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On the Modelling of Shallow-Water Waves with the Coriolis Effect

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Abstract
Consideration herein is a rotation-Camassa–Holm-type equation, which can be derived as an asymptotic model for the propagation of long-crested shallow-water waves in the equatorial ocean regions with the weak Coriolis effect due to the Earth's rotation, and is also related to the compressible hyperelastic rod model in the material science. This model equation has a formal Hamiltonian structure, and its solution corresponding to physically relevant initial perturbations is more accurate on a much longer time scale. It is shown that the solutions blow up in finite time in the sense of wave breaking. A refined analysis based on the local structure of the dynamics is performed to provide the wave-breaking phenomena. The effects of the Coriolis force caused by the Earth's rotation and nonlocal higher nonlinearities on blow-up criteria and wave-breaking phenomena are also investigated. Finally, a sufficient condition for global strong solutions to the equation in some special case is given.

Keywords Coriolis effect · Generalized rotation-Camassa–Holm equation · Shallow water · Wave breaking · Global existence

Mathematics Subject Classification 35Q53 · 35B30 · 35G25

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1 Introduction

It is known that some simple shallow-water model equations, such as the well-known Korteweg–de Vries (KdV) equation (Korteweg and de Vries 1895), have been proposed as approximations to the Euler equations in some specific physical regimes, which the appearance of waves with length is much greater than the depth of the water. There are other important physical phenomena for gravity waves in shallow water such as waves of maxima height (Amick and Toland 1981) or wave breaking, which means that the wave remains bounded while its slope becomes unbounded in finite time (Stokes 1880). These physical phenomena need a transition to full nonlinearity and motivate researchers to construct some new models for nonlinear shallow-water waves (Whitham 1973). To describe the long-wave regime, the amplitude parameter ϵ and the shallowness parameter μ are denoted, respectively, by

$$\epsilon = \frac{a}{h_0} \ll 1, \qquad \mu = \frac{h_0^2}{\lambda^2} \ll 1, \tag{1.1}$$

with the long wave length λ, small amplitude a and mean level of water surface h_0. It is known that the KdV (Korteweg and de Vries 1895) and Benjamin–Bona–Mahoney (BBM) (Benjamin et al. 1972) models give rise to good asymptotic approximations for the unidirectional solutions of the irrotational two-dimensional water waves problem in the weakly nonlinear regime (known as Boussinesq scaling)

$$\mu \ll 1, \qquad \epsilon = O(\mu). \tag{1.2}$$

However, for more accurate asymptotic approximations of these types waves which have more nonlinear behaviour than dispersive, larger values of ϵ are considered in the moderately nonlinear regime (known as the Camassa–Holm scaling)

$$\mu \ll 1, \qquad \epsilon = O(\sqrt{\mu}). \tag{1.3}$$

It involves some terms of higher order in ϵ or μ, which are neglected in the Boussinesq model, and could be useful for higher-order approximations of unidirectional waves. The Camassa–Holm (CH) (Camassa and Holm 1993; Camassa et al. 1994; Fuchssteiner and Fokas 1981) and Degasperis-Procesi (DP) (Constantin and Lannes 2009; Degasperis and Procesi 1999) equations arise in the moderately nonlinear regime. Stronger nonlinear effects are obtained with the CH scaling. That means the presence of breaking waves could be investigated with a higher nonlinearity.

One of our goals in the present paper is to derive a model equation of the shallow-water waves propagating mainly in one direction with the effect of the Earth's rotation by the formal asymptotic procedures in the equatorial zone. It is also known that the ocean dynamics near the Equator is quite different from that in nonequatorial regions since the meridional component of the Coriolis force (an effect of the Earth's rotation) vanishes at the Equator, so that the Equator acts as a wave guide, facilitating azimuthal wave propagation (Constantin and Johnson 2015, 2016). Moreover, the f-plane approximation is appropriate [cf. the considerations made in Constantin (2012)] and this shows that two-dimensional waves of the type studied here are physically relevant. Such a model equation with the Coriolis effect, referred to as the generalized rotation-Camassa–Holm (gr-CH) equation, can be derived from the incompressible and irrotational two-dimensional shallow water in the equatorial region. It can be written as

$$\begin{aligned} & u_t - \beta\mu u_{txx} + cu_x + 3\alpha\epsilon u u_x - \beta_0\mu u_{xxx} + \omega_1\epsilon^2 u^2 u_x + \omega_2\epsilon^3 u^3 u_x \\ & \quad = \sigma\alpha\beta\epsilon\mu(2u_x u_{xx} + u u_{xxx}), \end{aligned} \tag{1.4}$$

where the solution u of Eq. (1.4) represents the horizontal velocity field at height z_0, and after the re-scaling, it is required that $0 \leq z_0 \leq 1$,

$$z_0^2 = \frac{(3c^4 + 5c^2 + 3)\sigma - (6c^4 + 7c^2 + 10)}{3(\sigma - 3)(c^2 + 1)^2} \tag{1.5}$$

with $\sigma \neq 3$. The real dimensionless constant σ is a parameter to balance between nonlinear steepening and amplification in fluid convection due to stretching. The parameter Ω is the constant rotational frequency due to the Coriolis effect and $c = \sqrt{1 + \Omega^2} - \Omega$ is the wave speed. The other constants in the equation are defined by

$$\alpha = \frac{c^2}{1 + c^2}, \quad \omega_1 = \frac{-3c(c^2 - 1)(c^2 - 2)}{2(1 + c^2)^3}, \quad \omega_2 = \frac{(c^2 - 2)(c^2 - 1)^2(8c^2 - 1)}{2(1 + c^2)^5},$$

$$\beta = \begin{cases} \frac{3c^4 + 8c^2 - 1}{3(3 - \sigma)(c^2 + 1)^2}, & \sigma \neq 3, \\ 0, & \sigma = 3, \end{cases}$$

and

$$\beta_0 = \begin{cases} \frac{c^3(1 + c^2)\sigma + 5c^3 - c}{3(3 - \sigma)(1 + c^2)^2}, & \sigma \neq 3, \\ -\frac{c^3}{3(1 + c^2)}, & \sigma = 3. \end{cases}$$

In the vanishing Coriolis force limit $\Omega \to 0$ with $\sigma \neq 3$, we have

$$c \to 1, \quad \alpha \to \frac{1}{2}, \quad \beta \to \frac{5}{6(3 - \sigma)}, \quad \beta_0 \to \frac{\sigma + 2}{6(3 - \sigma)}, \quad \omega_1 \to 0, \quad \omega_2 \to 0.$$

It is known that some higher-order nonlocal model equations (Chen et al. 2018; Fan et al. 2016; Gui et al. 2018, 2019) with the Coriolis effect from full water-wave equations could be derived on the basis of the f-plane approximation in shallow water. The method used in these researches follows the classical idea of asymptotic perturbation analysis (Ivanov 2009; Johnson 2002). These models can be used to describe unidirectional waves in the moderately nonlinear scaling regime, resulting in the CH-type equations.

It is noted that Eq. (1.4) can be simply reduced to some classical model equations with different choices of those parameters, σ and Ω. For instances, when $\sigma = 1$, Eq. (1.4) is reduced to the rotation-Camassa–Holm equation (Gui et al. 2018), which is recently derived from the irrotational two-dimensional equatorial shallow-water equation with the Coriolis effects in a rotating frame. In the case that $\sigma = -7$, it is a model for the surface wave in the shallow water (Constantin and Lannes 2009). The rigorous justification of the these models approximate to the two-dimensional incompressible Euler equations is investigated in Constantin and Lannes (2009) (see also Chen et al. 2018).

If the Coriolis effect vanishes, i.e. $\Omega = 0$, the corresponding coefficients in the higher-power nonlinearities should be $\omega_1 = 0$ and $\omega_2 = 0$. Using the scaling transformation $u(t,x) \to \alpha\varepsilon u(\sqrt{\beta\mu}\ t, \sqrt{\beta\mu}\ x)$ and the Galilean transformation $u(t,x) \to u(t, x - \frac{3}{4}t) + \frac{1}{4}$, the gr-CH equation (1.4) can be written as the compressible hyper-elastic rod equation (Dai 1998; Dai and Huo 2000)

$$u_t - u_{xxt} + 3uu_x = \sigma(2u_x u_{xx} + uu_{xxx}), \tag{1.6}$$

where the solution u represents the radial stretch relative to a pre-stressed state, and the value of material index σ could range from -29.4760 to 3.4174. In particular, $\sigma = 1$, Eq. (1.6) is simply the classical CH equation (Camassa and Holm 1993; Camassa et al. 1994; Fuchssteiner and Fokas 1981) (see also Constantin and Lannes 2009),

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

where the solution u is the horizontal fluid velocity in the x direction at height $z_0 = \frac{1}{\sqrt{2}}$. In the case of $\sigma = 1$, $\beta = 0$ and $\omega_2 = 0$ in (1.4), we have the following integrable Gardner equation (Gardner et al. 1968)

$$u_t + cu_x + 3\alpha\varepsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x = 0.$$

Let $m = (1 - \beta\mu\partial_x^2)u$. Equation (1.4) can be rewritten with the evolution of the momentum density m in the following,

$$\begin{aligned} m_t \ &+ \sigma\alpha\epsilon(um_x + 2u_x m) + 3(1-\sigma)\alpha\epsilon uu_x + cu_x - \beta_0\mu u_{xxx} \\ &+ \omega_1\epsilon^2 u^2 u_x + \omega_2\epsilon^3 u^3 u_x = 0. \end{aligned} \tag{1.7}$$

It has at least three conserved quantities defined by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} u \mathrm{d}x, \\ H_1(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + \beta\mu u_x^2\right)\mathrm{d}x, \quad \text{and} \\ H_2(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(cu^2 + \alpha\epsilon u^3 + \sigma\alpha\epsilon\beta\mu uu_x^2 + \beta_0\mu u_x^2 + \frac{1}{6}\omega_1\epsilon^2 u^4 + \frac{1}{10}\omega_2\epsilon^3 u^5\right)\mathrm{d}x. \end{aligned}$$

Denote that

$$\begin{aligned}B_1 =&\partial_x(1-\beta\mu\partial_x^2), \quad \text{and}\\ B_2 =&\partial_x\left(\left(\sigma\alpha\epsilon m+\frac{c}{2}\right)\cdot\right)+\left(\sigma\alpha\epsilon m+\frac{c}{2}\right)\partial_x+(1-\sigma)\alpha\epsilon\partial_x(u\cdot)+(1-\sigma)\alpha\epsilon u\partial_x\\ &-\beta_0\mu\partial_x^3+\frac{2}{3}\omega_1\epsilon^2\partial_x(u\partial_x^{-1}(u\partial_x\cdot))+\frac{5}{8}\omega_2\epsilon^3\partial_x(u^{\frac{3}{2}}\partial_x^{-1}(u^{\frac{3}{2}}\partial_x\cdot)).\end{aligned}$$

With a simple calculation, it reveals that the gr-CH equation (1.7) can be written as

$$m_t=-B_1\frac{\delta H_2}{\delta m}=-B_2\frac{\delta H_1}{\delta m},$$

where B_1 and B_2 are two skew-symmetric differential operators.

It is also our purpose here to investigate the wave-breaking phenomena and to find out how the Coriolis effect manifests in the gr-CH model. To this end, assume $u_0\in H^s(\mathbb{R})$ for some $s>\frac{3}{2}$. Then for any $\sigma\neq 3$, the Cauchy problem for the gr-CH equation (1.4) is locally well posed. There exist a positive time $T>0$ and a unique solution $u\in C([0,T];H^s(\mathbb{R}))\cap C^1([0,T];H^{s-1}(\mathbb{R}))$. Due to the Hamiltonian $H_1(u)$, the horizontal velocity u is uniformly bounded by the Sobolev imbedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Let $0<T^*_{u_0}<\infty$ be the maximal time of existence, which enables us to show

$$0<T^*_{u_0}<\infty \;\;\Rightarrow\;\; \int_0^{T^*_{u_0}}\|u_x(\tau)\|_{L^\infty}\mathrm{d}\tau=\infty,$$

the solutions can be extended further in time.

To derive certain initial data for the finite-time wave-breaking phenomena, we consider the characteristic of the gr-CH equation (1.4),

$$\begin{cases}\frac{\partial q}{\partial t}=\sigma u(t,q), & 0<t<T,\\ q(0,x)=x, & x\in\mathbb{R}.\end{cases}$$

The gr-CH equation with a nonlocal structure can be expressed as a weak form of nonlinear nonlocal transport type. The dynamics of the blow-up quantity along the characteristics in the gr-CH equation includes three parts: a nonlocal term, a local nonlinearity, and a term resulting from the weak Coriolis forcing. The three parts interact with each other. The nonlocal term can help maintain the regularity during waves propagate and prevent them from blowing up, even when dispersion is weak or absent, such as the BBM equation (Benjamin et al. 1972). When the local nonlinearity becomes stronger and dominates over the dispersion, the nonlocal effects singularities may occur in the sense of wave breaking, such as the CH equation (Constantin and Escher 1998; Fuchssteiner and Fokas 1981) and the Whitham equation (Whitham 1973). The Coriolis effect can spread out waves and make them decay in time, which delays the threshold of wave-breaking phenomenon. Breaking waves are commonly observed in the ocean and are important for a variety of reasons, but surprisingly little is known about them. Indeed, breaking waves place large hydrodynamic loads on man-made structure, transfer horizontal momentum to surface currents, provide a source of turbulent energy to mix the upper layers of the ocean, move sediment in shallow water, and enhance the air-sea exchange of gases and particulate matter (Cokelet 1977).

Via the transport theory, the dynamics of the wave-breaking quantity along the characteristics is established by the Riccati-type differential inequality. Then the discussion is approached by the detailed analysis on the evolution of solution u and its gradient u_x. Based on the method of characteristic combined with the use of the conservation laws, we have an estimate of u_x, which says that σu_x is always uniformly bounded from above. Therefore, it sees that the only way wave breaking can occur is that σu_x tends to $-\infty$. Hence, we give a necessary and sufficient condition for the finite-time wave breaking

$$\liminf_{t\to T^-x\in\mathbb{R}} \sigma u_x(t,x) = -\infty.$$

Recently, it is shown in Brandolese and Cortez (2014) how local structure of the solution of the classical CH equation affects the finite-time blow-up. The argument relies heavily on the fact that the convolution terms are quadratic and positively definite. This method is now improved to more general cases to estimate the characteristic dynamics on the ratio of solution profile u to its slope u_x (Chen et al. 2016, 2015). As for the gr-CH equation (1.4), the convolution contains cubic even quadratic nonlinearities which do not have a lower bound in terms of the local terms. Because of the higher-order nonlinearities in the equation, it is difficult to generate finite-time wave breaking under the purely local condition of the initial data. In our case, the blow-up can be derived by the interplay between u and u_x. More precisely, inspired by Chen et al. (2016, 2015), it motivates us to give a refined analysis of the characteristic dynamics with $M = a(u-b) - u_x$ and $N = a(u-b) + u_x$. So the estimates of M and N can be closed with the following form

$$M'(t) \geq -cMN + c_1, \quad N'(t) \leq cMN + c_2,$$

where the nonlocal terms c_i $(i = 1, 2)$ are bounded in terms of certain order of conservation laws. Based on these Riccati-type differential inequalities, we can establish the monotonicity of M and N. Then it follows that the wave-breaking phenomena occur in finite time.

Having considered the wave-breaking phenomena with certain initial profiles, a natural issue to arise from this discuss is whether or not there exist permanent waves with certain initial data to the gr-CH model in the influence of the Coriolis frequency Ω and the balance parameter σ. To see this, one needs to control the low bound of the slope u_x in the local existence time $t \in [0, T)$. Indeed, using the conserved energy E_0, one crucial ingredient in the case $\sigma = 0$, for instance, is the following estimate for the solution $u(t)$, $t \in [0, T)$,

$$\|u_x(t,x)\|_{L^\infty} \approx \|u_{0,x}\|_{L^\infty} + C(E_0, \sigma, \Omega)t.$$

Our main results of the paper are the derivation of the asymptotic rotation-Camassa–Holm model equation from the incompressible two-dimensional Euler equations with the Coriolis effect in the equatorial region, wave-breaking data (Theorem 4.2), blow-up rate (Theorem 5.1), and global existence (Theorem 6.1).

The remainder of the paper is organized as follows. In Sect. 2, the gr-CH equation is derived as an asymptotic model in the CH regime to the f-plane geophysical governing equations in the equatorial region, which can be established from the incompressible and irrotational full water-wave equations with considering the Coriolis effect. In Sect. 3, the local well-posedness result and the blow-up criteria are presented. In Sect. 4, the wave-breaking criteria and wave-breaking data are established in details. Section 5 is devoted to the blow-up rate of the slope to a breaking wave for the gr-CH equation. Finally, a sufficient condition for the existence of global solutions is provided in Sect. 6.

Notation Throughout the paper, for $1 \le p < \infty$, the norms in the Lebesgue space $L^p(\mathbb{R})$ will be written $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$, the space $L^\infty(\mathbb{R})$ consists of all essentially bounded, while $\|\cdot\|_{H^s}$ will stand for the norm in the Sobolev space $H^s(\mathbb{R})$ $(s \ge 0)$. We use $[A, B] = AB - BA$ to denote the commutator operator between A and B. By denoting $p(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$ and the notation "$*''$ the convolution, we define the two convolution operators p_+ and p_- as

$$p_+ * f(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^y f(y)\mathrm{d}y,$$
$$p_- * f(x) = \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} f(y)\mathrm{d}y.$$

Then we have the relations $p = p_+ + p_-$ and $p_x = p_- - p_+$.

2 Derivation of the Generalized Rotation-Camassa–Holm Equation

2.1 The Governing Equations

For geophysical water waves, the forces with major influence are the gravity and the Coriolis force induced by the rotation of the Earth. It is assumed that the earth is a perfect sphere with the radius 6371 km and a constant rotational speed $\Omega_0 = 73 \times 10^{-6}$ rad/s round the polar axis towards east. In this section, we will derive the generalized rotation-Camassa–Holm equation following the method of the formal derivation of the CH equation with the Coriolis effect in the equatorial region. Suppose that the interface between the air and the water is a free surface, and the water flows are incompressible and inviscid with a constant density ρ and no surface tension. Under the influence of the gravity g and the Coriolis force due to the Earth's rotation, a motion of water flow occupying a domain $\mathcal{D}_t$ in $\mathbb{R}$ can be governed by the following Euler equations

$$\begin{aligned}
&\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla P + \vec{g}, \quad \vec{x}\in\mathcal{D}_t,\\
&\nabla\cdot\vec{u} = 0, \quad \vec{x}\in\mathcal{D}_t,\\
&\vec{u}|_{t=0} = \vec{u_0}, \quad \vec{x}\in\mathcal{D}_0.
\end{aligned} \tag{2.1}$$

Here waves are considered at the surface of water with a flat bed and $\mathcal{D}_t = \{\vec{x} = (x, y, z) : 0 < z < h_0+\eta(t, x, y)\}$ with h_0 the typical depth of the water and $\eta(t, x, y)$ measuring the deviation from the average level. $\vec{u} = (u, v, w)^T$ is the velocity of the fluid, and $\vec{\Omega} = (0, \Omega_0\cos\phi, \Omega_0\sin\phi)^T$ is the angular velocity vector which is directed along the axis of rotation of the rotating reference frame with the rotational frequency $\Omega_0 \approx 73\cdot 10^{-6}$ rad/s and the local latitude ϕ. The function $P(t, x, y, z)$ is the pressure in the fluid, and $\vec{g} = (0, 0, -g)^T$ is the gravity acceleration with $g \approx 9.8$ m/s^2 at the Earth's surface. Setting the x-axis selected horizontally due east, the y-axis horizontally due north and the z-axis upward, a rotating framework with the origin located at a point on the Earth's surface will be investigated in this paper. The background flow, which will model the flow with the Equatorial Undercurrent (EUC), is chosen to be a constant vorticity. In fact, the mathematical problem in the case of constant vorticity is very little different from the irrotational case. On the other hand, for oceanic motions of a limited meridional extent, within 2° from the Equator, it is adequate to use the f-plane approximation in the governing equations (LeBlond and Mysak 1978). In this setting, with the assumption of the constant density ρ, the motion of inviscid irrotational fluid near the Equator in the f-plane approximation ($\sin\phi \approx 0, \phi \ll 1$) and the region $0 < z < h_0 + \eta(t, x, y)$ can be demonstrated by the Euler equations (2.1)

The specification of constant density and incompressibility directly decouples the dynamics from the thermodynamics and reduces the mass conservation equation to the condition of incompressibility

$$u_x + v_y + w_z = 0, \tag{2.2}$$

and the condition of irrotationality

$$(w_y - v_z, u_z - w_x, v_x - u_y)^T = (0, 0, 0)^T. \tag{2.3}$$

The pressure can be written as

$$P(t, x, z) = P_{\text{atm}} + \rho g(h_0 - z) + p(t, x, y, z), \tag{2.4}$$

where P_{atm} is the constant atmosphere pressure, and p is a pressure variable measuring the hydrostatic pressure distribution. On the wave surface $z = h_0 + \eta$, the dynamic condition of the fluid pressure matches the atmosphere $P = P_{\text{atm}}$. Then it yields that

$$p = \rho g\eta. \tag{2.5}$$

Meanwhile, the free surface of the wave consists of the same fluid particles at each moment of time, so the kinematic boundary condition is given by

$$w = \eta_t + u\eta_x + v\eta_y \ \ on \ \ z = h_0 + \eta(t, x, y). \tag{2.6}$$

As the fluid bed is assumed impermeable, we pose the no-flux condition

$$w = 0 \ \ on \ \ z = 0. \tag{2.7}$$

In the two-dimensional condition, the flows travels in the zonal direction along the equator independent of the y-coordinate, that is $v \equiv 0$ throughout the flow. In this case, the irrotational condition will be simplified as follows

$$u_z - w_x = 0. \tag{2.8}$$

According to the above discussion, Eqs. (2.1)–(2.8) can be summarized as the following form

$$\begin{aligned}
&u_t + uu_x + wu_z + 2\Omega_0 w = -\frac{1}{\rho}P_x, && \text{in} \quad 0 < z < h_0 + \eta(t, x, y),\\
&w_t + uw_x + ww_z - 2\Omega_0 u = -\frac{1}{\rho}P_z - g, && \text{in} \quad 0 < z < h_0 + \eta(t, x, y),\\
&u_x + w_z = 0, && \text{in} \quad 0 < z < h_0 + \eta(t, x, y),\\
&u_z - w_x = 0, && \text{in} \quad 0 < z < h_0 + \eta(t, x, y),\\
&p = \rho g \eta, && \text{on} \quad z = h_0 + \eta(t, x, y),\\
&w = \eta_t + u\eta_x, && \text{on} \quad z = h_0 + \eta(t, x, y),\\
&w = 0, && \text{on} \quad z = 0.
\end{aligned} \tag{2.9}$$

2.2 Derivation of the Model

In this part, we will derive the generalized rotation-Camassa–Holm equation with effect of the Coriolis force. Due to the magnitude of the physical quantities, we first introduce dimensionless quantities of the variables as follows

$$\begin{aligned}
&x \mapsto \lambda x, \ \ z \mapsto h_0 z, \ \ \eta \mapsto a\eta, \ \ t \mapsto \frac{\lambda}{\sqrt{gh_0}}t,\\
&u \mapsto \sqrt{gh_0}u, \ \ w \mapsto \sqrt{\mu g h_0}w, \ \ p \mapsto \rho g h_0 p.
\end{aligned} \tag{2.10}$$

Under the influence of the Earth's rotation, we introduce

$$\Omega = \sqrt{\frac{h_0}{g}}\Omega_0. \tag{2.11}$$

By virtue of the following scaling around a laminar flow

$$u \mapsto \epsilon u, \quad w \mapsto \epsilon w, \quad p \mapsto \epsilon p, \tag{2.12}$$

the geophysical water-wave governing equations (2.9) transform into

$$\begin{aligned}
&u_t + \epsilon(uu_x + wu_z) + 2\Omega w = -\frac{1}{\rho}p_x, && \text{in } \ 0 < z < 1 + \epsilon\eta(t,x),\\
&\mu\{w_t + \epsilon(uw_x + ww_z)\} - 2\Omega u = -p_z, && \text{in } \ 0 < z < 1 + \epsilon\eta(t,x),\\
&u_x + w_z = 0, && \text{in } \ 0 < z < 1 + \epsilon\eta(t,x),\\
&u_z - \mu w_x = 0, && \text{in } \ 0 < z < 1 + \epsilon\eta(t,x),\\
&p = \eta, && \text{on } \ z = 1 + \epsilon\eta(t,x),\\
&w = \eta_t + \epsilon u\eta_x, && \text{on } \ z = 1 + \epsilon\eta(t,x),\\
&w = 0, && \text{on } \ z = 0,
\end{aligned} \tag{2.13}$$

where $\epsilon = \frac{a}{h_0}$ and $\mu = \frac{h_0^2}{\lambda^2}$ are the two dimensionless parameters with a being the typical amplitude of the wave and λ being the typical wavelength of the wave.

Following Johnson's method in (1997, 2002), a suitable far-field variable combined with a propagation problem

$$\xi = \epsilon^{\frac{1}{2}}(x - ct), \quad \tau = \epsilon^{\frac{3}{2}}t, \tag{2.14}$$

can be applied on the governing equations in order to be consistent with the equation of mass conservation with c the group speed of water waves. We also transform $w = \sqrt{\epsilon}W$. Then the governing equations (2.13) are written in the new scaling form

$$cu_\xi - \epsilon(u_\tau + uu_\xi + Wu_z) - 2\Omega W = p_\xi, \quad \text{in } \ 0 < z < 1 + \epsilon\eta, \tag{2.15}$$

$$\epsilon\mu\{cW_\xi - \epsilon(W_\tau + uW_\xi + WW_z)\} + 2\Omega u = p_z, \quad \text{in } \ 0 < z < 1 + \epsilon\eta, \tag{2.16}$$

$$u_\xi + W_z = 0, \quad \text{in } \ 0 < z < 1 + \epsilon\eta, \tag{2.17}$$

$$u_z - \epsilon\mu W_\xi = 0, \quad \text{in } \ 0 < z < 1 + \epsilon\eta, \tag{2.18}$$

$$p = \eta, \quad \text{on } \ z = 1 + \epsilon\eta, \tag{2.19}$$

$$W = -c\eta_\xi + \epsilon(\eta_\tau + u\eta_\xi, \quad \text{on } \ z = 1 + \epsilon\eta, \tag{2.20}$$

$$W = 0, \quad \text{on } \ z = 0. \tag{2.21}$$

In order to investigate solutions of the system (2.15), we introduce a double asymptotic expansion

$$f \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \epsilon^n \mu^m f_{nm},$$

and the Taylor expansion on the surface

$$q(z) = q(1) + \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} q^{(n)}(1),$$

as $\epsilon \to 0, \mu \to 0$, where f will be taken the scale functions u, p, η, and W. All the functions f_{nm} satisfy the far-field conditions $f_{nm} \to 0$ as $|\ \xi\ | \to \infty$ for all nonnegative integers n and m.

Substituting the asymptotic expansions of u, p, η, and W into Eqs. (2.15)–(2.21), we compute all the coefficients of the orders $O(\epsilon^i \mu^j)$ with all the nonnegative integers i and j.

We now establish the terms in Eqs. (2.15)–(2.21) in the orders of $O(\epsilon^0\mu^0)$, $O(\epsilon^1\mu^0)$, $O(\epsilon^0\mu^1)$, $O(\epsilon^2\mu^0)$, $O(\epsilon^3\mu^0)$, and $O(\epsilon^4\mu^0)$, respectively.

First of all, it follows from Eq. (2.18) that $u_{00} = u_{00}(\tau, \xi),\ u_{10} = u_{10}(\tau, \xi),\ u_{01} = u_{01}(\tau, \xi),\ u_{20} = u_{20}(\tau, \xi),\ u_{30} = u_{30}(\tau, \xi),\ u_{40} = u_{40}(\tau, \xi)$ are independent of z.

According to Eq. (2.17) and the boundary condition of W on $z = 0$, we can deduce:

$$\begin{aligned} W_{00} &= -\,zu_{00,\xi}, \quad W_{10} = -zu_{10,\xi}, \quad W_{01} = -zu_{01,\xi}, \\ W_{20} &= -\,zu_{20,\xi}, \quad W_{30} = -zu_{30,\xi}, \quad W_{40} = -zu_{40,\xi}, \end{aligned} \tag{2.22}$$

which along with the boundary condition of W on $z = 1$. This then implies

$$\begin{aligned} u_{00,\xi} &= c\eta_{00,\xi}, \quad u_{01,\xi} = c\eta_{01,\xi}, \\ u_{10,\xi} &= c\eta_{10,\xi} - \eta_{00,\tau} - (u_{00}\eta_{00})_\xi, \\ u_{20,\xi} &= c\eta_{20,\xi} - \eta_{10,\tau} - (u_{00}\eta_{10} + u_{10}\eta_{00})_\xi, \\ u_{30,\xi} &= c\eta_{30,\xi} - \eta_{20,\tau} - (u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi, \\ u_{40,\xi} &= c\eta_{40,\xi} - \eta_{30,\tau} - (u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi. \end{aligned} \tag{2.23}$$

On the other hand, from Eq. (2.16) and the boundary condition of p on $z = 1$, it appears that

$$\begin{aligned} p_{00} &= \eta_{00} + 2\Omega(z-1)u_{00}, \\ p_{01} &= \eta_{01} + 2\Omega(z-1)u_{01}, \\ p_{10} &= \eta_{10} - 2\Omega u_{00}\eta_{00} + 2\Omega(z-1)u_{10}, \\ p_{20} &= \eta_{20} - 2\Omega(u_{00}\eta_{10} + u_{10}\eta_{00}) + 2\Omega(z-1)u_{20}, \\ p_{30} &= \eta_{30} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00}) + 2\Omega(z-1)u_{30}, \\ p_{40} &= \eta_{40} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00}) + 2\Omega(z-1)u_{40}. \end{aligned} \tag{2.24}$$

Substituting expressions of W_{00} and p_{00} into Eq. (2.15), we can derive

$$(c^2 + 2\Omega c - 1)u_{00,\xi} = 0,$$

which follows that

$$c^2 + 2\Omega c - 1 = 0,$$

if assuming that u_{00} is an nontrivial velocity. By considering the waves move towards to the right side, we may obtain

$$c = \sqrt{1+\Omega^2} - \Omega. \tag{2.25}$$

Then combining Eq. (2.15) and Eqs. (2.22)–(2.24), it gives rise to

$$2(\Omega + c)\eta_{00,\tau} + 3c^2\eta_{00}\eta_{00,\xi} = 0, \tag{2.26}$$

$$2(\Omega + c)\eta_{10,\tau} + 3c^2(\eta_{00}\eta_{10})_\xi - \left(2c + \frac{4}{3}c_1\right)(c + c_1)(\eta_{00}^3)_\xi = 0, \tag{2.27}$$

$$2(\Omega + c)\eta_{20,\tau} + 3c^2\left(\eta_{00}\eta_{20} + \frac{1}{2}\eta_{10}^2\right)_\xi - 2(2c_1 + 3c)(c + c_1)(\eta_{00}^2\eta_{10})_\xi$$
$$- \frac{45c^2 + 64cc_1 + 24c_1^2 - 15}{12(\Omega + c)}(c + c_1)(\eta_{00}^4)_\xi = 0, \tag{2.28}$$

$$2(\Omega + c)\eta_{30,\tau} + 3c^2(\eta_{00}\eta_{30} + \eta_{10}\eta_{20})_\xi - 2(2c_1 + 3c)(c + c_1)(\eta_{00}^2\eta_{20} + \eta_{00}\eta_{10}^2)_\xi$$
$$- \frac{45c^2 + 64cc_1 + 24c_1^2 - 15}{3(\Omega + c)}(c + c_1)(\eta_{00}^3\eta_{10})_\xi + c_2(\eta_{00}^5)_\xi = 0, \tag{2.29}$$

with

$$c_1 = -\frac{3c^2}{4(\Omega + c)} = -\frac{3c^3}{2(c^2+1)}, \quad \text{and}$$
$$c_2 = \frac{c^2(c^2-2)(3c^{10} + 228c^8 - 540c^6 - 180c^4 - 13c^2 + 42)}{60(c^2+1)^6}.$$

Therefore, it transpires that

$$u_{00} = c\eta_{00}, \quad u_{01} = c\eta_{01},$$
$$u_{10} = c\eta_{10} - (c + c_1)\eta_{00}^2,$$
$$u_{20} = c\eta_{20} - 2(c + c_1)\eta_{00}\eta_{10} - \frac{2c_1 - 3\Omega}{3(c + \Omega)}(c + c_1)\eta_{00}^3, \quad \text{and}$$
$$u_{30} = c\eta_{30} - (c + c_1)(2\eta_{00}\eta_{20} + \eta_{00}^2) - \frac{2c_1 - 3\Omega}{3(c + \Omega)}(c + c_1)\eta_{00}^2\eta_{10}$$
$$- \frac{45c^2 + 64cc_1 + 24c_1^2 + 24\Omega^2 - 3}{24(c + \Omega)^2}(c + c_1)\eta_{00}^4, \tag{2.30}$$

with the far-field conditions $u_{00},\ u_{01},\ u_{10},\ u_{20},\ u_{30},\ \eta_{00},\ \eta_{10},\ \eta_{20} \to 0$ as $|\ \xi\ | \to \infty$.

For the terms in the orders of $O(\epsilon^1\mu^1)$ and $O(\epsilon^2\mu^1)$ of Eqs. (2.15)–(2.21), a direct computation shows from Eq. (2.18) that

$$u_{11,z} = -cz\eta_{00,\xi}, \quad \text{and} \quad u_{21,z} = z(2(c+c_1)(\eta_{00,\xi}^2 + \eta_{00}\eta_{00,\xi\xi}) - c\eta_{10,\xi\xi}),$$

or, what is the same,

$$u_{11} = -\frac{c}{2}z^2\eta_{00,\xi\xi} + H_{11}(\tau,\xi), \quad u_{21} = \frac{z^2}{2}\Phi_1 + H_{21}(\tau,\xi), \tag{2.31}$$

for some smooth functions $H_{11}(\tau,\xi)$ and $H_{21}(\tau,\xi)$ independent of z, where we denote

$$\Phi_1 = 2(c+c_1)(\eta_{00,\xi}^2 + \eta_{00}\eta_{00,\xi\xi}) - c\eta_{10,\xi\xi}.$$

Thanks to Eq. (2.17) and the boundary conditions of W_{11} and W_{21} at $z=0$, it is found that

$$W_{11} = \frac{c}{6}z^3\eta_{00,\xi\xi\xi} - z\partial_\xi H_{11}(\tau,\xi), \quad W_{21} = -\frac{z^3}{6}\Phi_{1,\xi} - z\partial_\xi H_{21}(\tau,\xi), \tag{2.32}$$

which along with the boundary conditions of W_{11} and W_{21} at $z=1$ leads to

$$\begin{aligned} \partial_\xi H_{11}(\tau,\xi) &= \frac{c}{6}\eta_{00,\xi\xi\xi} - (u_{00}\eta_{01} + u_{01}\eta_{00})_{\xi\xi} + c\eta_{11,\xi} - \eta_{01,\tau}, \\ \partial_\xi H_{21}(\tau,\xi) &= c\eta_{21,\xi} - \eta_{11,\tau} - \frac{1}{6}\Phi_{1,\xi} - \Phi_{2,\xi}\,|_{z=1}, \end{aligned} \tag{2.33}$$

where the function Φ_2 is defined by

$$\Phi_2 = u_{00}\eta_{11} + u_{01}\eta_{10} + u_{10}\eta_{01} + u_{11}\eta_{00}.$$

This in turn implies that

$$\begin{aligned} W_{11} &= \frac{c}{6}z(z^2-1)\eta_{00,\xi\xi\xi} + ((u_{00}\eta_{01} + u_{01}\eta_{00})_\xi + \eta_{01,\tau} - c\eta_{11,\xi}), \\ W_{21} &= \frac{z(1-z^2)}{6}\Phi_{1,\xi} - cz\eta_{21,\xi} + z\eta_{11,\tau} + z(\Phi_{2,\xi}\,|_{z=1}). \end{aligned} \tag{2.34}$$

It is then deduced from Eq. (2.16) and the boundary condition of p on $z=1$ that

$$\begin{aligned} p_{11} &= \eta_{11} - 2\Omega(u_{00}\eta_{01} + u_{01}\eta_{00}) - \frac{1}{6}(3c^2(z^2-1) + 2\Omega c(z^3-1))\eta_{00,\xi\xi} \\ &\quad + 2\Omega(z-1)H_{11}, \\ p_{21} &= \eta_{21} + \frac{1}{2}c(c+4c_1)(z^2-1)\eta_{00,\xi}^2 + \left(c^2 + \frac{1}{2}c(3c+4c_1)(z^2-1)\right)\eta_{00}\eta_{00,\xi\xi} \\ &\quad - \frac{1}{2}c^2(z^2-1)\eta_{10,\xi\xi} + 2\Omega\int_1^z u_{21}\mathrm{d}z'. \end{aligned} \tag{2.35}$$

In view of Eq. (2.16) and Eqs. (2.31)–(2.35), it is inferred that

$$2(\Omega+c)\eta_{01,\tau}+3c^2(\eta_{00}\eta_{01})_\xi+\frac{c^2}{3}\eta_{00,\xi\xi\xi}=0, \tag{2.36}$$

$$\begin{aligned}&2(\Omega+c)\eta_{11,\tau}+3c^2(\eta_{00}\eta_{11}+\eta_{10}\eta_{01})_\xi+\frac{c^2}{3}\eta_{10,\xi\xi\xi}\\&\quad-2(c+c_1)(3c+2c_1)(\eta_{00}^2\eta_{01})_\xi\\&\quad-\frac{1}{18}(3c^2+20cc_1+4c_1^2)(\eta_{00,\xi}^2)_\xi-\frac{1}{9}(3c^2+20cc_1+8c_1^2)(\eta_{00}\eta_{00,\xi\xi})_\xi=0,\end{aligned} \tag{2.37}$$

and then

$$u_{11}=c\eta_{11}-2(c+c_1)\eta_{00}\eta_{01}-\left(\frac{c}{2}z^2-\frac{c}{6}+\frac{2c_1}{9}\right)\eta_{00,\xi\xi}, \tag{2.38}$$

where use has been made of the fact that the far-field conditions $u_{11},\ \eta_{00},\ \eta_{00,\xi\xi},\ \eta_{01}$, and $\eta_{11}\to 0$ as $|\ \xi\ |\to\infty$.

Recall that $\eta=\eta_{00}+\epsilon\eta_{10}+\epsilon^2\eta_{20}+\epsilon^3\eta_{30}+\mu\eta_{01}+\epsilon\mu\eta_{11}+O(\epsilon^4,\mu^2)$. Multiplying Eqs. (2.26)–(2.29), (2.36), and (2.37) by 1, ϵ, ϵ^2, ϵ^3, μ, and $\epsilon\mu$, respectively, and then summating the results together, we arrive at the equation of η up to the order $O(\epsilon^4,\mu^2)$, that is,

$$\begin{aligned}&2(\Omega+c)\eta_\tau+3c^2\eta\eta_\xi+\frac{c^2}{3}\mu\eta_{\xi\xi\xi}+\epsilon A_1\eta^2\eta_\xi+\epsilon^2A_2\eta^3\eta_\xi+\epsilon^3A_3\eta^4\eta_\xi\\&\quad=\epsilon\mu(A_4\eta_\xi\eta_{\xi\xi}+A_5\eta\eta_{\xi\xi\xi})+O(\epsilon^4,\mu^2),\end{aligned} \tag{2.39}$$

where

$$\begin{aligned}&A_1=\frac{3c^2(c^2-2)}{(c^2+1)^2},\quad A_2=\frac{c^2(c^2-2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4},\\&A_3=\frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6},\\&A_4=-\frac{c^2(9c^4+16c^2-2)}{3(c^2+1)^2},\quad\text{and}\quad A_5=-\frac{c^2(3c^4+8c^2-1)}{3(c^2+1)^2}.\end{aligned}$$

On account of Eqs. (2.30) and (2.38), we have

$$\begin{aligned}&\eta_{00}=\frac{1}{c}u_{00},\quad \eta_{01}=\frac{1}{c}u_{01},\quad \eta_{11}=\frac{1}{c}u_{11}+2\alpha_1u_{00}u_{01}+\alpha_4u_{00,\xi\xi},\\&\eta_{10}=\frac{1}{c}u_{10}+\alpha_1u_{00}^2,\quad \eta_{20}=\frac{1}{c}u_{20}+2\alpha_1u_{00}u_{10}+\alpha_2u_{00}^3,\\&\eta_{30}=\frac{1}{c}u_{30}+\alpha_1u_{10}^2+2\alpha_1u_{00}u_{20}+3\alpha_2u_{00}^2u_{10}+\alpha_3u_{00}^4,\end{aligned}$$

where

$$\alpha_1 = \frac{2-c^2}{2c^2(c^2+1)}, \quad \alpha_2 = \frac{(c^2-1)(c^2-2)(2c^2+1)}{2c^3(c^2+1)^3},$$
$$\alpha_3 = \frac{(c^2-1)^2(2-c^2)(21c^4+16c^2+4)}{8c^4(c^2+1)^5}, \quad \text{and } \alpha_4 = \frac{z^2}{2c} - \frac{3c^2+1}{6c(c^2+1)} \tag{2.40}$$

Along with $u = u_{00} + \epsilon u_{10} + \epsilon^2 u_{20} + \epsilon^3 u_{30} + \mu u_{01} + \epsilon\mu u_{11} + O(\epsilon^4, \mu^2)$, it appears that

$$\eta = \frac{1}{c}u + \alpha_1 \epsilon u^2 + \alpha_2 \epsilon^2 u^3 + \alpha_3 \epsilon^3 u^4 + \alpha_4 \epsilon\mu u_{\xi\xi} + O(\epsilon^4, \mu^2), \tag{2.41}$$

where $\alpha_i (i = 1, 2, 3, 4)$ are defined in Eq. (2.40) and the parameter $z \in [0, 1]$.

Remark 2.1 In view of the above derivation, it is known that in the free-surface incompressible irrotational Euler equations within the equatorial region, the relation between the free surface η and the horizontal velocity u formally obeys the formula (2.41) with or without Coriolis effect. It also illustrates that, all the classical models, such as the classical KdV equation, the BBM equation, can be also formally derived from equation (1.4) with the KdV regime $\varepsilon = O(\mu)$.

It is then adduced from Eqs. (2.39) and (2.41) that

$$\begin{aligned} &u_\tau + 3\alpha_2 c\epsilon^2 u^2 u_\tau + 4\alpha_3 c\epsilon^3 u^3 u_\tau + \alpha_4 c\epsilon\mu u_{\tau\xi\xi} + \frac{2(c+c_1)}{c^2}\epsilon u u_\tau + \frac{3c}{2(\Omega+c)} u u_\xi \\ &+ \frac{cA_6}{2(\Omega+c)}\epsilon^2 u^3 u_\xi + \frac{cA_7}{2(\Omega+c)}\epsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{cA_8}{2(\Omega+c)}\epsilon^3 u^4 u_\xi \\ &+ \left(\frac{cA_9}{2(\Omega+c)} u_\xi u_{\xi\xi} + \frac{cA_{10}}{2(\Omega+c)} u u_{\xi\xi\xi}\right)\epsilon\mu = O(\epsilon^4, \epsilon^2\mu, \mu^2), \end{aligned} \tag{2.42}$$

where

$$\begin{aligned} A_6 &= 12c\alpha_2 + \frac{A_2}{c^4} + \frac{6(c+c_1)^2}{c^4} + \frac{4(c+c_1)A_2}{c^5}, \quad A_7 = \frac{A_1}{c^3} + \frac{9(c+c_1)}{c^2}, \\ A_8 &= 5\Big(3c\alpha_3 + \frac{3(c+c_1)}{c}\alpha_2 + \frac{A_1}{c^2}\alpha_2 + \frac{c_2}{c^5} + \frac{(c+c_1)A_2}{c^6} + \frac{(c+c_1)^2 A_1}{c^7}\Big), \\ A_9 &= 3c\alpha_4 + \frac{A_4}{c^2} + \frac{2(c+c_1)}{c}, \quad \text{and } A_{10} := 3c\alpha_4 - \frac{A_5}{c^2} + \frac{2(c+c_1)}{3c}. \end{aligned}$$

Hence, in view of Eq. (2.42), we have the following estimates

$$\begin{aligned} \epsilon u u_\tau = -\epsilon u\Big(&\frac{3c}{2(\Omega+c)} u u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{c^2 A_7 - 6(c+c_1)}{2c(\Omega+c)}\epsilon u^2 u_\xi \\ &+ \frac{c^2 A_6 - 2A_7(c+c_1) + 3c^2(\frac{4(c+c_1)^2}{c^4} - 3\alpha_2 c)}{2c(\Omega+c)}\epsilon^2 u^3 u_\xi\Big) + O(\epsilon^4, \mu^2), \end{aligned} \tag{2.43}$$

$$\epsilon^2 u^2 u_\tau = -\epsilon^2 u^2\Big(\frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2A_7 - 6(c+c_1)}{2c(\Omega+c)}\epsilon u^2 u_\xi\Big) + O(\epsilon^4, \epsilon^2\mu, \mu^2), \text{ and} \tag{2.44}$$

$$\epsilon^3 u^3 u_\tau = \frac{-3c}{2(\Omega+c)}\epsilon^3 u^4 u_\xi + O(\epsilon^4, \mu^2), \quad \epsilon\mu u_{\tau\xi\xi} = \frac{-3c}{2(\Omega+c)}\epsilon\mu(uu_\xi)_{\xi\xi} + O(\epsilon^4, \mu^2). \tag{2.45}$$

Decomposing $\epsilon\mu u_{\tau\xi\xi}$ into $\epsilon\mu(1-\nu)u_{\tau\xi\xi} + \epsilon\mu\nu u_{\tau\xi\xi}$ for some constant ν (to be determined later), it is inferred from (2.45) that

$$\epsilon\mu u_{\tau\xi\xi} = \epsilon\mu(1-\nu)u_{\tau\xi\xi} - \frac{3c\nu}{2(\Omega+c)}\epsilon\mu(uu_\xi)_{\xi\xi} + O(\epsilon^4, \mu^2). \tag{2.46}$$

Substituting Eqs. (2.43)–(2.46) into (2.42), it is found that

$$u_\tau + \frac{3c^2}{c^2+1}uu_\xi + \frac{c^3}{3(c^2+1)}\mu u_{\xi\xi\xi} + c\alpha_4(1-\nu)\epsilon\mu u_{\tau\xi\xi} + A_{11}\epsilon u^2 u_\xi + A_{12}\epsilon^2 u^3 u_\xi + A_{13}\epsilon^3 u^4 u_\xi + \epsilon\mu[A_{14}uu_{\xi\xi\xi} + A_{15}u_\xi u_{\xi\xi}] = O(\epsilon^4, \epsilon^2\mu, \mu^2), \tag{2.47}$$

where

$$\begin{aligned} A_{11} &= \frac{3c(c^2-1)(2-c^2)}{2(c^2+1)^3}, \quad A_{12} := \frac{(c^2-1)^2(c^2-2)(8c^2-1)}{2(c^2+1)^5},\\ A_{14} &= \frac{3c^3\alpha_4}{(c^2+1)}(1-\nu) + \frac{c^2(3c^4+8c^2-1)}{3(c^2+1)^3},\\ A_{15} &= \frac{3c^3\alpha_4}{(c^2+1)}(1-3\nu) + \frac{c^2(6c^4+19c^2+4)}{3(c^2+1)^3}, \quad \text{and}\\ A_{13} &= \frac{cA_8 - 3\alpha_2(c^2A_7 - 6(c+c_1)) - 12c^2\alpha_3}{2(\Omega+c)}\\ &\quad - \frac{(c+c_1)\left(c^2A_5 - 2A_6(c+c_1) + 3c^2\left(\frac{4(c+c_1)^2}{c^4} - 3c\alpha_2\right)\right)}{c^3(\Omega+c)}. \end{aligned}$$

In view of the variable transformation (2.14), we have

$$\frac{\partial}{\partial\xi} = \epsilon^{-\frac{1}{2}}\partial_x, \quad \frac{\partial}{\partial\tau} = \epsilon^{-\frac{3}{2}}(c\partial_x + \partial_t). \tag{2.48}$$

Then Eq. (2.47) can be transformed to

$$u_t + cu_x + \frac{3c^2}{c^2+1}\epsilon uu_x + A_{11}\epsilon^2 u^2 u_x + A_{12}\epsilon^3 u^3 u_x + c\alpha_4(1-\nu)\mu u_{txx} + \left(\frac{c^3}{3(c^2+1)} + c^2\alpha_4(1-\nu)\right)\mu u_{xxx} + \epsilon\mu(A_{14}uu_{xxx} + A_{15}u_x u_{xx}) = O(\epsilon^4, \mu^2). \tag{2.49}$$

In order to derive the gr-CH equation, the following relation is required

$$\frac{2c^2}{c^2+1}\sigma c\alpha_4(1-\nu) = 2A_{14} = A_{15}.$$

Or, it is the same,

$$\frac{(\sigma-3)c^3\alpha_4}{(c^2+1)}(1-\nu) = \frac{c^2(3c^4+8c^2-1)}{3(c^2+1)^3}.$$

In the case $\sigma = 3$, we have $3c^4 + 8c^2 - 1 = 0$. While $\sigma \neq 3$, it gives

$$\frac{2c^3}{c^2+1}\alpha_4 = \frac{c^2(3c^4 + (\sigma+5)c^2 + 2\sigma - 7)}{3(\sigma-3)(c^2+1)^3}, \tag{2.50}$$

and so

$$\frac{2c^2}{c^2+1}c\alpha_4(1-\nu) = 2A_{14} = A_{15} = \frac{2c^2(3c^4+8c^2-1)}{3(\sigma-3)(c^2+1)^3}.$$

Therefore, it enables us to construct the gr-CH equation in the following form

$$u_t - \beta\mu u_{txx} + cu_x + 3\alpha\epsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\epsilon^2 u^2 u_x + \omega_2\epsilon^3 u^3 u_x = \sigma\alpha\beta\epsilon\mu(2u_x u_{xx} + uu_{xxx}), \tag{2.51}$$

where the constants are defined as

$$\alpha = \frac{c^2}{1+c^2},\ \omega_1 = \frac{-3c(c^2-1)(c^2-2)}{2(1+c^2)^3},\ \omega_2 = \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5},$$

$$\beta = \begin{cases} \frac{3c^4+8c^2-1}{3(3-\sigma)(c^2+1)^2}, & \sigma \neq 3, \\ 0, & \sigma = 3, \end{cases}$$

and

$$\beta_0 = \begin{cases} \frac{c^3(1+c^2)\sigma+5c^3-c}{3(3-\sigma)(1+c^2)^2}, & \sigma \neq 3, \\ -\frac{c^3}{3(1+c^2)}, & \sigma = 3. \end{cases}$$

Combining Eqs. (2.50) and (2.40), it can be found that the height of the parameter z in α_4 may take the value

$$\begin{aligned} z_0^2 =& \frac{\sigma-2}{\sigma-3} - \frac{\sigma-5}{3(\sigma-3)(c^2+1)} + \frac{\sigma-9}{3(\sigma-3)(c^2+1)^2} \\ =& \frac{(3c^4+5c^2+3)\sigma-(6c^4+7c^2+10)}{3(\sigma-3)(c^2+1)^2}, \end{aligned} \tag{2.52}$$

with $\sigma \neq 3$.

It is natural to require that the constant $\beta > 0$ for the reason of the CH-type equation, that is

$$(3c^4+8c^2-1)(3-\sigma) > 0, \quad \sigma \in \mathbb{R}, \quad c = \sqrt{1+\Omega^2} - \Omega, \quad \Omega \geq 0. \tag{2.53}$$

Remark 2.2 When $\beta > 0$ and $0 \leq z_0 \leq 1$, it implies $\sigma \neq 3$. That is equivalent to the following conditions, either

$$\sqrt{\frac{1}{3}(\sqrt{19}-4)} < c \leq 1, \quad \sigma \leq \frac{6c^4+7c^2+10}{3c^4+5c^2+3}, \tag{I}$$

or

$$0 < c < \sqrt{\frac{1}{3}(\sqrt{19}-4)}, \quad \sigma \geq \frac{6c^4+7c^2+10}{3c^4+5c^2+3}. \tag{II}$$

In case (I), $\sigma \leq \frac{6c^4+7c^2+10}{3c^4+5c^2+3} = \min\left\{\frac{6c^4+7c^2+10}{3c^4+5c^2+3}, \frac{3c^4+11c^2-1}{c^2}\right\} < 3$ and $0 \leq \Omega < \sqrt{\frac{1}{6}(1+2\sqrt{19})}$.
In case (II), $\sigma \geq \frac{6c^4+7c^2+10}{3c^4+5c^2+3} = \max\left\{\frac{6c^4+7c^2+10}{3c^4+5c^2+3}, \frac{3c^4+11c^2-1}{c^2}\right\} > 3$ and $\Omega > \sqrt{\frac{1}{6}(1+2\sqrt{19})}$.

Remark 2.3 As $\beta \to 0$ or $\sigma \to 3$, the corresponding equation to (2.51) is now in the following form

$$\begin{aligned} & u_t + cu_x + \frac{3c^2}{c^2+1}\epsilon u u_x + \frac{c^3}{3(1+c^2)}\mu u_{xxx} - \frac{3c(c^2-1)(c^2-2)}{2(1+c^2)^3}\epsilon^2 u^2 u_x \\ & + \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5}\epsilon^3 u^3 u_x = 0, \end{aligned}$$

where the solution u is horizontal velocity at the height $z_0 = \frac{3c^4+5c^2+3}{3(c^2+1)^2} < 1$ and $c = \sqrt{\frac{1}{3}(\sqrt{19}-4)}$ or $\Omega = \sqrt{\frac{1}{6}(1+2\sqrt{19})}$. This model equation is a generalized KdV-type equation (Martel and Merle 2001; Merle 2001) or the Gardner-type equation

(Gardner et al. 1968). Global existence of solutions in the energy space H^1 is now well understood (Ginibre and Tsutsumi 1989; Kato 1983; Kenig et al. 2001, 1993). In the present paper, we only consider Eq. (2.51) for the case of $\beta > 0$.

For convenience, we now assume that the cases (I) and (II) in Remark 2.1 hold throughout the whole paper.

3 Blow-Up Criterion

In this section, we will derive blow-up criteria of the solutions in the Sobolev space $H^s(\mathbb{R}), s > \frac{3}{2}$ for the gr-CH equation. Applying the transformation

$$x \to \sqrt{\beta\mu}x, \qquad t \to \sqrt{\beta\mu}t, \qquad u \to \alpha\epsilon u,$$

we can transform the gr-CH equation (2.51) into the form

$$u_t - u_{txx} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = \sigma(2u_xu_{xx} + uu_{xxx}), \tag{3.1}$$

and the following form of equations,

$$\begin{cases} m_t + \sigma(um_x + 2u_xm) + 3(1-\sigma)uu_x + cu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 0, \\ m = u - u_{xx}. \end{cases} \tag{3.2}$$

It has at least three conservation laws

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} u\mathrm{d}x, \\ E(u) &= \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + u_x^2\right)\mathrm{d}x, \quad \text{and} \\ F(u) &= \frac{1}{2}\int_{\mathbb{R}}\left(cu^2 + u^3 + \sigma uu_x^2 + \frac{\beta_0}{\beta}u_x^2 + \frac{\omega_1}{6\alpha^2}u^4 + \frac{\omega_2}{10\alpha^3}u^5\right)\mathrm{d}x. \end{aligned}$$

Consider now the local well-posedness of the following Cauchy problem

$$\begin{cases} u_t - u_{txx} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = \sigma(2u_xu_{xx} + uu_{xxx}), \\ u|_{t=0} = u_0. \end{cases} \tag{3.3}$$

The local well-posedness result of the above Cauchy problem may now be enunciated.

Theorem 3.1 *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exist a positive time $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ to the Cauchy problem (3.3) with $u(0) = u_0$. Moreover, the solution u depends continuously on the initial value u_0. In addition, the Hamiltonians $I(u)$, $E(u)$, and $F(u)$ are independent of the existence time $t > 0$.*

We omit the proof of the above theorem, since it is similar to that in Constantin and Lannes (2009) and Danchin (2001) (up to a slight modification).

The gr-CH equation (3.1) can be rewritten in the following transport type, namely

$$u_t + \sigma u u_x + \frac{\beta_0}{\beta} u_x + \partial_x p * \left(\left(c - \frac{\beta_0}{\beta}\right) u + \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0, \tag{3.4}$$

where $p = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$ and "$*$" denotes the convolution.

By using the linear transport theory, the blow-up criterion for the gr-CH equation (3.1) can be formulated as the following result.

Theorem 3.2 (Blow-up criterion) *Let u be the solution of Eq. (3.3) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. If $T^*_{u_0}$ is the maximal time of existence, then*

$$T^*_{u_0} < \infty \quad \Rightarrow \quad \int_0^{T^*_{u_0}} \|u_x(\tau)\|_{L^\infty} \mathrm{d}\tau = \infty. \tag{3.5}$$

To prove this theorem, we need the following proposition to handle the regularity of solutions for Eq. (3.4), which can be proved via the Moser-type estimates and Littlewood-Paley analysis for the transport equation (Gui and Liu 2010). We recall this proposition for completeness.

Proposition 3.1 (Gui and Liu 2010) *Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0,T]; H^s)$, and $v, v_x \in L^1([0,T]; L^\infty)$ and that $f \in L^\infty([0,T]; H^s) \cap C([0,T]; \mathcal{S}')$ solves the following one-dimensional linear transport equation*

$$\begin{aligned} &f_t + v f_x = g, \\ &f(0,x) = f_0(x). \end{aligned} \tag{3.6}$$

Then $f \in C([0,T]; H^s)$. More precisely, there exists a constant C depending only on s such that the following estimate holds:

$$\|f(t)\|_{H^s} \le \|f_0\|_{H^s} + C \left(\int_0^t \|g(\tau)\|_{H^s} \mathrm{d}\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) \mathrm{d}\tau \right). \tag{3.7}$$

Hence

$$\|f(t)\|_{H^s} \le e^{CV(t)} \left(\|f_0\|_{H^s} + C \int_0^t \|g(\tau)\|_{H^s} \mathrm{d}\tau \right), \tag{3.8}$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) \mathrm{d}\tau$.

The following lemma and propositions will be also helpful in the proof of Theorem 3.2.

Lemma 3.1 (Commutator estimate) *Let Λ be a operator defined as $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. If f and g are smooth enough, then*

$$\|[\Lambda^s, f]g\|_{L^2} \le C(\|f\|_{H^s}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|g\|_{H^{s-1}}), \tag{3.9}$$

for all $s > \frac{3}{2}$ and $C > 0$.

Proposition 3.2 (1-D Moser-type estimates). *The following estimates hold.*

(1) *For $s \ge 0$,*

$$\|fg\|_{H^s(\mathbb{R})} \le C(\|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})}). \tag{3.10}$$

(2) *For $s > 0$,*

$$\|f\partial_x g\|_{H^s(\mathbb{R})} \le C(\|f\|_{H^{s+1}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|\partial_x g\|_{H^s(\mathbb{R})}). \tag{3.11}$$

(3) *For $s_1 \le \frac{1}{2}$, $s_2 > \frac{1}{2}$ and $s_1 + s_2 > 0$,*

$$\|fg\|_{H^{s_1}(\mathbb{R})} \le C\|f\|_{H^{s_1}(\mathbb{R})}\|g\|_{H^{s_2}(\mathbb{R})}, \tag{3.12}$$

where the constant C is independent of f and g.

Proposition 3.3 (Chemin 2004) *Let $m \in \mathbb{R}$ and f be an S^m-multiplier (that is, $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and satisfies that for all multi-index α, there exists a constant C_α such that $\forall \xi \in \mathbb{R}^d, |\partial^\alpha f(\xi) \le C_\alpha(1+|\xi|)^{m-|\alpha|}$). Then for all $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the operator $f(D)$ is continuous from the Besov spaces $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

Proof of Theorem 3.2 First of all, we apply the translation $u(t,x) \to u(t, x - \frac{\beta_0}{\beta}t)$ to Eq. (3.4),

$$u_t + \sigma u u_x = -\partial_x p * \left(\left(c - \frac{\beta_0}{\beta}\right)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \tag{3.13}$$

Consider the operator $\Lambda^s = (1-\partial_x^2)^{\frac{s}{2}}$. For all the constant coefficient skew-symmetric differential polynomial P, f and g smooth enough, a commutator process is

$$\Lambda^s(fPg) = fP\Lambda^s g + [\Lambda^s, f]Pg.$$

Applying Λ^s operator to Eq. (3.13) gives that

$$\begin{aligned}&\partial_t \Lambda^s u + \sigma \Lambda^s (u\partial_x u)\\ &\quad = \Lambda^s \left(-\partial_x p * \left(\left(c - \frac{\beta_0}{\beta}\right)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)\right).\end{aligned} \tag{3.14}$$

Then taking the inner product between Eq. (3.14) and $\Lambda^s u$ in L^2

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2\\ &\quad = -\frac{\sigma}{2}\int_{\mathbb{R}} u\partial_x(\Lambda^s u)^2 \mathrm{d}x - \sigma \int_{\mathbb{R}} [\Lambda^s, u]\partial_x u \Lambda^s u \mathrm{d}x\\ &\qquad + \int_{\mathbb{R}} \Lambda^s \Big(- \partial_x p * \Big(\Big(c - \frac{\beta_0}{\beta}\Big)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big)\Lambda^s u \mathrm{d}x\\ &\quad \le \frac{\sigma}{2}\|\partial_x u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2 + \sigma\|[\Lambda^s, u]\partial_x u\|_{L^2}\|\Lambda^s u\|_{L^2}\\ &\qquad + \Big\|\Lambda^s \Big(- \partial_x p * \Big(\Big(c - \frac{\beta_0}{\beta}\Big)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big)\Big\|_{L^2}\|\Lambda^s u\|_{L^2}.\end{aligned}$$

By using commutator estimate in Lemma 3.9 for $s > 0$, we get

$$\|[\Lambda^s, u]\partial u\|_{L^2} \le C(\|u\|_{H^s}\|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{H^{s-1}}\|\partial_x u\|_{L^\infty}) \le C\|u\|_{H^s}\|\partial_x u\|_{L^\infty}.$$

Then we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 \le & C\Big(\|u\|_{H^s}\|\partial_x u\|_{L^\infty}\\ & + \Big\|p_x * \Big(\Big(c - \frac{\beta_0}{\beta}\Big)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big\|_{H^s}\Big)\|u\|_{H^s}.\end{aligned} \tag{3.15}$$

Taking integration of (3.15) for the existence time from 0 to t gives that

$$\begin{aligned}\|u\|_{H^s} \le & \|u_0\|_{H^s} + C\int_0^t \|u(\tau)\|_{H^s}\|\partial_x u(\tau)\|_{L^\infty}\mathrm{d}\tau\\ & + C\int_0^t \Big\|p_x * \Big(\Big(c - \frac{\beta_0}{\beta}\Big)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big\|_{H^s}\mathrm{d}\tau.\end{aligned}$$

Thanks to the Moser-type estimate and Proposition 3.3, a direct computation reveals that

$$\begin{aligned}&\Big\| p_x * \Big(\Big(c - \frac{\beta_0}{\beta}\Big) u + \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \Big) \Big\|_{H^s} \\ &\quad \le C \Big\| \Big(c - \frac{\beta_0}{\beta}\Big) u + \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \Big\|_{H^{s-1}} \\ &\quad \le C \Big(\|u\|_{H^{s-1}} + \|u\|_{H^{s-1}} \|u\|_{L^\infty} + \|u_x\|_{H^{s-1}} \|u_x\|_{L^\infty} \\ &\qquad + \|u\|_{H^{s-1}} \|u\|_{L^\infty}^2 + \|u\|_{H^{s-1}} \|u\|_{L^\infty}^3 \Big).\end{aligned}$$

It is thereby inferred that

$$\|u\|_{H^s} \le \|u_0\|_{H^s} + C \int_0^t \|u(\tau)\|_{H^s} (1 + \|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^3) \mathrm{d}\tau.$$

Applying Gronwall's inequality yields

$$\|u\|_{H^s} \le \|u_0\|_{H^s} e^{C \int_0^t (1 + \|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^3) \mathrm{d}\tau} \le \|u_0\|_{H^s} e^{C \int_0^t \|u_x\|_{L^\infty}} d\tau, \tag{3.16}$$

where use has been made of the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ and invariant energy E. Suppose now the maximal existence time $T^*_{u_0} < \infty$ satisfies $\int_0^{T^*_{u_0}} ||u_x(\tau)||_{L^\infty} \mathrm{d}\tau < \infty$, it then follows from (3.16) that

$$\limsup_{t \to T^*_{u_0}} \|u(t)\|_{H^s} < \infty,$$

which contradicts the assumption on the maximal existence time $T^*_{u_0} < \infty$. □

4 Wave-Breaking Data for $\sigma \neq 0$

Having established the wave-breaking criteria for the gr-CH model, attention is now given to searching the initial wave-breaking data. We first consider the following associated Lagrangian scale form of the gr-CH equation (3.1), that is,

$$\begin{cases} \frac{\partial q}{\partial t} = \sigma u(t, q), & 0 < t < T, \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $u \in C^1([0, T), H^{s-1})$ is the solution of Eq. (3.1) with initial data $u_0 \in H^s (s > \frac{3}{2})$, and $T > 0$ is the maximal time of existence. A direct calculation shows that

$$q_{tx}(t, x) = \sigma u_x(t, q(t, x)) q_x(t, x).$$

Then, for $t > 0$, $x \in \mathbb{R}$,

$$q_x(t,x) = e^{\int_0^t \sigma u_x(\tau, q(\tau,x))d\tau} > 0,$$

implying that $q(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for each $t \in [0, T)$. Hence, the L^∞ norm of any function $v(t,\cdot) \in L^\infty$, $t \in [0, T)$ is preserved under the family of diffeomorphism $q(t,\cdot)$ with $t \in [0, T)$, that is,

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R})} = \|v(t, q(t,\cdot))\|_{L^\infty(\mathbb{R})}, \quad t \in [0, T).$$

Similarly, for $t \in [0, T)$, we get

$$\inf_{x\in\mathbb{R}} v(t,x) = \inf_{x\in\mathbb{R}} v(t, q(t,x)), \quad \sup_{x\in\mathbb{R}} v(t,x) = \sup_{x\in\mathbb{R}} v(t, q(t,x)). \tag{4.2}$$

Based on Theorem 3.2, we establish the following necessary and sufficient condition for the precise of the blow-up mechanism.

Theorem 4.1 (Wave-breaking criterion) *Let u be the solution of (3.3) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and T the maximal time of existence. Then the corresponding solution u blows up in finite time if and only if*

$$\liminf_{t\to T^-} \inf_{x\in\mathbb{R}} \sigma u_x(t,x) = -\infty. \tag{4.3}$$

To prove this wave-breaking criterion, we use the following lemma to show that indeed σu_x is uniformly bounded from above.

Lemma 4.1 *Let $\sigma \neq 0$ and u be the solution of (3.3) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and T the maximal time of existence.*

(1) *If $\sigma > 0$, then*

$$\sup_{x\in\mathbb{R}} u_x(t,x) \leq \|u_{0,x}\|_{L^\infty} + \frac{C_1}{\sqrt{\sigma}}. \tag{4.4}$$

(2) *If $\sigma < 0$, then*

$$\inf_{x\in\mathbb{R}} u_x(t,x) \geq -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}, \tag{4.5}$$

where the constants above are defined by

$$C_0^2 = \frac{2|\omega_1|}{3\alpha^2} E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3} E_0^2, \tag{4.6}$$

$$C_1^2 = \frac{1}{2}|c - \frac{\beta_0}{\beta}| + \left(|c - \frac{\beta_0}{\beta}| + |\sigma| + 2|3-\sigma|\right) E_0 + 2C_0^2, \tag{4.7}$$

$$C_2^2 = \frac{1}{2}|c - \frac{\beta_0}{\beta}| + \left(|c - \frac{\beta_0}{\beta}| + 6 - 3\sigma\right) E_0 + 2C_0^2, \quad \text{and} \tag{4.8}$$

$$E_0 = \frac{1}{2}\int_{\mathbb{R}} \left(u_0^2 + u_{0,x}^2\right)\mathrm{d}x. \tag{4.9}$$

Proof The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Also we may assume $u_0 \not\equiv 0$.

Differentiating Eq. (3.13) with respect to x and using the identity $-\partial_x^2 p * f = f - p * f$, we obtain

$$\begin{aligned} u_{tx} + \sigma u u_{xx} + \frac{\sigma}{2}u_x^2 =& \frac{3-\sigma}{2}u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & - p * \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned} \tag{4.10}$$

(1) When $\sigma > 0$, it is a fact that

$$\sup_{x\in\mathbb{R}}[v_x(t,x)] = -\inf_{x\in\mathbb{R}}[-v_x(t,x)]. \tag{4.11}$$

Given $x \in \mathbb{R}$, let $T > 0$ be the maximal existence time of the solution $u(t,\cdot)$ with initial data u_0

$$m_1(t) = u_x(t, q(t,x)), \quad t \in [0,T), \tag{4.12}$$

where $q(t,x)$ is defined by (4.1).

Therefore, we can deduce Eq. (4.10) at $x = q(t,x)$ that

$$m_1'(t) = -\frac{\sigma}{2}(m_1(t))^2 + f(t, q(t,x)), \tag{4.13}$$

for $t \in [0,T)$, where $f(t,q(t,x))$ is given by

$$\begin{aligned} f =& \frac{3-\sigma}{2}u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & - p * \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned} \tag{4.14}$$

In order to get the wave-breaking result later, we first derive the upper and lower bounds for f. Using that $\partial_x^2 p * u = p_x * u_x$, we have

$$
\begin{aligned}
f =&\frac{3-\sigma}{2}u^2 - p*\Big(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\Big)-\Big(c-\frac{\beta_0}{\beta}\Big)p_x*u_x\\
&+\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-p*\Big(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\Big)\\
\leq&\frac{3-\sigma}{2}u^2-|p*\Big(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\Big)|+|c-\frac{\beta_0}{\beta}|\,|p_x*u_x|\\
&+\Big|\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-p*\Big(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\Big)\Big|.
\end{aligned}
$$

Since

$$
\frac{3-\sigma}{4}u^2 \leq \frac{|3-\sigma|}{4}\int_{\mathbb{R}}\Big(u^2+u_x^2\Big)\mathrm{d}x = \frac{|3-\sigma|}{4}(\|u\|_{L^2}^2+\|u_x\|_{L^2}^2), \tag{4.15}
$$

$$
|p*\Big(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\Big)|\leq\frac{1}{2}\left\|\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\right\|_{L^1}\leq\frac{|3-\sigma|}{4}\|u\|_{L^2}^2+\frac{|\sigma|}{4}\|u_x\|_{L^2}^2, \tag{4.16}
$$

$$
|c-\frac{\beta_0}{\beta}|\,|p_x*u_x| \leq \frac{1}{2}|c-\frac{\beta_0}{\beta}|\|u_x\|_{L^2}\leq\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{1}{4}|c-\frac{\beta_0}{\beta}|\|u_x\|_{L^2}^2, \quad \text{and} \tag{4.17}
$$

$$
\begin{aligned}
&\Big|\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-p*\Big(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\Big)\Big|\\
&\leq\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}^3+\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^4+\frac{1}{2}\|u\|_{L^\infty}\left(\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^2}^2\right)+\frac{1}{2}\|u\|_{L^\infty}^2\left(\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^2}^2\right),
\end{aligned} \tag{4.18}
$$

and the fact that the conservation law implies that

$$
\|u\|_{L^2}^2\leq 2E_0 \quad \text{and} \quad \|u_x\|_{L^2}^2\leq 2E_0. \tag{4.19}
$$

Therefore, it is natural to bound $f(t,x)$ by

$$
\begin{aligned}
|f|\leq&\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{1}{2}|3-\sigma|\|u\|_{L^2}^2+\frac{1}{4}\Big(|c-\frac{\beta_0}{\beta}|+|3-\sigma|+|\sigma|\Big)\|u_x\|_{L^2}^2\\
&+\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}^3+\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^4+\frac{1}{2}\|u\|_{L^\infty}\Big(\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^2}^2\Big)+\frac{1}{2}\|u\|_{L^\infty}^2\Big(\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^2}^2\Big)\\
\leq&\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{1}{2}\Big(|c-\frac{\beta_0}{\beta}|+2|3-\sigma|+|\sigma|\Big)E_0+\frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{2\alpha^3}E_0^2=\frac{C_1^2}{2}.
\end{aligned} \tag{4.20}
$$

When $\sigma<0$, we have a finer estimate

$$
-f\leq\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{1}{2}\Big(|c-\frac{\beta_0}{\beta}|+6-3\sigma\Big)E_0+\frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{2\alpha^3}E_0^2=\frac{C_2^2}{2}. \tag{4.21}
$$

Since now $s \geq 3$, we have $u \in C_0^1(\mathbb{R})$. Thus,

$$\sup_{x \in \mathbb{R}} u_x(t,x) \leq 0, \quad \inf_{x \in \mathbb{R}} u_x(t,x) \geq 0, \quad t \in [0,T). \tag{4.22}$$

For any $x \in \mathbb{R}$, let us define

$$P_1(t) = m_1(t) - \|u_{0,x}\|_{L^\infty} - \frac{C_1}{\sqrt{\sigma}}. \tag{4.23}$$

Notice that $P_1(t)$ is a C^1-differential function in $[0,T)$ and satisfies

$$P_1(0) = m_1(0) - \|u_{0,x}\|_{L^\infty} - \frac{C_1}{\sqrt{\sigma}} \leq m_1(0) - \|u_{0,x}\|_{L^\infty} \leq 0. \tag{4.24}$$

We will show that

$$P_1(t) \leq 0, \quad \forall \ t \in [0,T). \tag{4.25}$$

If not, assuming there is a $t_0 \in [0,T)$ such that $P_1(t_0) > 0$. Let

$$t_1 = \max\{t < t_0 : P_1(t) = 0\}. \tag{4.26}$$

Then $P_1(t_1) = 0$ and $P_1'(t_1) \geq 0$, or equivalently,

$$m_1(t_1) = \|u_{0,x}\|_{L^\infty} + \frac{C_1}{\sqrt{\sigma}}, \quad \text{and} \quad m_1'(t_1) \geq 0. \tag{4.27}$$

On the other hand, we have

$$\begin{aligned} m_1'(t_1) &= -\frac{\sigma}{2}(m_1(t_1))^2 + f(t_1, q(t_1,x)) \\ &\leq -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty} + \frac{C_1}{\sqrt{\sigma}}\right]^2 + \frac{C_1^2}{2} < 0, \end{aligned}$$

which is a contradiction to (4.27). This verifies the estimate $P_1(t) \leq 0$ for $t \in [0,T)$. Therefore, the arbitrary chosen of x implies

$$\sup_{x \in \mathbb{R}} u_x(t,x) = \sup_{x \in \mathbb{R}} u_x(t, q(t,x)) \leq \|u_{0,x}\|_{L^\infty} + \frac{C_1}{\sqrt{\sigma}}.$$

(2) When $\sigma < 0$, we will derive the lower bound for u_x. Now we will consider the function $m_2(t)$

$$m_2(t) = u_x(t, q(t,x)), \quad t \in [0,T). \tag{4.28}$$

Similarly as case (1), there exists $x = q(t, x) \in \mathbb{R}$ such that

$$m_2'(t) = -\frac{\sigma}{2}(m_2(t))^2 + f(t, q(t, x)). \tag{4.29}$$

Now for any given $x \in \mathbb{R}$, we can define

$$P_2(t) = m_2(t) + \|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}. \tag{4.30}$$

Then $P_2(t)$ is also a C^1-differentiable on $[0, T)$ and satisfies

$$P_2(0) \geq m_2(0) + \|u_{0,x}\|_{L^\infty} \geq 0. \tag{4.31}$$

We now claim that $P_2(t) \geq 0$, for $t \in [0, T)$. If not, then suppose there is a $\bar{t_0} \in [0, T)$ such that $P_1(\bar{t_0}) < 0$. Define

$$t_2 = \max\{t < \bar{t_0} : P_2(t) = 0\}. \tag{4.32}$$

Then $P_2(t_2) = 0$ and $P_2'(t_2) \leq 0$, or equivalently,

$$m_2(t_2) = -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}, \quad \text{and} \quad m_2'(t_2) \leq 0. \tag{4.33}$$

On the other hand, we have

$$\begin{aligned} m_2'(t_2) &= -\frac{\sigma}{2}(m_2(t_2))^2 + f(t_2, q(t_2, x)) \\ &\geq -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}\right]^2 - \frac{C_2^2}{2} > 0. \end{aligned}$$

which is a contradiction to (4.33). This verifies the estimate $P_2(t) \geq 0$ for $t \in [0, T)$. Therefore, it is thereby inferred that

$$\inf_{x\in\mathbb{R}} u_x(t, x) = \inf_{x\in\mathbb{R}} u_x(t, q(t, x)) \geq -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}.$$

□

Proof of Theorem 4.1 Assume that $T < \infty$ and (4.3) is not valid. Then there is some positive number $M > 0$ such that

$$\sigma u_x(t, x) \geq -M, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Then it follows from Lemma 4.1 that

$$|u_x(t, x)| \leq C, \tag{4.34}$$

where $C = C(\Omega, \sigma, E_0)$. Therefore, Theorem 3.2 implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$.

Conversely, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > \frac{1}{2}$ implies that if (4.3) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 4.1. □

Inspired by the proof of Lemma 3.1 in Brandolese (2014), we now establish the following lemma to search for the wave-breaking data of the gr-CH equation in the case of $0 < \sigma < 3$ or $3 < \sigma \le 4$ later on.

Lemma 4.2 *Let* $0 < \sigma < 3$ *or* $3 < \sigma \le 4$, $0 \le \gamma \le 1$ *and* $\lambda = \frac{\sqrt{\sigma}}{4}(\sqrt{12-3\sigma} - \sqrt{\sigma})$. *Then*

$$(p \pm \gamma\partial_x p) * \left(\frac{3-\sigma}{2}u^2 - mu + \frac{\sigma}{2}u_x^2\right) \ge \lambda\left(u - \frac{m}{3-\sigma}\right)^2 - \frac{m^2}{2(3-\sigma)}. \tag{4.35}$$

Proof Denoting $\mathbf{1}_{\mathbb{R}^+}$ and $\mathbf{1}_{\mathbb{R}^-}$ the characteristic functions of $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Let $a \in \mathbb{R}$ (determined later).

$$\begin{aligned}
p\mathbf{1}_{\mathbb{R}^+}*&\Big(a^2\Big(u - \frac{m}{3-\sigma}\Big)^2 + u_x^2 - \frac{m^2}{\sigma(3-\sigma)}\Big)(x)\\
=&\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\xi}\Big(a^2\Big(u - \frac{m}{3-\sigma}\Big)^2 + u_x^2 - \frac{m^2}{\sigma(3-\sigma)}\Big)(\xi)\mathrm{d}\xi\\
\ge& ae^{-x}\int_{-\infty}^{x} e^{\xi}\Big(u - \frac{m}{3-\sigma}\Big)u_x \mathrm{d}\xi - \frac{m^2}{2\sigma(3-\sigma)}\\
=&\frac{a}{2}\Big(u - \frac{m}{3-\sigma}\Big)^2 - \frac{a}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}\Big(u - \frac{m}{3-\sigma}\Big)^2\mathrm{d}\xi - \frac{m^2}{2\sigma(3-\sigma)}\\
=&\frac{a}{2}\Big(u - \frac{m}{3-\sigma}\Big)^2 - a(p\mathbf{1}_{\mathbb{R}^+}) * \Big(u - \frac{m}{3-\sigma}\Big)^2 - \frac{m^2}{2\sigma(3-\sigma)}.
\end{aligned}$$

This leads to

$$p\mathbf{1}_{\mathbb{R}^+} * \Big((a^2 + a)\Big(u - \frac{m}{3-\sigma}\Big)^2 + u_x^2 - \frac{m^2}{\sigma(3-\sigma)}\Big) \ge \frac{a}{2}\Big(u - \frac{m}{3-\sigma}\Big)^2 - \frac{m^2}{2\sigma(3-\sigma)}.$$

Choosing a to be the largest real root of the second order equation (a will be negative for $3 < \sigma \le 4$), this leads to

$$a^2 + a = \frac{3-\sigma}{\sigma}.$$

We get $\sigma a = 2\lambda$, hence

$$p\mathbf{1}_{\mathbb{R}^+} * \left(\frac{3-\sigma}{2}u^2 - mu + \frac{\sigma}{2}u_x^2\right) \geq \frac{\lambda}{2}\left(u - \frac{m}{3-\sigma}\right)^2 - \frac{m^2}{4(3-\sigma)}.$$

The same computations also show that

$$p\mathbf{1}_{\mathbb{R}^-} * \left(\frac{3-\sigma}{2}u^2 - mu + \frac{\sigma}{2}u_x^2\right) \geq \frac{\lambda}{2}\left(u - \frac{m}{3-\sigma}\right)^2 - \frac{m^2}{4(3-\sigma)}.$$

In fact, both in the a.e. and the distributional sense give

$$\begin{aligned} p - \gamma\partial_x p &= (1-\gamma)p\mathbf{1}_{\mathbb{R}^-} + (1+\gamma)p\mathbf{1}_{\mathbb{R}^+}, \\ p + \gamma\partial_x p &= (1+\gamma)p\mathbf{1}_{\mathbb{R}^-} + (1-\gamma)p\mathbf{1}_{\mathbb{R}^+}. \end{aligned}$$

For $0 \leq \gamma \leq 1$, taking the linear combination in the two last inequalities gives the estimate in (4.35). This completes the proof of Lemma 4.2. □

Theorem 4.2 (Wave-breaking data) *Assume the cases (I) and (II) holds. Let* $0 < \sigma < 3$ *or* $3 < \sigma \leq 4$ *and* u *be the solution of* (3.4) *with initial data* $u_0 \in H^s$, $s > 3/2$, *and* $T > 0$ *be the maximal time of existence. Assume there is some* $x_0 \in \mathbb{R}$ *such that*

$$u_{0,x}(x_0) < -\gamma_\sigma \left| u_0(x_0) - \frac{1}{(3-\sigma)}\left(\frac{\beta_0}{\beta} - c\right)\right| - \sqrt{\frac{2}{\sigma}}C_0,$$

where $C_0 > 0$ *is defined by*

$$C_0^2 = \frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3}E_0^2, \tag{4.36}$$

and

$$E_0 = \frac{1}{2}\int_{\mathbb{R}}\left(u_0^2 + (\partial_x u_0)^2\right)\mathrm{d}x, \quad \gamma_\sigma^2 = \frac{3-\sigma}{\sigma} - \frac{2\lambda}{\sigma}.$$

Then the solution $u(t,x)$ *breaks down at the time*

$$T \leq \frac{2}{\sigma\sqrt{u_{0,x}^2(x_0) - \gamma_\sigma^2\left(u_0(x_0) - \frac{1}{(3-\sigma)}\left(\frac{\beta_0}{\beta} - c\right)\right)^2} - \sqrt{2\sigma}C_0}.$$

Remark 4.1 In the case of the rotation frequency parameter $\Omega = 0$, or the wave speed $c = 1$, the corresponding constant C_0 in (4.36) must be zero, because the parameters ω_1 and ω_2 vanish. The assumption on the wave breaking is then back to the case of the compressible hyper-elastic rod equation. Particularly, when $\sigma = 1$, it is simply the case of the classical CH equation.

Proof Along with the trajectory of $q(t,x)$ defined in (4.1), (3.13) and (4.10) become

$$\begin{aligned}\frac{\partial u(t,q)}{\partial t} &= -\,p_x * \left(\left(c-\frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right),\\ \frac{\partial u_x(t,q)}{\partial t} &= -\,\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \left(c-\frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\\ &\quad - p * \left(\left(c-\frac{\beta_0}{\beta}\right)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right).\end{aligned}$$

Denote at $(t, q(t, x_0))$,

$$\begin{aligned} M(t) &= \gamma_\sigma\left(u(t,q) - \frac{k}{3-\sigma}\right) - u_x(t,q) \quad \text{and}\\ N(t) &= \gamma_\sigma\left(u(t,q) - \frac{k}{3-\sigma}\right) + u_x(t,q),\end{aligned}$$

where $k = \frac{\beta_0}{\beta} - c$.

It then follows that at $(t, q(t, x_0))$,

$$\begin{aligned}\frac{\partial M}{\partial t} &= \frac{\sigma}{2}u_x^2 - \frac{3-\sigma}{2}u^2 + ku + (p - \gamma_\sigma p_x) * \left(-ku + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2\right)\\ &\quad - \frac{\omega_1}{3\alpha^2}u^3 - \frac{\omega_2}{4\alpha^3}u^4 + (p - \gamma_\sigma p_x) * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)\\ &\geq \frac{\sigma}{2}u_x^2 - \frac{3-\sigma}{2}\left(u - \frac{k}{3-\sigma}\right)^2 + \mu\left(u - \frac{k}{3-\sigma}\right)^2 - C_0^2\\ &= \frac{\sigma}{2}u_x^2 - \left(\frac{3-\sigma}{2} - \mu\right)\left(u - \frac{k}{3-\sigma}\right)^2 - C_0^2\\ &\geq \frac{\sigma}{2}\left(u_x^2 - \gamma_\sigma^2\left(u - \frac{k}{3-\sigma}\right)^2\right) - C_0^2\\ &= -\,\frac{\sigma}{2}MN - C_0^2, \quad \text{and}\\ \frac{\partial N}{\partial t} &= -\,\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 - ku - (p + \gamma_\sigma p_x) * \left(-ku + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2\right)\\ &\quad + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - (p + \gamma_\sigma p_x) * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)\\ &\leq -\,\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}\left(u - \frac{k}{3-\sigma}\right)^2 - \mu\left(u - \frac{k}{3-\sigma}\right)^2 + C_0^2\\ &= -\,\frac{\sigma}{2}u_x^2 + \left(\frac{3-\sigma}{2} - \mu\right)\left(u - \frac{k}{3-\sigma}\right)^2 + C_0^2\end{aligned}$$

$$\leq -\frac{\sigma}{2}\left(u_x^2-\gamma_\sigma^2\left(u-\frac{k}{3-\sigma}\right)^2\right)+C_0^2$$
$$=\frac{\sigma}{2}MN+C_0^2,$$

with $\gamma_\sigma^2=\frac{3-\sigma}{\sigma}-\frac{2\lambda}{\sigma}$. Here C_0^2 is the bound of the right sides in the above estimates for the terms with ω_1 and ω_2, that is,

$$\begin{aligned}
&\left|\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-(p-\gamma_\sigma p_x)*\left(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\right)\right|\\
&\quad\leq \frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}^3+\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^4+\|u\|_{L^\infty}\left(\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^2}^2\right)+\|u\|_{L^\infty}^2\left(\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^2}^2\right)\\
&\quad\leq \frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{2\alpha^3}E_0^2=C_0^2>0,
\end{aligned}$$

where we have used Lemma 4.2 and the fact that

$$\|p_\pm\|_{L^\infty}=\frac{1}{2},\quad \|p_\pm\|_{L^2}=\frac{1}{2\sqrt{2}}.$$

In consequence, we have

$$\begin{cases}\frac{\mathrm{d}M}{\mathrm{d}t}\geq -\frac{\sigma}{2}MN-C_0^2,\\ \frac{\mathrm{d}N}{\mathrm{d}t}\leq \frac{\sigma}{2}MN+C_0^2.\end{cases}\tag{4.37}$$

By the assumptions on $u_0(x_0)$, it can be easy to obtain that

$$M(0)=\gamma_\sigma\left(u_0(x_0)-\frac{k}{3-\sigma}\right)-u_{0,x}(x_0)>0,$$
$$N(0)=\gamma_\sigma\left(u_0(x_0)-\frac{k}{3-\sigma}\right)+u_{0,x}(x_0)<0,$$
$$\frac{\sigma}{2}M(0)N(0)+C_0^2<0.$$

By the continuity of $M(t)$ and $N(t)$, it then ensures that

$$\frac{\mathrm{d}M}{\mathrm{d}t}>0,\quad \frac{\mathrm{d}N}{\mathrm{d}t}<0,\quad \forall t\in[0,T).$$

This in turn implies that

$$M(t)>M(0)>0,\quad N(t)<N(0)<0,\quad \forall t\in[0,T).$$

Let $h(t)=\sqrt{-M(t)N(t)}$. It then follows from (4.37) that

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-M'(t)N(t)-M(t)N'(t)}{2h}\geq\frac{\left(-\frac{\sigma}{2}MN-C_0^2\right)(-N)-M\left(\frac{\sigma}{2}MN+C_0^2\right)}{2h}$$
$$=\frac{M-N}{2h}\left(-\frac{\sigma}{2}MN-C_0^2\right).$$

Using the estimate $\frac{M-N}{2h}\geq 1$ and the fact that $h+\sqrt{\frac{2}{\sigma}}C_0>h-\sqrt{\frac{2}{\sigma}}C_0>0$, we obtain the following differential inequalities

$$\frac{\mathrm{d}h}{\mathrm{d}t}\geq-\frac{\sigma}{2}MN-C_0^2=\frac{\sigma}{2}\left(h-\sqrt{\frac{2}{\sigma}}C_0\right)\left(h+\sqrt{\frac{2}{\sigma}}C_0\right)\geq\frac{\sigma}{2}\left(h-\sqrt{\frac{2}{\sigma}}C_0\right)^2.$$

Solving this inequality gives

$$T\leq\frac{2}{\sigma\sqrt{u_{0,x}^2(x_0)-\gamma_\sigma^2\left(u_0(x_0)-\frac{1}{(3-\sigma)}\left(\frac{\beta_0}{\beta}-c\right)\right)^2}-\sqrt{2\sigma}C_0}<\infty.$$

With the fact that $-u_x(t,q(t,x_0))=\frac{1}{2}(M-N)\geq h(t,q(t,x_0))$, this in turn implies there exists $T<\infty$, such that

$$\liminf_{t\uparrow T_{u_0},x\in\mathbb{R}}\partial_x u(t,x)=-\infty,$$

which completes the desired result as indicated above. □

Remark 4.2 Returning to the original scale, our assumption for the blow-up phenomena becomes

$$\sqrt{\beta\mu}u_{0,x}(\sqrt{\beta\mu}x_0)+\gamma_\sigma\left|u_0(\sqrt{\beta\mu}x_0)-\frac{1}{2(3-\sigma)\alpha\epsilon}\left(\frac{\beta_0}{\beta}-c\right)\right|<-\sqrt{\frac{2}{\sigma}}\frac{K_0}{\alpha\epsilon}.$$

Note that when Ω increases, α and β decrease. Then it can be observed that with effect of the Earth's rotation, worse initial data $u_0(x_0)$ are required to make the breaking wave happen. On the other hand, with the original scale, we have

$$T\leq\frac{2}{\sigma\alpha\epsilon\sqrt{\beta\mu u_{0,x}^2(\sqrt{\beta\mu}x_0)-\gamma_\sigma^2\left(u_0(\sqrt{\beta\mu}x_0)-\frac{1}{(3-\sigma)\alpha\epsilon}\left(\frac{\beta_0}{\beta}-c\right)\right)^2}-\sqrt{2\sigma}K_0},$$

where $K_0>0$ is defined by

$$K_0^2=\frac{2|\omega_1|\alpha\epsilon^3}{3}E^{\frac{3}{2}}+\frac{|\omega_2|\alpha\epsilon^4}{2}E^2,$$

$$E(u_0)=\frac{1}{\alpha^2\epsilon^2}E_0(\alpha\epsilon u_0(\sqrt{\beta\mu}x_0)) \quad \text{and} \quad \gamma_\sigma^2=\frac{3-\sigma}{\sigma}-\frac{2\lambda}{\sigma}.$$

For the case of $\sigma<0$ or $\sigma>4$, wave-breaking data can be obtained in the following theorem.

Theorem 4.3 *Let $\sigma<0$ or $\sigma>4$ and u be the solution of (3.4) with initial data $u_0\in H^s$, $s>3/2$, and $T>0$ be the maximal time of existence.*
(1) *If $\sigma>4$, assume there is some $x_0\in\mathbb{R}$ such that*

$$u_{0,x}(x_0)<-\frac{C_3}{\sqrt{\sigma}}, \tag{4.38}$$

where C_3 is defined by

$$C_3^2=\frac{1}{2}|c-\frac{\beta_0}{\beta}|+\left(\sigma-3+|c-\frac{\beta_0}{\beta}|\right)E_0+\frac{4|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{\alpha^3}E_0^2.$$

Then the corresponding solution u of Eq. (3.4) breaks down in finite time in the following sense: there exists a T_1 with

$$0<T_1\le\frac{1}{\sqrt{\sigma}C_1}\ln\left(\frac{u_{0,x}(x_0)-\frac{C_1}{\sqrt{\sigma}}}{u_{0,x}(x_0)+\frac{C_1}{\sqrt{\sigma}}}\right), \tag{4.39}$$

such that

$$\liminf_{t\uparrow T_1^-}\left(\inf_{x\in\mathbb{R}}u_x(t,x)\right)=-\infty. \tag{4.40}$$

(2) *If $\sigma<0$, assume there is some $x_0\in\mathbb{R}$ such that*

$$u_{0,x}(x_0)>\frac{C_4}{\sqrt{-\sigma}}, \tag{4.41}$$

where C_4 is defined by

$$C_4^2=\frac{1}{2}|c-\frac{\beta_0}{\beta}|+\left(3-\sigma+|c-\frac{\beta_0}{\beta}|\right)E_0+\frac{4|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{\alpha^3}E_0^2.$$

Then the corresponding solution u of Eq. (3.4) blows up in finite time in the following sense: there exists a T_2 with

$$0<T_2\le\frac{1}{\sqrt{-\sigma}C_4}\ln\left(\frac{u_{0,x}(x_0)+\frac{C_4}{\sqrt{-\sigma}}}{u_{0,x}(x_0)-\frac{C_4}{\sqrt{-\sigma}}}\right), \tag{4.42}$$

such that

$$\liminf_{t\uparrow T_2^-}\left(\sup_{x\in\mathbb{R}} u_x(t,x)\right)=\infty. \tag{4.43}$$

In order to prove Theorem 4.3, we need the following lemma due to Constantin and Escher (1998).

Lemma 4.3 *Let $T>0$ and $v\in C^1([0,T);H^1(\mathbb{R}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{R}$ with*

$$m(t):=\inf_{x\in\mathbb{R}}[v_x(t,x)]=v_x(t,\xi(t)).$$

The function $m(t)$ is absolutely continuous on $(0,T)$ with

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t}=v_{tx}(t,\xi(t))\quad \text{a. e. on } (0,T).$$

Proof of Theorem 4.3 Similar to the proof of Lemma 4.1, it suffices to consider $s\geq 3$. (1) When $\sigma>4$, with Lemma 4.3 and the fact that

$$\sup_{x\in\mathbb{R}}[v_x(t,x)]=-\inf_{x\in\mathbb{R}}[-v_x(t,x)],$$

we can consider $M_1(t)$ and $x_1(t)\in\mathbb{R}$ as follows

$$M_1(t):=u_x(t,x_1(t))=\inf_{x\in\mathbb{R}}[u_x(t,x)],\qquad t\in[0,T),$$

which implies $u_{xx}(t,x_1(t))=0$, a.e. on $(0,T)$.

Combined with the estimates of (4.15)–(4.19), it is estimated that

$$\begin{aligned}\frac{\mathrm{d}M_1(t)}{\mathrm{d}t}&=-\frac{\sigma}{2}(M_1(t))^2+\frac{3-\sigma}{2}u^2-p*\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2\right)+\left(c-\frac{\beta_0}{\beta}\right)u\\&\quad+\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-p*\left(\left(c-\frac{\beta_0}{\beta}\right)u+\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\right)\\&\leq-\frac{\sigma}{2}(M_1(t))^2-p*\left(\frac{3-\sigma}{2}u^2\right)+|c-\frac{\beta_0}{\beta}|\,|p_x*u_x|\\&\quad+\left|\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4-p*\left(\frac{\omega_1}{3\alpha^2}u^3+\frac{\omega_2}{4\alpha^3}u^4\right)\right|\\&\leq-\frac{\sigma}{2}(M_1(t))^2+\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{1}{2}\left(\sigma-3+|c-\frac{\beta_0}{\beta}|\right)E_0+\frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}\\&\quad+\frac{|\omega_2|}{2\alpha^3}E_0^2\\&=-\frac{\sigma}{2}(M_1(t))^2+\frac{C_3^2}{2},\end{aligned}$$

that is

$$M_1'(t) \leq -\frac{\sigma}{2}(M_1(t))^2 + \frac{1}{2}C_3^2, \quad \text{a. e. } t \in (0, T). \tag{4.44}$$

On account of the assumption in (4.38),

$$M_1(0) = u_{0,x}(x_0) < -\frac{C_3}{\sqrt{\sigma}},$$

we have that $M_1'(0) < 0$ and $M_1(t)$ is strictly decreasing over $[0, T)$.

As $M_1(t) < M_1(0) = u_{0,x}(x_0) < -\frac{C_3}{\sqrt{\sigma}} < 0$, solving the above inequality in (4.44) yields that

$$\frac{M_1(0) + \frac{C_3}{\sqrt{\sigma}}}{M_1(0) - \frac{C_3}{\sqrt{\sigma}}} e^{\sqrt{\sigma}C_3 t} - 1 \leq \frac{\frac{2C_3}{\sqrt{\sigma}}}{M_1(t) - \frac{C_3}{\sqrt{\sigma}}} \leq 0. \tag{4.45}$$

This in turn implies that

$$M_1(t) < \frac{C_3}{\sqrt{\sigma}} \to -\infty, \quad \text{when } \ t \to \frac{1}{\sqrt{\sigma}C_3} \ln\left(\frac{M_1(0) - \frac{C_3}{\sqrt{\sigma}}}{M_1(0) + \frac{C_3}{\sqrt{\sigma}}}\right). \tag{4.46}$$

Hence,

$$0 < T \leq \frac{1}{\sqrt{\sigma}C_3} \ln\left(\frac{u_{0,x}(x_0) - \frac{C_3}{\sqrt{\sigma}}}{u_{0,x}(x_0) + \frac{C_3}{\sqrt{\sigma}}}\right), \tag{4.47}$$

as advertised in (4.39).

(2) When $\sigma < 0$, similarly as in (1). Given $t \in [0, T)$, let $x_2(t) \in \mathbb{R}$ be such that

$$M_2(t) := u_x(t, x_2(t)) = \sup_{x\in\mathbb{R}}[u_x(t, x)], \quad t \in [0, T),$$

which implies $u_{xx}(t, x_2(t)) = 0$, a. e. on $(0, T)$.

In the same spirit of the case of $M_1(t)$, it arrives at

$$\begin{aligned}
\frac{\mathrm{d}M_2(t)}{\mathrm{d}t} &\geq -\frac{\sigma}{2}(M_2(t))^2 - p * \left(\frac{3-\sigma}{2}u^2\right) - |c - \frac{\beta_0}{\beta}|\,|p_x * u_x| \\
&\quad - \left|\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - p * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)\right| \\
&\geq -\frac{\sigma}{2}(M_2(t))^2 - \frac{1}{4}|c - \frac{\beta_0}{\beta}| - \frac{1}{2}\left(3 - \sigma + |c - \frac{\beta_0}{\beta}|\right) E_0 - \frac{2|\omega_1|}{3\alpha^2} E_0^{\frac{3}{2}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{|\omega_2|}{2\alpha^3} E_0^2 \\
&= -\frac{\sigma}{2}(M_1(t))^2 - \frac{C_4^2}{2},
\end{aligned}
$$

that is

$$M_2'(t) \geq -\frac{\sigma}{2}(M_2(t))^2 - \frac{C_4^2}{2}, \quad \text{a. e. } t \in (0, T). \tag{4.48}$$

From the assumption in (4.41),

$$M_2(0) = u_{0,x}(x_0) > \frac{C_4}{\sqrt{-\sigma}},$$

it is found that $M_2'(0) > 0$ and $M_2(t)$ is strictly increasing over $[0, T)$.

As $M_2(t) > M_2(0) = u_{0,x}(x_0) > \frac{C_4}{\sqrt{-\sigma}} > 0$, solving the above inequality in (4.44) yields that

$$\frac{M_2(0) + \frac{C_4}{\sqrt{-\sigma}}}{M_2(0) - \frac{C_4}{\sqrt{-\sigma}}} e^{-\sqrt{-\sigma}C_4 t} - 1 \geq \frac{\frac{2C_4}{\sqrt{-\sigma}}}{M_2(t) - \frac{C_4}{\sqrt{-\sigma}}} \geq 0. \tag{4.49}$$

Again,

$$M_2(t) > \frac{C_4}{\sqrt{-\sigma}} \to \infty, \quad \text{when } t \to \frac{1}{\sqrt{-\sigma}C_4} \ln\left(\frac{M_1(0) + \frac{C_4}{\sqrt{-\sigma}}}{M_2(0) - \frac{C_4}{\sqrt{-\sigma}}} \right). \tag{4.50}$$

In consequence,

$$0 < T \leq \frac{1}{\sqrt{-\sigma}C_4} \ln\left(\frac{u_{0,x}(x_0) + \frac{C_4}{\sqrt{-\sigma}}}{u_{0,x}(x_0) - \frac{C_4}{\sqrt{-\sigma}}} \right), \tag{4.51}$$

which gives the desired result in (4.42). This completes the proof of Theorem 4.3. □

5 Blow-Up Rate

We now in this section consider the blow-up rate of the slope to a breaking wave for Eq. (3.1). We have the following result.

Theorem 5.1 *Let $\sigma \neq 0$. If $T < \infty$ is the blow-up time of the solution u to Eq. (3.1) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$ satisfying the assumption of Theorems 4.2 and 4.3, then*

$$\lim_{t\to T^-}\left[\left(\inf_{x\in\mathbb{R}} u_x(t,x)\right)(T-t)\right] = -\frac{2}{\sigma}, \quad \text{when} \quad \sigma > 0, \tag{5.1}$$

$$\lim_{t\to T^-}\left[\left(\sup_{x\in\mathbb{R}} u_x(t,x)\right)(T-t)\right] = -\frac{2}{\sigma}, \quad \text{when} \quad \sigma < 0. \tag{5.2}$$

Proof The proof is divided into two cases for the parameter σ.
(a) For $\sigma > 0$, denote K by

$$K = \frac{1}{4}|c - \frac{\beta_0}{\beta}| + \frac{1}{2}\left(|\sigma| + 2|3-\sigma| + |c - \frac{\beta_0}{\beta}|\right) E_0 + \frac{2|\omega_1|}{3\alpha^2} E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3} E_0^2 = \frac{C_1^2}{2}. \tag{5.3}$$

In view of Eq. (4.44), we deduce the following relation

$$-\frac{1}{2}(M_1(t))^2 - K \le \frac{\mathrm{d}M_1(t)}{\mathrm{d}t} \le -\frac{1}{2}(M_1(t))^2 + K, \quad \text{a. e. } t \in [0,T). \tag{5.4}$$

Choose $0 < \theta < \sigma/2$. Since $t \to T^-$, $M_1(t) \to -\infty$, we can then find $t_0 \in (0,T)$ such that

$$M_1(t_0) < -\sqrt{2\sigma K + \frac{K}{\theta}}. \tag{5.5}$$

Note $M_1(t)$ is absolutely continuous on $[0,T)$. It is thereby inferred from the above differential inequality that $M_1(t)$ is strictly decreasing on $[t_0,T)$ and hence

$$M_1(t) < -\sqrt{2\sigma K + \frac{K}{\theta}} < -\sqrt{\frac{K}{\theta}}, \quad t \in [t_0,T). \tag{5.6}$$

Then using (5.4) gives that

$$\frac{\sigma}{2} - \theta < \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{M_1(t)}\right) < \frac{\sigma}{2} + \theta, \quad \text{a.e. } t \in [t_0,T). \tag{5.7}$$

Integrating the above inequality from over $[t,T)$ with $t \in [t_0,T)$, we obtain

$$\left(\frac{\sigma}{2} - \theta\right)(T-t) < -\frac{1}{M_1(t)} < \left(\frac{\sigma}{2} + \theta\right)(T-t), \quad t \in [t_0,T). \tag{5.8}$$

Since $\theta \in (0, \frac{\sigma}{2})$ is arbitrary, the above inequality implies (5.1).
(b) For $\sigma < 0$, it follows from (4.29) that

$$-\frac{\sigma}{2}(M_2(t))^2 - K \le \frac{\mathrm{d}M_2(t)}{\mathrm{d}t} \le -\frac{\sigma}{2}(M_2(t))^2 + K \quad \text{a. e. } t \in [0,T), \tag{5.9}$$

where K is defined in (5.3). Choose now θ with $0 < \theta < -\sigma/2$. It is noticed that $t \to T^-$, $M_2(t) \to \infty$. Again, $t_0 \in (0, T)$ can be chosen such that

$$M_2(t_0) > \sqrt{-2\sigma K + \frac{K}{\theta}}. \tag{5.10}$$

Since $M_2(t)$ is absolutely continuous on $[0, T)$, it can be inferred from the above differential inequality that $M_2(t)$ is strictly increasing on $[t_0, T)$ and hence

$$M_2(t) > \sqrt{2\sigma K + \frac{K}{\theta}} > \sqrt{\frac{K}{\theta}}, \quad t \in [t_0, T). \tag{5.11}$$

It then turns out from (5.9) that

$$\frac{\sigma}{2} - \theta < \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{M_2(t)}\right) < \frac{\sigma}{2} + \theta, \qquad \text{a.e. } t \in [t_0, T). \tag{5.12}$$

Integrating the above inequality with respect to time variable from over $[t, T)$ with $t \in [t_0, T)$, we obtain

$$\left(\frac{\sigma}{2} - \theta\right)(T - t) < -\frac{1}{M_2(t)} < \left(\frac{\sigma}{2} + \theta\right)(T - t), \quad t \in [t_0, T). \tag{5.13}$$

Since $\theta \in (0, -\frac{\sigma}{2})$ is arbitrary, the above inequality implies (5.2), thereby concluding the proof of Theorem 5.1. □

6 Global Solutions When $\sigma = 0$

Our attention in this section is now turned to the existence of the global solutions for the gr-CH equation (3.1). We give a sufficient condition of the global solution when the balance parameter $\sigma = 0$.

Theorem 6.1 (global existence) *Let $\sigma = 0$. Assume $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exists a unique solution $u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}))$ to the Cauchy problem (3.3) with $u(0) = u_0$. Moreover, the solution u depends continuously on the initial value u_0 and the Hamiltonians $E(u)$ and $F(u)$ are independent of the existence time $t > 0$.*

When $\sigma = 0$, Eq. (3.13) can be written as

$$u_t + \partial_x p * \left(\left(c - \frac{\beta_0}{\beta}\right)u + \frac{3}{2}u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) = 0, \tag{6.1}$$

where $p = \frac{1}{2}e^{-|x|}$.

To obtain Theorem 6.1 for global well-posedness of solutions, the following estimate for u_x is essential.

Lemma 6.1 *Let $\sigma = 0$. Assume u is the solution of Eq. (3.3) with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, and T is the maximal time of existence. Then*

$$\|u_x(t,x)\|_{L^\infty} \le \|u_{0,x}\|_{L^\infty} + C_3^2 t, \quad t \in [0,T), \tag{6.2}$$

where the above constant C_3 is defined by

$$C_3^2 = \frac{1}{4}|c - \frac{\beta_0}{\beta}| + \left(3 + \frac{1}{2}|c - \frac{\beta_0}{\beta}|\right) E_0^2 + C_0^2,$$

with $E_0 = \frac{1}{2}\int_{\mathbb{R}} \left(u_0^2 + u_{0,x}^2\right)\mathrm{d}x$ *and* $C_0^2 = \frac{2|\omega_1|}{3\alpha^2} E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3} E_0^2$.

Proof Differentiating Eq. (6.1) with respect to x and using the identity $-\partial_x^2 p * f = f - p * f$, it is found that

$$\begin{aligned} u_{tx} = & \frac{3}{2}u^2 + \Big(c - \frac{\beta_0}{\beta}\Big)u - p * \Big(\frac{3}{2}u^2 + \Big(c - \frac{\beta_0}{\beta}\Big)u\Big) + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & - p * \Big(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big). \end{aligned} \tag{6.3}$$

Denote the function f by

$$\begin{aligned} f(t,x) = & \frac{3}{2}u^2 + \Big(c - \frac{\beta_0}{\beta}\Big)u - p * \Big(\frac{3}{2}u^2 + \Big(c - \frac{\beta_0}{\beta}\Big)u\Big) + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & - p * \Big(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big). \end{aligned} \tag{6.4}$$

In order to estimate the bound of u_x, we first derive the upper and lower bounds for $f(t,x)$. Using the identity $\partial_x^2 p * u = p_x * u_x$, a simple computation reveals that

$$\begin{aligned} |f(t,x)| = & \Big|\frac{3}{2}u^2 - p * \Big(\frac{3}{2}u^2\Big) + \Big(c - \frac{\beta_0}{\beta}\Big)p_x * u_x \\ & + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - p * \Big(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big| \\ \le & \frac{3}{2}u^2 + |p * \Big(\frac{3}{2}u^2\Big)| + |\Big(c - \frac{\beta_0}{\beta}\Big)p_x * u_x| \\ & + \Big|\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - p * \Big(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\Big)\Big|. \end{aligned}$$

On the other hand,

$$\frac{3}{2}u^2 \le \frac{3}{4}\int_{\mathbb{R}} \left(u^2 + u_x^2\right)\mathrm{d}x = \frac{3}{4}(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2), \tag{6.5}$$

$$|p * \Big(\frac{3}{2}u^2\Big)| \le \frac{3}{4}\|u^2\|_{L^1} \le \frac{3}{4}\|u\|_{L^2}^2, \quad \text{and} \tag{6.6}$$

$$|c-\frac{\beta_0}{\beta}|\,|p_x * u_x| \le \frac{1}{2}|c-\frac{\beta_0}{\beta}|\|u_x\|_{L^2} \le \frac{1}{4}|c-\frac{\beta_0}{\beta}|\left(1+\|u_x\|_{L^2}^2\right). \tag{6.7}$$

The bound of $f(t,x)$ is then estimated by the inequalities (4.18) and (4.19), that is,

$$\begin{aligned}|f(t,x)| \le& \frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{3}{2}\|u\|_{L^2}^2+\left(\frac{1}{4}|c-\frac{\beta_0}{\beta}|+\frac{3}{4}\right)\|u_x\|_{L^2}^2+\frac{2|\omega_1|}{3\alpha^2}E_0^{\frac{3}{2}}+\frac{|\omega_2|}{2\alpha^3}E_0^2\\ \le& \frac{1}{4}|c-\frac{\beta_0}{\beta}|+\left(3+\frac{1}{2}|c-\frac{\beta_0}{\beta}|\right)E_0^2+C_0^2=C_3^2.\end{aligned} \tag{6.8}$$

It thus transpires that

$$-C_3^2 \le \frac{\partial u_x}{\partial t} = f(t,x) \le C_3^2. \tag{6.9}$$

Integrating the above inequality respect to time t over $[0, T]$ yields the desired result in (6.2). This completes the proof of Lemma 6.1. □

Proof of Theorem 6.1 In view of Lemma 6.1, it then follows from Theorem 3.2 that the local solution u obtained in Theorem 3.1 can be extended to all of the interval $[0, \infty)$. The proof of Theorem 6.1 is thus complete. □

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