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Nonlocal symmetry and exact solutions of the (2+1)-dimensional breaking soliton equation

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ABSTRACT

The nonlocal symmetry which is obtained from Lax pair and the residual symmetry relating to truncated Painlevé expansion are derived. The link between the residual symmetry and the nonlocal symmetry which is obtained from Lax pair is presented. The residual symmetry can be localized to Lie point symmetry by prolonging the original equation to a larger system. The finite transformation of the residual symmetry is equivalent to the second type of Darboux transformation. Furthermore, applying the standard Lie group approach to the prolonged system, new similarity reductions and the exact interaction solutions between solitons and cnoidal periodic waves are given, which is difficult to be found by other traditional methods.

1. Introduction

As we all know, symmetry analysis plays an important role in nonlinear mathematical physics [1–5]. Thanks to the classical or nonclassical Lie group method [3,4], one can reduce dimensions of differential equations and construct some exact solutions of these differential equations. Nevertheless, for the integrable model, there may exist nonlocal symmetries which are obtained by inverse recursion operators [6–8], Darboux transformation (DT) [9–11], Bäcklund transformation (BT) [12], the Möbious (conformal) invariant form [13], pseudopotential [14], potential system [4,5], negative hierarchies [15,16], the self-consistent sources [17] and so on.

After the derivation of nonlocal symmetry, it is necessary to inquire whether nonlocal symmetries can transformed into local ones. The general localization method was proposed by Krasil'shchik and Vinogradov [18]. Bluman introduced the concept of potential symmetry [5] which possess close prolongation to obtain solutions of differential system. Galas [14] derived the nonlocal Lie–Bäcklund symmetries by introducing the pseudopotentials as an auxiliary system. Recently, Lou et al. [19,20] found that the residual symmetry of the truncated Painlevé expansion is a nonlocal symmetry. The residual symmetry can be localized to find finite transformation and obtain new symmetry reduction solutions.

We consider the (2+1)-dimensional breaking soliton equation [21]

$$v_t + v_{xxy} - 4vv_y - 2v_x\partial_x^{-1}v_y = 0, \tag{1}$$

with $\partial_x^{-1} = \int \cdot dx$. Setting $v = u_x$, Eq. (1) becomes

$$u_{xt} + u_{xxxy} - 4u_xu_{xy} - 2u_{xx}u_y = 0, \tag{2}$$

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which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis [22]. For $y = x$, Eq. (1) is reduced to the KdV equation. The Painlevé property, the Lax pair, the Hamiltonian structure, and various exact solutions have been studied [23–34]. In this paper, the analytic interaction solutions between solitons and cnoidal periodic waves for the (2+1)-dimensional breaking soliton equation are shown by means of the nonlocal symmetry method.

The paper is arranged as follows. In Section 2, for the (2+1)-dimensional breaking equation, we derive the nonlocal symmetry which is obtained from Lax pair and the residual symmetry relating to truncated Painlevé expansion. The link between the residual symmetry and the nonlocal symmetry which is obtained from Lax pair is presented. The residual symmetry is localized to Lie point symmetry by prolonging the original equation to a larger system. For the prolonged system, the finite symmetry transformation is obtained by using Lie's first theorem in Section 3. In Section 4, some new exact solutions are derived via the similarity reductions of the prolonged system. In the last section, some conclusions and discussions are given.

2. Nonlocal symmetry and its localization

Eq. (2) possesses the Lax pair [25]

$$\psi_{xx} - u_x\psi = 0, \tag{3}$$

$$\psi_t - 2u_y\psi_x + u_{xy}\psi - \lambda\psi = 0. \tag{4}$$

A symmetry σ^u of Eq. (2) is defined as a solution of its linearized equation

$$\sigma^u_{xt} + \sigma^u_{xxxy} - 4u_x\sigma^u_{xy} - 2u_y\sigma^u_{xx} - 4u_{xy}\sigma^u_x - 2u_{xx}\sigma^u_y = 0, \tag{5}$$

which means Eq. (2) is form invariant under the infinitesimal transformation

$$u \to u + \epsilon\sigma^u, \tag{6}$$

with the infinitesimal parameter ϵ.

Proposition 1. *Eq. (2) has a nonlocal symmetry given by*

$$\sigma^u = \mathrm{e}^{-2\lambda t}\psi^2, \tag{7}$$

where ψ satisfies the Lax pair (3)–(4).

Proof. By direct calculation.

It is seen that the symmetry (7) is a local symmetry of the system (2)–(3). The linearized equation of the Eq. (3) reads

$$\sigma^\psi_{xx} - \psi\sigma^u_x - u_x\sigma^\psi = 0, \tag{8}$$

with σ^u given by (7).

It is not difficult to verify that the solution of Eq. (8) with (7) has the following form:

$$\sigma^\psi = \frac{1}{2}\psi\phi, \tag{9}$$

where the new quantity ϕ is defined as

$$\phi_x = \mathrm{e}^{-2\lambda t}\psi^2. \tag{10}$$

One compatibility condition of (10) is worth to be mentioned here

$$\phi_t = 2\mathrm{e}^{-2\lambda t}(2\psi_x\psi_y - 2\psi\psi_{xy} + u_y\psi^2), \tag{11}$$

which means the condition $\phi_{xt} = \phi_{tx}$ is satisfied identically. Furthermore, the linearized equation of its symmetry σ^ϕ reads

$$\sigma^\phi_x = 2\mathrm{e}^{-2\lambda t}\psi\sigma^\psi, \tag{12}$$

and a straightforward calculation shows that σ^ϕ has the simple form

$$\sigma^\phi = \frac{1}{2}\phi^2. \tag{13}$$

The results (9) and (13) reveal the nonlocal symmetry (7) in the original space $\{x, y, t, u\}$ has been successfully localized to a Lie point symmetry in the enlarged space $\{x, y, t, u, \psi, \phi\}$.

Here, it should be emphasized that the differential equation that quantity ϕ need to be satisfied is nothing but the Schwartzian form of Eq. (2)

$$C_x + KS_x + 2SK_x + K_{xxx} = 0, \tag{14}$$

where

$$C = \frac{\phi_t}{\phi_x}, \quad K = \frac{\phi_y}{\phi_x}, \quad S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}. \tag{15}$$

The above Schwartzian equation is consistent with the result of [35].

Now, applying the following transformation:

$$\psi = e^{\lambda t}\sqrt{\phi_x}, \tag{16}$$

to Lax pair (3)–(4), we obtain

$$u_x = \frac{1}{2}\frac{\phi_{xxx}}{\phi_x} - \frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}, \tag{17}$$

$$\phi_{xt} + \phi_{xxxy} - \frac{3\phi_{xx}\phi_{xxy}}{\phi_x} - \frac{\phi_{xx}\phi_t}{\phi_x} - \frac{\phi_{xy}\phi_{xxx}}{\phi_x} + \frac{3\phi_{xy}\phi_{xx}^2}{\phi_x^2} = 0, \tag{18}$$

and the equivalent nonlocal symmetry of u

$$\sigma^u = \phi_x. \quad \square \tag{19}$$

Remark 1. Eq. (18) is just the Schwartzian equation (14). What more interesting here is that the symmetry (19) is the residual symmetry of Eq. (2), and Eq. (17) is a nonauto-BT which transforms the original Eq. (2) into its Schwartzian equation (14). The transformation Eq. (16) is the link between the nonlocal symmetry (7) which is obtained form Lax pair and the residual symmetry (19).

Here, we can obtain the residual symmetry of Eq. (2) by the truncated painlevé analysis. Balancing the nonlinear and dispersive terms in the Eq. (2), its truncated Painlevé expansion can be expressed as

$$u = \frac{u_0}{\phi} + u_1, \tag{20}$$

where u_0, u_1 and ϕ are arbitrary functions with respect to x, y and t. substituting Eq. (20) into Eq. (2) and vanishing coefficients of the different powers $\frac{1}{\phi}$, we obtain

$$u_0 = -2\phi_x, \quad u_1 = \frac{1}{2}\frac{\phi_{xx}}{\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\mathrm{d}x. \tag{21}$$

Consequently,

$$u = -\frac{2\phi_x}{\phi} + \frac{1}{2}\frac{\phi_{xx}}{\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\mathrm{d}x \tag{22}$$

is a solution of Eq. (2) with ϕ satisfying Schwartzian equation (14). The Schwarzian equation (14) is invariant under the Möbious transformation

$$\phi \to \frac{a+b\phi}{c+d\phi}, \quad (ad \neq bc), \tag{23}$$

which means (13) is the symmetry of Schwartzian equation (14).

The residual of truncated Painlevé expansion (20) with the singular manifold ϕ, i.e., u_0 is a symmetry of Eq. (2) with the solution u_1. Thus, Eq. (19) is the residual symmetry of Eq. (2), and Eq. (17) is a nonauto-BT. It is worthy to mention that the residual symmetry (19) is just related to the the Möbious transformation symmetry (13) by the linearized equation of nonauto-BT (17).

As we know, the nonlocal symmetries can not be used to find explicit solutions for differential equations directly. Thus, the next step is to transform the nonlocal symmetries into local ones, one may extend the original system to a closed prolonged system which possesses a Lie point symmetry that is equivalent to the nonlocal symmetry.

The nonlocal residual symmetry of Eq. (2) can be localized to Lie point symmetry

$$\sigma^u = g, \quad \sigma^g = \phi g, \quad \sigma^\phi = \frac{1}{2}\phi^2, \tag{24}$$

for the prolonged system

$$\begin{aligned} &u_{xt} + u_{xxxy} - 4u_x u_{xy} - 2u_{xx}u_y = 0, \\ &u_x = \frac{1}{2}\frac{\phi_{xxx}}{\phi_x} - \frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}, \\ &g = \phi_x. \end{aligned} \tag{25}$$

Finally, the prolonged system (25) is closed after covering dependent variables u, g and ϕ with the vector form

$$V = g\frac{\partial}{\partial u} + \phi g\frac{\partial}{\partial g} + \frac{1}{2}\phi^2\frac{\partial}{\partial \phi}. \tag{26}$$

3. Finite symmetry transformation

In this section, we study the finite symmetry transformation of Lie point symmetry (26). According to Lie's first theorem, by solving the following initial value problem:

$$\begin{aligned}
&\frac{d\hat{u}(\varepsilon)}{d\varepsilon} = \hat{g}(\varepsilon), \quad \hat{u}(0) = u, \\
&\frac{d\hat{g}(\varepsilon)}{d\varepsilon} = \hat{\phi}(\varepsilon)\hat{g}(\varepsilon), \quad \hat{g}(0) = g, \\
&\frac{d\hat{\phi}(\varepsilon)}{d\varepsilon} = \frac{1}{2}\hat{\phi}(\varepsilon)^2, \quad \hat{\phi}(0) = \phi,
\end{aligned} \tag{27}$$

we arrive at the symmetry group transformation theorem as follows:

Theorem 1. *If $\{u, g, \phi\}$ is a solution of the prolonged system (25), then so is $\{\hat{u}, \hat{g}, \hat{\phi}\}$ with*

$$\hat{u}(\varepsilon) = u + \frac{2g\varepsilon}{2-\phi\varepsilon}, \quad \hat{g}(\varepsilon) = \frac{4g}{(2-\phi\varepsilon)^2}, \quad \hat{\phi}(\varepsilon) = \frac{2\phi}{2-\phi\varepsilon}, \tag{28}$$

for an arbitrary group parameter ϵ.

The finite transformation (28) provide a way to generate a new solution from given one. It is necessary to point out that this finite transformation is distinct from the usual DT. Actually, it is equivalent to the Levi transformation, i.e., the second type of DT. In addition, the last equation of (28) is nothing but the corresponding Möbious transformation about the Schwartzian equation (14).

4. New similarity reductions

To seek corresponding new similarity reductions related to the residual symmetry of Eq. (2), we employ the classical Lie point symmetry method to study the prolonged system (25) and assume that the symmetries have the vector form

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + G\frac{\partial}{\partial g} + \Phi\frac{\partial}{\partial \phi}, \tag{29}$$

where X, Y, T, U, G and Φ are the functions with respect to $\{x, y, t, u, g, \phi\}$, which means that the prolonged system (25) is invariant under the transformations

$$\{x, y, t, u, g, \phi\} \to \{x + \varepsilon X, y + \varepsilon Y, t + \varepsilon T, u + \varepsilon U, g + \varepsilon G, \phi + \varepsilon\Phi\}, \tag{30}$$

with a small parameter ε. Equivalently, the symmetries in the vector form (29) can be written as a function form

$$\begin{aligned}
&\sigma^u = Xu_x + Yu_y + Tu_t - U, \\
&\sigma^g = Xg_x + Yg_y + Tg_t - G, \\
&\sigma^\phi = X\phi_x + Y\phi_y + T\phi_t - \Phi.
\end{aligned} \tag{31}$$

Moreover, σ^u, σ^g and σ^ϕ satisfy the linearized equations for the prolonged system (25)

$$\begin{aligned}
&\sigma^u_{xt} + \sigma^u_{xxxy} - 4u_x\sigma^u_{xy} - 2u_y\sigma^u_{xx} - 4u_{xy}\sigma^u_x - 2u_{xx}\sigma^u_y = 0, \\
&2\phi_x^3\sigma^u - \phi_x^2\sigma^\phi_{xxx} + \phi_x\phi_{xx}\sigma^\phi_{xx} + (\phi_x\phi_{xxx} - \phi_{xx}^2)\sigma^\phi_x = 0, \\
&\sigma^g = \sigma^\phi_x.
\end{aligned} \tag{32}$$

Substituting Eq. (31) into Eq. (32) and eliminating u_{xt}, g_{xx}, g_t, ϕ_x and ϕ_t in terms of the prolonged system (25), we obtain an over-determined set of equations for the functions X, Y, T, U, G and Φ. Solving the determining equations, the general solutions of them take the form

$$\begin{aligned}
&X = \frac{1}{2}(c_1 - c_3)x + f_1, \\
&Y = c_3y + c_4, \quad T = c_1t + c_2, \\
&U = -\frac{1}{2}(c_1 - c_3)u + f_2g - \frac{1}{2}f_{1t}y + f_3, \\
&G = (f_2\phi + f_4)g, \\
&\Phi = \frac{1}{2}f_2\phi^2 + \frac{1}{2}(c_1 - c_3 + 2f_4)\phi + f_5,
\end{aligned} \tag{33}$$

where $f_2 \equiv f_2(y)$ is an arbitrary function of y, $f_1 \equiv f_1(t)$ and $f_3 \equiv f_3(t)$ are arbitrary functions of t, $f_4 \equiv f_4(y, t)$ and $f_5 \equiv f_5(y, t)$ are arbitrary functions of $\{y, t\}$ and c_1, c_2, c_3 and c_4 are arbitrary constants. Especially, when $f_1 = f_3 = f_4 = f_5 = c_1 = c_2 = c_3 = c_4 = 0$

Table 1
Lie bracket.

$[V_i, V_j]$	V_1	V_2	V_3	V_4	$V_5(f_1)$	$V_6(f_2)$	$V_7(f_3)$	$V_8(f_4)$	$V_9(f_5)$
V_1	0	$-V_2$	0	0	$V_5(\tilde{f}_1)$	$V_6(\frac{1}{2}f_2)$	$V_7(\tilde{f}_2)$	$V_8(tf_{4t})$	$V_9(\tilde{f}_3)$
V_2		0	0	0	$V_5(f_{1t})$	0	$V_7(f_{3t})$	$V_8(f_{4t})$	$V_9(f_{5t})$
V_3			0	$-V_4$	$V_5(\frac{1}{2}f_1)$	$V_6(\tilde{f}_4)$	$V_7(-\frac{1}{2}f_3)$	$V_8(yf_{4y})$	$V_9(\tilde{f}_5)$
V_4				0	$V_7(-\frac{1}{2}f_{1t})$	$V_6(f_{2y})$	0	$V_8(f_{4y})$	$V_9(f_{5y})$
$V_5(\hat{f}_1)$					0	0	0	0	0
$V_6(\hat{f}_2)$						0	0	$V_6(-f_2f_4)$	$V_8(-f_2f_5)$
$V_7(\hat{f}_3)$							0	0	0
$V_8(\hat{f}_4)$								0	$V_9(-f_4f_5)$
$V_9(\hat{f}_5)$									0

and $f_2 = 1$, the degenerated symmetry is just Eq. (24), and when $f_2 = 0$, the related symmetry is only the general Lie point symmetry of the original Eq. (2).

From (33), we can obtain that the symmetry algebra of the prolonged system (25) is generated by the four vector fields

$$\begin{aligned} V_1 &= \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u} + \frac{1}{2}\phi\frac{\partial}{\partial \phi}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= -\frac{1}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}u\frac{\partial}{\partial u} - \frac{1}{2}\phi\frac{\partial}{\partial \phi}, \\ V_4 &= \frac{\partial}{\partial y}, \end{aligned} \tag{34}$$

and the infinite-dimensional subalgebra

$$\begin{aligned} V_5(f_1) &= f_1\frac{\partial}{\partial x} - \frac{1}{2}f_{1t}y\frac{\partial}{\partial u}, \\ V_6(f_2) &= f_2 g\frac{\partial}{\partial u} + f_2 g\phi\frac{\partial}{\partial g} + \frac{1}{2}f_2\phi^2\frac{\partial}{\partial \phi}, \\ V_7(f_3) &= f_3\frac{\partial}{\partial u}, \\ V_8(f_4) &= f_4 g\frac{\partial}{\partial g} + f_4\phi\frac{\partial}{\partial \phi}, \\ V_9(f_5) &= f_5\frac{\partial}{\partial \phi}. \end{aligned} \tag{35}$$

Applying the commutator operators $[V_m, V_n] = V_mV_n - V_nV_m$, we obtain the commutator table presented in Table 1 with the (i,j)th entry indicating $[V_i, V_j]$, where

$$\tilde{f}_1 = tf_{1t} - \frac{1}{2}f_1, \quad \tilde{f}_2 = tf_{3t} + \frac{1}{2}f_3, \quad \tilde{f}_3 = tf_{5t} - \frac{1}{2}f_5, \quad \tilde{f}_4 = yf_{2y} - \frac{1}{2}f_2, \quad \tilde{f}_5 = yf_{5y} + \frac{1}{2}f_5.$$

Some corresponding group invariant solutions can be given by solving the characteristic equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{du}{U} = \frac{dg}{G} = \frac{d\phi}{\Phi}. \tag{36}$$

In the following part of the paper, two nontrivial cases are discussed in detail.

Case 1. The first type of special soliton–cnoidal waves solution

Without loss of generality, we assume $c_1 = c_2 = c_3 = f_3 = 0$, $c_4 = 1$, $f_1 = \frac{1}{k}$, $f_2 = c_6$, $f_4 = c_7$ and $f_5 = c_8$, and redefine $\Delta^2 = c_7^2 - 2c_6c_8$. By solving (36), the group invariant solutions read

$$\begin{aligned} u &= U - \frac{2c_6}{\Delta}G\tanh\left[\frac{1}{2}k\Delta(x+\Phi)\right], \\ g &= -G\operatorname{sech}^2\left[\frac{1}{2}k\Delta(x+\Phi)\right], \\ \phi &= -\frac{c_7}{c_6} - \frac{\Delta}{c_6}\tanh\left[\frac{1}{2}k\Delta(x+\Phi)\right], \end{aligned} \tag{37}$$

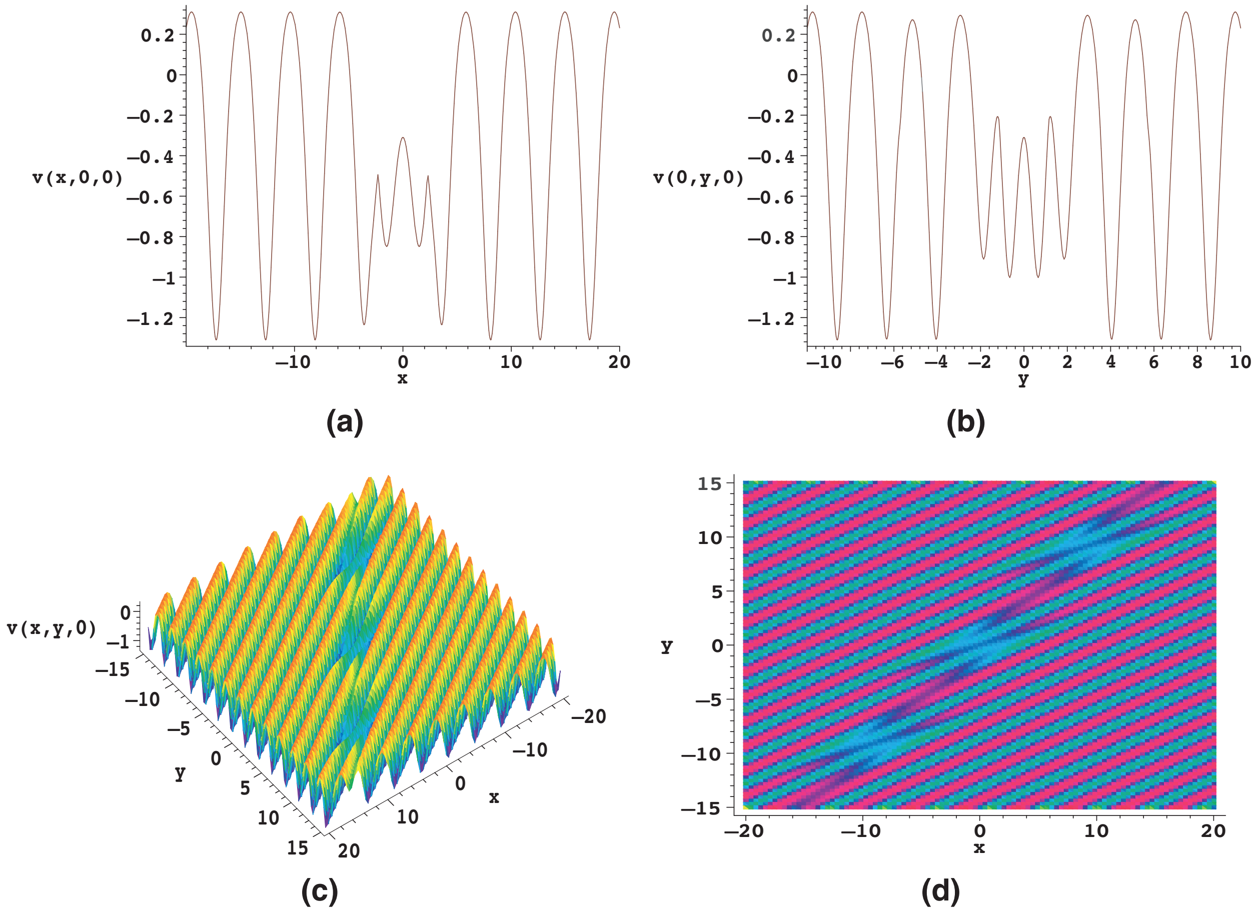

Fig. 1. The first type of special soliton–cnoidal wave interaction solution for Eq. (1) expressed by Eq. (41), with the parameters $k = 0.5$, $c = r_1 = 2$, $m = 0.9$ and $n = 0.5$. (a) The profile of the special structure at $t = 0$ and $y = 0$; (b) the profile of the special structure at $t = 0$ and $x = 0$; (c) perspective view of the wave; (d) overhead view of the wave.

where $U \equiv U(\xi,\eta)$, $G \equiv G(\xi,\eta)$ and $\Phi \equiv \Phi(\xi,\eta)$ are the group invariant functions while $\xi = -kx + y$ and $\eta = t$ are the similarity variables.

Substituting Eq. (37) into the prolonged system (25) yields

$$\begin{aligned} G &= -\frac{k\Delta^2}{2c_6}(k\Phi_\xi - 1), \\ U_\xi &= -\frac{1}{4}k\Delta^2(k\Phi_\xi - 1)^2 - \frac{1}{4}k^2\frac{2k\Phi_{\xi\xi\xi}\Phi_\xi - 2\Phi_{\xi\xi\xi} - k\Phi_{\xi\xi}^2}{(k\Phi_\xi - 1)^2}, \end{aligned} \tag{38}$$

where Φ satisfies the following reduction equation

$$2k^2\Phi_{\xi\xi\xi} + 3k^3\Phi_{\xi\xi}^2 + k^5\Delta^2\Phi_\xi^4 - 4k^4\Delta^2\Phi_\xi^3 + 6k^3\Delta^2\Phi_\xi^2 - 2k^2(k\Phi_\xi^3 + 2\Delta^2)\Phi_\xi - 2(k\Phi_\xi - 1)\Phi_\eta + k\Delta^2 = 0. \tag{39}$$

We just write a special solution of the reduction equation (39) in the form

$$\Phi = r_0\xi + \omega_0\eta + cE_\pi(\mathrm{sn}(r_1\xi + \omega_1\eta, m), n, m). \tag{40}$$

It leads to the soliton–cnoidal wave solution of the Eq. (1) as follows:

$$v = \frac{c^2k^4r_1^2n^2\Delta^2S^4T^2}{2(nS^2-1)^2} - \frac{2ck^3r_1^2n\Delta SCDT}{(nS^2-1)^2} + \frac{k^2r_1^2(2nC^2+n-1)D^2}{(nS^2-1)^2} + \frac{k^2r_1^2m^2C^2}{nS^2-1} - \frac{c^2k^4r_1^2n^2\Delta^2S^4}{4(nS^2-1)^2}, \tag{41}$$

where $\{c, k, r_1, m, n\}$ are arbitrary constant, $S \equiv \mathrm{sn}(-kr_1x + r_1y + \omega_1t, m)$, $C \equiv \mathrm{cn}(-kr_1x + r_1y + \omega_1t, m)$, $D \equiv \mathrm{dn}(-kr_1x + r_1y + \omega_1t, m)$, $T \equiv \tanh\{\frac{1}{2}k\Delta[(1 - kr_0)x + r_0y + \omega_0t + cE_\pi(S, n, m)]\}$ and

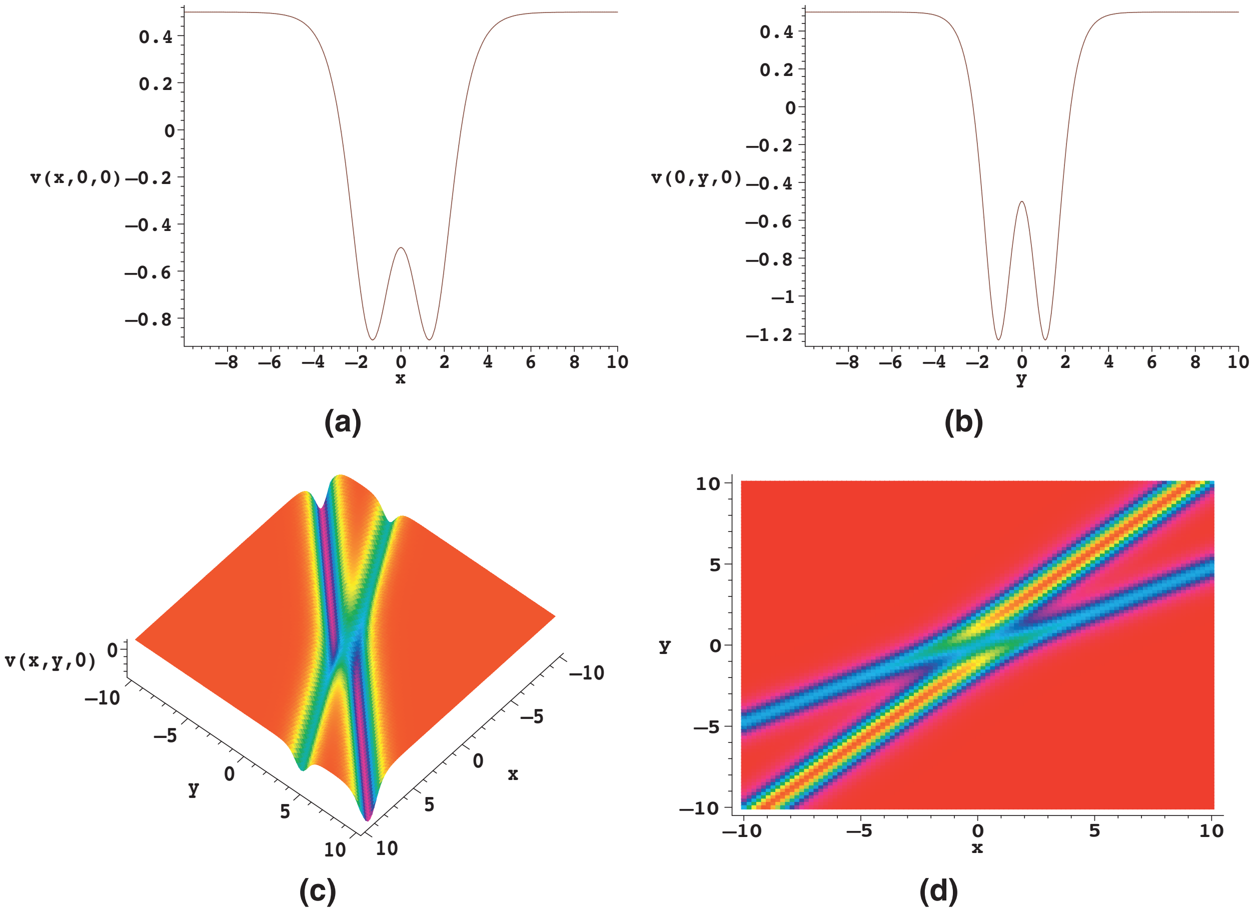

Fig. 2. Two-soliton solution for Eq. (1) expressed by Eq. (41), with the parameters $c = 1.2$, $k = r_1 = 1$, $m = 1$ and $n = 0.5$. (a) The profile of the special structure at $t = 0$ and $y = 0$; (b) the profile of the special structure at $t = 0$ and $x = 0$; (c) perspective view of the wave; (d) overhead view of the wave.

$$\begin{aligned} r_0 &= \frac{1 - ckr_1}{k}, \\ \omega_0 &= 2ck^2r_1^3(-n + m^2 + 1), \\ \omega_1 &= 2k^2r_1^3(3n - m^2 - 1), \\ \Delta^2 &= \frac{4(n-1)(n-m^2)}{c^2k^2n}. \end{aligned} \tag{42}$$

In solution (41), $E_\pi(\zeta, n, m)$ is the third type of incomplete elliptic integral.

The solution given in (41) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave. In Fig. 1, we plot the interaction solution between the solitary wave and the cnoidal wave when the value of the Jacobi elliptic function modulus $m \neq 1$. We can see that a dark soliton propagates on a cnoidal wave background instead of on the plane continuous wave background. This kind of solution can be easily applicable to the analysis of physically interesting processes. If setting the modulus $m = 1$, the soliton–cnoidal wave interaction solution reduces back to the two-dark-soliton solution, whose interaction behavior is displayed in Fig. 2.

Case 2. The second type of special soliton–cnoidal waves solution

In this case, we let $c_1 = c_3 = f_3 = 0$, $f_2 = c_6$, $f_4 = c_7$, $f_5 = c_8$ and $f_1 = h_t$ with $h \equiv h(t)$, and redefine $k_1 = \frac{c_2}{c_4}$, $k_2 = \frac{1}{c_2}$ and $\Delta^2 = c_7^2 - 2c_6c_8$. Following the similar procedure as Case 1, we derive the group invariant solutions

$$\begin{aligned} u &= U + \frac{k_2(h - k_1yh_t - C_1)}{2k_1} - \frac{2c_6}{\Delta}G\left\{\tanh\left[\frac{1}{2}k_1k_2\Delta(y+\Phi)\right] - 1\right\}, \\ g &= -G\,\mathrm{sech}^2\left[\frac{1}{2}k_1k_2\Delta(y+\Phi)\right], \end{aligned} \tag{43}$$

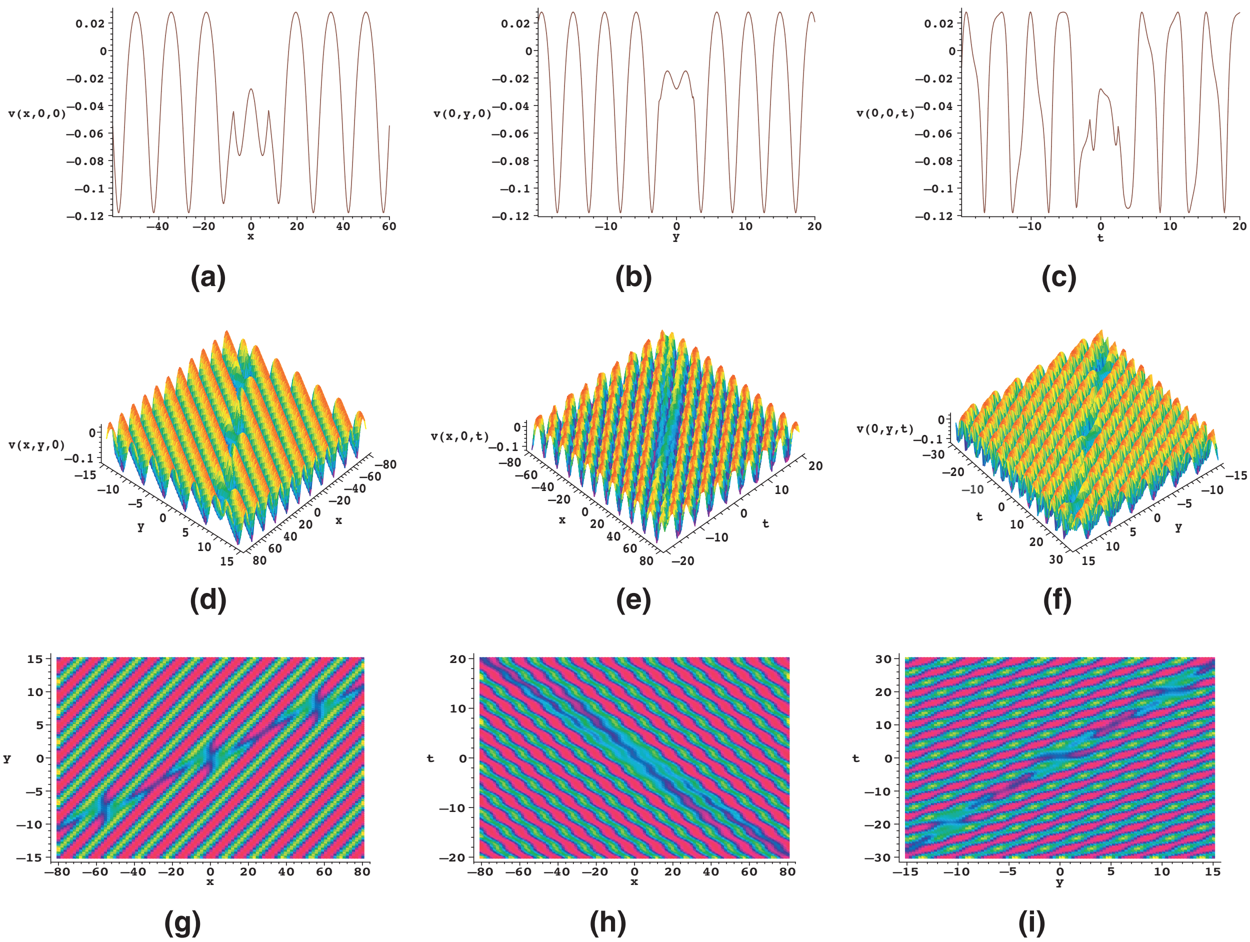

Fig. 3. The second type of special soliton–cnoidal wave interaction solution for Eq. (1) given by Eq. (47), with the parameters $c = k_1 = r_1 = 1$, $k_2 = 1.2$, $\omega_1 = 0.3$, $m = 0.9$, $n = 0.5$ and $h = 2\sin^2(t)$. (a) One-dimensional image at $t = 0$ and $y = 0$; (b) one-dimensional image at $t = 0$ and $x = 0$; (c) one-dimensional image at $x = 0$ and $y = 0$; (d) two-dimensional image at $t = 0$; (e) two-dimensional image at $y = 0$; (f) two-dimensional image at $x = 0$; (g) overhead view of (d); (h) overhead view of (e); (i) overhead view of (f).

$$\phi = -\frac{c_7}{c_6} - \frac{\Delta}{c_6}\tanh\left[\frac{1}{2}k_1k_2\Delta(y+\Phi)\right],$$

where C_1 is arbitrary constant, $U \equiv U(\xi,\eta)$, $G \equiv G(\xi,\eta)$ and $\Phi \equiv \Phi(\xi,\eta)$ are the group invariant functions while $\xi = -k_1y + t$ and $\eta = x - k_2h$ are the similarity variables.

Substituting Eq. (43) into the prolonged system (25) leads to

$$G = \frac{k_1k_2\Delta^2}{2c_6}\Phi_\eta,$$
$$U_\eta = -k_1k_2\Delta\Phi_{\eta\eta} + \frac{1}{4}k_1^2k_2^2\Delta^2\Phi_\eta^2 + \frac{1}{2}\frac{\Phi_{\eta\eta\eta}}{\Phi_\eta} - \frac{1}{4}\frac{\Phi_{\eta\eta}^2}{\Phi_\eta^2}, \tag{44}$$

where Φ is a solution of the reduction equation

$$k_1\Phi_{\xi\eta}\left(k_1^2k_2^2\Delta^2\Phi_\eta^4 - 3\Phi_{\eta\eta}^2\right) + \Phi_\eta^2(\Phi_{\xi\eta} - k_1\Phi_{\xi\eta\eta\eta}) + \Phi_\eta(3k_1\Phi_{\eta\eta}\Phi_{\xi\eta\eta} + k_1\Phi_{\eta\eta\eta}\Phi_{\xi\eta} - \Phi_\xi\Phi_{\eta\eta}) = 0. \tag{45}$$

We take a special solution of the reduction equation (45) in the form

$$\Phi = r_0\xi + \omega_0\eta + cE_\pi(\mathrm{sn}(r_1\xi + \omega_1\eta, m), n, m), \tag{46}$$

which leads to the soliton–cnoidal wave solution of the Eq. (1):

$$v = \frac{k_1^2k_2^2\omega_0^2n^2\Delta^2S^4T^2}{2(nS^2-1)^2} - \frac{2ck_1k_2\omega_1^2n\Delta SCDT}{(nS^2-1)^2} - \frac{\omega_1^2(2nC^2+n-1)D^2}{(nS^2-1)^2} - \frac{k_1^2k_2^2\omega_0^2n^2\Delta^2S^4}{4(nS^2-1)^2} + \frac{\omega_1^2m^2C^2}{nS^2-1}, \tag{47}$$

where $\{c, k_1, k_2, r_1, \omega_1, m, n\}$ are arbitrary constant, $S \equiv \mathrm{sn}(\omega_1x - k_1r_1y + r_1t - k_2\omega_1h, m)$, $C \equiv \mathrm{cn}(\omega_1x - k_1r_1y + r_1t - k_2\omega_1h, m)$, $D \equiv \mathrm{dn}(\omega_1x - k_1r_1y + r_1t - k_2\omega_1h, m)$, $T \equiv \tanh\{\frac{1}{2}k_1k_2\Delta[\omega_0x + (1-k_1r_0)y + r_0t - k_2\omega_0h + cE_\pi(S, n, m)]\}$ and

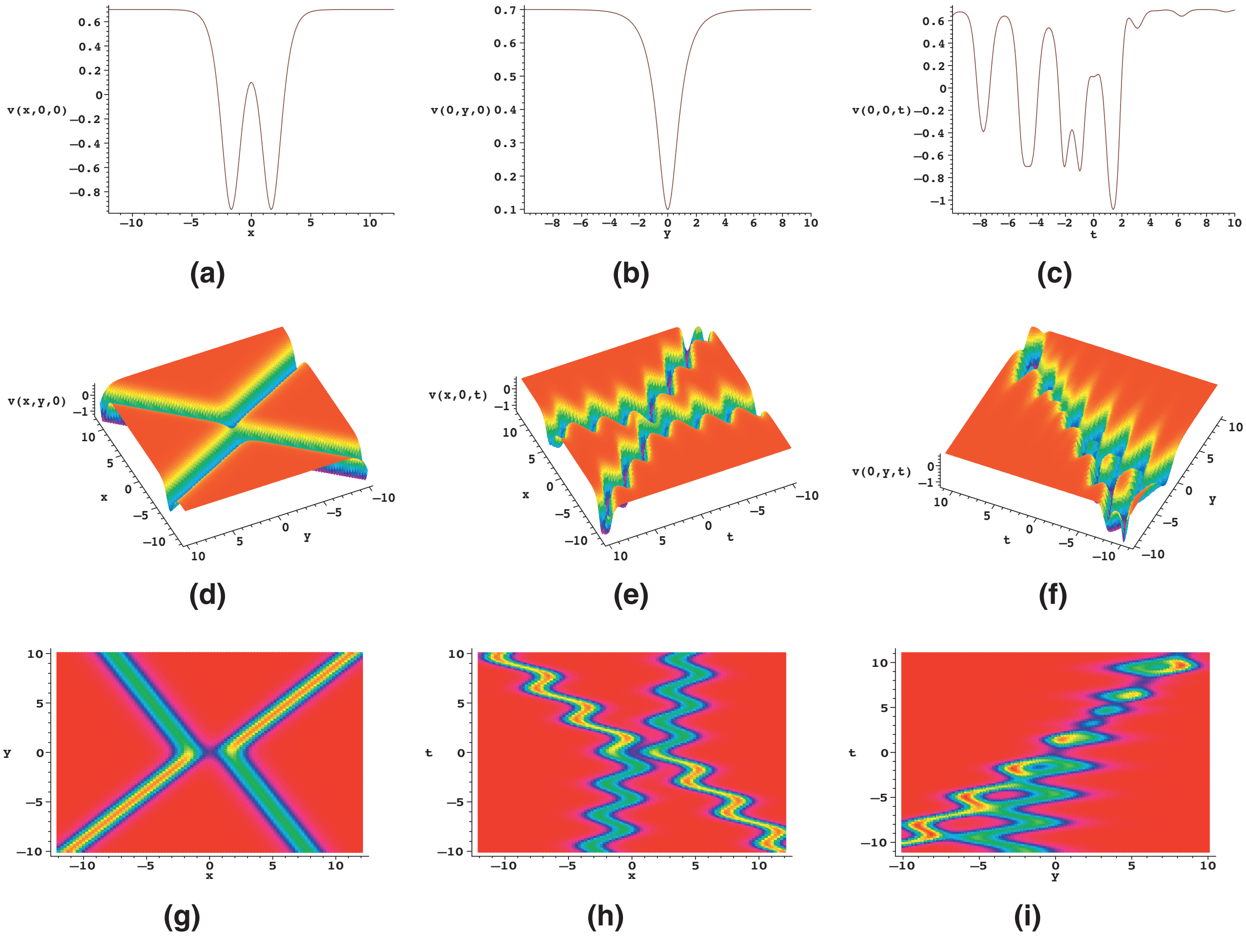

Fig. 4. Two-soliton solution for Eq. (1) given by Eq. (47), with the parameters $c = k_1 = r_1 = \omega_1 = 1,\ k_2 = 1.2,\ m = 1,\ n = 0.7$ and $h = 2\sin^2(t)$. (a) One-dimensional image at $t = 0$ and $y = 0$; (b) one-dimensional image at $t = 0$ and $x = 0$; (c) one-dimensional image at $x = 0$ and $y = 0$; (d) two-dimensional image at $t = 0$; (e) two-dimensional image at $y = 0$; (f) two-dimensional image at $x = 0$; (g) overhead view of (d); (h) overhead view of (e); (i) overhead view of (f).

$$\begin{aligned} r_0 &= -cr_1(1 + 4k_1\omega_1^2 n), \\ \omega_0 &= -c\omega_1, \\ \Delta^2 &= \frac{4(n-1)(n-m^2)}{c^2k_1^2k_2^2 n}. \end{aligned} \tag{48}$$

In solution (47) , $E_\pi(\zeta, n, m)$ is the third type of incomplete elliptic integral. In order to study the properties of this solution, we give some pictures as shown below in Figs. 3 and 4.

Fig. 3 displays the second type of special soliton–cnoidal wave structure of v determined by (47) when the vaule of the Jacobi elliptic function modulus $m \neq 1$. One can see that a dark soliton propagates on a cnoidal wave background. When the modulus $m = 1$, the soliton–cnoidal wave interaction solution reduces back to the two-dark-soliton solution, whose interaction behavior is exhibited in Fig. 4.

5. Summary and discussion

In conclusion, the nonlocal symmetry of the (2+1)-dimensional breaking soliton equation is derived from the Lax pair. Under the transformation $\psi = e^{\lambda t}\sqrt{\phi_x}$, this nonlocal symmetry becomes residual symmetry which is obtained by the truncated painlevé analysis. Then, the residual symmetry is readily localized to Lie point symmetry by prolonging the original equation to a larger system. Meanwhile, we observe that the residual symmetry is just related to the Möbious transformation symmetry by the linearized equation of nonauto-Bäcklund transformation.

The standard Lie point symmetry approach is used to study the finite symmetry transformation and similarity reductions of the prolonged system. Two types of special interaction solution between the soliton and the cnoidal periodic wave are presented. This kind of solution can be easily applicable to the analysis of physically interesting processes.

The method presented here could be applied to other physical interested models, especially for supersymmetric models and discrete ones. The link between the residual symmetry and other kinds of nonlocal symmetries, which can be obtained from Bäcklund transformation, negative hierarchies, and the self-consistent sources, etc., is also an interesting topic. The above topics will be discussed in our future research work.

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