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High-order Soliton Matrix for the Third-order Flow Equation of the Gerdjikov-Ivanov Hierarchy Through the Riemann-Hilbert Method

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Abstract The Gerdjikov-Ivanov (GI) hierarchy is derived via recursion operator, in this article, we mainly investigate the third-order flow GI equation. In the framework of the Riemann-Hilbert method, the soliton matrices of the third-order flow GI equation with simple zeros and elementary high-order zeros of Riemann-Hilbert problem are constructed through the standard dressing process. Taking advantage of this result, some properties and asymptotic analysis of single soliton solution and two soliton solution are discussed, and the simple elastic interaction of two soliton are proved. Compared with soliton solution of the classical second-order flow, we find that the higher-order dispersion term affects the propagation velocity, propagation direction and amplitude of the soliton. Finally, by means of a certain limit technique, the high-order soliton solution matrix for the third-order flow GI equation is derived.

Keywords Gerdjikov-Ivanov hierarchy; third-order flow GI equation; Riemann-Hilbert method; high-order soliton

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1 Introduction

As is well known, completely integrable equations have widespread applications in plasma physics, water wave, field theory and nonlinear optics (see [5, 16, 18]). In these integrable systems, the nonlinear Schrödinger (NLS) equation is a general mathematical model for controlling weak nonlinearity and dispersive wave packets in one-dimensional physical systems. Another well-known NLS-type integrable system is the derivative NLS equation

$$iu_t + u_{xx} - iu^2u_x^* + \frac{1}{2}u^3u^{*2} = 0, \tag{1.1}$$

the symbol '$*$' denotes complex conjugation. Eq.(1.1) was first discovered by Gerdjikov and Ivanov in Ref.[12], also known as the GI equation. It can be seen as a generalization of NLS when some higher-order nonlinear effects are considered, which is also known as DNLS III. In fact, there are three famous DNLS equations, the other two types of derivative NLS equations are the Kaup-Newell (KN) equation[17]

$$iu_t + u_{xx} + i\left(u^2u^*\right)_x = 0, \tag{1.2}$$

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which is a typical dispersion equation derived from the magnetohydrodynamic equations with Hall effect and usually called DNLS I, and the Chen-Lee-Liu (CLL) equation[3]

$$iu_t + u_{xx} + iuu^* u_x = 0, \tag{1.3}$$

which describes ultra-short pulses in the optical model, also known as DNLS II. Ref.[9] gives the unified expression of KN, CLL and GI equations.

The GI equation (1.1) is a model of Alfvén waves propagating parallel to the surrounding magnetic field in plasma physics. In recent years, many academics have done a lot of research on classical GI equation, such as its Hamiltonian structures, Darboux transformation[7, 8], algebra-geometric solutions[4], breather solution[28], the rogue wave and the long-time asymptotic behavior of its solution[25, 27]. With the development of research, the importance of higher-order nonlinear effects in plasma physics and other fields promote us to consider integrable models with higher nonlinearity.

In this work, we mainly study the soliton solutions and high-order solutions of the third-order flow GI equation

$$u_t = -\frac{1}{2}u_{xxx} + \frac{3}{2}iuu_x u_x^* - \frac{3}{4}|u|^4 u_x \tag{1.4}$$

with the help of the Riemann-Hilbert method. It has been proved in Ref.[9] that Eq.(1.4) is Liouville integrable and has multiple Hamiltonian structures.

It is well known that the classical inverse scattering transform (IST) method is a general method to obtain soliton solutions[1, 2, 11], which was initially solved by using the Gel'Fand-Levitan-Marchenko (GLM) integral equation. Although the GLM equation can be used to solve the equation, the solution process is very complex. After that, Shabat used Riemann-Hilbert problem (RHP) to reconstruct the inverse scattering method[23].As an improved IST method, Riemann-Hilbert method has become a more popular method to study soliton solutions and long-time asymptotic behavior of integrable systems in recent years[6, 14, 15, 19, 26, 30, 32, 33, 36].

Higher-order soliton is also an important exact solution of NLS type equation, it can describe the weak bound state of soliton, which may appear in the study of soliton train transmission with specific chirp and almost equal velocity and amplitude[10]. Therefore, it is valuable to investigate the higher-order soliton solutions of DNLS equation.

In this paper, we construct GI hierarchy based on recursive operator. In the framework of the RHP, soliton matrices corresponding to simple zeros and elementary higher-order zeros of the third-order flow GI equation are constructed through a standard dressing transformation method. Different from the NLS-type equation, in this case, a pair of zeros are always treated at the same time. On the basis of the determinant solution, asymptotic analysis and some properties of single soliton solution and two soliton solutions are studied. Compared with the classical second-order flow GI equation, it is find that the higher-order dispersion term has a great influence on the direction, velocity and amplitude of solitons. In the situation of elementary higher-order zeros, the higher-order soliton matrix of the third-order flow GI equation is derived by using the limit process of spectral parameters.

The article is arranged as follows. In Section 2, we derive the GI hierarchy with recursion operator. The RHP based on the Jost solutions to the Lax pair of the third-order flow GI equation and scattering data are constructed in section 3. In Section 4, taking advantage of the Plemelj formula to discuss the solutions of regular and non-regular RHP. In Section 5, we reconstruct the potential u through the inverse problem, and derive the N-solitons formula of the third-order flow GI equation by considering the simple zeros in the RHP. In Section 6, the higher-order soliton matrix corresponding to the elementary higher-order zeros in the RHP is constructed. The conclusion and discussion are given in the final section.

2 The GI Hierarchy and Recursion Operator

In the theory of integrable system, an important assignment is to construct new integrable equations and solve them with the IST method. In this section, we will associate with recursion operator, and construct the GI hierarchy of integrable equations. The GI hierarchy has the following spectral problem:

$$Y_x = MY, \qquad M = \begin{pmatrix} -i\lambda^2 - \dfrac{i}{2}uv & \lambda u \\ \lambda v & i\lambda^2 + \dfrac{i}{2}uv \end{pmatrix}, \tag{2.1}$$

$$Y_t = NY, \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{2.2}$$

where λ is the spectral parameter, $u = u(x,t)$ and $v = v(x,t)$ are field variables, A, B and C are the quantities relating to field variables and their derivatives and λ.

Theorem 2.1. *According to the consistency of space part* (2.1) *and time part* (2.2) *of spectral problem, infinite hierarchy of GI system can be obtained by recursive operator:*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Big(-\frac{1}{2}\Big)^{n-2} (i\sigma_3)^{n-1} (L_1\partial_x + L_2)^{n-1} \begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad n = 2, 3, \cdots, \tag{2.3}$$

where

$$L_1 = \begin{pmatrix} -1 + iu\partial^{-1}v & iu\partial^{-1}u \\ -iv\partial^{-1}v & -1 - iv\partial^{-1}u \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$L_2 = \begin{pmatrix} -iuv - u\partial^{-1}uv^2 & u\partial^{-1}u^2v \\ v\partial^{-1}uv^2 & iuv - v\partial^{-1}u^2v \end{pmatrix}.$$

Proof. The compatibility condition of Eq.(2.1) and Eq.(2.2) leads to

$$M_t - N_x + [M, N] = 0, \tag{2.4}$$

which can lead

$$\frac{i}{2}(uv)_t + A_x - \lambda uC + \lambda vB = 0, \tag{2.5}$$

$$\lambda u_t - B_x - 2i\lambda^2 B - iuvB - 2\lambda uA = 0, \tag{2.6}$$

$$\lambda v_t - C_x + 2i\lambda^2 C + iuvC + 2\lambda vA = 0. \tag{2.7}$$

From those equations, we can get

$$A = \frac{-i}{2\lambda}\partial^{-1}\big(vB_x + uC_x + iuv^2B - iu^2vC\big) + A_0, \tag{2.8}$$

where ∂^{-1} is an integral in x which can be regarded as either $\partial^{-1} = \int_{-\infty}^{x} dy$ or $\partial^{-1} = -\int_{x}^{+\infty} dy$, and A_0 are $x-$independent.

Using (2.8), Eqs.(2.6) and (2.7) may be rewritten as

$$\lambda\begin{pmatrix} u \\ v \end{pmatrix}_t + L_1\begin{pmatrix} B_x \\ C_x \end{pmatrix} - 2i\lambda^2\begin{pmatrix} B \\ -C \end{pmatrix} + L_2\begin{pmatrix} B \\ C \end{pmatrix} - 2\lambda A_0\begin{pmatrix} u \\ -v \end{pmatrix} = 0, \tag{2.9}$$

where

$$L_1 = \begin{pmatrix} -1 + iu\partial^{-1}v & iu\partial^{-1}u \\ -iv\partial^{-1}v & -1 - iv\partial^{-1}u \end{pmatrix}, \qquad L_2 = \begin{pmatrix} -iuv - u\partial^{-1}uv^2 & u\partial^{-1}u^2v \\ v\partial^{-1}uv^2 & iuv - v\partial^{-1}u^2v \end{pmatrix}.$$

In order to get the evolution equations, we can expand B and C in the following form

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n}\begin{pmatrix} b_j \\ c_j \end{pmatrix}(\lambda)^{2j-1}. \tag{2.10}$$

Let $A_0 = -2i\lambda^{2n}$, substituting (2.10) into (2.9), and taking advantage of the equality of the values of the same power terms of λ to obtain the following recurrence

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + L_1\begin{pmatrix} b_1 \\ c_1 \end{pmatrix}_x + L_2\begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = 0, \tag{2.11}$$

and

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = 2\begin{pmatrix} u \\ v \end{pmatrix}, \quad L_1\begin{pmatrix} b_j \\ c_j \end{pmatrix}_x - 2i\sigma_3\begin{pmatrix} b_{j-1} \\ c_{j-1} \end{pmatrix} + L_2\begin{pmatrix} b_j \\ c_j \end{pmatrix} = 0, \qquad j = 2, \cdots, n. \tag{2.12}$$

Eq.(2.11) and (2.12) are used to iterate and derive the GI hierarchy

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \Big(-\frac{1}{2}\Big)^{n-2}(i\sigma_3)^{n-1}(L_1\partial_x + L_2)^{n-1}\begin{pmatrix} u_x \\ v_x \end{pmatrix}, \qquad n = 2, 3, \cdots,$$

the Theorem can eventually be proved. □

Remark 2.2. In the zero curvature equation of GI equation, the derivative of M principal diagonal to t is not zero, which leads to GI hierarchy recursive form more complex than KN hierarchy[13] and the AKNS hierarchy[30].

Taking $n = 2$, the first nontrivial flow in the hierarchy (2.3) is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} iu_{xx} - u^2v_x + \dfrac{i}{2}u^3v^2 \\ -iv_{xx} - v^2u_x - \dfrac{i}{2}u^2v^3 \end{pmatrix}, \tag{2.13}$$

which form a coupled GI system. The reduction $v = -u^*$ for (2.13) yields the DNLS III Eq. (1.1). When $n = 3$, the second nontrivial flow in the hierarchy (2.3) is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -\dfrac{1}{2}u_{xxx} - \dfrac{3i}{2}uu_xv_x - \dfrac{3}{4}(u^2v^2)u_x \\ -\dfrac{1}{2}v_{xxx} + \dfrac{3i}{2}u_xv_xv - \dfrac{3}{4}(u^2v^2)v_x \end{pmatrix}. \tag{2.14}$$

Taking $v=-u^*$, the Eq.(2.14) is simplified as

$$u_t=-\frac{1}{2}u_{xxx}+\frac{3}{2}iuu_xu_x^*-\frac{3}{4}|u|^4u_x,$$

which corresponding to Eq.(1.4). In this case, the specific forms of b_j and c_j are

$$\begin{aligned}&b_3=2u,\qquad c_3=-2u^*,\qquad b_2=iu_x,\qquad c_2=iu_x^*,\\&b_1=-\frac{1}{2}u_{xx}+\frac{1}{2}iu^2u_x^*-\frac{1}{4}u^3u^{*2},\\&c_1=\frac{1}{2}u_{xx}^*+\frac{1}{2}iu^{*2}u_x+\frac{1}{4}u^{*3}u^2.\end{aligned}\tag{2.15}$$

For convenience, we present the spectral problem of Eq.(1.4)

$$Y_x=MY,\qquad M=-i\lambda^2\sigma_3+\lambda Q-\frac{i}{2}Q^2\sigma_3,\tag{2.16}$$

$$Y_t=NY,\qquad N=-2i\lambda^6\sigma_3+Z_5\lambda^5+Z_4\lambda^4+Z_3\lambda^3+Z_2\lambda^2+Z_1\lambda+Z_0,\tag{2.17}$$

where

$$Q=\begin{pmatrix}0&u\\-u^*&0\end{pmatrix},\tag{2.18}$$

$$\begin{aligned}&Z_5=2Q,\quad Z_4=-iQ^2\sigma_3,\quad Z_3=i\sigma_3Q_x,\\&Z_2=-\frac{1}{2}QQ_x+\frac{1}{2}Q_xQ+\frac{1}{4}iQ^4\sigma_3,\\&Z_1=-\frac{1}{2}Q_{xx}+\frac{i}{2}QQ_xQ\sigma_3-\frac{1}{4}Q^5,\\&Z_0=\frac{i}{4}(QQ_{xx}+Q_{xx}Q)\sigma_3-\frac{i}{4}Q_x^2\sigma_3+\frac{i}{8}Q^6\sigma_3.\end{aligned}\tag{2.19}$$

It's easy to find that

$$Q^{\mathcal{H}}=-Q,\qquad \sigma_3Q\sigma_3=-Q,$$

where the symbol $'\mathcal{H}'$ denotes the Hermitian of a matrix.

3 The Construction of RHP

This section mainly constructs the RHP of Eq.(1.4). In our analysis, we mainly consider the zero boundary condition, i.e.,

$$u(x,0)\to 0,\qquad x\to\pm\infty,$$

which belongs to Schwartz space. Therefore, it is easy to take the form of the solution of Eqs.(2.16) and (2.17) as

$$Y=Je^{(-i\lambda^2x-2i\lambda^6t)\sigma_3}.\tag{3.1}$$

We can know

$$J(x,t,\lambda)\to I,\qquad x\to\pm\infty.\tag{3.2}$$

The Lax pair of Eq.(2.16)–(2.17) becomes

$$J_x+i\lambda^2[\sigma_3,J]=(\lambda Q-\frac{i}{2}Q^2\sigma_3)J,\tag{3.3}$$

$$J_t + 2i\lambda^6[\sigma_3, J] = (Z_5\lambda^5 + Z_4\lambda^4 + Z_3\lambda^3 + Z_2\lambda^2 + Z_1\lambda + Z_0)J, \tag{3.4}$$

where Q, Z_i $(i = 0, \cdots, 5)$ has been given by Eq.(2.19),(2.18).

Next, considing the scattering problem at first. In this situation , the time t is fixed, so it will be limited in our symbol. Then it is evident that $J(x, \lambda)$ satisfies the Volterra integral equations

$$J_-(x,\lambda) = I + \int_{-\infty}^{x} e^{i\lambda^2\sigma_3(y-x)}(\lambda Q(y) - \frac{i}{2}Q^2\sigma_3)J_- e^{i\lambda^2\sigma_3(x-y)}dy, \tag{3.5}$$

$$J_+(x,\lambda) = I - \int_{x}^{+\infty} e^{i\lambda^2\sigma_3(y-x)}(\lambda Q(y) - \frac{i}{2}Q^2\sigma_3)J_+ e^{i\lambda^2\sigma_3(x-y)}dy. \tag{3.6}$$

It is easy to prove the uniqueness and existence of Jost solution for the above integral equation by using the standard iterative method.

Splitting $J_\pm$ into columns as $J = (J^{[1]}, J^{[2]})$, due to the structure Eqs.(3.5) and (3.6) of the potential Q, we have

Proposition 3.1. *The column vectors $J_-^{(1)}$ and $J_+^{(2)}$ are analytic for $\lambda \in D_+$ and continuous for $\lambda \in D_+ \cup \mathbb{R} \cup i\mathbb{R}$, while the columns $J_+^{(1)}$ and $J_-^{(2)}$ are analytical for $\lambda \in D_-$ and continuous for $\lambda \in D_- \cup \mathbb{R} \cup i\mathbb{R}$, where*

$$D_+ = \Big\{\lambda \mid \arg\lambda \in \Big(0, \frac{\pi}{2}\Big) \cup \Big(\pi, \frac{3\pi}{2}\Big)\Big\}, \qquad D_- = \Big\{\lambda \mid \arg\lambda \in \Big(\frac{\pi}{2}, \pi\Big) \cup \Big(\frac{3\pi}{2}, 2\pi\Big)\Big\}.$$

The distribution area of $D_\pm$ is shown in Figure (3.1).

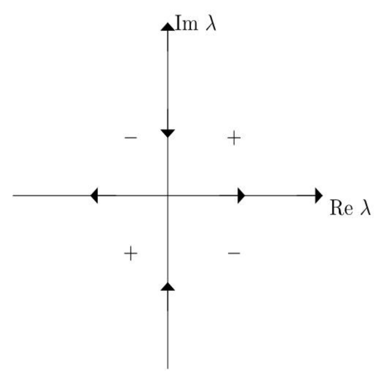


Figure 3.1. The jump contour in the complex λ-plane. Region $D_+ = \{\lambda \in \mathbb{C} | \mathrm{Re}\lambda \mathrm{Im}\lambda > 0\}$, region $D_- = \{\lambda \in \mathbb{C} | \mathrm{Re}\lambda \mathrm{Im}\lambda < 0\}$.

In fact, the J_+E and J_-E are the simultaneous solutions of the Lax pair (3.3). Therefore, they have following linear relation by the constant matrix $S(\lambda)$

$$J_-E = J_+ES(\lambda), \qquad \lambda \in \mathbb{R} \cup i\mathbb{R}, \tag{3.7}$$

where $E = e^{-i\lambda^2 x\sigma_3}$ and $S(\lambda) = (s_{ij})_{2\times 2}$. According to Abel's formula and $\mathrm{tr}(Q) = 0$, we obtain that the determinant of J is independent of x, then considering the conditions (3.2), we can get

$$\det J = 1.$$

Further, $\det S(\lambda) = 1$ can be easily obtained according to Eq.(3.7).

Proposition 3.2. *In the scattering matrix, s_{11} can be analytically extended to D_+, and s_{22} allows analytic extended to D_-.*

Proof. According to the relation (3.7) we have

$$S(\lambda) = \lim_{x\to+\infty} E^{-1}J_-E = I + \int_{-\infty}^{+\infty} E^{-1}\Big(\lambda Q(y) - \frac{i}{2}Q^2\sigma_3\Big)J_-Edx, \qquad \lambda\in\mathbb{R}\cup i\mathbb{R}. \quad (3.8)$$

So

$$S(\lambda) = I + \lambda\begin{pmatrix} \int_{-\infty}^{+\infty} uJ_-^{21}dx & \int_{-\infty}^{+\infty} uJ_-^{22}e^{2i\lambda^2x}dx \\ -\int_{-\infty}^{+\infty} u^*J_-^{11}e^{-2i\lambda^2x}dx & -\int_{-\infty}^{+\infty} u^*J_-^{12}dx \end{pmatrix}, \quad (3.9)$$

i,e,

$$s_{11} = 1 + \lambda\int_{-\infty}^{+\infty} uJ_-^{21}dx, \qquad s_{22} = 1 - \lambda\int_{-\infty}^{+\infty} u^*J_-^{12}dx,$$

through the analytic property of J_-, it's easy to know that s_{11} can be analytically extended to D_+, and s_{22} allows analytic extended to D_-. □

In order to construct the RHP, introducing the notation

$$P_+ = (J_-^{[1]}, J_+^{[2]}) = J_-H_1 + J_+H_2 = J_+E\begin{pmatrix} s_{11} & 0 \\ s_{21} & 1 \end{pmatrix}E^{-1}, \quad (3.10)$$

where $H_1 = \mathrm{diag}\{1,0\}$ and $H_2 = \mathrm{diag}\{0,1\}$. Through the previous analysis, we can see that P_+ is analytic in D_+ and $\det(P_+) = s_{11}$. In order to find the boundary condition of P_+ as $\lambda\to\infty$, the following asymptotic expansion can be considered

$$P_+ = P_+^{(0)} + \frac{1}{\lambda}P_+^{(1)} + \frac{1}{\lambda^2}P_+^{(2)} + O\Big(\frac{1}{\lambda^3}\Big). \quad (3.11)$$

Substituting (3.11) into (3.3) and equating terms with like powers of λ, which lead to

$$P_{+x}^{(0)} = 0, \quad (3.12)$$

without losing generality, we can set $P_+^{(0)} = I$. This means

$$P_+ \to I, \qquad \lambda\in D_+\to\infty. \quad (3.13)$$

To receive the analytic correspondence of P_+ in D_-, the adjoint scattering equation of Eq.(3.3) is considered

$$\Phi_x = -i\lambda^2[\sigma_3,\Phi] - \lambda\Phi Q + \frac{i}{2}\Phi Q^2\sigma_3, \quad (3.14)$$

using the formula,

$$0 = \left(JJ^{-1}\right)_x = J_xJ^{-1} + J\left(J^{-1}\right)_x,$$

it is easy to get that J^{-1} satisfies Eq.(3.14), that is to say, J^{-1} is the solution of Eq.(3.14), and the boundary condition is also $J^{-1}\to I$ as $x\to\pm\infty$. Let $(J_\pm^{-1})^{[k]}$ be the k-th row of the matrices $J_\pm^{-1}$,

$$J_\pm^{-1} = \left((J_\pm^{-1})^{[1]}, (J_\pm^{-1})^{[2]}\right)^T. \quad (3.15)$$

Adopting the same technology as above, we can prove that

$$P_-^{-1} = H_1 J_-^{-1} + H_2 J_+^{-1} = E\begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ 0 & 1 \end{pmatrix} E^{-1} J_+^{-1} \tag{3.16}$$

analytic for D_-, where

$$J_-^{-1} = ES^{-1}E^{-1}J_+^{-1}, \qquad \hat{S} = S^{-1} = \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{21} & \hat{s}_{22} \end{pmatrix},$$

and $\det P_-^{-1} = \hat{s}_{11}$. Through direct calculation, we can get that P_-^{-1} also satisfies the same boundary condition (3.13), as $\lambda \to \infty$. i.e.

$$P_-^{-1}(x,\lambda) \to I, \qquad \lambda \in \mathbb{C}_- \to \infty. \tag{3.17}$$

Therefore, two matrix functions $P_\pm(x,\lambda)$, which are analytic functions for $\lambda \in D_\pm$, have been constructed. Thus the RHP can be constructed as follow by P_+, P_-^{-1}:

$$P_-^{-1}(x,\lambda)P_+(x,\lambda) = G(x,\lambda) = E\begin{pmatrix} 1 & \hat{s}_{12} \\ s_{21} & 1 \end{pmatrix} E^{-1}, \qquad \lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{3.18}$$

with boundary condition

$$P_\pm \to I, \qquad \lambda \to \infty. \tag{3.19}$$

In the end of this section, the time evolution of scattering matrices $S(\lambda)$ and $\hat{S}(\lambda)$ are considered. Since J satisfies the temporal Eq.(3.4) of the Lax pair and the relation (3.7), then reviewing the evolution property (3.7) and $Q \to 0,\ Z_i (i = 1, \cdots, 5) \to 0$ as $|x| \to \infty$, we have

$$S_t + 2i\lambda^6 [\sigma_3, S] = 0.$$

The time evolution of $\hat{S}(\lambda)$ can be similarly obtained as

$$\hat{S}_t + 2i\lambda^6 [\sigma_3, \hat{S}] = 0.$$

These two equations lead that

$$s_{11,t} = \hat{s}_{11,t} = 0,$$

$$s_{12}(t;\lambda) = s_{12}(0;\lambda)\exp\big(-4i\lambda^6 t\big), \tag{3.20}$$

$$\hat{s}_{21}(t;\lambda) = \hat{s}_{21}(0;\lambda)\exp\big(4i\lambda^6 t\big). \tag{3.21}$$

4 Solution of the RHP

In this section, we discuss how to solve the matrix RHP (3.18) in the complex λ plane. When $\det(P_+) = s_{11} \neq 0$ and $\det(P_-^{-1}) = \hat{s}_{11} \neq 0$, the RHP (3.18) constructed above is regular, and RHP is called nonregular when $\det(P_+) = 0$ and $\det(P_-^{-1}) = 0$ are in some discrete positions. In fact, nonregular RHP can be converted into regular RHP, so we consider regular case at first.

4.1 Solution to the Regular RHP

In this subsection, we consider the regular RHP of (3.18), i.e., in their analytic domain. Rewriting Eq.(3.18) as

$$(P^+)^{-1}(\lambda) - P_-^{-1}(\lambda) = \widehat{G}(\lambda)(P^+)^{-1}(\lambda), \qquad \lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}, \tag{4.1}$$

where

$$\widehat{G} = I - G = -E\begin{pmatrix} 0 & \hat{s}_{12} \\ s_{21} & 0 \end{pmatrix}E^{-1}. \tag{4.2}$$

On the basis of Plemelj formula, the formal solution of this problem can be given

$$\left(P^+\right)^{-1}(\lambda) = I + \frac{1}{2\pi i}\int_T \frac{\widehat{G}(\xi)\left(P^+\right)^{-1}(\xi)}{\xi - \lambda}d\xi, \qquad \lambda \in D_+, \tag{4.3}$$

and $T = (-i\infty, 0] \cup (i\infty, 0] \cup [0, -\infty) \cup [0, \infty)$.

Under the canonical normalization condition (3.19), the solution to this regular RHP is unique. Assume that (3.18) has two solutions $P_\pm$ and $\tilde{P}_\pm$. So there is

$$P_-^{-1}(\lambda)P_+(\lambda) = \tilde{P}_-^{-1}(\lambda)\tilde{P}_+(\lambda),$$

and thus

$$\tilde{P}_-(\lambda)P_-^{-1}(\lambda) = \tilde{P}_+(\lambda)P_+^{-1}(\lambda), \qquad \lambda \in \mathbb{R} \cup i\mathbb{R}. \tag{4.4}$$

Because of $\tilde{P}_+(\lambda)P_+^{-1}(\lambda)$ is analytic in D_+, $\tilde{P}_-(\lambda)P_-^{-1}(\lambda)$ is analytic in D_-, and they are equal in $\mathbb{R} \cup i\mathbb{R}$. So they define a matrix function together, which is resolves on the entire plane of λ. With the boundary condition (3.19), by applying Liouville's theorem, we obtain

$$\tilde{P}_-^{-1}(\lambda)P_-(\lambda) = \tilde{P}_+(\lambda)P_+^{-1}(\lambda) = I \tag{4.5}$$

for all λ. In other words, $\tilde{P}_\pm = P_\pm$, which means that the solution of RHP (3.18) is unique.

4.2 Solution to the Nonregular RHP

In the more general case, the RHP (3.18) is not regular, i.e., $\det P_+(\lambda) = s_{11}(\lambda) = 0$ and $\det P_-^{-1}(\lambda) = \hat{s}_{11}(\lambda) = 0$ at certain discrete locations. For the sake of research the non-regular RHP, we need to consider the symmetry of these zeros. Note that $s_{11}(\lambda)$ and $\hat{s}_{11}(\lambda)$ are independent of t, so the roots of $s_{11}(\lambda)$ and $\hat{s}_{11}(\lambda)$ are also independent of t.

The Hermitian of the spectral Eq.(3.3) is

$$(J^{\mathcal{H}})_x = -i\lambda^2\left[\sigma_3, J^{\mathcal{H}}\right] - \lambda J^{\mathcal{H}}Q + \frac{i}{2}J^{\mathcal{H}}Q^2\sigma_3, \tag{4.6}$$

where $Q^{\mathcal{H}} = -Q$ is considered. It is easy to find that $J^{\mathcal{H}}(x, \lambda^*)$ also satisfies the adjoint scattering Eq.(3.14), so $J^{\mathcal{H}}(x, \lambda^*)$ and $J^{-1}(x, \lambda)$ are definitely linearly related. Looking back at the boundary conditions of the Jost solution J, we further see that $J^{\mathcal{H}}(x, \lambda^*)$ and $J^{-1}(x, \lambda)$ possess the same boundary conditions as $x \to \pm\infty$, and therefore they have to be equal,

$$J^{-1}(x, \lambda) = J^{\mathcal{H}}(x, \lambda^*), \tag{4.7}$$

so there is

$$P_-^{-1}(\lambda) = \left(P_+\right)^{\mathcal{H}}(\lambda^*). \tag{4.8}$$

Besides, reviewing the scattering relation (3.7) between J_+ and J_-, it is easy to know that the involution property of $S(\lambda)$ satisfies is

$$S^{-1}(\lambda) = S^{\mathcal{H}}(\lambda^*). \tag{4.9}$$

Moreover , from the symmetric property $\sigma_3 Q \sigma_3 = -Q$ and $\sigma_3 Q^2 \sigma_3 = Q$, we conclude that

$$J(\lambda) = \sigma_3 J(-\lambda)\sigma_3. \tag{4.10}$$

It follows that

$$P_\pm(-\lambda) = \sigma_3 P_\pm(\lambda)\sigma_3, \tag{4.11}$$

and

$$S(-\lambda) = \sigma_3 S(\lambda)\sigma_3. \tag{4.12}$$

From the (4.9) and (4.12), we obtain the relations

$$\hat{s}_{11}(\lambda) = s_{11}^*(\lambda^*), \quad \hat{s}_{12}(\lambda) = s_{21}^*(\lambda^*), \quad \hat{s}_{21}(\lambda) = s_{12}^*(\lambda^*), \qquad \lambda \in \mathbb{R}\cup i\mathbb{R}, \tag{4.13}$$

and

$$s_{11}(\lambda) = s_{11}(-\lambda), \quad s_{12}(-\lambda) = -s_{12}(\lambda), \quad s_{21}(-\lambda) = -s_{21}(\lambda), \quad s_{22}(\lambda) = s_{22}(-\lambda). \tag{4.14}$$

Thus $s_{11}(\lambda)$ is an even function, and each zero λ_k of s_{11} is accompanied with zero $-\lambda_k$. Similarly, $\hat{s}_{11}(\lambda)$ has two zeros $\pm\bar{\lambda}_k$.

Here we first consider the case of simple zeros $\{\pm\lambda_k \in D_+,\ 1 \le k \le N\}$ and $\{\pm\bar{\lambda}_k \in D_-,\ 1 \le k \le N\}$, where N is the number of these zeros. According to the involution property (4.13), the following involution relation is obtained

$$\bar{\lambda}_k = \lambda_k^*. \tag{4.15}$$

It follows that symmetry relation (4.13) and (4.14), in this case, the kernels of ker $\big(P_+\big(\pm\lambda_k\big)\big)$ and $\ker(P_-^{-1}\big(\pm\bar{\lambda}_k\big))$ contain only a single column vector $\big|v_k\big\rangle$ and row vector $\big\langle v_k\big|$,

$$P_+\big(\lambda_k\big)\big|v_k\big\rangle = 0, \qquad \big\langle v_k\big|P_-^{-1}\big(\bar{\lambda}_k\big) = 0, \qquad 1 \le k \le N. \tag{4.16}$$

On the basis of the symmetry relation (4.8), which can lead to

$$\big|v_k\big\rangle = \big\langle v_k\big|^{\mathcal{H}}. \tag{4.17}$$

Next, we discuss the evolution of kernel function with time and space, the first equation of differential (4.16) with respect to both sides of x and t, and reviewing the Lax (3.3)–(3.4), we get

$$P_+(\lambda_k;x)\Big(\frac{d|v_k\rangle}{dx} + i\lambda^2\sigma_3|v_k\rangle\Big) = 0, \qquad P_+(\lambda_k;x)\Big(\frac{d|v_k\rangle}{dt} + 2i\lambda^6\sigma_3|v_k\rangle\Big) = 0.$$

It concludes that

$$\big|v_k\big\rangle = e^{-i\lambda_k^2\sigma_3 x - 2i\lambda_k^6\sigma_3 t}\big|v_{k0}\big\rangle,$$

where $v_{k0} = v_k\big|_{x=0}$.

On the basis of the above analysis, we obtain the following theorem for the solution of non-regular RHP under normalization condition (3.19).

Theorem 4.1. *The solution to the nonregular RHP (3.18) with simple zeros with the canonical normalized condition (3.13) and (3.17) is*

$$P_+ = \hat{P}_+\Gamma, \qquad P_-^{-1} = \Gamma^{-1}\hat{P}_-^{-1}, \tag{4.18}$$

where

$$\Gamma(\lambda) = \Gamma_N(\lambda)\Gamma_{N-1}(\lambda)\cdots\Gamma_1(\lambda), \qquad \Gamma^{-1}(\lambda) = \Gamma_1^{-1}(\lambda)\Gamma_2^{-1}(\lambda)\cdots\Gamma_N(\lambda),$$

$$\Gamma_k(\lambda) = I + \frac{A_k}{\lambda-\lambda_k^*} - \frac{\sigma_3 A_k \sigma_3}{\lambda+\lambda_k^*}, \tag{4.19}$$

$$\Gamma_k^{-1}(\lambda) = I + \frac{A_k^{\mathcal{H}}}{\lambda-\lambda_k} - \frac{\sigma_3 A_k^{\mathcal{H}} \sigma_3}{\lambda+\lambda_k}, \qquad k = 1, 2, \ldots, N, \tag{4.20}$$

$$A_k = \frac{\lambda_k^{*2}-\lambda_k^2}{2}\begin{pmatrix} \alpha_k^* & 0 \\ 0 & \alpha_k \end{pmatrix}\big|w_k\big\rangle\big\langle w_k\big|, \tag{4.21}$$

$$\alpha_k^{-1} = \langle w_k|\begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k^* \end{pmatrix}|w_k\rangle, \tag{4.22}$$

and

$$\det\Gamma_k(\lambda) = \frac{\lambda^2-\lambda_k^2}{\lambda^2-\lambda_k^{*2}}, \qquad \big|w_k\big\rangle = \Gamma_{k-1}(\lambda_k)\cdots\Gamma_1(\lambda_k)\big|v_k\big\rangle, \qquad \big\langle w_k\big| = \big|w_k\big\rangle^{\mathcal{H}}. \tag{4.23}$$

Therefore, $\Gamma(x,t,\lambda)$ *and* $\Gamma^{-1}(x,t,\lambda)$ *collect all zero of the RHP, and* $\hat{P}_\pm^{-1}$*is the only solution to the following regular RHP:*

$$\hat{P}_-^{-1}(\lambda)\hat{P}_+(\lambda) = \Gamma(\lambda)G(\lambda)\Gamma^{-1}(\lambda), \qquad \lambda \in \mathbb{R}\cup i\mathbb{R}, \tag{4.24}$$

and the boundary condition $\hat{P}_\pm \to I$ *as* $\lambda\to\infty$*, where* $\hat{P}_\pm$ *are analytic in* $D_\pm$ *respectively.*

5 The Inverse Problem

The ultimate purpose of inverse scattering is to obtain the potential u. Based on (3.11), substitute it into the Eq.(2.16) and compare the coefficients of the λ to get

$$Q = i[\sigma_3, P_+^{(1)}], \tag{5.1}$$

from this formula, we can get the potential

$$u = 2i(P_+^{(1)})_{12}. \tag{5.2}$$

As we all know that the soliton solutions correspond to the disappearance of scattering coefficients, $G = I, \hat{G} = 0$. Thus, we intend to solve the corresponding RHP(4.24). In the inverse scattering transform method, the product representations $\Gamma(\lambda)$ and $\Gamma^{-1}(\lambda)$ is not convenient for subsequent calculations ,so it needs to be simplified the expression of $\Gamma(\lambda)$ and its inverse, in fact,

$$\Gamma(\lambda) = I + \sum_{j=1}^{N}\Big[\frac{B_j}{\lambda-\lambda_j^*} - \frac{\sigma_3 B_j \sigma_3}{\lambda+\lambda_j^*}\Big], \tag{5.3}$$

and

$$\Gamma^{-1}(\lambda) = I + \sum_{j=1}^{N}\Big[\frac{B_j^{\mathcal{H}}}{\lambda-\lambda_j} - \frac{\sigma_3 B_j^{\mathcal{H}}\sigma_3}{\lambda+\lambda_j}\Big]$$

with $B_j = \big|z_j\big\rangle\big\langle v_j\big|$. To determine the form of matrix B_j, we take advantage of the equation $\Gamma(\lambda)\Gamma(\lambda)^{-1} = I$. Considering the residue condition at λ_j, $\mathrm{Res}_{\lambda=\lambda_j}\Gamma(\lambda)\Gamma^{-1}(\lambda) = \Gamma(\lambda_j)B_j^{\mathcal{H}} = 0$, and it yields

$$\Big[I + \sum_{k=1}^{N}\Big(\frac{\big|z_k\big\rangle\big\langle v_k\big|}{\lambda_j-\lambda_k^*} - \frac{\sigma_3\big|z_k\big\rangle\big\langle v_k\big|\sigma_3}{\lambda_j+\lambda_k^*}\Big)\Big]\big|v_j\big\rangle = 0, \qquad j=1,2,\cdots N, \tag{5.4}$$

it's easy to figure out

$$\big|z_k\big\rangle_1 = \sum_{j=1}^{N}(M^{-1})_{jk}\big|v_j\big\rangle_1, \tag{5.5}$$

where $\big|z_k\big\rangle_l$ denotes the $l-$th element of $\big|z_k\big\rangle$, matrix M is defined as

$$(M)_{jk} = \frac{\big\langle v_k\big|\sigma_3\big|v_j\big\rangle}{\lambda_j+\lambda_k^*} - \frac{\big\langle v_k \mid v_j\big\rangle}{\lambda_j-\lambda_k^*}. \tag{5.6}$$

Guided by these equations, we can get

$$P_+^{(1)} = \sum_{j=1}^{N}(B_j - \sigma_3 B_j\sigma_3),$$

by Eq.(5.1), we can obtain that the potential function u is

$$u = 2i\Big[\sum_{j=1}^{N}(B_j - \sigma_3 B_j \sigma_3)_{12}\Big], \tag{5.7}$$

and substituting above expressions of $\big|z_k\big\rangle_l$ and $\big|v_j\big\rangle_l$ into Eq.(5.7) to obtain

$$u = -4i\frac{\det F}{\det M}, \tag{5.8}$$

where M defined as (5.6), and

$$F = \begin{pmatrix} M_{11} & \cdots & M_{1N} & \big|v_1\big\rangle_1 \\ \vdots & \ddots & \vdots & \vdots \\ M_{N1} & \cdots & M_{NN} & \big|v_N\big\rangle_1 \\ \big\langle v_1\big|_2 & \cdots & \big\langle v_N\big|_2 & 0 \end{pmatrix}.$$

According to the dressing method [21], it can be directly verified that (5.8) satisfies the third-order flow GI equation.

Next, our major work is to obtain the soliton solutions of the third-order flow GI equation. For the sake of obtain an explicit N-soliton solutions, we can

$$\big|v_k\big\rangle = \begin{pmatrix} c_k e^{\theta_k} \\ e^{-\theta_k} \end{pmatrix}, \qquad \big\langle v_k\big| = (c_k^* e^{\theta_k^*} \ \ e^{-\theta_k^*}),$$

where $\theta_k = -i\lambda_k^2 x - 2i\lambda_k^6 t$. Let $\lambda_j = \xi_j + i\eta_j$, then $z_j = 2m_j(x-(8m_j^2-6\beta_j^2)t)$, $\phi_j = -\beta_j x - 2(\beta_j^3 - 12m_j^2\beta_j)t$, $m_j = \xi_j\eta_j$, $\beta_j = \xi_j^2 - \eta_j^2$, where z_j, ϕ_j are the real and imaginary parts of θ_j. Next, we will study the properties of soliton solutions in more detail by taking single-soliton and two-soliton solutions as examples.

5.1 Single-soliton Solution

Taking $N = 1$ in formula (5.8), the single soliton solution is

$$u(x,t) = -2i(\lambda_1^2 - \lambda_1^{*2}) \frac{c_1 e^{\theta_1 - \theta_1^*}}{\lambda_1 e^{-(\theta_1 + \theta_1^*)} + \lambda_1^* |c_1|^2 e^{\theta_1 + \theta_1^*}}. \tag{5.9}$$

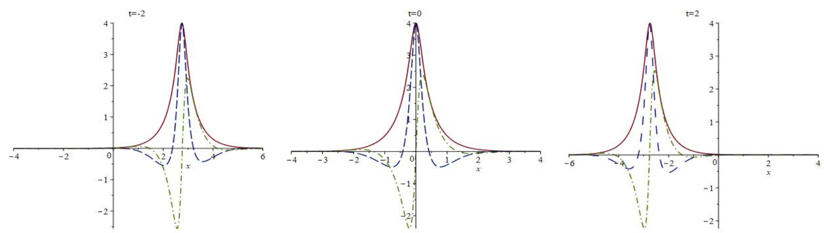
Therefore, it is easy to receive the velocity of the single soliton solution is $v_1 = 8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$, and its behavior occurring along the line

$$x - v_1 t + \frac{1}{4m_1} \ln |c_1| = 0.$$

Let $c_1 = 1$, the intensity profile for $|u|^2$ are given by

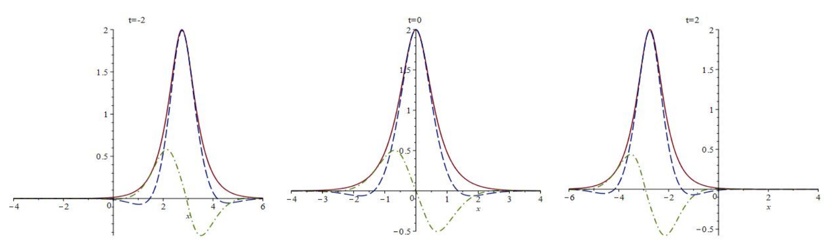
$$|u(x,t)|^2 = \frac{64\xi_1^2\eta_1^2}{2|\lambda_1|^2 \cos h(4z_1) + \lambda_1^2 + \lambda_1^{*2}}.$$

More, $\xi_1\eta_1 > 0$ if $\lambda_1 \in D_+$. By selecting the same parameters as in Ref.[20], in the subregion $\xi_1 > \eta_1$ and $\xi_1 < \eta_1$ of D_+, the single-soliton is a left traveling wave (see Figure (5.1) and Figure (5.2)). On the line $\xi_1 = \eta_1$, the single-soliton is a right traveling wave (see Figure (5.3)).When $\xi_1 = \pm\sqrt{3}\eta_1$ or $\xi_1 = \pm\frac{\sqrt{3}}{3}\eta_1$, it is a stable wave(see Figure (5.4)).



(a) (b) (c)

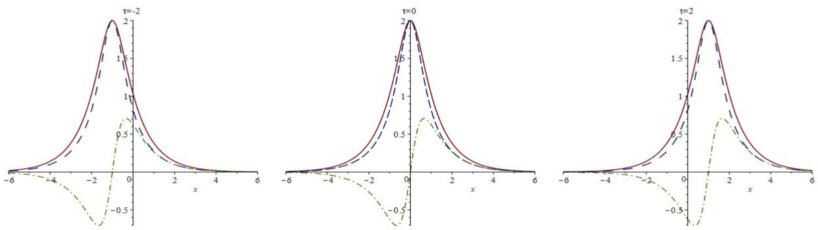
Figure 5.1. Single-soliton $u(x,t)$ in (5.9) with the parameters chosen as $\xi_1 = \frac{1}{2}, \eta_1 = 1, c_1 = 1$. Red line absolute value of u, blue line real part of u and green line imaginary of u.



(a) (b) (c)

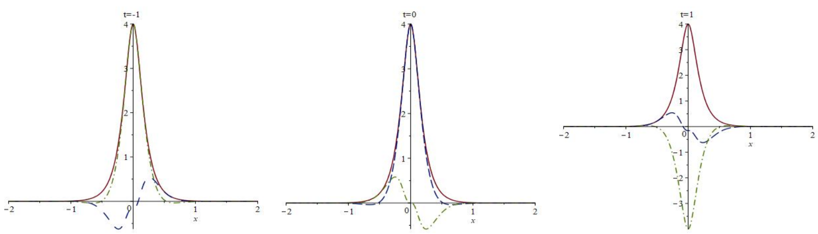
Figure 5.2. Single-soliton solution for $|u|$, where $\xi_1 = 1,\ \eta_1 = \frac{1}{2},\ c_1 = 1$.

Compared with the classical second-order flow GI Eq.(1.1), the position of the stable wave changes. Mainly because the wave propagation speed has changed, the classical GI equation velocity is: $4(\xi_1^2 - \eta_1^2)$, and the velocity of third-order flow GI equation is: $8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$. Besides, the amplitude of the soliton solution of the higher-order GI equation is also affected. Compared with the classical GI, the amplitude of the third-order flow GI equation becomes higher. That is to say, the introduction of third-order dispersion and fifth-order nonlinearity will affect the velocity, position and amplitude of solution.



(a) (b) (c)

Figure 5.3. Single-soliton solution for $|u|$ with $\xi_1 = \frac{1}{2}, \eta_1 = \frac{1}{2}, c_1 = 1$.



(a) (b) (c)

Figure 5.4. Single-soliton solution for $|u|$, with the parameters as $\xi_1 = \sqrt{3}, \eta_1 = 1, c_1 = 1$.

5.2 Two-soliton Solution

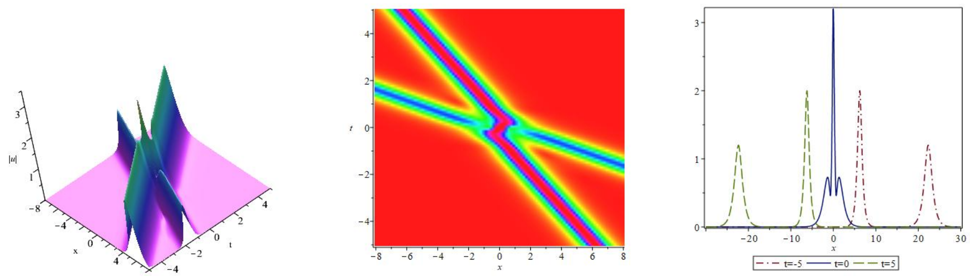
When $N = 2$, the two-soliton solution of the third-order flow GI equation also can be written out as follows

$$u(x,t) = \frac{a_1 e^{\Theta_1^{'} - \Theta_2} + a_2 e^{\Theta_1^{'} + \Theta_2} + a_3 e^{-\Theta_1 + \Theta_2^{'}} + a_4 e^{\Theta_1 + \Theta_2^{'}}}{b_1 e^{-\Theta_1 - \Theta_2} + b_2 e^{\Theta_1 + \Theta_2} + b_3 e^{\Theta_1^{'} - \Theta_2^{'}} + b_4 e^{-\Theta_1^{'} + \Theta_2^{'}} + b_5 e^{\Theta_1 - \Theta_2} + b_6 e^{-\Theta_1 + \Theta_2}}, \tag{5.10}$$

where

$$\begin{aligned}
&\Theta_1 = \theta_1 + \theta_1^*;\ \Theta_1^{'} = \theta_1 - \theta_1^*;\ \Theta_2 = \theta_2 + \theta_2^*;\ \Theta_2^{'} = \theta_2 - \theta_2^*;\\
&a_1 = c_1 \lambda_2 (\lambda_1^2 - \lambda_1^{*2})(\lambda_1^2 - \lambda_2^{*2})(\lambda_2^{*2} - \lambda_1^{*2});\\
&a_2 = c_1 |c_2|^2 \lambda_2^* (\lambda_1^2 - \lambda_1^{*2})(\lambda_2^2 - \lambda_1^{*2})(\lambda_1^2 - \lambda_2^2);\\
&a_3 = c_2 \lambda_1 (\lambda_2^2 - \lambda_2^{*2})(\lambda_2^2 - \lambda_1^{*2})(\lambda_1^{*2} - \lambda_2^{*2});\\
&a_4 = |c_1|^2 c_2 \lambda_1^* (\lambda_2^2 - \lambda_2^{*2})(\lambda_1^2 - \lambda_2^{*2})(\lambda_2^2 - \lambda_1^2);\\
&b_1 = 2\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)(\lambda_2^{*2} - \lambda_1^{*2});\\
&b_2 = 2|c_1|^2 |c_2|^2 \lambda_1^* \lambda_2^* (\lambda_1^2 - \lambda_2^2)(\lambda_2^{*2} - \lambda_1^{*2});\\
&b_3 = -2c_1 c_2^* |\lambda_2|^2 (\lambda_1^2 - \lambda_1^{*2})(\lambda_2^2 - \lambda_2^{*2});\\
&b_4 = -2c_1^* c_2 |\lambda_1|^2 (\lambda_1^2 - \lambda_1^{*2})(\lambda_2^2 - \lambda_2^{*2});\\
&b_5 = 2|c_1|^2 \lambda_1^* \lambda_2 (\lambda_1^2 - \lambda_2^{*2})(\lambda_2^2 - \lambda_1^{*2});\\
&b_6 = 2|c_2|^2 \lambda_1 \lambda_2^* (\lambda_1^2 - \lambda_2^{*2})(\lambda_2^2 - \lambda_1^{*2}).
\end{aligned}$$

We show the two-soliton solution behaviors in Figure (5.5) with $\lambda_1 = 1 + \frac{3}{10}i, c_1 = 1, \lambda_2 = 1 + \frac{1}{2}i, c_2 = 1$. From Figure (5.5)(a) we can see that the solution consists of two single solitons that are far apart from each other and moving toward each other as $t \to -\infty$. When they collide, the interaction gets stronger. But when $t \to +\infty$, these solitons reappear from the interaction without any change in shape or speed, and no energy radiates into the far field. Thus the interaction of two single solitons is elastic. In effect, each soliton gains a position shift and a phase shift after the interaction. The position of each soliton always moves forward, as if the soliton accelerates during interactions.



(a) (b) (c)

Figure 5.5. (Color online) Two soliton solutions for $|u|$, (a) Three dimensional plot; (b) The density plot; (c) The plot for the two-soliton solutions evolution. where $\lambda_1 = 1 + \frac{3}{10}i, c_1 = 1, \lambda_2 = -1 + \frac{1}{2}i, c_2 = 1$.

To better illustrate this fact, we analyze the asymptotic states of the solution (5.8) as $t \to \pm\infty$. On the premise of generality, suppose $\xi_i\eta_i > 0$ and $v_1 < v_2$. This means that soliton-1 is located on the right side of soliton-2 and moves slowly at $t \to -\infty$, , and the two solitons are in the moving frame with velocity $v_i = 8\xi_i^2\eta_i^2 - 6(\xi_i^2 - \eta_i^2)^2$. Note that $z_1 = 2m_1(x - v_1t), z_2 = 2m_2(x - v_2t)$, it yields

$$m_2z_1 - m_1z_2 = 2m_1m_2(v_2 - v_1)t.$$

When $t \to -\infty,\ |z_1| < \infty, z_2 \to +\infty$. In this case, the asymptotic state of the solution Eq.(5.10) is

$$u(x,t) \to -2i(\lambda_1^2 - \lambda_1^{*2})\frac{c_1^- e^{\theta_1 - \theta_1^*}}{\lambda_1 e^{-(\theta_1+\theta_1^*)} + \lambda_1^*|c_1^-|^2 e^{\theta_1+\theta_1^*}}, \qquad t \to -\infty,$$

where $c_1^- = c_1\frac{(\lambda_1^2-\lambda_2^2)}{(\lambda_1^2-\lambda_2^{*2})}$. Comparing this expression with Eq.(5.9), we find that this asymptotic solution is a single-soliton solution with velocity $8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$ and the intensity profile of $|u|^2$ is $\frac{64\xi_1^2\eta_1^2}{2|\lambda_1|^2 \cos h(4z_1)+\lambda_1^2+\lambda_1^{*2}}$.

When $t \to +\infty,\ |z_1| < \infty,\ z_2 \to -\infty$. In this case, the asymptotic state is

$$u(x,t) \to -2i(\lambda_1^2 - \lambda_1^{*2})\frac{c_1^+ e^{\theta_1 - \theta_1^*}}{\lambda_1 e^{-(\theta_1+\theta_1^*)} + \lambda_1^*|c_1^+|^2 e^{\theta_1+\theta_1^*}}, \qquad t \to +\infty,$$

where $c_1^+ = c_1\frac{(\lambda_1^2-\lambda_2^{*2})}{(\lambda_1^2-\lambda_2^2)}$, which is also a single-soliton solution with velocity $8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$ and peak amplitude $\frac{64\xi_1^2\eta_1^2}{2|\lambda_1^2|+\lambda_1^2+\lambda_1^{*2}}$. This shows that the shape and velocity of the soliton will not change after collision. However, its phase and position have changed, as shown in Figure (5.5)(b). Here we give the specific offset of the position

$$\Delta x_{01} = -\frac{1}{8\xi_1\eta_1}\big(\ln|c_1^+| - \ln|c_1^-|\big) = \frac{1}{4\xi_1\eta_1}\ln\Big|\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^{*2}}\Big|,$$

and the phase shift is

$$\Delta\sigma_{01} = \arg\left(c_1^+\right) - \arg\left(c_1^-\right) = -2\arg\Big(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2-\lambda_2^{*2}}\Big).$$

After the similar calculation, we find that the asymptotic solutions of soliton-2 are both single-soliton with the same velocity and intensity profile as $t\to\pm\infty$, the relation of soliton constant $c_2^\pm$ before and after collision is as follows

$$c_2^+ = c_2^- \frac{(\lambda_1^2-\lambda_2^2)^2}{(\lambda_1^{*2}-\lambda_2^2)^2}.$$

Therefore, after the collision, the position shift of the soliton-2 is

$$\Delta x_{02} = -\frac{1}{8\xi_2\eta_2}\big(\ln|c_2^+| - \ln|c_2^-|\big) = -\frac{1}{4\xi_2\eta_2}\ln\Big|\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^{*2}-\lambda_2^2}\Big|,$$

and a phase shift

$$\Delta\sigma_{02} = \arg(c_2^+) - \arg(c_2^-) = 2\arg\Big(\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^{*2}-\lambda_2^2}\Big).$$

6 Soliton Matrix for High-order Zeros

In this section, we will further consider the higher-order zeros in the RHP of the third-order flow GI equation. We suppose that $\det P_+(\lambda)$ has high-order zeros $\{\pm\lambda_j\}_{j=1}^N$, according to the symmetries (4.2) and (4.14), $\{\pm\lambda_j^*\}_{j=1}^N$ are high-order zeros of $\det P_-^{-1}(\lambda)$. Therefore, $\det P_+(\lambda)$ and $\det P_-^{-1}(\lambda)$ can be expanded as follows:

$$\begin{aligned}\det P_+(\lambda) &= s_{11}(\lambda) = \big(\lambda^2-\lambda_1^2\big)^{n_1}\big(\lambda^2-\lambda_2^2\big)^{n_2}\cdots\big(\lambda^2-\lambda_N^2\big)^{n_N}s_0(\lambda),\\ \det P_-^{-1}(\lambda) &= \hat{s}_{11}(\lambda) = \big(\lambda^2-\lambda_1^{*2}\big)^{n_1}\big(\lambda^2-\lambda_2^{*2}\big)^{n_2}\cdots\big(\lambda^2-\lambda_N^{*2}\big)^{n_N}\hat{s}_0(\lambda),\end{aligned}$$

where $s_0(\lambda)\neq 0$ for all $\lambda\in D_+$, and $\hat{s}_0(\lambda)\neq 0$ for all $\lambda\in D_-$.

First of all, we let functions $P_+(\lambda)$ and $P_-^{-1}(\lambda)$ from above RHP have only one pair of zero of order n_1, i.e. $\{\lambda_1,-\lambda_1\}$ and $\{\lambda_1^*,-\lambda_1^*\}$. Hence, it is necessary to construct a matrix $\Gamma(\lambda)$ whose determinant is $\frac{(\lambda^2-\lambda_1^2)^{n_1}}{(\lambda^2-\lambda_1^{*2})^{n_1}}$. For multiple zeros, its kernel vector will no longer be one. The geometric multiplicity of $\pm\lambda_i(\pm\lambda_i^*)$ is defined as the number of the null vectors in the kernel of $\det P_+(\,\det P_-^{-1})$. It is easy to prove that the order of zero is always greater than or equal to its geometric multiplicity.

For an elementary high-order zero, we derive a matrix $\Gamma(\lambda)$ and its inverse, the results are presented in the lemma below.

Lemma 6.1 ([29], Lemma 1)**.** *Consider a pair of elementary high-order zeros of order n: $\{\lambda_1,-\lambda_1\}$ in D_+ and $\{\lambda_1^*,-\lambda_1^*\}$ in D_-. The corresponding soliton matrix $\Gamma(\lambda)$ and its inverse can be constructed in the following form*

$$\begin{aligned}\Gamma^{-1}(\lambda) &= I + \big(|p_1\rangle,\cdots,|\tilde{p}_n\rangle\big)\mathcal{D}(\lambda)\begin{pmatrix}\langle q_n|\\ \vdots\\ \langle\tilde{q}_1|\end{pmatrix},\\ \Gamma(\lambda) &= I + \big(|\bar{q}_n\rangle,\cdots,|\bar{\tilde{q}}_1\rangle\big)\overline{\mathcal{D}}(\lambda)\begin{pmatrix}\langle\bar{p}_1|\\ \vdots\\ \langle\bar{\tilde{p}}_n|\end{pmatrix},\end{aligned} \tag{6.1}$$

where the matrices $\mathcal{D}(\lambda)$ *and* $\overline{\mathcal{D}}(\lambda)$ *are defined as*

$$\mathcal{D}(\lambda)=\begin{pmatrix}\mathcal{K}^{+}\big(\lambda-\lambda_1\big) & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathcal{K}^{+}\big(\lambda+\lambda_1\big)\end{pmatrix},\qquad \overline{\mathcal{D}}(\lambda)=\begin{pmatrix}\mathcal{K}^{-}\big(\lambda-\lambda_1^*\big) & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathcal{K}^{-}\big(\lambda+\lambda_1^*\big)\end{pmatrix},$$

$\mathcal{K}^{+}(f),\mathcal{K}^{-}(f)$ *are upper triangular and lower triangular Toeplitz matrices defined as:*

$$\mathcal{K}^{+}(f)=\begin{pmatrix} f^{-1} & f^{-2} & \cdots & f^{-n}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & f^{-1} & f^{-2}\\ 0 & \cdots & 0 & f^{-1}\end{pmatrix},\qquad \mathcal{K}^{-}(f)=\begin{pmatrix} f^{-1} & 0 & \cdots & 0\\ f^{-2} & f^{-1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ f^{-n} & \cdots & f^{-2} & f^{-1}\end{pmatrix},$$

and vectors $|p_j\rangle,|\tilde{p}_j\rangle,\langle\bar{p}_j|,\langle q_j|,|\bar{q}_j\rangle,|\bar{\tilde{q}}_j\rangle(j=1,\cdots,n)$ *are independent of* λ.

The rest of the vector parameters in (6.1) can be derived to calculate the poles of each order in the identity $\Gamma(\lambda)\Gamma^{-1}(\lambda)=I$ at $\lambda=\lambda_1$ and $\lambda=-\lambda_1$,

$$\Gamma(\lambda_1)\begin{pmatrix}|p_1\rangle\\ \vdots\\ |p_n\rangle\end{pmatrix}=0,\qquad \Gamma(-\lambda_1)\begin{pmatrix}|\tilde{p}_1\rangle\\ \vdots\\ |\tilde{p}_n\rangle\end{pmatrix}=0,$$

where

$$\Gamma(\lambda)=\begin{pmatrix}\Gamma(\lambda) & 0 & \cdots & 0\\ \frac{d}{d\lambda}\Gamma(\lambda) & \Gamma(\lambda) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ \frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}\Gamma(k) & \cdots & \frac{d}{d\lambda}\Gamma(\lambda) & \Gamma(\lambda)\end{pmatrix}.$$

Hence, on the part of the independent vector parameters, the results (6.1) can be expressed in a more compact form as described in Ref.[24], and we are omitting these overlaps here. Using this method, the process of solving soliton solution is very complex. In the following, we derive dressing matrix of higher-order poles via the method of unipolar point limit. The specific results are given by the following theorem.

Lemma 6.2. *For one pair of elementary high-order zero, the dressing matrix of the third-order flow GI equation can be expressed as:*

$$\Gamma=\Gamma_1^{[n-1]}\cdots\Gamma_1^{[0]},\qquad \Gamma^{-1}=\Gamma_1^{[0]-1}\cdots\Gamma_1^{[n-1]-1},$$

where

$$\Gamma_1^{[j]}=I+\frac{A_1^{[j]}}{\lambda-\lambda_1^*}-\frac{\sigma_3A_1^{[j]}\sigma_3}{\lambda+\lambda_1^*},$$
$$\Gamma_1^{[j]-1}=I+\frac{A_1^{\mathcal{H}[j]}}{\lambda-\lambda_1}-\frac{\sigma_3A_1^{\mathcal{H}[j]}\sigma_3}{\lambda+\lambda_1},$$
$$A_1^{[j]}=\frac{\lambda_1^2-\lambda_1^{*2}}{2}\begin{pmatrix}\alpha_1^{[j]} & 0\\ 0 & \alpha_1^{*[j]}\end{pmatrix}|v_1^{[j]}\rangle\langle v_1^{[j]}|,$$

$$(\alpha_1^{[j]})^{-1} = \langle v_1^{[j]}| \begin{pmatrix} \lambda_1^* & 0 \\ 0 & \lambda_1 \end{pmatrix} |v_1^{[j]}\rangle,$$

and

$$|v_1^{[j]}\rangle = \lim_{\delta\to 0} \frac{(\Gamma_1^{[j-1]}\cdots\Gamma_1^{[0]})|_{\lambda=\lambda_1+\delta}}{\delta^j}|v_1\rangle(\lambda_1+\delta),$$
$$\langle v_1^{[j]}| = \lim_{\delta\to 0}\langle v_1|(\lambda_1^*+\delta)\frac{(\Gamma_1^{[0]-1}\cdots\Gamma_1^{[j-1]-1}|)_{\lambda=\lambda_1^*+\delta}}{\delta^j}.$$

Then by techniques similar to those used above, we can get

$$u = 2i\Big(\sum_{j=0}^{n-1}[B_1^{[j]} - \sigma_3 B_1^{[j]}\sigma_3]_{12}\Big).$$

As before, the above formulas also could be rewritten with the determinant form

$$u = -4i\frac{\det\hat{F}}{\det\hat{M}}, \tag{6.2}$$

where

$$\hat{F} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{12} & \cdots & \hat{M}_{1n} & |v_1\rangle_1^{[0]} \\ \hat{M}_{21} & \hat{M}_{22} & \cdots & \hat{M}_{2n} & |v_1\rangle_1^{[1]} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{M}_{n1} & \hat{M}_{n2} & \cdots & \hat{M}_{nn} & |v_1\rangle_1^{[n-1]} \\ \langle v_1|_2^{[0]} & \langle v_1|_2^{[1]} & \cdots & \langle v_1|_2^{[n-1]} & 0 \end{pmatrix},$$

and

$$\hat{M}_{jk} = \frac{1}{(j-1)!(k-1)!}\frac{\partial^{j+k-2}}{\partial\lambda^{*j-1}\partial\lambda^{k-1}}\frac{\langle v_1 \mid v_1\rangle}{\lambda-\lambda^*} - \frac{\langle v_1|\sigma_3|v_1\rangle}{\lambda+\lambda^*}|_{\lambda=\lambda_1,\lambda^*=\lambda_1^*}.$$

Where $|v_1\rangle^{[l]}, \langle v_1|^{[l]}$ can be written as follows

$$|v_1\rangle^{[l]} = \frac{1}{(l)!}\frac{\partial^l}{\partial(\lambda)^l}|v_1\rangle|_{\lambda=\lambda_1}, \qquad \langle v_1|^{[l]} = \frac{1}{(l)!}\frac{\partial^j}{\partial(\lambda)^l}\langle v_1|_{\lambda=\lambda_1^*}. \tag{6.3}$$

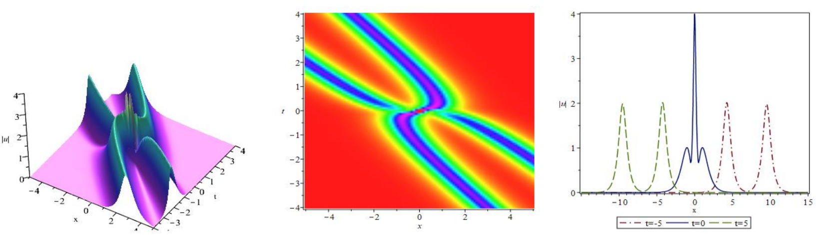
Thus, the elementary higher-order zeros solution formula is derived from Eq.(6.2). When $N=1$, it corresponds to a single soliton solution, and when $N\geq 2$, it corresponds to higher-order soliton. It should be noted that that the general expression of the high-order soliton solution of Eq.(6.2) is very complicated and is not explicitly given.

Here, we give the simplest form of higher-order 1-soliton solution with reference to Ref.[34]. Explicitly, taking $N=2, \lambda=\xi+i\eta$ and $c_1=1$ in Eq.(6.2), and the form of the corresponding solution is

$$u(x,t) = \frac{u_n}{u_d}\exp(60it\eta^2\xi^4 - 60it\eta^4\xi^2 - 4it\xi^6 + 4it\eta^6 + 2ix\eta^2 - 2ix\xi^2), \tag{6.4}$$

where

$$\begin{aligned}
u_n &= h_1 e^{\vartheta} + g_1 e^{-\vartheta};\\
u_d &= h_2 e^{2\vartheta} + g_2 e^{-2\vartheta} + \varpi;\\
\vartheta &= 4x\eta\xi + 24t\eta^5\xi - 80t\eta^3\xi^3 + 24t\eta\xi^5;\\
h_1 &= -64i\eta\xi(i\xi-\eta)(144i\eta^6t\xi^2 - 480i\eta^4t\xi^4 + 144i\eta^2t\xi^6 + 24\eta^7t\xi - 360\eta^5t\xi^3\\
&\qquad + 360\eta^3t\xi^5 - 24\eta t\xi^7 + 8i\eta^2x\xi^2 + 4\eta^3x\xi - 4\eta x\xi^3 - i\eta\xi - \eta^2 + \xi^2);\\
g_1 &= -64i\eta\xi(i\xi+\eta)(144i\eta^6t\xi^2 - 480i\eta^4t\xi^4 + 144i\eta^2t\xi^6 - 24\eta^7t\xi + 360\eta^5t\xi^3\\
&\qquad - 360\eta^3t\xi^5 + 24\eta t\xi^7 + 8i\eta^2x\xi^2 - 4\eta^3x\xi + 4\eta x\xi^3 + i\eta\xi - \eta^2 + \xi^2);\\
h_2 &= 4(\xi+i\eta)(\xi-i\eta)^3;\\
g_2 &= 4(\xi+i\eta)^3(\xi-i\eta);\\
\varpi &= 8\eta^4 + 8\xi^4 - 768it\xi^8\eta^2 + 11520it\xi^6\eta^4 - 11520it\xi^4\eta^6 + 768it\xi^2\eta^8 - 128ix\xi^4\eta^2\\
&\quad + 128ix\xi^2\eta^4 + 3072t\xi^{10}x\eta^2 - 12288t\xi^8x\eta^4 - 30720t\xi^6x\eta^6 - 12288t\xi^4x^8\\
&\quad + 3072t\xi^2x\eta^{10} + 256\xi^6x^2\eta^2 + 512\xi^4x^2\eta^4 + 256\xi^2x^2\eta^6 + 9216t^2\xi^2\eta^{14} + 9216t^2\xi^{14}\eta^2\\
&\quad + 55296t^2\xi^{12}\eta^4 + 138240t^2\xi^{10}\eta^6 + 184320t^2\xi^8\eta^8 + 138240t^2\xi^6\eta^{10} + 55296t^2\xi^4\eta^{12}.
\end{aligned}$$



(a) (b) (c)

Figure 6.1. high-order 1-soliton solution for $|u|$.(a) Three dimensional plot; (b) The density plot; (c) The plot for the high-order 1-soliton solution evolution. where $\lambda = 1 + \frac{1}{2}i,\ c_1 = 1$.

Further, $\xi = 1$ and $\eta = \frac{1}{2}$ are selected, and the solution is

$$u(x,t) = \frac{d_1 e^{\frac{11}{4}t+2x} + d_2 e^{-\frac{11}{4}t-2x}}{d_3 e^{-\frac{11}{2}t-4x} + d_4 e^{\frac{11}{2}t+4x} + d_5} e^{\frac{117}{16}it - \frac{3}{2}ix},$$

where

$$\begin{aligned}
d_1 &= 570t + 615it + 32 - 4i - 80x + 40ix;\\
d_2 &= -570t + 615it + 32 + 4i + 80x + 40ix;\\
d_3 &= \frac{15}{4} + 5i;\\
d_4 &= \frac{15}{4} - 5i;\\
d_5 &= \frac{140625}{16}t^2 + 351it - 525tx - 24ix + 100x^2 + \frac{17}{2},
\end{aligned}$$

which is plotted in Figure (6.1). In addition, with the help of Maple, Mathlab and other computer software, double-pole solution of different orders can be obtained by using formula (6.2).

7 Conclusion

In summary, the GI hierarchy is derived by using recursive operator. The recursive operator here contains two operators, which is more complex than the form of AKNS hierarchy and KN hierarchy. The main reason is that the derivative of the main diagonal of M to t is not 0, but a function related to the potential functions u and v, which leads to the complex expression of A. Then, the inverse scattering method is applied to the third-order flow GI equation. By considering the related RHP, a simple representation of the determinant form of N-solitons is obtained successfully. Because of the symmetry properties of Jost solution and scattering data, the corresponding zeros in the RHP of higher-order GI equation always appear in pairs, which is the same as the 3×3 Sasa-Satsuma equation[29]. Later, taking single soliton solution and two-soliton solutions as examples, the long-time behavior of the solution is studied. Compared with the classical GI direction of the second-order flow, it is found that the motion direction and wave height of the soliton solution are affected by the third-order dispersion and the fifth-order nonlinearity. These analysis results have important reference value for the study of GI hierarchy or other nonlinear integrable dynamic systems of higher order flow equations, and provide a theoretical basis for possible experimental research and application. Finally, the corresponding higher-order soliton solution matrix is derived by analyzing the limiting behavior of spectral parameters.

In recent years, there are many achievements in the study of the classical second-order flow GI equation with non-zero boundary conditions[22, 31, 35]. This paper only considers the simple zeros and a pair of elementary higher-order zeros of the third-order flow GI equation with vanishing boundary conditions. Whether the behavior of soliton solutions with non-zero boundary and more multiplicity will have more abundant forms and long time behavior can be studied in the future.

Conflict of Interest

The authors declare no conflict of interest.

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