] = B[\zeta_2] = 1$, we can obtain the explicit 2-double-pole solution and give out relevant plots in Fig. 3. Figs. 3 (a) and (b) exhibit the three-dimensional and density diagrams for the exact 2-double-pole soliton solution of the TOFKN equation with ZBCs, which is equivalent to the interaction of two 1-double-pole soliton solutions. Fig. 3 (c) displays the distinct profiles of the exact 2-double-pole soliton solution at $t = \pm 20, 0$.

3. THE IST WITH NZBCS AND DOUBLE POLES SOLUTION

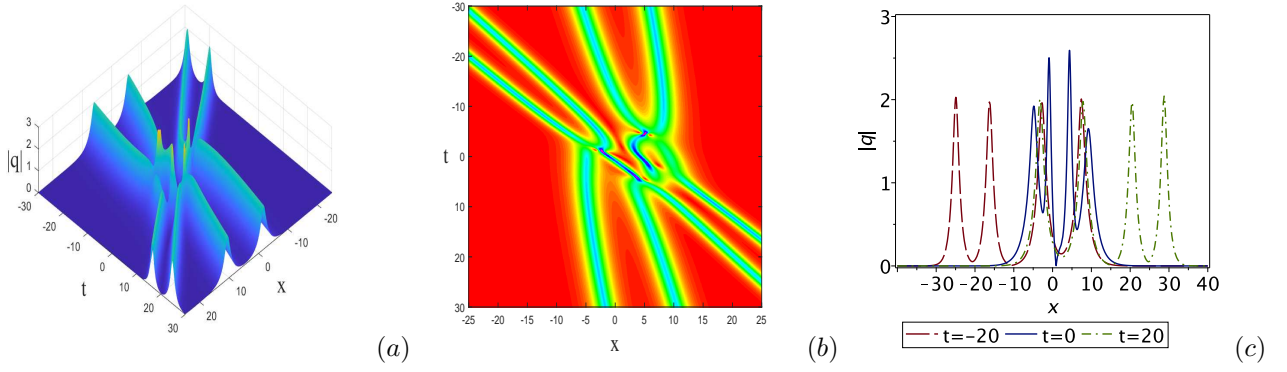


FIGURE 3. (Color online) The 2-double-pole soliton solution of TOFKN equation (1.1) with ZBCs and $N = 2$, $\zeta_1 = \frac{1}{2} + \frac{1}{2}i$, $A[\zeta_1] = B[\zeta_1] = 1$, $\zeta_2 = \frac{1}{3} + \frac{1}{2}i$, $A[\zeta_2] = B[\zeta_2] = 1$. (a) The three-dimensional plot; (b) The density plot; (c) The sectional drawings at $t = -20$ (dashed line), $t = 0$ (solid line), and $t = 20$ (dash-dot line).

In this section, we will find the double-pole solution $q(x, t)$ for the TOFKN equation (1.1) with the NZBCs

$$q(x, t) \sim q_{\pm}, \text{ as } x \rightarrow \pm\infty, \quad (3.1)$$

where $|q_{\pm}| = q_0 > 0$, and q_{\pm} are independent of x, t .

3.1. Direct Scattering with NZBCs and Double Poles.

In this subsection, we will investigate direct scattering with NZBCs and the case of double poles for the modified Zakharov-Shabat eigenvalue problem of TOFKN equation (1.1) in detail.

3.1.1. Jost Solutions, Analyticity, and Continuity.

As $x \rightarrow \pm\infty$, we consider the asymptotic scattering problem of the modified Zakharov-Shabat eigenvalue problem (1.2)-(1.3):

$$\begin{aligned} \Psi_x &= X_{\pm} \Psi, \quad \Psi_t = T_{\pm} \Psi, \quad X_{\pm} = i\lambda^2 \sigma_3 + \lambda Q_{\pm}, \\ T_{\pm} &= \left(4\lambda^4 + \frac{3}{2}q_0^4 - 2\lambda^2 q_0^2 \right) X_{\pm}, \quad Q_{\pm} = \begin{pmatrix} 0 & q_{\pm} \\ -q_{\pm}^* & 0 \end{pmatrix}. \end{aligned} \quad (3.2)$$

Proposition 12. *the fundamental matrix solution of Eq. (3.2) is presented as follows:*

$$\Psi_{\pm}^{\text{bg}}(\lambda; x, t) = \begin{cases} Y_{\pm}(\lambda) e^{i\theta(\lambda; x, t) \sigma_3}, & \lambda \neq \pm i q_0, \text{ and } \lambda + k(\lambda) \neq 0, \\ I + \left(x + \frac{15}{2} q_0^4 t \right) X_{\pm}(\lambda), & \lambda = \pm i q_0, \end{cases} \quad (3.3)$$

where

$$\begin{aligned} Y_{\pm}(\lambda) &= \begin{pmatrix} 1 & \frac{i q_{\pm}}{z} \\ \frac{i q_{\pm}^*}{z} & 1 \end{pmatrix} = I + \frac{i}{z} \sigma_3 Q_{\pm}, \quad z = \lambda + k(\lambda), \\ \theta(\lambda; x, t) &= \lambda k(\lambda) \left[x + \left(4\lambda^4 + \frac{3}{2} q_0^4 - 2\lambda^2 q_0^2 \right) t \right], \quad k^2(\lambda) = \lambda^2 + q_0^2. \end{aligned} \quad (3.4)$$

Proof. For the case of $\lambda = \pm i q_0$, we can directly calculate and obtain $\Psi_{\pm}^{\text{bg}}(\lambda; x, t) = I + \left(x + \frac{15}{2} q_0^4 t \right) X_{\pm}(\lambda)$. However, when $\lambda \neq \pm i q_0$, based on eigenvalues $\pm i \lambda k(\lambda)$, one can derive $k^2(\lambda) = \lambda^2 + q_0^2$, and the eigenvector matrix of X_{\pm} and T_{\pm} can be written

$$Y_{\pm}(\lambda) = \begin{pmatrix} 1 & \frac{i q_{\pm}}{\lambda + k(\lambda)} \\ \frac{i q_{\pm}^*}{\lambda + k(\lambda)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{i q_{\pm}}{z} \\ \frac{i q_{\pm}^*}{z} & 1 \end{pmatrix} = I + \frac{i}{z} \sigma_3 Q_{\pm}, \quad z = \lambda + k(\lambda) \neq 0. \quad (3.5)$$

Now, X_{\pm} and T_{\pm} can be diagonalized by the eigenvector matrix (3.5) as

$$X_{\pm} = Y_{\pm} [i \lambda k(\lambda) \sigma_3] Y_{\pm}^{-1}, \quad T_{\pm} = Y_{\pm} \left[\left(4\lambda^4 + \frac{3}{2} q_0^4 - 2\lambda^2 q_0^2 \right) i \lambda k(\lambda) \sigma_3 \right] Y_{\pm}^{-1}, \quad (3.6)$$

and direct calculation shows that

$$\det Y_{\pm} = 1 + \frac{q_0^2}{z^2} \triangleq \eta, \quad (3.7)$$

and

$$Y_{\pm}^{-1} = \frac{1}{\eta} \begin{pmatrix} 1 & -\frac{iq_{\pm}}{z} \\ -\frac{iq_{\pm}^*}{z} & 1 \end{pmatrix} = \frac{1}{\eta} \left(I - \frac{i}{z} \sigma_3 Q_{\pm} \right). \quad (3.8)$$

By substituting (3.6) into (3.2), we have

$$(Y_{\pm}^{-1}\Psi)_x = i\lambda k(\lambda)\sigma_3(Y_{\pm}^{-1}\Psi), \quad (Y_{\pm}^{-1}\Psi)_t = \left(4\lambda^2 + \frac{3}{2}q_0^4 - 2\lambda^2 q_0^2 \right) i\lambda k(\lambda)\sigma_3(Y_{\pm}^{-1}\Psi), \quad (3.9)$$

from which we can derive the solution of the asymptotic spectral problem (3.2)

$$\Psi = \Psi_{\pm}^{\text{bg}}(\lambda; x, t) = Y_{\pm}(\lambda) e^{i\theta(\lambda; x, t)\sigma_3}, \quad (3.10)$$

where $\theta(\lambda; x, t)$ is given by (3.4). This completes the proof. \square

Since $k(\lambda)$ stands for a two-sheeted Riemann surface. Therefore, in order to avoid multi-valued case of eigenvalue $k(\lambda)$, we define a new uniformization variable: $z = \lambda + k(\lambda)$, which was first introduced by Faddeev and Takhtajan in 1987 [49]. Next, we will illustrate the scattering problem on a standard z -plane instead of the two-sheeted Riemann surface by utilizing two single-valued inverse mappings:

$$k(\lambda) = \frac{1}{2} \left(z + \frac{q_0^2}{z} \right), \quad \lambda = \frac{1}{2} \left(z - \frac{q_0^2}{z} \right), \quad (3.11)$$

where the inverse mapping evidently satisfy the relation $k^2(\lambda) = \lambda^2 + q_0^2$. It means that the Jost solutions $\psi_{\pm}(z; x, t)$ of the Lax pairs (1.2)-(1.3) possess the following asymptotics:

$$\psi_{\pm}(z; x, t) \sim Y_{\pm}(z) e^{i\theta(z; x, t)\sigma_3}, \quad \text{as } x \rightarrow \pm\infty, \quad (3.12)$$

and the modified Jost solutions $\mu_{\pm}(z; x, t)$ are derived by making the transform

$$\mu_{\pm}(z; x, t) = \psi_{\pm}(z; x, t) e^{-i\theta(z; x, t)\sigma_3}, \quad (3.13)$$

such that

$$\mu_{\pm}(z; x, t) \sim Y_{\pm}(z), \quad \text{as } x \rightarrow \pm\infty. \quad (3.14)$$

On the other hand, the modified Jost solutions $\mu_{\pm}(z; x, t)$ satisfy the following equivalent Lax pair:

$$\left[Y_{\pm}^{-1}(z) \mu_{\pm}(z; x, t) \right]_x + i\lambda k(\lambda) \left[Y_{\pm}^{-1}(z) \mu_{\pm}(z; x, t), \sigma_3 \right] = Y_{\pm}^{-1}(z) \Delta X_{\pm} \mu_{\pm}(z; x, t), \quad (3.15)$$

$$\left[Y_{\pm}^{-1}(z) \mu_{\pm}(z; x, t) \right]_t + i\lambda k(\lambda) \left(4\lambda^4 + \frac{3}{2}q_0^4 - 2\lambda^2 q_0^2 \right) \left[Y_{\pm}^{-1}(z) \mu_{\pm}(z; x, t), \sigma_3 \right] = Y_{\pm}^{-1}(z) \Delta T_{\pm} \mu_{\pm}(z; x, t), \quad (3.16)$$

where $\Delta X_{\pm} = X(\lambda(z); x, t) - X_{\pm}(\lambda(z); x, t) = \lambda(z)[Q(x, t) - Q_{\pm}(x, t)] := \lambda(z)\Delta Q_{\pm}(x, t)$ and $\Delta T_{\pm} = T(\lambda(z); x, t) - T_{\pm}(\lambda(z); x, t)$. Equations (3.15) and (3.16) can be written in full derivative form

$$d(e^{-i\theta(z; x, t)\sigma_3} Y_{\pm}^{-1}(z) \mu_{\pm}(z; x, t)) = e^{-i\theta(z; x, t)\sigma_3} [Y_{\pm}^{-1}(z) (\Delta X_{\pm} dx + \Delta T_{\pm} dt) \mu_{\pm}(z; x, t)], \quad (3.17)$$

which lead to Volterra integral equations

$$\mu_{\pm}(z; x, t) = Y_{\pm}(z) + \begin{cases} \lambda(z) \int_{\pm\infty}^x Y_{\pm}(y) e^{ik(z)\lambda(z)(x-y)\sigma_3} [Y_{\pm}^{-1}(y) \Delta Q_{\pm}(y, t) \mu_{\pm}(y; y, t)] dy, & z \neq 0, \pm iq_0, \\ \lambda(z) \int_{\pm\infty}^x [I + (x-y)X_{\pm}(y)] \Delta Q_{\pm}(y, t) \mu_{\pm}(y; y, t) dy, & z = \pm iq_0. \end{cases} \quad (3.18)$$

Since the exponential function contains $k(z)\lambda(z)$, we define the z plane by the positive and negative values of $\text{Im}(k(z)\lambda(z))$, that is

$$k(z)\lambda(z) = \frac{1}{4} \left(z^2 - \frac{q_0^4}{z^2} \right) = \frac{|z|^4 z^2 - q_0^4 z^2 + q_0^4 z^2 - q_0^4 (z^*)^2}{4|z|^4} = \frac{1}{4|z|^4} [(|z|^4 - q_0^4)z^2 + 4iq_0^4 \text{Re}z \text{Im}z],$$

$$\text{Im}[k(z)\lambda(z)] = \frac{1}{4|z|^4} [(2|z|^4 - 2q_0^4) \text{Re}z \text{Im}z + 4q_0^4 \text{Re}z \text{Im}z] = \frac{1}{4|z|^4} [(2|z|^4 + 2q_0^4) \text{Re}z \text{Im}z],$$

and

$$\begin{aligned} \text{Im}[k(z)\lambda(z)] &> 0, \quad \text{that is } \text{Re}z \text{Im}z > 0, \\ \text{Im}[k(z)\lambda(z)] &< 0, \quad \text{that is } \text{Re}z \text{Im}z < 0, \\ \text{Im}[k(z)\lambda(z)] &= 0, \quad \text{that is } z^4 = q_0^4 \Rightarrow \begin{cases} z = \pm iq_0, \\ z = \pm q_0. \end{cases} \end{aligned}$$

Therefore, from the above mapping relation between the λ -plane and z -plane, we define Σ and D^{\pm} on the z -plane as $\Sigma := \mathbb{R} \cup i\mathbb{R} \setminus \{0\}$, $D^{\pm} := \{z \in \mathbb{C} \mid \pm (\text{Re}z)(\text{Im}z) > 0\}$, which are shown in Fig. 4.

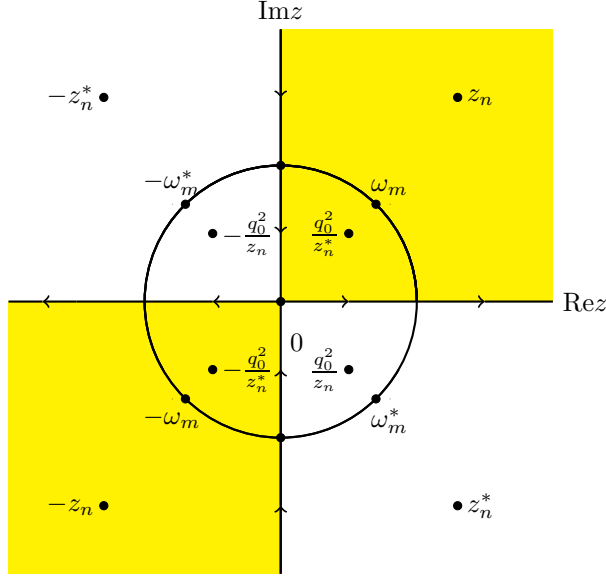


FIGURE 4. (Color online) Distribution of the discrete spectrum and jumping curves for the Riemann-Hilbert problem on complex z -plane, Region $D^+ = \{k \in \mathbb{C} | \text{Re} z \text{Im} z > 0\}$ (yellow region), region $D^- = \{z \in \mathbb{C} | \text{Re} z \text{Im} z < 0\}$ (white region).

Then, according to the Volterra integral equations (3.18) for the case of $z \neq 0, \pm i q_0$, the following analyticity of the (modified) Jost solution can be derived as follows:

Proposition 13. Suppose $(1+|x|)(q(x, t) - q_{\pm}) \in L^1(\mathbb{R}^{\pm})$. Then, Jost solution $\psi(z; x, t)$ (modified Jost solution $\mu(z; x, t)$) have the following properties:

- Eq. (1.2) has the unique solution $\psi(z; x, t)$ ($\mu(z; x, t)$) satisfying Eq. (3.12) (Eq. (3.14)) on Σ .
- The column vectors $\psi_{+1}(z; x, t)$ ($\mu_{+1}(z; x, t)$) and $\psi_{-2}(z; x, t)$ ($\mu_{-2}(z; x, t)$) can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$.
- The column vectors $\psi_{+2}(z; x, t)$ ($\mu_{+2}(z; x, t)$) and $\psi_{-1}(z; x, t)$ ($\mu_{-1}(z; x, t)$) can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$.

Proof. We define modified Jost solutions $\mu_{\pm} = (\mu_{\pm 1}, \mu_{\pm 2}) = \begin{pmatrix} \mu_{\pm 11} & \mu_{\pm 21} \\ \mu_{\pm 12} & \mu_{\pm 22} \end{pmatrix}$, in which $\mu_{\pm 1}$ and $\mu_{\pm 2}$ is the first and second columns of μ_{\pm} , respectively. Then, taking μ_- as an example, Eq. (3.18) (here $z \neq 0, \pm i q_0$) can be rewritten as

$$\begin{pmatrix} \mu_{-11} & \mu_{-21} \\ \mu_{-12} & \mu_{-22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{i q_-}{z} \\ \frac{i q_-}{z} & 1 \end{pmatrix} + \lambda \int_{-\infty}^x \begin{pmatrix} -\frac{1}{q_0^2 + z^2} [\Xi_1 e^{-2ik(z)\lambda(z)(x-y)} + \Xi_2] & \frac{1}{q_0^2 + z^2} [\Xi_3 e^{2ik(z)\lambda(z)(x-y)} + \Xi_4] \\ -\frac{1}{q_0^2 + z^2} [\Xi_5 e^{-2ik(z)\lambda(z)(x-y)} + \Xi_6] & \frac{1}{q_0^2 + z^2} [\Xi_7 e^{2ik(z)\lambda(z)(x-y)} + \Xi_8] \end{pmatrix} dy,$$

where

$$\begin{aligned} \Xi_1 &= i q^* \mu_{-11} q_- z - i \mu_{-11} q_0^2 z - q \mu_{-12} q_0^2 + \mu_{-12} q_0^2 q_-, \\ \Xi_2 &= -i q^* \mu_{-11} q_- z + i \mu_{-11} q_0^2 z - q \mu_{-12} z^2 + \mu_{-12} q_- z^2, \\ \Xi_3 &= i q^* \mu_{-21} q_- z - i \mu_{-21} q_0^2 z + q \mu_{-22} z^2 - \mu_{-22} q_- z^2, \\ \Xi_4 &= -i q^* \mu_{-21} q_- z + i \mu_{-21} q_0^2 z + q \mu_{-22} q_0^2 - \mu_{-22} q_0^2 q_-, \\ \Xi_5 &= i q \mu_{-12} q_-^* z - i \mu_{-12} q_0^2 z + q^* \mu_{-11} z^2 - \mu_{-11} q_-^* z^2, \\ \Xi_6 &= -i q \mu_{-12} q_-^* z + i \mu_{-12} q_0^2 z + q^* \mu_{-11} q_0^2 - \mu_{-11} q_0^2 q_-^*, \\ \Xi_7 &= i q \mu_{-22} q_-^* z - i \mu_{-22} q_0^2 z - q^* \mu_{-21} q_0^2 + \mu_{-21} q_0^2 q_-^*, \\ \Xi_8 &= -i q \mu_{-22} q_-^* z + i \mu_{-22} q_0^2 z - q^* \mu_{-21} z^2 + \mu_{-21} q_-^* z^2. \end{aligned}$$

Note that $\mu_{-1} = \begin{pmatrix} 1 \\ \frac{i q_-}{z} \end{pmatrix} + \lambda(z) \int_{-\infty}^x \begin{pmatrix} -\frac{1}{q_0^2 + z^2} [\Xi_1 e^{-2ik(z)\lambda(z)(x-y)} + \Xi_2] \\ -\frac{1}{q_0^2 + z^2} [\Xi_5 e^{-2ik(z)\lambda(z)(x-y)} + \Xi_6] \end{pmatrix} dy$, where $e^{-2ik(z)\lambda(z)(x-y)} = e^{-2i(x-y)\text{Re}[k(z)\lambda(z)]} e^{2(x-y)\text{Im}[k(z)\lambda(z)]}$. Since $x - y > 0$ and $\text{Im}[k(z)\lambda(z)] < 0$, that is $\text{Re} z \text{Im} z < 0$, it indicates that the first column of μ_- is analytically extended to D^- . In addition, since $\text{Im}[k(z)\lambda(z)] = 0$ when $z \in \Sigma$, it demonstrate that the first column of μ_- is continuously extended to $D^- \cup \Sigma$. In the same way, we also obtain the analyticity and continuity of μ_{-2}, μ_{+1} and μ_{+2} . Similarly, the analyticity and continuity for the Jost solutions $\psi_{\pm}(\lambda; x, t)$ can be simply presented via the analyticity and continuity of $\mu_{\pm}(\lambda; x, t)$ and the relation (3.13). This completes the proof. \square

Similar to Proposition 2 in Sec. 2 with ZBCs, one can also confirm that the Jost solutions $\psi(z; x, t)$ are the simultaneous solutions for both parts of the modified Zakharov-Shabat eigenvalue problem Eqs. (1.2)-(1.3) with NZBCs.

3.1.2. Scattering Matrix, Analyticity, and Continuity.

Since $\psi_{\pm}(z; x, t)$ are two fundamental matrix solutions of the modified Zakharov-Shabat eigenvalue problem Eqs. (1.2)-(1.3) in $z \in \Sigma \setminus \{\pm iq_0\}$, there exists a linear relation between $\psi_+(z; x, t)$ and $\psi_-(z; x, t)$, so one can define the constant scattering matrix $S(z)(s_{ij}(z))_{2 \times 2}$ such that

$$\psi_+(z; x, t) = \psi_-(z; x, t)S(z), \quad z \in \Sigma \setminus \{\pm iq_0\}. \quad (3.19)$$

According to the (2.11) of the derivation process, and combining with formula $\det(\psi_{\pm}(z; x, t)) = \det(Y_{\pm}) = \eta$, the scattering coefficients shown as:

$$\begin{aligned} s_{11}(z) &= \frac{\det(\psi_{+1}(z; x, t), \psi_{-2}(z; x, t))}{\eta}, & s_{12}(z) &= \frac{\det(\psi_{+2}(z; x, t), \psi_{-2}(z; x, t))}{\eta}, \\ s_{21}(z) &= \frac{\det(\psi_{-1}(z; x, t), \psi_{+1}(z; x, t))}{\eta}, & s_{22}(z) &= \frac{\det(\psi_{-1}(z; x, t), \psi_{+2}(z; x, t))}{\eta}. \end{aligned} \quad (3.20)$$

From these determinant representations, one can extend the analytical regions of $s_{11}(z)$ and $s_{22}(z)$.

Proposition 14. *Suppose that $q(x, t) - q_{\pm} \in L^1(\mathbb{R}^{\pm})$. Then, $s_{11}(\lambda)$ can be analytically extended to D^+ and continuously extended to $D^+ \cup (\Sigma \setminus \{\pm iq_0\})$, while $s_{22}(\lambda)$ can be analytically extended to D^- and continuously extended to $D^- \cup (\Sigma \setminus \{\pm iq_0\})$. Moreover, both $s_{12}(\lambda)$ and $s_{21}(\lambda)$ are continuous in $\Sigma \setminus \{\pm iq_0\}$.*

Proof. The proposition can be verified by using Proposition (13) and Eq. (3.20). \square

Proposition 15. *Suppose that $(1 + |x|)(q(x, t) - q_{\pm}) \in L^1(\mathbb{R}^{\pm})$. Then, $k(z)s_{11}(\lambda)$ can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$, while $k(z)s_{22}(\lambda)$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$. Moreover, both $k(z)s_{12}(\lambda)$ and $k(z)s_{21}(\lambda)$ are continuous in Σ .*

Proof. The proposition can be verified by using Proposition (13) and Eqs. (3.7), (3.11) and (3.20). \square

In order to study the RH problem in the inverse process, we focus on the potential without spectral singularity, and suppose $s_{ij}(z)(i, j = 1, 2)$ are continuous in the branch points $\{\pm iq_0\}$. The reflection coefficients $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ are defined as follows:

$$\rho(z) = \frac{s_{21}(z)}{s_{11}(z)}, \quad \tilde{\rho}(z) = \frac{s_{12}(z)}{s_{22}(z)}, \quad z \in \Sigma, \quad (3.21)$$

which will be utilized in the inverse scattering problem.

3.1.3. Symmetry Structures.

Compared with the case of ZBCs, The symmetries of $X(z; x, t), T(z; x, t)$, Jost solutions, modified Jost solutions, scattering matrix and reflection coefficients of the case of NZBCs will be more complicated.

Proposition 16. *For the case of NZBCs, the Jost solutions, modified Jost solutions, scattering matrix and reflection coefficients admit following three kinds of reduction conditions on the z -plane:*

- *The first symmetry reduction*

$$\begin{aligned} X(z; x, t) &= \sigma_2 X(z^*; x, t)^* \sigma_2, & T(z; x, t) &= \sigma_2 T(z^*; x, t)^* \sigma_2, \\ \psi_{\pm}(z; x, t) &= \sigma_2 \psi_{\pm}(z^*; x, t)^* \sigma_2, & \mu_{\pm}(z; x, t) &= \sigma_2 \mu_{\pm}(z^*; x, t)^* \sigma_2, \\ S(z) &= \sigma_2 S(z^*)^* \sigma_2, & \rho(z) &= -\tilde{\rho}(z^*)^*. \end{aligned} \quad (3.22)$$

- *The second symmetry reduction*

$$\begin{aligned} X(z; x, t) &= \sigma_1 X(-z^*; x, t)^* \sigma_1, & T(z; x, t) &= \sigma_1 T(-z^*; x, t)^* \sigma_1, \\ \psi_{\pm}(z; x, t) &= \sigma_1 \psi_{\pm}(-z^*; x, t)^* \sigma_1, & \mu_{\pm}(z; x, t) &= \sigma_1 \mu_{\pm}(-z^*; x, t)^* \sigma_1, \\ S(z) &= \sigma_1 S(-z^*)^* \sigma_1, & \rho(z) &= \tilde{\rho}(-z^*)^*. \end{aligned} \quad (3.23)$$

• *The third symmetry reduction*

$$\begin{aligned} X(z; x, t) &= X\left(-\frac{q_0^2}{z}; x, t\right), \quad T(z; x, t) = T\left(-\frac{q_0^2}{z}; x, t\right), \\ \psi_{\pm}(z; x, t) &= \frac{i}{z}\psi_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3 Q_{\pm}, \quad \mu_{\pm}(z; x, t) = \frac{i}{z}\mu_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3 Q_{\pm}, \\ S(z) &= (\sigma_3 Q_-)^{-1}S\left(-\frac{q_0^2}{z}\right)\sigma_3 Q_+, \quad \rho(z) = \frac{q_-^*}{q_-}\bar{\rho}\left(-\frac{q_0^2}{z}\right). \end{aligned} \quad (3.24)$$

Proof. The proof of the first symmetry reduction and the second symmetry reduction are similar to Proposition (4). We only need to replace λ in Proposition (4) with z . Next, we mainly prove the third symmetry reduction.

Since $\lambda(z) = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right)$, one can derive $\lambda\left(-\frac{q_0^2}{z}\right) = \frac{1}{2}\left(-\frac{q_0^2}{z} + z\right) = \lambda(z)$, we have

$$\begin{aligned} X(\lambda(z); x, t) &= X\left(\lambda\left(-\frac{q_0^2}{z}\right); x, t\right) \Rightarrow X(z; x, t) = X\left(-\frac{q_0^2}{z}; x, t\right), \\ T(\lambda(z); x, t) &= T\left(\lambda\left(-\frac{q_0^2}{z}\right); x, t\right) \Rightarrow T(z; x, t) = T\left(-\frac{q_0^2}{z}; x, t\right). \end{aligned} \quad (3.25)$$

Then $\psi_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)C$ is also the solution of Eqs. (1.2)-(1.3), for any 2×2 matrix C independent of x and t . Hence, according to (3.12), it is apparent that

$$\psi_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)C \sim Y_{\pm}\left(-\frac{q_0^2}{z}\right)e^{i\theta\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3}C = Y_{\pm}\left(-\frac{q_0^2}{z}\right)e^{-i\theta(z; x, t)\sigma_3}C, \text{ as } x \rightarrow \pm\infty, \quad (3.26)$$

where $\theta\left(-\frac{q_0^2}{z}; x, t\right) = -\theta(z; x, t)$, and noting that

$$\frac{i}{z}Y_{\pm}\left(-\frac{q_0^2}{z}\right)e^{-i\theta(z; x, t)\sigma_3}\sigma_3 Q_{\pm} = Y_{\pm}(z)e^{i\theta(z; x, t)\sigma_3},$$

which together with (3.26) implies that $C = \frac{i}{z}\sigma_3 Q_{\pm}$, we then obtain the symmetric relation

$$\psi_{\pm}(z; x, t) = \frac{i}{z}\psi_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3 Q_{\pm}, \quad (3.27)$$

and by using the transformation (3.13), we have

$$\begin{aligned} \mu_{\pm}(z; x, t) &= \frac{i}{z}\mu_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)e^{i\theta\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3}\sigma_3 Q_{\pm}e^{-i\theta(z; x, t)\sigma_3} = \frac{i}{z}\mu_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)e^{-i\theta(z; x, t)\sigma_3}\sigma_3 Q_{\pm}e^{-i\theta(z; x, t)\sigma_3} \\ &= \frac{i}{z}\mu_{\pm}\left(-\frac{q_0^2}{z}; x, t\right)\sigma_3 Q_{\pm}. \end{aligned}$$

In addition, By using Eq. (3.19), we have

$$\psi_+\left(-\frac{q_0^2}{z}; x, t\right) = \psi_-\left(-\frac{q_0^2}{z}; x, t\right)S\left(-\frac{q_0^2}{z}\right). \quad (3.28)$$

and according the symmetry of Jost solutions (3.27), one has following formulas:

$$\psi_+\left(-\frac{q_0^2}{z}; x, t\right) = \frac{z}{i}\psi_+(z; x, t)(\sigma_3 Q_+)^{-1}, \quad \psi_-\left(-\frac{q_0^2}{z}; x, t\right) = \frac{z}{i}\psi_-(z; x, t)(\sigma_3 Q_-)^{-1}, \quad (3.29)$$

then substituting (3.29) into Eq. (3.28), one can obtain

$$\frac{z}{i}\psi_+(z; x, t)(\sigma_3 Q_+)^{-1} = \frac{z}{i}\psi_-(z; x, t)(\sigma_3 Q_-)^{-1}S\left(-\frac{q_0^2}{z}\right) \Rightarrow \psi_+(z; x, t) = \psi_-(z; x, t)(\sigma_3 Q_-)^{-1}S\left(-\frac{q_0^2}{z}\right)\sigma_3 Q_+, \quad (3.30)$$

combining (3.30) with (3.19), and considering the same asymptotic behavior,

$$S(z), \quad S\left(-\frac{q_0^2}{z}\right) \sim I, \text{ as } x \rightarrow \pm\infty,$$

one can lead to

$$S(z) = (\sigma_3 Q_-)^{-1}S\left(-\frac{q_0^2}{z}\right)\sigma_3 Q_+. \quad (3.31)$$

According to the symmetry reduction (3.31) of scattering matrix, we have

$$\begin{aligned} \begin{pmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{pmatrix} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & q_- \\ -q_-^* & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} s_{11}\left(-\frac{q_0^2}{z}\right) & s_{12}\left(-\frac{q_0^2}{z}\right) \\ s_{21}\left(-\frac{q_0^2}{z}\right) & s_{22}\left(-\frac{q_0^2}{z}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & q_+ \\ -q_+^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{q_+^*}{q_-^*} s_{22}\left(-\frac{q_0^2}{z}\right) & \frac{q_+}{q_-^*} s_{21}\left(-\frac{q_0^2}{z}\right) \\ \frac{q_+^*}{q_-^*} s_{12}\left(-\frac{q_0^2}{z}\right) & \frac{q_+}{q_-^*} s_{11}\left(-\frac{q_0^2}{z}\right) \end{pmatrix}, \end{aligned}$$

we then obtain these symmetric relations

$$s_{11}(z) = \frac{q_+^*}{q_-^*} s_{22}\left(-\frac{q_0^2}{z}\right), \quad s_{12}(z) = \frac{q_+}{q_-^*} s_{21}\left(-\frac{q_0^2}{z}\right), \quad s_{21}(z) = \frac{q_+^*}{q_-^*} s_{12}\left(-\frac{q_0^2}{z}\right), \quad s_{22}(z) = \frac{q_+}{q_-^*} s_{11}\left(-\frac{q_0^2}{z}\right). \quad (3.32)$$

Since $\rho(z) = \frac{s_{21}(z)}{s_{11}(z)} = \frac{\frac{q_+^*}{q_-^*} s_{12}\left(-\frac{q_0^2}{z}\right)}{\frac{q_+^*}{q_-^*} s_{22}\left(-\frac{q_0^2}{z}\right)} = \frac{q_-^*}{q_-} \tilde{\rho}\left(-\frac{q_0^2}{z}\right)$, the symmetry reduction of reflection coefficient is $\rho(z) = \frac{q_-^*}{q_-} \tilde{\rho}\left(-\frac{q_0^2}{z}\right)$. This completes the proof. \square

3.1.4. Asymptotic Behaviors.

In order to propose and solve the matrix RH problem for the inverse problem in the next part, it is necessary to discuss the asymptotic behaviors of the modified Jost solutions and scattering matrix as $z \rightarrow 0$ and $z \rightarrow \infty$, which differ from the case of ZBCs. The asymptotic behaviors of the modified Jost solution are deduce with the aid of the general WKB expansions.

Proposition 17. *The asymptotic behaviors of the modified Jost solutions are exhibited as*

$$\mu_{\pm}(z; x, t) = \begin{cases} \frac{i}{z} e^{i\nu_{\pm}(x, t)\sigma_3} \sigma_3 Q_{\pm} + O(1), & \text{as } z \rightarrow 0, \\ e^{i\nu_{\pm}(x, t)\sigma_3} + O\left(\frac{1}{z}\right), & \text{as } z \rightarrow \infty, \end{cases} \quad (3.33)$$

where

$$\nu_{\pm}(x, t) = \frac{1}{2} \int_{\pm\infty}^x (|q(y, t)|^2 - q_0^2) dy. \quad (3.34)$$

Proof. First, we consider the following asymptotic WKB expansion of the modified Jost solutions $\mu_{\pm}(z; x, t)$ as $z \rightarrow 0$

$$\mu_{\pm}(z; x, t) = \sum_{i=-1}^n \mu_{\pm}^{[i]}(x, t) z^i + O(z^{n+1}) = \frac{\mu_{\pm}^{[-1]}(x, t)}{z} + \mu_{\pm}^{[0]}(x, t) + \mu_{\pm}^{[1]}(x, t) z + \cdots. \quad (3.35)$$

Substituting (3.35) into the equivalent Lax pair (3.15), and comparing the coefficients of different powers of z , we have

$$O(z^{-4}) : \frac{q_0^4}{4} [\sigma_3 Q_{\pm} \mu_{\pm}^{[-1]}(x, t), \sigma_3] = 0, \Rightarrow \left(\mu_{\pm}^{[-1]}(x, t) \right)^{\text{diag}} = 0,$$

where $\left(\mu_{\pm}^{[-1]}(x, t) \right)^{\text{diag}}$ represent the diagonal parts of $\mu_{\pm}^{[-1]}(x, t)$.

$$\begin{aligned} O(z^{-3}) : -\frac{q_0^4}{4} i \left[\mu_{\pm}^{[-1]}(x, t), \sigma_3 \right] - \frac{q_0^4}{4} \left[\sigma_3 Q_{\pm} \mu_{\pm}^{[0]}(x, t), \sigma_3 \right] &= \frac{q_0^2}{2} i \sigma_3 Q_{\pm} (Q - Q_{\pm}) \mu_{\pm}^{[-1]}(x, t), \\ \Rightarrow \left(\mu_{\pm}^{[0]}(x, t) \right)^{\text{diag}} &= \text{diag} \left(\frac{-iq}{q_0^2} \mu_{\pm 21}^{[-1]}(x, t), \frac{-iq^*}{q_0^2} \mu_{\pm 12}^{[-1]}(x, t) \right), \end{aligned}$$

where $\mu_{\pm 21}^{[-1]}(x, t)$ and $\mu_{\pm 12}^{[-1]}(x, t)$ represent the second row and first column element and the first row and second column element of matrix $\mu_{\pm}^{[-1]}(x, t)$, respectively. Moreover,

$$\text{diag} \left(\frac{-iq}{q_0^2} \mu_{\pm 21}^{[-1]}(x, t), \frac{-iq^*}{q_0^2} \mu_{\pm 12}^{[-1]}(x, t) \right) = \begin{pmatrix} \frac{-iq}{q_0^2} \mu_{\pm 21}^{[-1]}(x, t) & 0 \\ 0 & \frac{-iq^*}{q_0^2} \mu_{\pm 12}^{[-1]}(x, t) \end{pmatrix}.$$

$$O(z^{-2}) : \left(-i\sigma_3 Q_{\pm} \mu_{\pm}^{[-1]}(x, t) \right)_x - \frac{q_0^4}{4} i \left[\mu_{\pm}^{[0]}(x, t), \sigma_3 \right] - \frac{q_0^4}{4} \left[\sigma_3 Q_{\pm} \mu_{\pm}^{[-1]}(x, t), \sigma_3 \right] = -\frac{q_0^2}{2} (Q - Q_{\pm}) \mu_{\pm}^{[-1]}(x, t) + \frac{q_0^2}{2} i \sigma_3 Q_{\pm} (Q - Q_{\pm}) \mu_{\pm}^{[0]}(x, t), \Rightarrow \left(\mu_{\pm}^{[-1]}(x, t) \right)_x = \frac{1}{2} i (|q(x, t)|^2 - q_0^2) \sigma_3 \mu_{\pm}^{[-1]}(x, t).$$

Therefore, we obtain that $\mu_{\pm}^{[-1]}(x, t) = e^{\int_{\pm\infty}^x \frac{1}{2}(|q(y, t)|^2 - q_0^2) dy} \sigma_3 C$, where C is off-diagonal constant matrix. Let $\nu_{\pm}(x, t)$ satisfy Eq. (3.34), one can obtain $\mu_{\pm}^{[-1]}(x, t) = e^{i\nu_{\pm}(x, t)\sigma_3} C$ and $\lim_{x \rightarrow \pm\infty} \mu_{\pm}^{[-1]}(x, t) = C$. According to the expansion (3.35), we have

$$\lim_{x \rightarrow \pm\infty} z \mu_{\pm}(z; x, t) = z Y_{\pm}(z) = z \left(I + \frac{i}{z} \sigma_3 Q_{\pm} \right) = \lim_{x \rightarrow \pm\infty} \left(\mu_{\pm}^{[-1]}(x, t) + z \mu_{\pm}^{[0]}(x, t) + \dots \right),$$

so we have $C = i\sigma_3 Q_{\pm}$ and $\mu_{\pm}^{[-1]}(x, t) = i e^{i\nu_{\pm}(x, t)\sigma_3} \sigma_3 Q_{\pm}$. Finally, we obtain the asymptotic behaviors

$$\mu_{\pm}(z; x, t) = \frac{i}{z} e^{i\nu_{\pm}(x, t)\sigma_3} \sigma_3 Q_{\pm} + O(1), \text{ as } z \rightarrow 0.$$

Next, we consider the asymptotic expansions of the Jost solutions $\mu_{\pm}(z; x, t)$ as $z \rightarrow \infty$

$$\mu_{\pm}(z; x, t) = \sum_{i=0}^n \frac{\mu_{\pm}^{[i]}(x, t)}{z^i} + O\left(\frac{1}{z^{n+1}}\right) = \mu_{\pm}^{[0]}(x, t) + \frac{\mu_{\pm}^{[1]}(x, t)}{z} + \frac{\mu_{\pm}^{[2]}(x, t)}{z^2} + \dots \quad (3.36)$$

By utilizing same way, substituting (3.36) into the equivalent Lax pair (3.15), and comparing the coefficients of different powers of z , we have

$$O(z^2) : \frac{i}{4} \left[\mu_{\pm}^{[0]}(x, t), \sigma_3 \right] = 0, \Rightarrow \left(\mu_{\pm}^{[0]}(x, t) \right)^{\text{off}} = 0,$$

where $\left(\mu_{\pm}^{[0]}(x, t) \right)^{\text{off}}$ represent the off-diagonal parts of $\mu_{\pm}^{[0]}(x, t)$.

$$O(z) : \frac{i}{4} \left[\mu_{\pm}^{[1]}(x, t), \sigma_3 \right] + \frac{1}{4} \left[\sigma_3 Q_{\pm} \mu_{\pm}^{[0]}(x, t), \sigma_3 \right] = \frac{1}{2} (Q - Q_{\pm}) \mu_{\pm}^{[0]}(x, t), \Rightarrow \left(\mu_{\pm}^{[1]}(x, t) \right)^{\text{off}} = \text{off} \left(i q \mu_{\pm 22}^{[0]}(x, t), i q^* \mu_{\pm 11}^{[0]}(x, t) \right),$$

$$\text{where } \text{off} \left(i q \mu_{\pm 22}^{[0]}(x, t), i q^* \mu_{\pm 11}^{[0]}(x, t) \right) = \begin{pmatrix} 0 & i q \mu_{\pm 22}^{[0]}(x, t) \\ i q^* \mu_{\pm 11}^{[0]}(x, t) & 0 \end{pmatrix}.$$

$$O(1) : \mu_{\pm, x}^{[0]}(x, t) + \frac{i}{4} \left[\mu_{\pm}^{[2]}(x, t), \sigma_3 \right] + \frac{1}{4} \left[\sigma_3 Q_{\pm} \mu_{\pm}^{[1]}(x, t), \sigma_3 \right] = \frac{1}{2} (Q - Q_{\pm}) \mu_{\pm}^{[1]}(x, t) - \frac{i}{2} \sigma_3 Q_{\pm} (Q - Q_{\pm}) \mu_{\pm}^{[0]}(x, t), \\ \Rightarrow \left(\mu_{\pm}^{[0]}(x, t) \right)_x = \frac{i}{2} (|q(x, t)|^2 - q_0^2) \sigma_3.$$

Therefore, we obtain that $\mu_{\pm}^{[0]}(x, t) = \tilde{C} e^{\int_{\pm\infty}^x \frac{1}{2}(|q(y, t)|^2 - q_0^2) dy} \sigma_3$, where \tilde{C} is diagonal constant matrix. Let $\nu_{\pm}(x, t)$ satisfy Eq. (3.34), one can obtain $\mu_{\pm}^{[0]}(x, t) = \tilde{C} e^{i\nu_{\pm}(x, t)\sigma_3}$ and $\lim_{x \rightarrow \pm\infty} \mu_{\pm}^{[0]}(x, t) = \tilde{C}$. According to the expansion (3.36), we have

$$\lim_{\substack{x \rightarrow \pm\infty \\ z \rightarrow \infty}} \mu_{\pm}(z; x, t) = \lim_{z \rightarrow \infty} Y_{\pm}(z) = \lim_{z \rightarrow \infty} \left(I + \frac{i}{z} \sigma_3 Q_{\pm} \right) = I,$$

so we have $\tilde{C} = I$ and $\mu_{\pm}^{[0]}(x, t) = e^{i\nu_{\pm}(x, t)\sigma_3}$. Finally, we get the asymptotic behaviors

$$\mu_{\pm}(z; x, t) = e^{i\nu_{\pm}(x, t)\sigma_3} + O\left(\frac{1}{z}\right), \text{ as } z \rightarrow \infty.$$

This completes the proof. \square

The asymptotic behaviors of the scattering matrix can be directly yielded by exploiting the determinant representations of the scattering matrix and the asymptotic behaviors of the modified Jost solutions.

Proposition 18. *The asymptotic behaviors of the scattering matrix are as follows*

$$S(z) = \begin{cases} \text{diag} \left(\frac{q_-}{q_+}, \frac{q_+}{q_-} \right) e^{i\nu(t)\sigma_3} + O(z), & \text{as } z \rightarrow 0, \\ e^{-i\nu(t)\sigma_3} + O\left(\frac{1}{z}\right), & \text{as } z \rightarrow \infty, \end{cases} \quad (3.37)$$

where

$$\nu(t) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q(y, t)|^2 - q_0^2) dy. \quad (3.38)$$

Proof. From the relationship (3.13) between Jost solutions and modified Jost solutions, the determinant representations (3.20) of the scattering matrix and the asymptotic behaviors (3.33), for $z \rightarrow 0$, we have

$$\begin{aligned} s_{11}(z) &= \frac{\det(\psi_{+1}(z; x, t), \psi_{-2}(z; x, t))}{\eta} = \frac{\det(\mu_{+1}(z; x, t), \mu_{-2}(z; x, t))}{1 + \frac{q_0^2}{z^2}} = \\ &= \frac{\det \begin{pmatrix} O(1) & \frac{i}{z} q_- e^{-i\nu_-(x, t)} + O(1) \\ \frac{i}{z} q_+^* e^{i\nu_+(x, t)} + O(1) & O(1) \end{pmatrix} \begin{pmatrix} \frac{z^2}{q_0^2} - \frac{z^4}{q_0^4} + \dots \end{pmatrix}}{\left\{ \frac{q_- q_+^*}{z^2} e^{-i[\nu_-(x, t) - \nu_+(x, t)]} + O(z) \right\} \left(\frac{z^2}{q_0^2} - \frac{z^4}{q_0^4} + \dots \right)} = \frac{q_-}{q_+} e^{-i\nu(t)} + O(z), \end{aligned}$$

where $\frac{1}{1 + \frac{q_0^2}{z^2}} = \frac{z^2}{z^2 + q_0^2} = \frac{z^2}{q_0^2} \cdot \frac{1}{1 - \left(-\frac{z^2}{q_0^2}\right)} = \frac{z^2}{q_0^2} - \frac{z^4}{q_0^4} + \dots$, and $\nu(t) = \nu_-(x, t) - \nu_+(x, t) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q(y, t)|^2 - q_0^2) dy$.

Similarly, one can obtain

$$s_{12} = O(z), \quad s_{21} = O(z), \quad s_{22} = \frac{q_+}{q_-} e^{i\nu(t)} + O(z),$$

so we have the asymptotic behavior (3.37) as $z \rightarrow 0$.

When $z \rightarrow \infty$, by using the same way, we have

$$\begin{aligned} s_{11}(z) &= \frac{\det(\psi_{+1}(z; x, t), \psi_{-2}(z; x, t))}{\eta} = \frac{\det(\mu_{+1}(z; x, t), \mu_{-2}(z; x, t))}{1 + \frac{q_0^2}{z^2}} = \det \begin{pmatrix} e^{i\nu_+(x, t)} + O(z^{-1}) & O(z^{-1}) \\ O(z^{-1}) & e^{-i\nu_-(x, t)} + O(z^{-1}) \end{pmatrix} \\ &\quad \left(1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} + \dots \right) = \left\{ e^{-i[\nu_-(x, t) - \nu_+(x, t)]} + O(z^{-1}) \right\} \left(1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} + \dots \right) = e^{-i\nu(t)} + O(z^{-1}), \end{aligned}$$

where $\frac{1}{1 + \frac{q_0^2}{z^2}} = \frac{1}{1 - \left(-\frac{q_0^2}{z^2}\right)} = 1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} + \dots$, and

$$s_{12} = O(z^{-1}), \quad s_{21} = O(z^{-1}), \quad s_{22} = e^{i\nu(t)} + O(z^{-1}),$$

Hence, we also obtain the asymptotic behavior (3.37) as $z \rightarrow \infty$.

Moreover, on the one hand, substituting the WKB expansion $\mu_{\pm}(\lambda; x, t) = \sum_{i=-1}^n \mu_{\pm}^{[i]}(x, t) z^i + O(z^{n+1})$ (as $z \rightarrow 0$) into the time part of equivalent Lax pairs (3.16) and matching the $O(\lambda^i)$ ($i = -6, -5, -4, -3, -2, -1, 0$) in order, combining the

$$\begin{aligned} O(1) : & -\frac{13}{16} i q_0^{10} [\mu_{\pm}^{[2]}(x, t), \sigma_3] + \frac{39}{16} q_0^8 \sigma_3 Q_{\pm} [\mu_{\pm}^{[1]}(x, t), \sigma_3] + \frac{39}{16} i q_0^8 [\mu_{\pm}^{[0]}(x, t), \sigma_3] - \\ & \frac{27}{16} q_0^6 \sigma_3 Q_{\pm} [\mu_{\pm}^{[-1]}(x, t), \sigma_3] + i \sigma_3 Q_{\pm} \mu_{\pm, t}^{[-1]}(x, t) = i \sigma_3 Q_{\pm} (T - T_{\pm}) \mu_{\pm}^{[-1]}(x, t) \end{aligned}$$

and $(\mu_{\pm}^{[-1]}(x, t))^{\text{diag}} = 0$, one can yield $\mu_{\pm, t}^{[-1]}(x, t) \neq 0$, $\nu_{\pm, t}(x, t) \neq 0$ and $\nu_t \neq 0$ as $x \rightarrow \pm\infty$. That is, ν is dependent on the variable t .

On the other hand, substituting the WKB expansion $\mu_{\pm}(\lambda; x, t) = \sum_{i=0}^n \frac{\mu_{\pm}^{[i]}(x, t)}{z^i} + O\left(\frac{1}{z^{n+1}}\right)$ (as $z \rightarrow \infty$) into the time part of equivalent Lax pairs (3.16) and matching the $O(\lambda^i)$ ($i = 6, 5, 4, 3, 2, 1, 0$) in order, combining the

$$\begin{aligned} O(1) : & \frac{1}{16} i [\mu_{\pm}^{[6]}(x, t), \sigma_3] + \frac{1}{16} \sigma_3 Q_{\pm} [\mu_{\pm}^{[5]}(x, t), \sigma_3] - \frac{3}{16} i q_0^2 [\mu_{\pm}^{[4]}(x, t), \sigma_3] - \\ & \frac{13}{16} q_0^2 \sigma_3 Q_{\pm} [\mu_{\pm}^{[3]}(x, t), \sigma_3] + \frac{5}{2} i q_0^4 [\mu_{\pm}^{[2]}(x, t), \sigma_3] + \frac{5}{2} q_0^4 \sigma_3 Q_{\pm} [\mu_{\pm}^{[1]}(x, t), \sigma_3] - \\ & \frac{39}{16} i q_0^6 [\mu_{\pm}^{[0]}(x, t), \sigma_3] + \mu_{\pm, t}^{[0]}(x, t) = (T - T_{\pm}) \mu_{\pm}^{[0]}(x, t) \end{aligned}$$

and $(\mu_{\pm}^{[0]}(x, t))^{\text{off}} = 0$, one can yield $\mu_{\pm, t}^{[0]}(x, t) \neq 0$, $\nu_{\pm, t}(x, t) \neq 0$ and $\nu_t \neq 0$ as $x \rightarrow \pm\infty$. That is, ν also is dependent on the variable t . In summary, ν is a function about time t . Proof complete. \square

3.1.5. Discrete Spectrum with Double Zeros.

In this section, we consider the case of $s_{11}(z)$ with double zeros and suppose that $s_{11}(z)$ has N_1 double zeros in $Z_0 = \{z \in \mathbb{C} | \text{Re} z > 0, \text{Im} z > 0, |z| > q_0\}$ denoted by z_n , and N_2 double zeros on the circle $W_0 = \{z \in \mathbb{C} | z = q_0 e^{i\phi}, 0 < \phi < \frac{\pi}{2}\}$ denoted by ω_m , that is, $s_{11}(z_0) = s'_{11}(z_0) = 0$ and $s''_{11}(z_0) \neq 0$ if z_0 is a double zero of $s_{11}(z)$. From the

symmetries of the scattering matrix presented in Proposition (16), the $4(2N_1 + N_2)$ discrete spectrum can be given by the set

$$Z = \left\{ \pm z_n, \pm z_n^*, \pm \frac{q_0^2}{z_n}, \pm \frac{q_0^2}{z_n^*} \right\}_{n=1}^{N_1} \bigcup \{ \pm \omega_m, \pm \omega_m^* \}_{m=1}^{N_2}, \quad (3.39)$$

the distributions are shown in Figure 4. When a given $z_0 \in Z \cap D^+$, one can obtain that $\psi_{+1}(z_0; x, t)$ and $\psi_{-2}(z_0; x, t)$ are linearly dependent by combining Eq. (3.20) and $s_{11}(z_0) = 0$. Similarly, when a given $z_0 \in Z \cap D^-$, one can obtain that $\psi_{+2}(z_0; x, t)$ and $\psi_{-1}(z_0; x, t)$ are linearly dependent by combining Eq. (3.20) and $s_{22}(z_0) = 0$. For convenience, we introduce the norming constant $b[z_0]$ such that

$$\begin{aligned} \psi_{+1}(z_0; x, t) &= b[z_0]\psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \\ \psi_{+2}(z_0; x, t) &= b[z_0]\psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^-. \end{aligned} \quad (3.40)$$

For a given $z_0 \in Z \cap D^+$, according to $s_{11}(z_0) = \frac{\det(\psi_{+1}(z; x, t), \psi_{-2}(z; x, t))}{\eta}$ in Eq. (3.20), and taking derivative respect to z (here $z = z_0$) on both sides of this equation, combining $s_{11}(z_0) = 0$, $s'_{11}(z_0) = 0$ and Eq. (3.40), we have

$$\begin{aligned} &\det(\psi'_{+1}(z_0; x, t), \psi_{-2}(z_0; x, t)) + \det(\psi_{+1}(z_0; x, t), \psi'_{-2}(z_0; x, t)) \\ &= \det(\psi'_{+1}(z_0; x, t), \psi_{-2}(z_0; x, t)) + \det(b[z_0]\psi_{-2}(z_0; x, t), \psi'_{-2}(z_0; x, t)) \\ &= \det(\psi'_{+1}(z_0; x, t), \psi_{-2}(z_0; x, t)) - \det(b[z_0]\psi'_{-2}(z_0; x, t), \psi_{-2}(z_0; x, t)) \\ &= \det(\psi'_{+1}(z_0; x, t) - b[z_0]\psi'_{-2}(z_0; x, t), \psi_{-2}(z_0; x, t)) = 0, \end{aligned}$$

it is evidence that $\psi'_{+1}(z_0; x, t) - b[z_0]\psi'_{-2}(z_0; x, t)$ and $\psi_{-2}(z_0; x, t)$ are linearly dependent. Similarly, when a given $z_0 \in Z \cap D^-$, one can obtain that $\psi'_{+2}(z_0; x, t) - b[z_0]\psi'_{-1}(z_0; x, t)$ and $\psi_{-1}(z_0; x, t)$ are linearly dependent by combining Eq. (3.20) and $s'_{22}(z_0) = 0$. For convenience, we define another norming constant $d[z_0]$ such that

$$\begin{aligned} \psi'_{+1}(z_0; x, t) - b[z_0]\psi'_{-2}(z_0; x, t) &= d[z_0]\psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \\ \psi'_{+2}(z_0; x, t) - b[z_0]\psi'_{-1}(z_0; x, t) &= d[z_0]\psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^-. \end{aligned} \quad (3.41)$$

On the other hand, we notice that $\psi_{+1}(z; x, t)$ and $s_{11}(z)$ are analytic on D^+ . Suppose z_0 is the double zeros of $s_{11}(z)$, let $\psi_{+1}(z; x, t)$ and $s_{11}(z)$ carry out Taylor expansion at $z = z_0$, we have

$$\begin{aligned} \frac{\psi_{+1}(z; x, t)}{s_{11}(z)} &= \frac{\psi_{+1}(z_0; x, t) + \psi'_{+1}(z_0; x, t)(z - z_0) + \frac{1}{2!}\psi''_{+1}(z_0; x, t)(z - z_0)^2 + \dots}{s_{11}(z_0) + s'_{11}(z_0)(z - z_0) + \frac{1}{2!}s''_{11}(z_0)(z - z_0)^2 + \frac{1}{3!}s'''_{11}(z_0)(z - z_0)^3 + \dots} \\ &= \frac{\psi_{+1}(z_0; x, t) + \psi'_{+1}(z_0; x, t)(z - z_0) + \frac{1}{2!}\psi''_{+1}(z_0; x, t)(z - z_0)^2 + \dots}{\frac{1}{2!}s''_{11}(z_0)(z - z_0)^2 + \frac{1}{3!}s'''_{11}(z_0)(z - z_0)^3 + \dots} \\ &= \frac{2\psi_{+1}(z_0; x, t)}{s''_{11}(z_0)}(z - z_0)^{-2} + \left(\frac{2\psi'_{+1}(z_0; x, t)}{s'_{11}(z_0)} - \frac{2\psi_{+1}(z_0; x, t)s'''_{11}(z_0)}{3s''_{11}(z_0)^2} \right)(z - z_0)^{-1} + \dots, \end{aligned}$$

Then, one has the compact form

$$P_{-2} \left[\frac{\psi_{+1}(z; x, t)}{s_{11}(z)} \right]_{z=z_0} = \frac{2\psi_{+1}(z_0; x, t)}{s''_{11}(z_0)} = \frac{2b[z_0]\psi_{-2}(z_0; x, t)}{s''_{11}(z_0)}, \text{ as } z_0 \in Z \cap D^+,$$

where $P_{-2}[f(z; x, t)]$ denotes the coefficient of $O((z - z_0)^{-2})$ term in the Laurent series expansion of $f(z; x, t)$ at $z = z_0$. and

$$\begin{aligned} \text{Res}_{z=z_0} \left[\frac{\psi_{+1}(z; x, t)}{s_{11}(z)} \right] &= \frac{2\psi'_{+1}(z_0; x, t)}{s''_{11}(z_0)} - \frac{2\psi_{+1}(z_0; x, t)s'''_{11}(z_0)}{3s''_{11}(z_0)^2} = \frac{2(b[z_0]\psi'_{-2}(z_0; x, t) + d[z_0]\psi_{-2}(z_0; x, t))}{s''_{11}(z_0)} \\ &= \frac{2b[z_0]\psi_{-2}(z_0; x, t)s'''_{11}(z_0)}{3s''_{11}(z_0)^2} = \frac{2b[z_0]\psi'_{-2}(z_0; x, t)}{s''_{11}(z_0)} + \frac{2d[z_0]\psi_{-2}(z_0; x, t)}{s''_{11}(z_0)} - \frac{2b[z_0]\psi_{-2}(z_0; x, t)s'''_{11}(z_0)}{3s''_{11}(z_0)^2} \\ &= \frac{2b[z_0]\psi'_{-2}(z_0; x, t)}{s''_{11}(z_0)} + \frac{2b[z_0]}{s''_{11}(z_0)} \cdot \frac{d[z_0]}{b[z_0]}\psi_{-2}(z_0; x, t) - \frac{2b[z_0]}{s''_{11}(z_0)} \cdot \frac{s'''_{11}(z_0)}{3s''_{11}(z_0)^2}\psi_{-2}(z_0; x, t) \\ &= \frac{2b[z_0]\psi'_{-2}(z_0; x, t)}{s''_{11}(z_0)} + \left[\frac{2b[z_0]}{s''_{11}(z_0)} \left(\frac{d[z_0]}{b[z_0]} - \frac{s'''_{11}(z_0)}{3s''_{11}(z_0)^2} \right) \right] \psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \end{aligned}$$

where $\text{Res}_{z=z_0}[f(z; x, t)]$ denotes the coefficient of $O((z - z_0)^{-1})$ term in the Laurent series expansion of $f(z; x, t)$ at $z = z_0$.

Similarly, for the case of $\psi_{+2}(z; x, t)$ and $s_{22}(z)$ are analytic on D^- , we repeat the above process and obtain

$$\begin{aligned} P_{-2} \left[\frac{\psi_{+2}(z; x, t)}{s_{22}(z)} \right] &= \frac{2\psi_{+2}(z_0; x, t)}{s_{22}''(z_0)} = \frac{2b[z_0]\psi_{-1}(z_0; x, t)}{s_{22}''(z_0)}, \text{ as } z_0 \in Z \cap D^-, \\ \text{Res}_{z=z_0} \left[\frac{\psi_{+2}(z; x, t)}{s_{22}(z)} \right] &= \frac{2b[z_0]\psi_{-1}'(z_0; x, t)}{s_{22}''(z_0)} + \left[\frac{2b[z_0]}{s_{22}''(z_0)} \left(\frac{d[z_0]}{b[z_0]} - \frac{s_{22}'''(z_0)}{3s_{22}''(z_0)} \right) \right] \psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^-. \end{aligned}$$

Moreover, let

$$A[z_0] = \begin{cases} \frac{2b[z_0]}{s_{11}''(z_0)}, & \text{as } z_0 \in Z \cap D^+, \\ \frac{2b[z_0]}{s_{22}''(z_0)}, & \text{as } z_0 \in Z \cap D^-, \end{cases} \quad B[z_0] = \begin{cases} \frac{d[z_0]}{b[z_0]} - \frac{s_{11}'''(z_0)}{3s_{11}''(z_0)}, & \text{as } z_0 \in Z \cap D^+, \\ \frac{d[z_0]}{b[z_0]} - \frac{s_{22}'''(z_0)}{3s_{22}''(z_0)}, & \text{as } z_0 \in Z \cap D^-. \end{cases} \quad (3.42)$$

Then, we have

$$\begin{aligned} P_{-2} \left[\frac{\psi_{+1}(z; x, t)}{s_{11}(z)} \right] &= A[z_0]\psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \\ P_{-2} \left[\frac{\psi_{+2}(z; x, t)}{s_{22}(z)} \right] &= A[z_0]\psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^-, \\ \text{Res}_{z=z_0} \left[\frac{\psi_{+1}(z; x, t)}{s_{11}(z)} \right] &= A[z_0][\psi_{-2}'(z_0; x, t) + B[z_0]\psi_{-2}(z_0; x, t)], \text{ as } z_0 \in Z \cap D^+, \\ \text{Res}_{z=z_0} \left[\frac{\psi_{+2}(z; x, t)}{s_{22}(z)} \right] &= A[z_0][\psi_{-1}'(z_0; x, t) + B[z_0]\psi_{-1}(z_0; x, t)], \text{ as } z_0 \in Z \cap D^-. \end{aligned} \quad (3.43)$$

Proposition 19. For a $z_0 \in Z$, the three symmetry relations for $A[z_0]$ and $B[z_0]$ are deduced as follow

- The first symmetry relation $A[z_0] = -A[z_0^*]^*$, $B[z_0] = B[z_0^*]^*$.
- The second symmetry relation $A[z_0] = A[-z_0^*]^*$, $B[z_0] = -B[-z_0^*]^*$.
- The third symmetry relation $A[z_0] = \frac{z_0^4 q^*}{q_0^4 q^-} A \left[-\frac{q_0^2}{z_0} \right]$, $B[z_0] = \frac{q_0^2}{z_0^2} B \left[-\frac{q_0^2}{z_0} \right] + \frac{2}{z_0}$.

Proof. For the first symmetry relation, a given $z_0 \in Z$, $s_{11}(z_0) = s_{22}(z_0) = 0$, hence substituting $S(z_0) = \begin{pmatrix} 0 & s_{12}(z_0) \\ s_{21}(z_0) & 0 \end{pmatrix}$ into $\psi_+(z_0; x, t) = \psi_-(z_0; x, t)S(\lambda_0)$, one can obtain

$$\begin{aligned} \psi_{+1}(z_0; x, t) &= s_{21}(z_0)\psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \\ \psi_{+2}(z_0; x, t) &= s_{12}(z_0)\psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^-. \end{aligned}$$

Therefore, when $z_0 \in Z \cap D^+$ ($z_0 \in Z \cap D^-$), one has obtain $b[z_0] = s_{21}(z_0)$ ($b[z_0] = s_{12}(z_0)$). According to $S(z_0) = \sigma_2 S(z_0^*)^* \sigma_2$ in Eq. (3.22), one can obtain

$$s_{11}(z_0) = s_{22}(z_0^*)^*, \quad s_{12}(z_0) = -s_{21}(z_0^*)^*, \quad s_{21}(z_0) = -s_{12}(z_0^*)^*, \quad s_{22}(z_0) = s_{11}(z_0^*)^*, \quad (3.44)$$

so we have $b[z_0^*]^* = s_{12}(z_0^*)^* = -s_{21}(z_0) = -b[z_0]$ (where $z_0^* \in Z \cap D^-$), and $A[z_0] = \frac{2b[z_0]}{s_{11}''(z_0)} = \frac{-2b[z_0^*]^*}{s_{22}''(z_0^*)^*} = -A[z_0^*]^*$. In addition, for a given $z_0 \in Z \cap D^+$ ($z_0 \in Z \cap D^-$), by taking derivative respect to z (here $z = z_0$) both sides of $\psi_{+1}(z_0; x, t) = b[z_0]\psi_{-2}(z_0; x, t)$ ($\psi_{+2}(z_0; x, t) = b[z_0]\psi_{-1}(z_0; x, t)$), we have

$$\begin{aligned} \psi_{+1}'(z_0; x, t) - b[z_0]\psi_{-2}'(z_0; x, t) &= b'[z_0]\psi_{-2}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+, \\ \psi_{+2}'(z_0; x, t) - b[z_0]\psi_{-1}'(z_0; x, t) &= b'[z_0]\psi_{-1}(z_0; x, t), \text{ as } z_0 \in Z \cap D^+. \end{aligned}$$

Combining with Eq. (3.41), one can obtain

$$\begin{aligned} b[z_0] = s_{21}(z_0) &\Rightarrow d[z_0] = b'[z_0] = s_{21}'(z_0), \text{ as } z_0 \in Z \cap D^+, \\ b[z_0] = s_{12}(z_0) &\Rightarrow d[z_0] = b'[z_0] = s_{12}'(z_0), \text{ as } z_0 \in Z \cap D^-, \end{aligned}$$

then lead to

$$\begin{aligned} B[z_0] &= \frac{d[z_0]}{b[z_0]} - \frac{s_{11}'''(z_0)}{3s_{11}''(z_0)} = \frac{s_{21}'(z_0)}{s_{21}(z_0)} - \frac{s_{22}'''(z_0^*)^*}{3s_{22}''(z_0^*)^*} = \frac{-s_{12}'(z_0^*)^*}{-s_{12}(z_0^*)^*} - \frac{s_{22}'''(z_0^*)^*}{3s_{22}''(z_0^*)^*} \\ &= \frac{-d[z_0^*]^*}{-b[z_0^*]^*} - \frac{s_{22}'''(z_0^*)^*}{3s_{22}''(z_0^*)^*} = \frac{d[z_0^*]^*}{b[z_0^*]^*} - \frac{s_{22}'''(z_0^*)^*}{3s_{22}''(z_0^*)^*} = B[z_0^*]^*. \end{aligned}$$

Similarly, by repeating the above process, we can demonstrate the second symmetry relation. Next, we will prove the third symmetry relation.

For a given $z_0 \in Z$, $s_{11}(z_0) = s_{22}(z_0) = 0$, According to symmetry reduction $S(z_0) = (\sigma_3 Q_-)^{-1} S \left(-\frac{q_0^2}{z_0} \right) \sigma_3 Q_+$ in Eq. (3.24), one can obtain

$$\begin{pmatrix} s_{11}(z_0) & s_{12}(z_0) \\ s_{21}(z_0) & s_{22}(z_0) \end{pmatrix} = \begin{pmatrix} \frac{q_+^*}{q_-^*} s_{22} \left(-\frac{q_0^2}{z_0} \right) & \frac{q_+}{q_-^*} s_{21} \left(-\frac{q_0^2}{z_0} \right) \\ \frac{q_+}{q_-^*} s_{12} \left(-\frac{q_0^2}{z_0} \right) & \frac{q_+}{q_-^*} s_{11} \left(-\frac{q_0^2}{z_0} \right) \end{pmatrix},$$

and

$$s_{11}(z_0) = \frac{q_+^*}{q_-^*} s_{22} \left(-\frac{q_0^2}{z_0} \right) = 0, \quad s_{12}(z_0) = \frac{q_+}{q_-^*} s_{21} \left(-\frac{q_0^2}{z_0} \right), \quad s_{21}(z_0) = \frac{q_+^*}{q_-} s_{12} \left(-\frac{q_0^2}{z_0} \right), \quad s_{22}(z_0) = \frac{q_+}{q_-} s_{11} \left(-\frac{q_0^2}{z_0} \right) = 0,$$

so we have $b \left[-\frac{q_0^2}{z_0} \right] = s_{12} \left(-\frac{q_0^2}{z_0} \right) = \frac{q_+}{q_-^*} s_{21}(z_0) = \frac{q_+}{q_-^*} b[z_0]$ (where $z_0 \in Z \cap D^+$, $-\frac{q_0^2}{z_0} \in Z \cap D^-$).

Due to the formula $s_{11}(z_0) = s_{22} \left(-\frac{q_0^2}{z_0} \right) = 0$, note that

$$\begin{aligned} s'_{11}(z_0) &= \frac{q_+^*}{q_-^*} s'_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^2}{z_0^2}, \\ s''_{11}(z_0) &= \frac{q_+^*}{q_-^*} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^4}{z_0^4} + \frac{q_+^*}{q_-^*} s'_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot (-2) \cdot \frac{q_0^2}{z_0^3} = \frac{q_+^*}{q_-^*} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^4}{z_0^4}, \text{ here } s'_{22} \left(-\frac{q_0^2}{z_0} \right) = 0, \\ s'''_{11}(z_0) &= \frac{q_+^*}{q_-^*} s'''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^6}{z_0^6} + \frac{q_+^*}{q_-^*} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot (-4) \cdot \frac{q_0^4}{z_0^5} + \frac{q_+^*}{q_-^*} s'_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot (-2) \cdot \frac{q_0^4}{z_0^5} + \frac{q_+^*}{q_-^*} s'_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot 6 \cdot \frac{q_0^2}{z_0^4} \\ &= \frac{q_+^*}{q_-^*} s'''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^6}{z_0^6} + \frac{q_+^*}{q_-^*} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot (-4) \cdot \frac{q_0^4}{z_0^5} + \frac{q_+^*}{q_-^*} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \cdot (-2) \cdot \frac{q_0^4}{z_0^5}, \text{ here } s'_{22} \left(-\frac{q_0^2}{z_0} \right) = 0, \end{aligned}$$

we have

$$A[z_0] = \frac{2b[z_0]}{s'_{11}(z_0)} = \frac{2 \frac{q_+^*}{q_-^*} b \left[-\frac{q_0^2}{z_0} \right]}{\frac{q_+^*}{q_-^*} \frac{q_0^4}{z_0^4} s''_{22} \left(-\frac{q_0^2}{z_0} \right)} = \frac{z_0^4 q_-^*}{q_0^4 q_-} \frac{b \left[-\frac{q_0^2}{z_0} \right]}{s''_{22} \left(-\frac{q_0^2}{z_0} \right)} = \frac{z_0^4 q_-^*}{q_0^4 q_-} A \left[-\frac{q_0^2}{z_0} \right].$$

Furthermore

$$d[z_0] = b'[z_0] = s'_{21}(z_0) = \frac{q_+^*}{q_-} s'_{12} \left(-\frac{q_0^2}{z_0} \right) \cdot \frac{q_0^2}{z_0^2} = \frac{q_+^*}{q_-} b' \left[-\frac{q_0^2}{z_0} \right] \cdot \frac{q_0^2}{z_0^2} = \frac{q_+^*}{q_-} d \left[-\frac{q_0^2}{z_0} \right] \cdot \frac{q_0^2}{z_0^2}, \text{ where } z_0 \in Z \cap D^+, -\frac{q_0^2}{z_0} \in Z \cap D^-,$$

so one can obtain

$$\begin{aligned} B[z_0] &= \frac{d[z_0]}{b[z_0]} - \frac{s'''_{11}(z_0)}{3s''_{11}(z_0)} = \frac{\frac{q_+^*}{q_-} d \left[-\frac{q_0^2}{z_0} \right] \cdot \frac{q_0^2}{z_0^2}}{\frac{q_+^*}{q_-} b \left[-\frac{q_0^2}{z_0} \right]} - \frac{\frac{q_+^*}{q_-} s'''_{22} \left(-\frac{q_0^2}{z_0} \right) \frac{q_0^6}{z_0^6} - 6 \frac{q_+^*}{q_-} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \frac{q_0^4}{z_0^5}}{3 \frac{q_+^*}{q_-} s''_{22} \left(-\frac{q_0^2}{z_0} \right) \frac{q_0^4}{z_0^4}} \\ &= \frac{q_0^2}{z_0^2} \left(\frac{d \left[-\frac{q_0^2}{z_0} \right]}{b \left[-\frac{q_0^2}{z_0} \right]} - \frac{s'''_{22} \left(-\frac{q_0^2}{z_0} \right)}{3s''_{22} \left(-\frac{q_0^2}{z_0} \right)} \right) + \frac{2}{z_0} = \frac{q_0^2}{z_0^2} B \left[-\frac{q_0^2}{z_0} \right] + \frac{2}{z_0}. \end{aligned}$$

This completes the proof. \square

Proposition 20. For $n = 1, 2, \dots, N_1$ and $m = 1, 2, \dots, N_2$, we have

$$\begin{aligned} A[z_n] &= -A[z_n^*]^* = A[-z_n^*]^* = -A[-z_n] = \frac{z_n^4 q_-^*}{q_0^4 q_-} A \left[-\frac{q_0^2}{z_n} \right] = \frac{z_n^4 q_-^*}{q_0^4 q_-} A \left[\frac{q_0^2}{z_n^*} \right]^* \\ &= -\frac{z_n^4 q_-^*}{q_0^4 q_-} A \left[-\frac{q_0^2}{z_n^*} \right]^* = -\frac{z_n^4 q_-^*}{q_0^4 q_-} A \left[\frac{q_0^2}{z_n} \right], \quad z_n \in Z_0, \\ A[\omega_m] &= -A[\omega_m^*]^* = A[-\omega_m^*]^* = -A[-\omega_m] = \frac{\omega_m^4 q_-^*}{q_0^4 q_-} A[\omega_m]^*, \quad \omega_m \in W_0, \\ B[z_n] &= B[z_n^*]^* = -B[-z_n^*]^* = -B[-z_n] = \frac{q_0^2}{z_n^2} B \left[-\frac{q_0^2}{z_n} \right] + \frac{2}{z_n} = \frac{q_0^2}{z_n^2} B \left[-\frac{q_0^2}{z_n^*} \right]^* + \frac{2}{z_n} \\ &= -\frac{q_0^2}{z_n^2} B \left[\frac{q_0^2}{z_n^*} \right]^* + \frac{2}{z_n} = -\frac{q_0^2}{z_n^2} B \left[\frac{q_0^2}{z_n} \right] + \frac{2}{z_n}, \quad z_n \in Z_0, \\ B[\omega_m] &= B[\omega_m^*]^* = -B[-\omega_m^*]^* = -B[-\omega_m] = -\frac{q_0^2}{\omega_m^2} B[\omega_m]^* + \frac{2}{\omega_m}, \quad \omega_m \in W_0. \end{aligned} \tag{3.45}$$

Proof. From the proposition (19), the symmetry relation $A[z_n] = -A[-z_n]$ can be derived by combining $A[z_0] = -A[z_0^*]^*$ and $A[z_0] = A[-z_0^*]^*$, and we directly obtain $A[z_n] = -A[z_n^*]^* = A[-z_n^*]^* = \frac{z_0^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n}\right]$ and $\frac{z_0^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n}\right] = \frac{z_0^4 q_-^*}{q_0^4 q_-} A\left[\frac{q_0^2}{z_n^*}\right]^* = -\frac{z_0^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n^*}\right]^* = -\frac{z_0^4 q_-^*}{q_0^4 q_-} A\left[\frac{q_0^2}{z_n}\right]$. In conclusion, we obtain the following identity

$$\begin{aligned} A[z_n] &= -A[z_n^*]^* = A[-z_n^*]^* = -A[-z_n] = \frac{z_n^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n}\right] = \frac{z_n^4 q_-^*}{q_0^4 q_-} A\left[\frac{q_0^2}{z_n^*}\right]^* \\ &= -\frac{z_n^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n^*}\right]^* = -\frac{z_n^4 q_-^*}{q_0^4 q_-} A\left[\frac{q_0^2}{z_n}\right], \quad z_n \in Z_0. \end{aligned}$$

Furthermore, here let $z_n = \omega_m = q_0 e^{i\phi}$, substituting it into $A[z_n] = \frac{z_n^4 q_-^*}{q_0^4 q_-} A\left[-\frac{q_0^2}{z_n}\right]$, one can obtain

$$A[\omega_m] = \frac{\omega_m^4 q_-^*}{q_0^4 q_-} A\left[-q_0 e^{-i\phi}\right] = \frac{\omega_m^4 q_-^*}{q_0^4 q_-} A[-\omega_m^*] = \frac{\omega_m^4 q_-^*}{q_0^4 q_-} A[\omega_m]^*, \quad \omega_m \in W_0,$$

and

$$A[\omega_m] = -A[\omega_m^*]^* = A[-\omega_m^*]^* = -A[-\omega_m] = \frac{\omega_m^4 q_-^*}{q_0^4 q_-} A[\omega_m]^*, \quad \omega_m \in W_0.$$

Similarly, the identities about $B[z_n]$ and $B[\omega_m]$ can be derived by repeating above process. End of proof. \square

3.2. Inverse Scattering Problem with NZBCs and Double Poles.

In the following subsections, we will propose an inverse problem with NZBCs and solve it to obtain explicit double poles solutions for the TOFKN equation (1.1).

3.2.1. The Matrix Riemann-Hilbert Problem with NZBCs and Double Poles.

Similar to the case of ZBCs in Section 2, the matrix RH problem for the NZBCs can also be established. In order to pose and solve the RH problem conveniently, we define $\tilde{\zeta}_n = -\frac{q_0^2}{\zeta_n}$ with

$$\zeta_n = \begin{cases} z_n, & n = 1, 2, \dots, N_1, \\ -z_{n-N}, & n = N_1 + 1, N_1 + 2, \dots, 2N_1, \\ \frac{q_0^2}{z_{n-2N_1}^*}, & n = 2N_1 + 1, 2N_1 + 2, \dots, 3N_1, \\ -\frac{q_0^2}{z_{n-3N_1}^*}, & n = 3N_1 + 1, 3N_1 + 2, \dots, 4N_1, \\ \omega_{n-4N_1}, & n = 4N_1 + 1, 4N_1 + 2, \dots, 4N_1 + N_2, \\ -\omega_{n-4N_1-N_2}, & n = 4N_1 + N_2 + 1, 4N_1 + N_2 + 2, \dots, 4N_1 + 2N_2. \end{cases} \quad (3.46)$$

Then, we can proposed a matrix RH problem as follows.

Proposition 21. Define the sectionally meromorphic matrices

$$M(z; x, t) = \begin{cases} M^+(z; x, t) = \left(\frac{\mu_{+1}(z; x, t)}{s_{11}(z)}, \mu_{-2}(z; x, t) \right), & \text{as } z \in D^+, \\ M^-(z; x, t) = \left(\mu_{-1}(z; x, t), \frac{\mu_{+2}(z; x, t)}{s_{22}(z)} \right), & \text{as } z \in D^-, \end{cases} \quad (3.47)$$

where $\lim_{\substack{z' \rightarrow z \\ z' \in D^\pm}} M(z'; x, t) = M^\pm(z; x, t)$. Then, the multiplicative matrix Riemann-Hilbert problem is given below:

- *Analyticity:* $M(z; x, t)$ is analytic in $D^+ \cup D^- \setminus Z$ and has the double poles in Z , whose principal parts of the Laurent series at each double pole ζ_n or $\tilde{\zeta}_n$, are determined as

$$\begin{aligned} \text{Res}_{z=\zeta_n} M(z; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \{ \mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)] \mu_{-2}(\zeta_n; x, t) \}, 0 \right), \\ \text{P}_{-2} M(z; x, t) &= \left(A[\zeta_n] e^{-2i\theta(\zeta_n; x, t)} \mu_{-2}(\zeta_n; x, t), 0 \right), \\ \text{Res}_{z=\tilde{\zeta}_n} M(z; x, t) &= \left(0, A[\tilde{\zeta}_n] e^{2i\theta(\tilde{\zeta}_n; x, t)} \{ \mu'_{-1}(\tilde{\zeta}_n; x, t) + [B[\tilde{\zeta}_n] + 2i\theta'(\tilde{\zeta}_n; x, t)] \mu_{-1}(\tilde{\zeta}_n; x, t) \} \right), \\ \text{P}_{-2} M(z; x, t) &= \left(0, A[\tilde{\zeta}_n] e^{2i\theta(\tilde{\zeta}_n; x, t)} \mu_{-1}(\tilde{\zeta}_n; x, t) \right). \end{aligned} \quad (3.48)$$

• *Jump condition:*

$$M^-(z; x, t) = M^+(z; x, t)[I - J(z; x, t)], \text{ as } z \in \Sigma, \quad (3.49)$$

where

$$J(z; x, t) = e^{i\theta(z; x, t)\widehat{\sigma}_3} \begin{pmatrix} 0 & -\tilde{\rho}(z) \\ \rho(z) & \rho(z)\tilde{\rho}(z) \end{pmatrix}. \quad (3.50)$$

• *Asymptotic behavior:*

$$M(z; x, t) = \begin{cases} \frac{i}{z} e^{i\nu_-(x, t)\sigma_3} \sigma_3 Q_- + O(1), & \text{as } z \rightarrow 0, \\ e^{i\nu_-(x, t)\sigma_3} + \left(\frac{1}{z}\right), & \text{as } z \rightarrow \infty. \end{cases} \quad (3.51)$$

Proof. For the analyticity of $M(z; x, t)$, It follows from Eqs. (3.13) and (3.43) that for each double poles $\zeta_n \in D^+$ or $\tilde{\zeta}_n \in D^-$. Now, we consider $\zeta_n \in D^+$, and obtain

$$\begin{aligned} \text{Res}_{z=\zeta_n} \left[\frac{\psi_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= \text{Res}_{z=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} e^{i\theta(\zeta_n; x, t)} \right] = A[\zeta_n][\psi'_{-2}(\zeta_n; x, t) + B[\zeta_n]\psi_{-2}(\zeta_n; x, t)] \\ &= A[\zeta_n]\{[\mu_{-2}(\zeta_n; x, t)e^{-i\theta(\zeta_n; x, t)}] + B[\zeta_n]\mu_{-2}(\zeta_n; x, t)e^{-i\theta(\zeta_n; x, t)}\}, \Rightarrow \\ \text{Res}_{z=\zeta_n} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= A[\zeta_n]e^{-2i\theta(\zeta_n; x, t)}\{\mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)]\mu_{-2}(\zeta_n; x, t)\}, \Rightarrow \\ \text{Res}_{z=\zeta_n} M(z; x, t) &= \left(A[\zeta_n]e^{-2i\theta(\zeta_n; x, t)}\{\mu'_{-2}(\zeta_n; x, t) + [B[\zeta_n] - 2i\theta'(\zeta_n; x, t)]\mu_{-2}(\zeta_n; x, t)\}, 0 \right), \end{aligned}$$

where the “'” denotes the partial derivative with respect to z (here $z = \zeta_n$), and

$$\begin{aligned} P_{-2} \left[\frac{\psi_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= P_{-2} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} e^{i\theta(\zeta_n; x, t)} \right] = A[\zeta_n]\mu_{-2}(\zeta_n; x, t)e^{-i\theta(\zeta_n; x, t)}, \Rightarrow \\ P_{-2} \left[\frac{\mu_{+1}(\zeta_n; x, t)}{s_{11}(\zeta_n)} \right] &= A[\zeta_n]e^{-2i\theta(\zeta_n; x, t)}\mu_{-2}(\zeta_n; x, t), \Rightarrow \\ P_{-2} M(\lambda; x, t) &= \left(A[\zeta_n]e^{-2i\theta(\zeta_n; x, t)}\mu_{-2}(\zeta_n; x, t), 0 \right). \end{aligned}$$

Similarly, we also can obtain the analyticity for $\tilde{\zeta}_n \in D^-$. It follows from Eqs. (3.13) and (3.19) that

$$\begin{cases} \mu_{+1}(z; x, t) = \mu_{-1}(z; x, t)s_{11}(z) + \mu_{-2}(z; x, t)e^{-2i\theta(z; x, t)}s_{21}(z), \\ \mu_{+2}(z; x, t) = \mu_{-1}(z; x, t)e^{2i\theta(z; x, t)}s_{12}(z) + \mu_{-2}(z; x, t)s_{22}(z), \end{cases}$$

by combining formula (3.21), one can obtain

$$\begin{aligned} M^+(z; x, t) &= \left(\frac{\mu_{+1}(z; x, t)}{s_{11}(z)}, \mu_{-2}(z; x, t) \right) = \left(\mu_{-1}(z; x, t) + \mu_{-2}(z; x, t)e^{-2i\theta(z; x, t)}\rho(z), \mu_{-2}(z; x, t) \right), \\ M^-(z; x, t) &= \left(\mu_{-1}(z; x, t), \frac{\mu_{+2}(z; x, t)}{s_{22}(z)} \right) = \left(\mu_{-1}(z; x, t), \mu_{-1}(z; x, t)e^{2i\theta(z; x, t)}\tilde{\rho}(z) + \mu_{-2}(z; x, t) \right), \end{aligned}$$

and

$$\begin{aligned} \left(\mu_{-1}(z; x, t), \frac{\mu_{+2}(z; x, t)}{s_{22}(z)} \right) &= \left(\frac{\mu_{+1}(z; x, t)}{s_{11}(z)}, \mu_{-2}(z; x, t) \right) \begin{pmatrix} 1 & e^{2i\theta(z; x, t)}\tilde{\rho}(z) \\ -e^{-2i\theta(z; x, t)}\rho(z) & 1 - \rho(z)\tilde{\rho}(z) \end{pmatrix}, \\ M^-(z; x, t) &= M^+(z; x, t)(I - J(z; x, t)), \end{aligned}$$

where $J(z; x, t)$ is given by Eq. (3.50). The asymptotic behaviors of the modified Jost solutions $\mu_{\pm}(z; x, t)$ and scattering matrix $S(z)$ given in Propositions (17) and (18) can easily lead to the asymptotic behavior of $M(z; x, t)$.

Specifically, when $z \rightarrow 0$, we have

$$\begin{aligned}
M^+(z; x, t) &= \left(\frac{\mu_{+1}(z; x, t)}{s_{11}(z)}, \mu_{-2}(z; x, t) \right) = \begin{pmatrix} 0 & \frac{i}{z} q_- e^{i\nu_-(x, t)} \\ \frac{iq_+^* e^{-i\nu_+(x, t)}}{z \frac{q_-}{q_+} e^{i\nu(t)}} & 0 \end{pmatrix} + O(1) \\
&= \frac{i}{z} \begin{pmatrix} 0 & q_- e^{i\nu_-(x, t)} \\ q_-^* e^{-i\nu_-(x, t)} & 0 \end{pmatrix} + O(1) = \frac{i}{z} e^{i\nu_-(x, t)\sigma_3} \sigma_3 Q_- + O(1), \\
M^-(z; x, t) &= \left(\mu_{-1}(z; x, t), \frac{\mu_{+2}(z; x, t)}{s_{22}(z)} \right) = \begin{pmatrix} 0 & \frac{\frac{i}{z} q_+ e^{i\nu_+(x, t)}}{\frac{q_+}{q_-} e^{-i\nu(t)}} \\ \frac{i}{z} q_-^* e^{-i\nu_-(x, t)} & 0 \end{pmatrix} + O(1) \\
&= \frac{i}{z} \begin{pmatrix} 0 & q_- e^{i\nu_-(x, t)} \\ q_-^* e^{-i\nu_-(x, t)} & 0 \end{pmatrix} + O(1) = \frac{i}{z} e^{i\nu_-(x, t)\sigma_3} \sigma_3 Q_- + O(1),
\end{aligned}$$

it can obtain that $M(z; x, t) = \frac{i}{z} e^{i\nu_-(x, t)\sigma_3} \sigma_3 Q_- + O(1)$ as $z \rightarrow 0$. Moreover, when $z \rightarrow \infty$, we have

$$\begin{aligned}
M^+(z; x, t) &= \left(\frac{\mu_{+1}(z; x, t)}{s_{11}(z)}, \mu_{-2}(z; x, t) \right) = \begin{pmatrix} \frac{e^{i\nu_+(x, t)}}{e^{-i\nu(t)}} & 0 \\ 0 & e^{-i\nu_-(x, t)} \end{pmatrix} + O\left(\frac{1}{z}\right) \\
&= \begin{pmatrix} e^{i\nu_-(x, t)} & 0 \\ 0 & e^{-i\nu_-(x, t)} \end{pmatrix} + O\left(\frac{1}{z}\right) = e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{z}\right), \\
M^-(z; x, t) &= \left(\mu_{-1}(z; x, t), \frac{\mu_{+2}(z; x, t)}{s_{22}(z)} \right) = \begin{pmatrix} e^{i\nu_-(x, t)} & 0 \\ 0 & \frac{e^{-i\nu_+(x, t)}}{e^{i\nu(t)}} \end{pmatrix} + O\left(\frac{1}{z}\right) \\
&= \begin{pmatrix} e^{i\nu_-(x, t)} & 0 \\ 0 & e^{-i\nu_-(x, t)} \end{pmatrix} + O\left(\frac{1}{z}\right) = e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{z}\right),
\end{aligned}$$

it can obtain that $M(z; x, t) = e^{i\nu_-(x, t)\sigma_3} + O\left(\frac{1}{z}\right)$. Therefore, we obtain the asymptotic behavior (3.51) of $M(z; x, t)$ when $z \rightarrow 0$ and $z \rightarrow \infty$. This completes the proof. \square

Proposition 22. *The solution of the matrix Riemann-Hilbert problem with double poles can be expressed as*

$$\begin{aligned}
M(z; x, t) &= e^{i\nu_-(x, t)\sigma_3} \left(I + \frac{i}{z} \sigma_3 Q_- \right) + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(\xi; x, t) J(\xi; x, t)}{\xi - z} d\xi + \\
&\quad \sum_{n=1}^{4N_1+2N_2} \left(C_n(z) \left[\mu'_{-2}(\zeta_n; x, t) + \left(D_n + \frac{1}{z - \zeta_n} \right) \mu_{-2}(\zeta_n; x, t) \right], \right. \\
&\quad \left. \tilde{C}_n(z) \left[\mu'_{-1}(\tilde{\zeta}_n; x, t) + \left(\tilde{D}_n + \frac{1}{z - \tilde{\zeta}_n} \right) \mu_{-1}(\tilde{\zeta}_n; x, t) \right] \right),
\end{aligned} \tag{3.52}$$

where \int_{Σ} stands for an integral along the oriented contour displayed in Fig. 4,

$$\begin{aligned}
C_n(z) &= \frac{A[\zeta_n]}{z - \zeta_n} e^{-2i\theta(\zeta_n; x, t)}, \quad \tilde{C}_n(z) = \frac{A[\tilde{\zeta}_n]}{z - \tilde{\zeta}_n} e^{2i\theta(\tilde{\zeta}_n; x, t)}, \\
D_n &= B[\zeta_n] - 2i\theta'(\zeta_n; x, t), \quad \tilde{D}_n = B[\tilde{\zeta}_n] + 2i\theta'(\tilde{\zeta}_n; x, t),
\end{aligned} \tag{3.53}$$

$\mu_{-2}(\zeta_n; x, t)$ and $\mu'_{-2}(\zeta_n; x, t)$ are determined via $\mu_{-1}(\tilde{\zeta}_n; x, t)$ and $\mu'_{-1}(\tilde{\zeta}_n; x, t)$ as

$$\mu_{-2}(\zeta_n; x, t) = \frac{iq_-}{\zeta_n} \mu_{-1}(\tilde{\zeta}_n; x, t), \quad \mu'_{-2}(\zeta_n; x, t) = -\frac{iq_-}{\zeta_n^2} \mu_{-1}(\tilde{\zeta}_n; x, t) + \frac{iq_- q_0^2}{\zeta_n^3} \mu'_{-1}(\tilde{\zeta}_n; x, t), \tag{3.54}$$

and $\mu_{-1}(\tilde{\zeta}_n; x, t)$ and $\mu'_{-1}(\tilde{\zeta}_n; x, t)$ satisfy the linear system of $8N_1 + 4N_2$ as below:

$$\begin{aligned} & \sum_{n=1}^{4N_1+2N_2} \left\{ \tilde{C}_n(\zeta_s) \mu'_{-1}(\tilde{\zeta}_n; x, t) + \left[\tilde{C}_n(\zeta_s) \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s} \delta_{s,n} \right] \mu_{-1}(\tilde{\zeta}_n; x, t) \right\} = \\ & - e^{i\nu_-(x,t)\sigma_3} \left(\frac{iq_-}{\zeta_s} \right) - \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t) J(\xi; x, t))_2}{\xi - \zeta_s} d\xi, \\ & \sum_{n=1}^{4N_1+2N_2} \left\{ \left(\frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} + \frac{iq_- q_0^2}{\zeta_s^3} \delta_{s,n} \right) \mu'_{-1}(\tilde{\zeta}_n; x, t) + \left[\frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} \left(\tilde{D}_n + \frac{2}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s^2} \delta_{s,n} \right] \mu_{-1}(\tilde{\zeta}_n; x, t) \right\} = \\ & - e^{i\nu_-(x,t)\sigma_3} \left(\frac{iq_-}{\zeta_s^2} \right) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t) J(\xi; x, t))_2}{(\xi - \zeta_s)^2} d\xi, \end{aligned} \quad (3.55)$$

where $s = 1, 2, \dots, 4N_1 + 2N_2$ and $\delta_{s,n}$ are the Kronecker δ -symbol.

Proof. Similar to the proof of Proposition (9) for the case of ZBCs. In order to regularize the RH problem established in Proposition (21) for the case of NZBCs, one has to subtract out the asymptotic values as $z \rightarrow 0$ and $z \rightarrow \infty$ given by Eq. (3.51) and the singular contributions. Then, the jump condition (3.49) becomes

$$\begin{aligned} & M^-(z; x, t) - \frac{i}{z} e^{i\nu_-(x,t)\sigma_3} \sigma_3 Q_- - e^{i\nu_-(x,t)\sigma_3} - \\ & \sum_{n=1}^{4N_1+2N_2} \left[\frac{P_{-2} M(z; x, t)}{(z - \zeta_n)^2} + \frac{\text{Res } M(z; x, t)}{z - \zeta_n} + \frac{P_{-2} M(z; x, t)}{(z - \tilde{\zeta}_n)^2} + \frac{\text{Res } M(z; x, t)}{z - \tilde{\zeta}_n} \right] = M^+(z; x, t) - \frac{i}{z} e^{i\nu_-(x,t)\sigma_3} \sigma_3 Q_- - \\ & e^{i\nu_-(x,t)\sigma_3} - \sum_{n=1}^{4N_1+2N_2} \left[\frac{P_{-2} M(z; x, t)}{(z - \zeta_n)^2} + \frac{\text{Res } M(z; x, t)}{z - \zeta_n} + \frac{P_{-2} M(z; x, t)}{(z - \tilde{\zeta}_n)^2} + \frac{\text{Res } M(z; x, t)}{z - \tilde{\zeta}_n} \right] - M^+(z; x, t) J(z; x, t), \end{aligned} \quad (3.56)$$

where $P_{-2} M(z; x, t)$, $\text{Res } M(z; x, t)$, $P_{-2} M(z; x, t)$, $\text{Res } M(z; x, t)$ have given in Eq. (3.48). By using Plemelj's formula, one can obtain the solution (3.52) with formula (3.53) of the matrix RH problem. According to the symmetry reduction $\mu_{\pm}(z; x, t) = \frac{i}{z} \mu_{\pm} \left(-\frac{q_0^2}{z}; x, t \right) \sigma_3 Q_{\pm}$, we can obtain

$$\mu_{-2}(z; x, t) = \frac{i}{z} q_- \mu_{-1} \left(-\frac{q_0^2}{z}; x, t \right),$$

and take derivative respect to z on both sides of above equation at the same time. Then, let $z = \zeta_n$, one have following formula

$$\mu'_{-2}(\zeta_n; x, t) = -\frac{iq_-}{\zeta_n^2} \mu_{-1}(\tilde{\zeta}_n; x, t) + \frac{iq_- q_0^2}{\zeta_n^3} \mu'_{-1}(\tilde{\zeta}_n; x, t).$$

From Eq. (3.47), when we take $z = \zeta_s$, $\mu_{-2}(\zeta_s; x, t)$ is the second column element of matrix $M(\zeta_s; x, t)$, that is

$$\begin{aligned} \mu_{-2}(\zeta_s; x, t) &= \left(\frac{i}{\zeta_s} e^{i\nu_-(x,t)q_-} \right) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t) J(\xi; x, t))_2}{\xi - \zeta_s} d\xi + \\ & \sum_{n=1}^{4N_1+2N_2} \left\{ \tilde{C}_n(\zeta_s) \left[\mu'_{-1}(\tilde{\zeta}_n; x, t) + \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} \right) \mu_{-1}(\tilde{\zeta}_n; x, t) \right] \right\}, \end{aligned} \quad (3.57)$$

where $s = 1, 2, \dots, 4N_1 + 2N_2$ and ζ_s is equivalent to ζ_n . Then, taking the derivative respect to z (here $z = \zeta_s$) on both sides of Eq. (3.57) at the same time, we have

$$\begin{aligned} \mu'_{-2}(\zeta_s; x, t) &= \left(-\frac{i}{\zeta_s^2} q_- e^{i\nu_-(x,t)} \right) + \frac{1}{2\pi i} \int_{\Sigma} \frac{(M^+(\xi; x, t) J(\xi; x, t))_2}{(\xi - \zeta_s)^2} d\xi - \\ & \sum_{n=1}^{4N_1+2N_2} \tilde{C}_n(\zeta_s) \left\{ \frac{1}{\zeta_s - \tilde{\zeta}_n} \left[\mu'_{-1}(\tilde{\zeta}_n; x, t) + \left(\tilde{D}_n + \frac{2}{\zeta_s - \tilde{\zeta}_n} \right) \mu_{-1}(\tilde{\zeta}_n; x, t) \right] \right\}. \end{aligned} \quad (3.58)$$

After that, substituting Eqs. (3.57)-(3.58) into Eq. (3.54) to eliminate $\mu_{-2}(\zeta_s; x, t)$ and $\mu'_{-2}(\zeta_s; x, t)$, by merging and simplifying, one can derived Eq. (3.55). Completing the proof. \square

3.2.2. Reconstruction Formula of the Potential.

From the solution (3.52) of the matrix RH problem, we have

$$M(z; x, t) = e^{i\nu_-(x,t)\sigma_3} + \frac{M^{[1]}(x, t)}{z} + O\left(\frac{1}{z^2}\right), \text{ as } z \rightarrow \infty, \quad (3.59)$$

where

$$\begin{aligned} M^{[1]}(x, t) = & ie^{i\nu_-(x,t)\sigma_3}\sigma_3Q_- - \frac{1}{2\pi i} \int_{\Sigma} M^+(\xi; x, t)J(\xi; x, t)d\xi + \sum_{n=1}^{4N_1+2N_2} \{A[\tilde{\zeta}_n]e^{-2i\theta(\tilde{\zeta}_n; x, t)}[\mu'_{-2}(\tilde{\zeta}_n; x, t) + D_n\mu_{-2}(\tilde{\zeta}_n; x, t)], \\ & A[\tilde{\zeta}_n]e^{2i\theta(\tilde{\zeta}_n; x, t)}[\mu'_{-1}(\tilde{\zeta}_n; x, t) + \tilde{D}_n\mu_{-1}(\tilde{\zeta}_n; x, t)]\}. \end{aligned} \quad (3.60)$$

Substituting Eq. (3.59) into Eq. (3.15) and matching $O(z)$ term, we have

$$O(z) : \frac{i}{2}[M^{[1]}(x, t), \sigma_3] + \frac{1}{2}[\sigma_3Q_{\pm}e^{i\nu_-(x,t)\sigma_3}, \sigma_3] = (Q - Q_{\pm})e^{i\nu_-(x,t)\sigma_3},$$

then, by expanding the above equation, one can find the reconstruction formula of the double poles solution (potential) for the TOFKN equation with NZBCs as follows

$$q(x, t) = -ie^{i\nu_-(x,t)}M_{12}^{[1]}(x, t), \quad (3.61)$$

where $M_{12}^{[1]}(x, t)$ represents the first row and second column element of the matrix $M^{[1]}(x, t)$, and

$$M_{12}^{[1]}(x, t) = ie^{i\nu_-(x,t)}q_- - \frac{1}{2\pi i} \int_{\Sigma} [M^+(\xi; x, t)J(\xi; x, t)]_{12}d\xi + \sum_{n=1}^{4N_1+2N_2} \{A[\tilde{\zeta}_n]e^{2i\theta(\tilde{\zeta}_n; x, t)}[\mu'_{-11}(\tilde{\zeta}_n; x, t) + \tilde{D}_n\mu_{-11}(\tilde{\zeta}_n; x, t)]\}, \quad (3.62)$$

when taking row vector $\alpha = (\alpha^{(1)}, \alpha^{(2)})$ and column vector $\gamma = (\gamma^{(1)}, \gamma^{(2)})^T$, where

$$\begin{aligned} \alpha^{(1)} &= \left(A[\tilde{\zeta}_n]e^{2i\theta(\tilde{\zeta}_n; x, t)} \right)_{1 \times (4N_1+2N_2)}, \quad \alpha^{(2)} = \left(A[\tilde{\zeta}_n]e^{2i\theta(\tilde{\zeta}_n; x, t)}\tilde{D}_n \right)_{1 \times (4N_1+2N_2)}, \\ \gamma^{(1)} &= \left(\mu'_{-11}(\tilde{\zeta}_n; x, t) \right)_{1 \times (4N_1+2N_2)}, \quad \gamma^{(2)} = \left(\mu_{-11}(\tilde{\zeta}_n; x, t) \right)_{1 \times (4N_1+2N_2)}, \end{aligned} \quad (3.63)$$

we can obtain a more concise reconstruction formulation of the double-pole solution (potential) for the TOFKN equation with NZBCs as follows

$$q(x, t) = e^{i\nu_-(x,t)} \left[q_- e^{i\nu_-(x,t)} - i\alpha\gamma + \frac{1}{2\pi} \int_{\Sigma} (M^+(\xi; x, t)J(\xi; x, t))_{12}d\xi \right]. \quad (3.64)$$

3.2.3. Trace Formulae and Theta Condition.

The so-called trace formulae are that the scattering coefficients $s_{11}(z)$ and $s_{22}(z)$ are formulated in terms of the discrete spectrum Z and reflection coefficients $\rho(z)$ and $\tilde{\rho}(z)$. We know that $s_{11}(z), s_{22}(z)$ are analytic on D^+, D^- , respectively. The discrete spectral points ζ_n 's are the double zeros of $s_{11}(z)$, while $\tilde{\zeta}_n$'s are the double zeros of $s_{22}(z)$. Define the functions $\beta^{\pm}(z)$ as follows:

$$\begin{aligned} \beta^+(z) &= s_{11}(z) \prod_{n=1}^{4N_1+2N_2} \left(\frac{z - \tilde{\zeta}_n}{z - \zeta_n} \right)^2 e^{i\nu}, \\ \beta^-(z) &= s_{22}(z) \prod_{n=1}^{4N_1+2N_2} \left(\frac{z - \zeta_n}{z - \tilde{\zeta}_n} \right)^2 e^{-i\nu}. \end{aligned} \quad (3.65)$$

Then, $\beta^+(z)$ and $\beta^-(z)$ are analytic and have no zero in D^+ and D^- , respectively. Furthermore, we have the relation $\beta^+(z)\beta^-(z) = s_{11}(z)s_{22}(z)$ and the asymptotic behaviors $\beta^{\pm}(z) \rightarrow 1$, as $z \rightarrow \infty$.

According to $\det S(z) = s_{11}s_{22} - s_{21}s_{12} = 1$, we can derive

$$\frac{1}{s_{11}(z)s_{22}(z)} = 1 - \frac{s_{21}(z)s_{12}(z)}{s_{11}(z)s_{22}(z)} = 1 - \rho(z)\tilde{\rho}(z),$$

by taking logarithm on both sides of the above equation at the same time, we have

$$-\log(s_{11}(z)s_{22}(z)) = \log[1 - \rho(z)\tilde{\rho}(z)] \Rightarrow \log[\beta^+(z)\beta^-(z)] = -\log[1 - \rho(z)\tilde{\rho}(z)],$$

then employing the Plemelj' formula such that one has

$$\log\beta^{\pm}(z) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(z)\tilde{\rho}(z)]}{\xi - z} d\xi, \quad z \in D^{\pm}. \quad (3.66)$$

Substituting Eq. (3.66) into Eq. (3.65), we can obtain the trace formulae

$$\begin{aligned} s_{11}(z) &= \exp\left(-\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(z)\tilde{\rho}(z)]}{\xi - z} d\xi\right) \prod_{n=1}^{4N_1+2N_2} \left(\frac{z - \zeta_n}{z - \tilde{\zeta}_n}\right)^2 e^{-i\nu}, \\ s_{22}(z) &= \exp\left(\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 - \rho(z)\tilde{\rho}(z)]}{\xi - z} d\xi\right) \prod_{n=1}^{4N_1+2N_2} \left(\frac{z - \tilde{\zeta}_n}{z - \zeta_n}\right)^2 e^{i\nu}. \end{aligned} \quad (3.67)$$

As $z \rightarrow 0$, we consider formulas (3.67) and the asymptotic behavior of the scattering matrix (3.37), then the so-called theta condition is obtained. That is to say, there exists $i \in \mathbb{Z}$ such that

$$\arg\left(\frac{q_-}{q_+}\right) + 2\nu = 16 \sum_{n=1}^{N_1} \arg(z_n) + 8 \sum_{n=1}^{N_2} \arg(\omega_n) + 2\pi i + \frac{1}{2\pi} \int_{\Sigma} \frac{\log[1 - \rho(\xi)\tilde{\rho}(\xi)]}{\xi} d\xi. \quad (3.68)$$

3.2.4. Reflectionless Potential: Double-Pole Soliton Solutions.

For the case of the reflectionless potential: $\rho(z) = \tilde{\rho}(z) = 0$, the part jump matrix $J(z; x, t)$ in Eq. (3.50) can be simplified as $J(z; x, t) = 0_{2 \times 2}$. From the Volterra integral equation (72), one can derive $\psi_{\pm}(q_0; x, t) = Y_{\pm} q_0$. Combining with the definition of scattering matrix, one has $S(q_0) = I$ and $q_+ = q_-$. From the theta condition, there exists $i \in \mathbb{Z}$ lead to

$$\nu = 8 \sum_{n=1}^{N_1} \arg(z_n) + 4 \sum_{n=1}^{N_2} \arg(\omega_n) + \pi i. \quad (3.69)$$

Then Eqs. (3.55) and (3.64) with $J(z; x, t) = 0_{2 \times 2}$ become

$$\begin{aligned} &\sum_{n=1}^{4N_1+2N_2} \left\{ \tilde{C}_n(\zeta_s) \mu'_{-1}(\tilde{\zeta}_n; x, t) + \left[\tilde{C}_n(\zeta_s) \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s} \delta_{s,n} \right] \mu_{-1}(\tilde{\zeta}_n; x, t) \right\} = \\ &- e^{i\nu_-(x,t)\sigma_3} \begin{pmatrix} \frac{iq_-}{\zeta_s} \\ 1 \end{pmatrix}, \\ &\sum_{n=1}^{4N_1+2N_2} \left\{ \left(\frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} + \frac{iq_- q_0^2}{\zeta_s^3} \delta_{s,n} \right) \mu'_{-1}(\tilde{\zeta}_n; x, t) + \left[\frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} \left(\tilde{D}_n + \frac{2}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s^2} \delta_{s,n} \right] \mu_{-1}(\tilde{\zeta}_n; x, t) \right\} = \\ &- e^{i\nu_-(x,t)\sigma_3} \begin{pmatrix} \frac{iq_-}{\zeta_s^2} \\ 0 \end{pmatrix}, \end{aligned} \quad (3.70)$$

and

$$q(x, t) = e^{i\nu_-(x,t)} \left(q_- e^{i\nu_-(x,t)} - i\alpha\gamma \right). \quad (3.71)$$

Theorem 23. *The explicit expression for the double-pole solution of the TOFKN equation (1.1) with NZBCs is given by determinant formula*

$$q(x, t) = q_- \left(\frac{\det(\tilde{R})}{\det(\tilde{G})} \right)^2 \frac{\det(G)}{\det(R)} \left(1 + \frac{\det(G)}{\det(R)} \right), \quad (3.72)$$

where

$$\begin{aligned} R &= \begin{pmatrix} 0 & \alpha \\ \tau & G \end{pmatrix}, \quad \tilde{R} = \begin{pmatrix} 0 & \alpha \\ \tau & \tilde{G} \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau^{(1)} \\ \tau^{(2)} \end{pmatrix}, \\ \tau^{(1)} &= \begin{pmatrix} 1 \\ \zeta_s \end{pmatrix}_{(4N_1+2N_2) \times 1}, \quad \tau^{(2)} = \begin{pmatrix} 1 \\ \zeta_s^2 \end{pmatrix}_{(4N_1+2N_2) \times 1}, \end{aligned} \quad (3.73)$$

and the $(8N_1+4N_2) \times (8N_1+4N_2)$ partitioned matrix $G = \begin{pmatrix} G^{(1,1)} & G^{(1,2)} \\ G^{(2,1)} & G^{(2,2)} \end{pmatrix}$ with $G^{(i,j)} = \left(g_{s,n}^{(i,j)} \right)_{(4N_1+2N_2) \times (4N_1+2N_2)}$ ($i, j = 1, 2$) given by

$$\begin{aligned} g_{s,n}^{(1,1)} &= \tilde{C}_n(\zeta_s), \quad g_{s,n}^{(1,2)} = \tilde{C}_n(\zeta_s) \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s} \delta_{s,n}, \\ g_{s,n}^{(2,1)} &= \frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} + \frac{iq_- q_0^2}{\zeta_s^3} \delta_{s,n}, \quad g_{s,n}^{(2,2)} = \frac{\tilde{C}_n(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} \left(\tilde{D}_n + \frac{2}{\zeta_s - \tilde{\zeta}_n} \right) - \frac{iq_-}{\zeta_s^2} \delta_{s,n}, \end{aligned}$$

and the $(8N_1+4N_2) \times (8N_1+4N_2)$ partitioned matrix $\tilde{G} = \begin{pmatrix} \tilde{G}^{(1,1)} & \tilde{G}^{(1,2)} \\ \tilde{G}^{(2,1)} & \tilde{G}^{(2,2)} \end{pmatrix}$ with $\tilde{G}^{(i,j)} = \left(\tilde{g}_{s,n}^{(i,j)} \right)_{(4N_1+2N_2) \times (4N_1+2N_2)}$ ($i, j = 1, 2$) given by

$$\begin{aligned} \tilde{g}_{s,n}^{(1,1)} &= \frac{\lambda(\zeta_s)}{\lambda(\tilde{\zeta}_n)} \tilde{C}_n(\zeta_s), \quad \tilde{g}_{s,n}^{(1,2)} = \frac{\lambda(\zeta_s)}{\lambda(\tilde{\zeta}_n)} \tilde{C}_n(\zeta_s) \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} - \frac{\lambda'(\tilde{\zeta}_n)}{\lambda(\tilde{\zeta}_n)} \right) - \frac{iq_-}{\zeta_s} \delta_{s,n}, \\ \tilde{g}_{s,n}^{(2,1)} &= \frac{\tilde{C}_n(\zeta_s)}{\lambda(\tilde{\zeta}_n)} \left(\frac{\lambda(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} - \lambda'(\zeta_s) \right) + \frac{iq_- q_0^2}{\zeta_s^3} \delta_{s,n}, \\ \tilde{g}_{s,n}^{(2,2)} &= \left[\frac{\lambda(\zeta_s)}{\zeta_s - \tilde{\zeta}_n} \left(\tilde{D}_n + \frac{2}{\zeta_s - \tilde{\zeta}_n} - \frac{\lambda'(\tilde{\zeta}_n)}{\lambda(\tilde{\zeta}_n)} \right) - \lambda'(\zeta_s) \left(\tilde{D}_n + \frac{1}{\zeta_s - \tilde{\zeta}_n} - \frac{\lambda'(\tilde{\zeta}_n)}{\lambda(\tilde{\zeta}_n)} \right) \right] - \frac{iq_-}{\zeta_s^2} \delta_{s,n}. \end{aligned}$$

Proof. From the Eqs. (3.63), (3.70) and (3.71), the reflectionless potential is deduced by determinants:

$$q(x, t) = q_- e^{2i\nu_-(x, t)} \left(1 + \frac{\det(R)}{\det(G)} \right). \quad (3.74)$$

However, this formula (3.74) is implicit since $\nu_-(x, t)$ is included. One needs to derive an explicit form for the reflectionless potential. From the trace formulae (3.67) and Volterra integral equation (3.18) as $x \rightarrow -\infty$, one derives that

$$\begin{aligned} M(z; x, t) = Y_-(z) + \lambda(z) \sum_{n=1}^{4N_1+2N_2} & \left[\frac{P_{-2}(M(\lambda(z); x, t)/\lambda(z))}{z=\tilde{\zeta}_n} \frac{1}{(z - \tilde{\zeta}_n)^2} + \frac{\text{Res}(M(\lambda(z); x, t)/\lambda(z))}{z=\tilde{\zeta}_n} \frac{1}{z - \tilde{\zeta}_n} + \right. \\ & \left. \frac{P_{-2}(M(\lambda(z); x, t)/\lambda(z))}{z=\tilde{\zeta}_n} \frac{1}{(z - \tilde{\zeta}_n)^2} + \frac{\text{Res}(M(\lambda(z); x, t)/\lambda(z))}{z=\tilde{\zeta}_n} \frac{1}{z - \tilde{\zeta}_n} \right], \end{aligned} \quad (3.75)$$

which can yield the γ given by Eq. (3.63) exactly. Then, substituting γ into the formula of the potential, one yields

$$q(x, t) = q_- e^{i\nu_-(x, t)} \left(e^{i\nu_-(x, t)} + \frac{\det(\tilde{R})}{\det(\tilde{G})} \right), \quad (3.76)$$

then, by combining Eq. (3.74) with Eq. (3.76), we can obtain Eq. (3.72), and complete the proof. \square

For example, we exhibit the double-pole solutions which contain three kind of types for the TOFKN equation with NZBCs:

- When taking parameters $N_1 = 0, N_2 = 1, q_{\pm} = 1, \omega_1 = e^{\frac{\pi}{4}i}, A[\omega_1] = i, B[\omega_1] = 1 + (1 - \sqrt{2})i$, we can obtain the explicit double-pole dark-bright soliton solution $q(x, t) = \frac{E_1}{E_2}$ with

$$\begin{aligned} E_1 &= -i \left\{ (3\sqrt{2}it + 3\sqrt{2}t + \sqrt{2} - 1 - i)^2 (e^{2x+7t})^2 + \left[2(12\sqrt{2}it - 36it^2 + 2\sqrt{2}i - 12it - 4i)(3\sqrt{2}it + 3\sqrt{2}t + \sqrt{2} - 1 - i) \right. \right. \\ &\quad \left. \left. e^{x+\frac{7}{2}t} + 2(-3\sqrt{2}it - \sqrt{2}i + 3\sqrt{2}t - 1 + i)(3\sqrt{2}it + 3\sqrt{2}t + \sqrt{2} - 1 - i) \right] e^{2x+7t} + (12\sqrt{2}it - 36it^2 + 2\sqrt{2}i - 12it - 4i)^2 \right. \\ &\quad \left. \left(e^{x+\frac{7}{2}t} \right)^2 + 2(-3\sqrt{2}it - \sqrt{2}i + 3\sqrt{2}t - 1 + i)(12\sqrt{2}it - 36it^2 + 2\sqrt{2}i - 12it - 4i) e^{x+\frac{7}{2}t} + (-3\sqrt{2}it - \sqrt{2}i + 3\sqrt{2}t - 1 \right. \right. \\ &\quad \left. \left. + i)^2 \right\} \left[(12\sqrt{2}it - 36it^2 + 2\sqrt{2}i - 12it + 4\sqrt{2} - 24t - 4) e^{2x+7t} + (2\sqrt{2}i - 6\sqrt{2}t - 2\sqrt{2} + 2) e^{x+\frac{7}{2}t} + (2\sqrt{2}i - 6\sqrt{2}t + 2) \right. \\ &\quad \left. e^{3x+\frac{21}{2}t} + i e^{4x+14t} + i \right], \\ E_2 &= \left[(12\sqrt{2}t - 36t^2 + 2\sqrt{2} - 12t - 4) e^{2x+7t} + (-6\sqrt{2}t - 2\sqrt{2} + 2) e^{x+\frac{7}{2}t} + (6\sqrt{2}t - 2) e^{3x+\frac{21}{2}t} - e^{4x+14t} - 1 \right] \\ &\quad \left[(3\sqrt{2}it + \sqrt{2}i - 3\sqrt{2}t + 1 - i) e^{2x+7t} + (12\sqrt{2}it - 36it^2 + 2\sqrt{2}i - 12it + 2\sqrt{2} - 2 - 2i - 12t) e^{x+\frac{7}{2}t} - \right. \\ &\quad \left. 3\sqrt{2}it - 3\sqrt{2}t - \sqrt{2} + 1 + i \right]^2, \end{aligned}$$

and give out relevant plots in Fig. 5. Fig. 5 (a) and (b) exhibit the three-dimensional and density diagrams for the exact double-pole dark-bright soliton solution of the TOFKN equation with NZBCs. Fig. 5 (c) displays the distinct profiles of the exact double-pole soliton for $t = \pm 3, 0$. It is a semi rational soliton, which is different from the simple pole solution usually expressed by exponential function, even if the double-pole dark-bright soliton solution shows the interaction of dark soliton and bright soliton.

Compared with the classical second-order flow Kaup-Newell system which also be called the DNLS equation in Ref. [44], according to the density diagrams of the double-pole dark-bright soliton solutions for the TOFKN equation and the DNLS equation in Ref. [44] shows that the trajectories of solutions are different obviously. Moreover, by

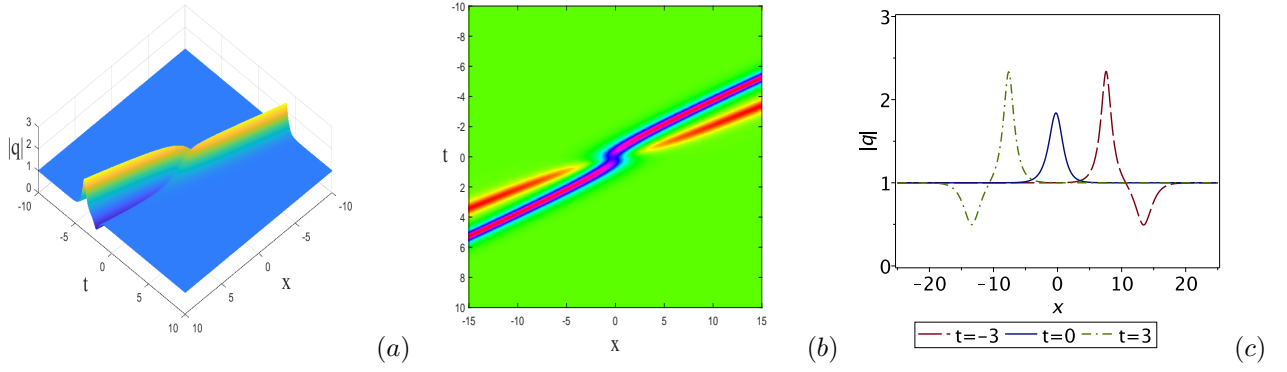


FIGURE 5. (Color online) The double-pole dark-bright soliton solution of TOFKN equation (1.1) with NZBCs and $N_1 = 0, N_2 = 1, q_{\pm} = 1, \omega_1 = e^{\frac{\pi}{4}i}, A[\omega_1] = i, B[\omega_1] = 1 + (1 - \sqrt{2})i$. (a) The three-dimensional plot; (b) The density plot; (c) The sectional drawings at $t = -3$ (dashed line), $t = 0$ (solid line), and $t = 3$ (dash-dot line).

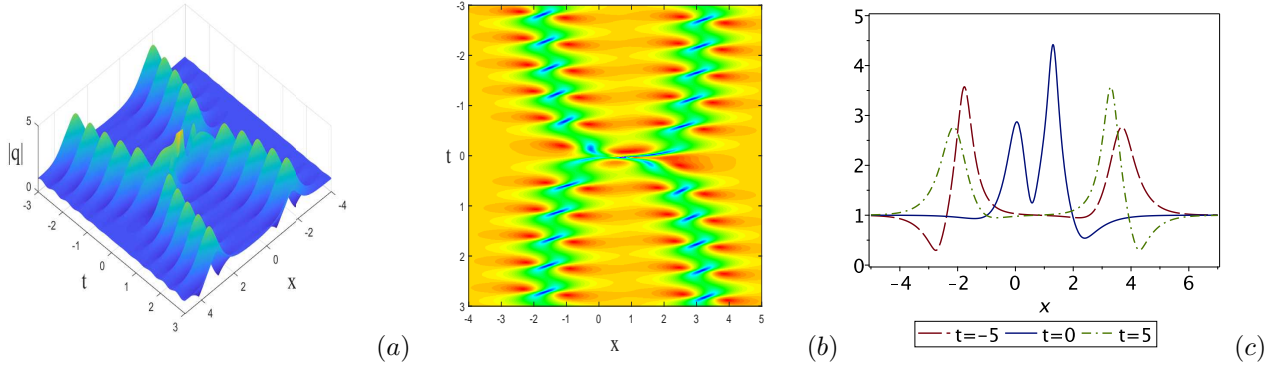


FIGURE 6. (Color online) The double-pole breather-breather solution of TOFKN equation (1.1) with NZBCs and $N_1 = 1, N_2 = 0, q_{\pm} = 1, \zeta_1 = 2e^{\frac{\pi}{4}i}, A[\zeta_1] = i, B[\zeta_1] = i$. (a) The three-dimensional plot; (b) The density plot; (c) The sectional drawings at $t = -5$ (dashed line), $t = 0$ (solid line), and $t = 5$ (dash-dot line).

observing the form of exact double-pole dark-bright soliton solution, the average wave velocity of the double-pole dark-bright soliton solution of the DNLS equation from Ref. [44] is about -2 , while the average wave velocity of the double-pole dark-bright soliton solution in this paper is about $-7/2$. It further validates the introduction of third-order dispersion and quintic nonlinear term of Kaup-Newell systems can affect the trajectories and the speed of solutions.

Remark 24. The parameter selection of the double-pole dark-bright soliton solution shown in Fig. 5. is consistent with that in reference [44].

- When taking parameters $N_1 = 1, N_2 = 0, q_{\pm} = 1, \zeta_1 = 2e^{\frac{\pi}{4}i}, A[\zeta_1] = i, B[\zeta_1] = i$, we can obtain the explicit double-pole breather-breather solution and give out relevant plots in Fig. 6. Figs. 6 (a) and (b) exhibit the three-dimensional and density diagrams for the exact double-pole breather-breather solution of the TOFKN equation with NZBCs. Fig. 6 (c) displays the distinct profiles of the exact double-pole breather-breather solution for $t = \pm 5, 0$. Moreover, from the density plot Fig. 6 (b), we can find that the propagation of the double-pole breather-breather solution is localized in space x and periodic in time t . Obviously, from the Fig. 6 (a) and (b), we can find that the double-pole breather-breather solution breathes 10 times at $t = [-3, 3]$ and collides once at near $t = 0$.

- When taking parameters $N_1 = 1, N_2 = 1, q_{\pm} = 1, \zeta_1 = 2e^{\frac{\pi}{4}i}, \omega_1 = e^{\frac{\pi}{4}i}, A[\zeta_1] = B[\zeta_1] = i, A[\omega_1] = B[\omega_1] = i$, we can obtain the explicit double-pole breather-breather-dark-bright solution and give out relevant plots in Fig. 7. Figs. 7 (a) and (b) exhibit the three-dimensional and density diagrams for the exact double-pole breather-breather-dark-bright solution of the TOFKN equation with NZBCs, which is equivalent to a combination of breather-breather solution and dark-bright soliton solutions. Fig. 7 (c) displays the distinct profiles of the exact double-pole breather-breather-dark-bright solution for $t = \pm 5, 0$. It is obvious that the double-pole breather-breather-dark-bright solution is an interaction between the double-pole breather-breather solution and the double-pole dark-bright soliton solution. Moreover, We can find that the amplitude at $t = 0$ in Fig. 7 (c) is between that the amplitude at $t = 0$ in Fig. 5 (c) and the amplitude at $t = 0$ in Fig. 6 (c), which also in accordance with the energy conservation law of the interaction between two waves.

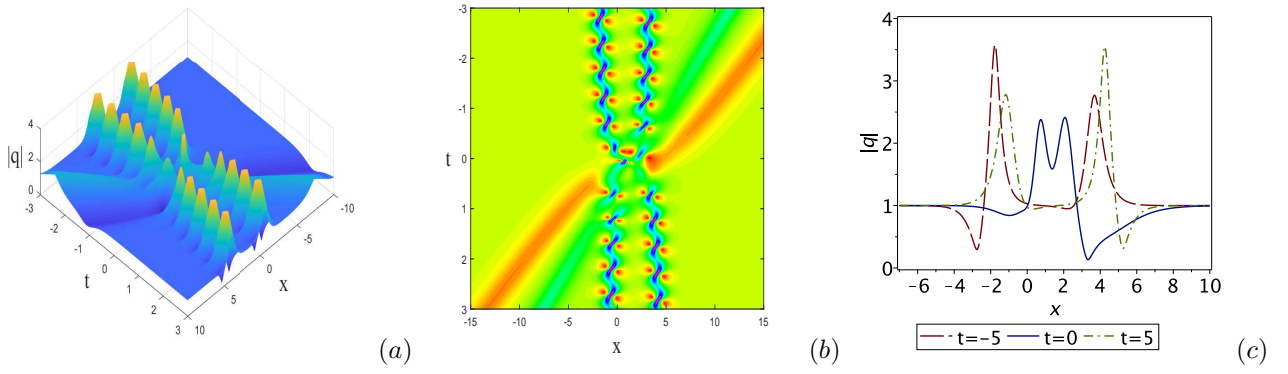


FIGURE 7. (Color online) The double-pole breather-breather-dark-bright solution of TOFKN equation (1.1) with NZBCs and $N_1 = 1, N_2 = 1, q_{\pm} = 1, \zeta_1 = 2e^{\frac{\pi}{4}i}, \omega_1 = e^{\frac{\pi}{4}i}, A[\zeta_1] = B[\zeta_1] = i, A[\omega_1] = B[\omega_1] = i$. (a) The three-dimensional plot; (b) The density plot; (c) The sectional drawings at $t = -5$ (dashed line), $t = 0$ (solid line), and $t = 5$ (dash-dot line).

4. CONCLUSIONS AND DISCUSSIONS

In this paper, we reveal the corresponding parameter reduction from the coupled TOFKN equation to the general form of the TOFKN equation (1.1) specifically. the inverse scattering transforms for the TOFKN equation under ZBCs and NZBCs have been built by applying RH method. We not only construct the direct scattering problem which illustrates the analyticity, symmetries, asymptotic behaviors and discrete spectrum, but also establish and solve the inverse problem which can derive the trace formula and reflectionless potential with the aid of a matrix RH problem. For the case of ZBCs, from the asymptotic behavior for the scattering matrix in the direct scattering problem, we connect the constant ν with the conservation of mass (also called power) I_1 of the modified Zakharov-Shabat spectral problem and exhibit essential proof in details. Due to the multi-valued case of eigenvalue $k(\lambda)$ in the case of NZBCs, we illustrate the scattering problem on a standard z -plane by utilizing two single-valued inverse mappings. Since the introduction of z -plane, the inverse scattering transform with NZBCs is more complex than the case of ZBCs. In particular, the reflectionless potentials with double poles which also be called the double-pole solutions of the TOFKN equation (1.1) are deduced explicitly via the determinants. Based on the RH method, we further improve the rigorous theory of inverse scattering transforms of higher-order flow KN system, and also provides a valuable reference for the inverse scattering transforms of the higher-order flow equations of KN systems or other nonlinear integrable dynamical systems.

Vivid illustration and dynamic behavior for some representative double-pole solutions have been given out in details. In the case of ZBCs, we exhibit the dynamic behaviors of the N -double-pole solution when taking $N = 1$ and $N = 2$ respectively, and discover the exact 1-double-pole soliton solution is equivalent to the elastic collisions of two bright-bright solitons. Furthermore, the exact 2-double-pole soliton solution is equivalent to the interaction of two 1-double-pole soliton solutions. In the case of NZBCs, when N_1 and N_2 take different values, we give out the dynamic behaviors of abundant double-pole solutions which contain the double-pole dark-bright soliton solution as $N_1 = 0, N_2 = 1$, the double-pole breather-breather solution as $N_1 = 1, N_2 = 0$ and the double-pole breather-breather-dark-bright solution as $N_1 = 1, N_2 = 1$.

In addition, by comparing the solutions of the TOFKN equation and the DNLS equation, it is found that the third-order dispersion and quintic nonlinear term of the KN system can affect both the trajectory and the speed of the solutions. However, the third-order dispersion and quintic nonlinear term of KN system have little influence on the maximum amplitude of the solution at a certain moment. These conclusions are consistent with those in Ref. [46]. These analytical results obtained in this work might have vital reference value for the study of the higher-order flow equations of KN systems or other nonlinear integrable dynamical systems, and provide a theoretical basis for possible experimental research and applications. At present, there are more and more researches on the triple poles solutions and even the N -pole solutions, and the long-time asymptotic behaviors have attracted more and more attention. In the future, we will study the multipole solutions and long-time asymptotic behaviors for the higher-order flow equations of KN systems or other nonlinear integrable dynamical systems.

REFERENCES

- [1] Lax, P.D.: Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure. Appl. Math.* **21**, 467-490 (1968)
- [2] Ablowitz, M.J., Fokas, A.S.: *Complex Variables: Introduction and Applications*. Cambridge University Press, Cambridge (2003)
- [3] Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. *Appl. Phys. Lett.* **23**, 142-144 (1973)
- [4] Yu, S.J., Toda, K., Fukuyama, T.: N-soliton solutions to a (2+1)-dimensional integrable equation. *J. Phys. A Math. Gen.* **31**, 10181-10186 (1998)
- [5] Iwao, M., Hirota, R.: Soliton solutions of a coupled modified KdV equations. *J. Phys. Soc. Japan* **66**, 577-588 (1997)

- [6] Osman, M.S., Ghanbari, B., Machado, J.A.T.: New complex waves in nonlinear optics based on the complex Ginzburg-Landau equation with Kerr law nonlinearity. *Eur. Phys. J. Plus* **134**, 20 (2019)
- [7] Raissi, M., Perdikaris, P., Karniadakis, G.E.: Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *J. Comput. Phys.* **378**, 686-707 (2019)
- [8] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M.: Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.* **19**, 1095-1097 (1967)
- [9] Hirota, R.: *Direct Methods in Soliton Theory*. Springer-verlag, Berlin (2004)
- [10] Matveev, V.B., Salle, M.A.: *Darboux Transformations and Solitons*. Springer-Verlag, Berlin (1991)
- [11] Satsuma, J.: Higher conservation laws for the Korteweg-de Vries equation through Bäcklund transformation. *Progr. Theoret. Phys.* **52**(4), 1396-1397 (1974)
- [12] Olver, P.J.: *Applications of Lie Groups to Differential Equations*. Springer, New York (1993)
- [13] Zakharov, V., Shabat, A.: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**, 62-69 (1972)
- [14] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **53**, 249-315 (1974)
- [15] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Nonlinear-evolution equations of physical significance. *Phys. Rev. Lett.* **31**, 125-127 (1973)
- [16] Wadati, M.: The modified Korteweg-de Vries equation. *J. Phys. Soc. Japan* **34**, 1289-1296 (1973)
- [17] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Method for solving the sine-Gordon equation. *Phys. Rev. Lett.* **30**, 1262-1264 (1973)
- [18] Fokas, A., Ablowitz, M.: The inverse scattering transform for the Benjamin-Ono equation pivot to multidimensional problems. *Stud. Appl. Math.* **68**, 1-10 (1983)
- [19] Ablowitz, M., Yaacov, D.B., Fokas, A.: On the inverse scattering transform for the Kadomtsev-Petviashvili equation. *Stud. Appl. Math.* **69**, 135-143 (1983)
- [20] Beals, R., Coifman, R.R.: Scattering and inverse scattering for first order systems. *Comm. Pure Applied Math.* **XXXVII**, 39-90 (1984)
- [21] Zakharov, V.E., Manakov, S.V., Novikov, S.P., Pitaevskii, L.P.: *The Theory of Solitons: The Inverse Scattering Method*. Consultants Bureau, New York (1984)
- [22] Deift, P., Zhou, X.: A steepest descent method for oscillatory Riemann-Hilbert problems. *Ann. Math.* **137**, 295-368 (1993)
- [23] Yang, B., Chen, Y.: High-order soliton matrices for Sasa-Satsuma equation via local Riemann-Hilbert problem. *Nonlinear Anal. Real World Appl.* **45**, 918-941 (2019)
- [24] Geng, X.G., Wu, J.P.: Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation. *Wave Motion* **60**, 62-72 (2016)
- [25] Zhang, X.E., Chen, Y.: Inverse scattering transformation for generalized nonlinear Schrödinger equation. *Appl. Math. Lett.* **98**, 306-313 (2019)
- [26] Ma, W.X.: Application of the Riemann-Hilbert approach to the multicomponent AKNS integrable hierarchies. *Nonlinear Anal. Real World Appl.* **47**, 1-17 (2019)
- [27] Guo, B.L., Ling, L.M.: Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation. *J. Math. Phys.* **53**, 133-3966 (2012)
- [28] Peng, W.Q., Tian, S.F., Wang, X.B., Zhang, T.T., Fang, Y.: Riemann-Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations. *J. Geom. Phys.* **146**, 103508 (2019)
- [29] Zhang, B., Fan, E.G.: Riemann-Hilbert approach for a Schrödinger-type equation with nonzero boundary conditions. *Modern Phys. Lett. B* **35**(12), 2150208 (2020)
- [30] Lenells, J., Fokas, A.S.: The unified method: II. NLS on the half-line t-periodic boundary conditions. *J. Phys. A Math. Theor.* **45**, 195202 (2012)
- [31] Bilman, D., Miller, P.: A robust inverse scattering transformation for the focusing nonlinear Schrödinger equation. *Comm. Pure Appl. Math.* **LXXII**, 1722-1805 (2019)
- [32] Xu, J., Fan, E.G.: Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: without solitons. *J. Differ. Equ.* **259**, 1098-1148 (2015)
- [33] Wang, D.S., Guo, B.L., Wang, X.L.: Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions. *J. Differ. Equ.* **266**(9), 5209-5253 (2019)
- [34] Kaup, D.J., Newell, A.C.: An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **19**(4), 798-801 (1978)
- [35] Xu, J., Fan E.G.: Long-time asymptotic behavior for the complex short pulse equation. *J. Differ. Equ.* **269**, 10322-10349 (2020)
- [36] Mio, K., Ogino, T., Minami, K., Takeda, S.: Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Japan* **41**, 265-271 (1976)
- [37] Xu, M.J., Xia, T.C., Hu, B.B.: Riemann-Hilbert approach and N-soliton solutions for the Chen-Lee-Liu equation. *Modern Phys. Lett. B* **33**, 1950002 (2019)
- [38] Wen, L.L., Zhang, N., Fan, E.G.: N-Soliton Solution of The Kundu-Type Equation Via Riemann-Hilbert Approach. *Acta Mathematica Scientia* **40**, 113-126 (2020)
- [39] Zhang, Z.C., Fan, E.G.: Inverse scattering transform for the Gerdjikov-Ivanov equation with nonzero boundary conditions. *Z. Angew. Math. Phys.* **71**, 149 (2020)
- [40] Zhang, G.Q., Yan, Z.Y.: Focusing and defocusing mKdV equations with nonzero boundary conditions: Inverse scattering transforms and soliton interactions. *Physica D* **410**, 132521 (2020)
- [41] Peng, W.Q., Chen, Y.: Double poles soliton solutions for the Gerdjikov-Ivanov type of derivative nonlinear schrödinger equation with zero/nonzero boundary couditions. *arXiv: arXiv:2104.12073v1*
- [42] Zhang, G.Q., Chen, S.Y., Yan, Z.Y.: Focusing and defocusing Hirota equations with non-zero boundary conditions: Inverse scattering transforms and soliton solutions. *Physica D* **80**, 104927 (2020)
- [43] Pichler, M., Biondini, G.: On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles. *IMA J. Appl. Math.* **82**, 131-151 (2017)
- [44] Zhang, G.Q., Yan, Z.Y.: The derivative nonlinear Schrödinger equation with zero/nonzero boundary conditions: inverse scattering transforms and N-double-pole solutions. *J. Nonlinear Sci.* **30**, 3089-3127 (2020)
- [45] Imai, K.: Generalization of the Kaup-Newell inverse scattering formulation and Darboux transformation. *J. Phys. Soc. Jpn.* **68**, 355-359 (1999)

- [46] Lin, H.A., He, J.S., Wang, L.H., Mihalache, D.: Several categories of exact solutions of the third-order flow equation of the Kaup-Newell system. *Nonlinear Dyn.* **100**, 2839-2858 (2020)
- [47] Biondini, G., Kovačič, G.: Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions. *J. Math. Phys.* **55**, 031506 (2014)
- [48] Yang, J.K.: *Nonlinear Waves in Integrable and Nonintegrable Systems*. Philadelphia Press, SIAM (2010)
- [49] Faddeev, L.D., Takhtajan, L.A.: *Hamiltonian Methods in the Theory of Solitons*. Springer, Berlin (1987)

(JP) SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI KEY LABORATORY OF PURE MATHEMATICS AND MATHEMATICAL PRACTICE, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, PEOPLE'S REPUBLIC OF CHINA

(YC) SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI KEY LABORATORY OF PURE MATHEMATICS AND MATHEMATICAL PRACTICE, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, PEOPLE'S REPUBLIC OF CHINA

(YC) COLLEGE OF MATHEMATICS AND SYSTEMS SCIENCE, SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, QINGDAO 266590, PEOPLE'S REPUBLIC OF CHINA

(YC) DEPARTMENT OF PHYSICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA 321004, PEOPLE'S REPUBLIC OF CHINA
Email address: `ychen@sei.ecnu.edu.cn`