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ORIGINAL PAPER

Novel solitons and higher-order solitons for the nonlocal generalized Sasa–Satsuma equation of reverse-space-time type

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Abstract The general soliton solutions and higher-order soliton solutions for the nonlocal generalized Sasa–Satsuma (SS) equation of reverse-space-time type are explored. Firstly, a novel nonlocal generalized SS equation is derived, and the infinitely many conserved quantities and conservation laws are considered. Secondly, some novel symmetry properties and nonlocal constraints for eigenvalues, eigenvectors and scattering data are obtained, which is quite different from the local ones. Then, in the framework of the Riemann–Hilbert problem and by the special nonlocal properties, the N-soliton formula with determinant and the higher-order soliton formulas are constructed for the nonlocal generalized SS equation by a limit technique. Thirdly, some new patterns and unusual dynamical behaviors of the N-soliton and the higher-order soliton solutions for the nonlocal generalized SS equation are exhibited and explored. The general single soliton is always collapsing periodically whether the eigenvalues are pure imaginary or not, but when the absolute value of the eigenvalue approaches to zero, the solution tends to be a standing solution, which does not move with time. Besides, some novel interesting physical patterns for the two-soliton solution are obtained, such as a singular wave in the periodical background and two-soliton solution with two singular branches. It is worth mentioning that the two-soliton solution does not degenerate into a bounded breathing soliton instead of a breathing singular wave when $\lambda_2 = -\lambda_1^*$. And the higher-order soliton with one double zero is singular and collapsing periodically while the soliton with triple zero is nonsingular when the eigenvalue is purely imaginary. query Please check the edit made in the article title.

Keywords Nonlocal generalized Sasa–Satsuma equation · Reverse-space-time · Riemann–Hilbert method · Solitons · Dynamic behavior

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1 Introduction

PT symmetry was proposed by Bender [1] in 1998, where they obtained that the energy spectrum of non-Hermitian Hamilton operators with *PT* symmetric potential are all real and positive. Then, *PT* symmetry aroused wide applications in many areas [2–4], especially in optical physics. In 2015, Cham Jorge listed

PT symmetry in optics as one of the ten physics discoveries of the past decades [5]. And *PT* symmetry was realized experimentally by Ruter et al. in 2015 [6], where the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential was observed firstly. In addition, Zhang et al. [7] experimentally proved a *PT* symmetric optical lattice with controllable periodic gain and loss characteristics in a four-level N type atomic system in 2016. *PT* symmetric system possesses many new and intriguing properties [8–12], which laid new foundations and a better understanding of the special physical properties of Hermitian particles.

Muslimani et al. introduced *PT* symmetric potential into integrable nonlinear Schrödinger (NLS) equation in 2008, and they obtained a stable soliton solution under Scarf-II potential [4]. After that, Ablowitz et al. introduced a new integrable nonlocal NLS equation from a direct *PT* symmetric reduction of the AKNS system; the inverse scattering transformation (IST) for the nonlocal system was developed [13,15]. Meanwhile, Ablowitz et al. developed the IST problem via a left–right RH problem for the nonlocal NLS equation with zero boundary conditions [15]. After the nonlocal *PT* symmetric integrable equation was proposed by a special reduction of a general system [13], then in 2017, new types of nonlocal nonlinear integrable equations with reverse space-time and reverse time reduction were introduced [14]. Subsequently, due to the vital roles that nonlocal integrable equations played in mathematics and physics, increasing number of nonlocal integrable equations were proposed and studied by Darboux transformation [16–21], Riemann–Hilbert(RH) method [22–27], Hirota direct bilinear and KP reduction method [28,29].

The establishment of the IST associated with the spectral problem signified the formulation of much information such as the analytic properties, asymptotic properties and the symmetry of the eigenvalues, eigenvectors and other scattering data. Zakharov and Shabat [30] formulated a more general method to solve the spectral problem of the integrable system, the RH method, which is the modern version and an extension of the IST method. Compared to the traditional IST with solving the integral Gelfand–Levitan–Marchenko equation, in the framework of the RH problem, the higher-order spectral problems can also be solved [31–33]. Yang [34] developed the RH formulation for the NLS equation and extended to the higher-order matrix spectral problem. Then in 2018, the general solitons for nonlocal NLS equations from the RH solutions of the 2×2 AKNS hierarchy were derived [43]. In the same year, a nonlocal Manakov system, which corresponded to the 3×3 linear problem corresponded, was proposed [26], where they derived one- and two-solitons in the framework of RH problem. Then, some general new N-soliton solution for a generalized nonlocal NLS equation has been explored in [36] via RH approach with a 2×2 scattering problem, where the symmetry relations of the scattering data which involve the reverse-space, reverse-time and reverse-space-time reductions were studied. The N-soliton solutions to the nonlocal complex reverse-space-time-modified KdV hierarchies were obtained from the reflectionless transforms by building the associated RH problems [24]. However, there is not much research on nonlocal 3×3 matrix RH problem; we will investigate a nonlocal 3×3 spectral problem in the framework of the RH problem. Based on the RH formulation, the N-soliton solutions corresponded to the case that all discrete spectra are simple. As for the RH problem with multiple poles, new classes of higher-order soliton have been considered [37,38]. Ling et al. obtained the general higher-order soliton solutions in the framework of IST and by Darboux transformation [39,40]. The multiple-pole solitons by N-fold application of Darboux transformations for the focusing NLS equation were obtained starting from the zero background [41]. For the focusing Ablowitz–Ladik equation, the double- and triple-pole solutions were derived from its multi-soliton solutions via some limit technique [40]. Zhang et al. obtained the formulate of one higher-order pole solitons and multiple higher-order poles solitons based on the IST method for NLS equation [42] with the decaying initial value conditions. Later, we presented the higher-order solitons with multiple poles for nonlocal NLS equation [43] and Sasa–Satsuma (SS) equation [44]. But for the higher-order soliton solution for the nonlocal integrable equation, especially for the nonlocal integrable equation which is compatible with a 3×3 linear spectral problem, there has not been a lot of research on it. This work will focus on the following nonlocal generalized SS equation of reverse-space-time type with decaying initial-value condition

$$
\begin{aligned}
&q_t(x,t)+q_{xxx}(x,t)\\
&\quad+3\alpha q(x,t)\left[V_x+2q(-x,-t)q_x(x,t)\right]\\
&\quad-6\beta q^2(x,t)q_x(x,t)+3\beta^* q(-x,-t)V_x=0 \qquad (1)
\end{aligned}
$$

where $V(x,t)=q(x,t)q(-x,-t)$, which meets the symmetry property $V(x,t)=V(-x,-t)$. q is a complex function of (x,t). α is real and β is complex number with $|\alpha|\neq|\Re(\beta)|$. The general solitons of Eq. (1) will be investigated with the following decaying initial-value condition

$$
q(x,0)=q_0(x),\quad q_0(x)\in\mathbb{S}(\mathbb{R}) \qquad (2)
$$

where $\mathbb{S}(\mathbb{R}) = \{s\in C^\infty(\mathbb{R}), \|s\|_{\alpha,\beta}=\sup_{x\in\mathbb{R}} \left|x^\alpha\partial^\beta s(x)\right|<\infty,\ \ \alpha,\beta\in\mathbb{Z}_+\}$ is the Schwartz space. Generalized SS equation was proposed by Geng et al. [45] in 2016 and where the corresponding RH problem with vanishing initial value was formulated. The equation can be reduced to complex modified KdV equation and SS equation with suitable parameter. It is natural to investigate the combination of the nonlocal complex modified KdV equation and the nonlocal SS equation, i.e., the nonlocal generalized SS equation. It is interesting to research the IST and soliton solutions for this equation. Inspired by Ablowitz et al. [13,14], through an apparently direct reduction, the nonlocal generalized SS equation is obtained, which will be described in detail later in Sect. 2. The inverse scattering process in the framework of the RH problem will be imposed for the nonlocal generalized SS equation. The nonlocal generalized SS Eq. (1) corresponded to a 3×3 linear spectral problem. But for the 3×3 matrix RH problem of the nonlocal generalized SS Eq. (1), these symmetry relations are quite complex to derive. Novel symmetry relationships for eigenvalues and eigenvectors will be carried out. As a result, the new symmetry relations of the scattering matrix and the scattering data will be imposed. In the framework of the RH problem, it is vital to investigate the symmetry relations of the scattering data. For a 2×2 matrix RH problem, the derivation is relatively simple. But as for a 3×3 spectral problem, the symmetry is hard to derive. For these motivations, we will investigate general solitons in the framework of the RH problem with simple and multiple poles for the nonlocal generalized SS Eq. (1). Some new dynamical behaviors of the general soliton solutions and the higher-order soliton solutions will be explored. query Please check the clarity of the sentence ‘It is natural that ... the nonlocal generalized SS equation’. The outline of this paper is as follows: In Sect. 2, the nonlocal generalized SS equation is proposed and the compatibility condition is given. In Sect. 3, the corresponding infinitely many conserved quantities and conservation laws for the nonlocal generalized SS Eq. (1) are exhibited. Section 4 develops the inverse scattering theory and N soliton solutions for the coupled generalized SS Eq. (5) via RH problem formula. Besides, the symmetry relations corresponding to the scattering data are concluded for the coupled generalized SS Eq. (5). Section 5 focuses on the nonlocal constraints of the eigenvectors and scattering data for Eq. (1). Then, in Sect. 6, we study the dynamic behaviors of the soliton solutions with simple zeros and N multiple zeros. The exact expressions of single- and two-soliton solutions, higher-order soliton solutions with double and triple zero are given. Then, the higher-order soliton solution of the corresponding RH problem with multiple poles is derived for the nonlocal generalized SS equation and studies the dynamical behaviors of the higher-order soliton. The last section is the conclusion.

2 The nonlocal generalized SS equation and the compatibility condition

First of all, we consider the following 3×3 spectral problem

$$
Y_x=\mathbf{U}Y=(i\lambda\Lambda+Q)Y, \qquad (3)
$$

where $Y(x,t;\lambda) = (y_1(\lambda;x,t), y_2(\lambda;x,t), y_3(\lambda;x,t))^T$ is a vector of three components and

$$
Q=\begin{pmatrix} 0 & 0 & q\\ 0 & 0 & r\\ \alpha r+\beta q & \alpha q+\beta^* r & 0\end{pmatrix},
$$

$$
\Lambda=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},
$$

with λ is the spectral parameter and q,r are complex functions of (x,t). Define the time part of the Lax pair

$$
Y_t=\mathbf{V}Y=\left(4i\lambda^3\Lambda+\widetilde{Q}\right)Y, \qquad (4)
$$

with $\widetilde{Q}=4\lambda^2Q+2i\lambda\left(Q^2+Q_x\right)\Lambda+Q_xQ-QQ_x-Q_{xx}+2Q^3$. As a consequence of the compatible condition of (3) and (4), one can observe that the potential functions $q(x,t)$ and $r(x,t)$ yield the following coupled generalized SS equations

$$\begin{cases} q_t + q_{xxx} - 3\alpha q\left[(qr)_x + 2rq_x\right] \\ \quad -6\beta q^2 q_x - 3\beta^* r(qr)_x = 0, \\ r_t + r_{xxx} - 3\alpha r\left[(qr)_x + 2qr_x\right] \\ \quad -6\beta^* r^2 r_x - 3\beta q(qr)_x = 0. \end{cases} \tag{5}$$

Successively, imposing the nonlocal reverse-space-time potential reduction [14]

$$r(x,t) = -q(-x,-t) \tag{6}$$

on the coupled system (5), thus the nonlocal generalized SS Eq. (1) can be ontained. It is natural that the corresponding Lax pair of Eq. (1) is:

$$Y_x = i\lambda\Lambda Y + QY = \begin{pmatrix} i\lambda & 0 & q(x,t), \\ 0 & i\lambda & -q(x,-t) \\ \beta q(x,t) - \alpha q(-x,-t) & \alpha q(x,t) - \beta^* q(-x,-t) & -i\lambda \end{pmatrix},$$

$$Y_t = 4i\lambda^3\Lambda Y + \widetilde{Q}Y.$$

with

$$Q = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & -q(x,-t) \\ -\alpha q(x,-t) + \beta q & \alpha q - \beta^* q(x,-t) & 0 \end{pmatrix}.$$

Thus, the integrability of the nonlocal generalized SS Eq. (1) can be guaranteed. The generalized SS equation in [45] can be derived when $r = q^*$ in (5). The nonlocal generalized SS Eq. (1) of reverse-space-time type is a combination of the nonlocal complex modified KdV equation and nonlocal SS equation. When $\alpha = 0$ and $\beta = -1$, Eq. (1) reduces to the nonlocal complex modified KdV equation

$$q_t(x,t) + q_{xxx}(x,t) + 3(q(x,t)q(-x,-t))_x q(-x,-t) \\ -6q(x,t)^2 q_x(x,t) = 0.$$

Besides, the nonlocal SS equation can be obtained with $\alpha = -1$ and $\beta = 0$:

$$q_t(x,t) + q_{xxx}(x,t) - 3(q(x,t)q(-x,-t))_x q(x,t) \\ -6q(x,t)q_x(x,t)q(-x,-t) = 0.$$

It is necessary to illustrate that the other nonlocal generalized SS equation can also be obtained by imposing the corresponding reverse-space and reverse-space-time reductions. Here we list some other nonlocal generalized SS equations. The complex reverse-space-time nonlocal generalized SS equation

$$q_t(x,t) + q_{xxx}(x,t) + 6q(x,t)q_x(x,t) \\ \left[\beta q(x,t) - \alpha q^*(-x,-t)\right] \\ +3\left[q(x,t)q^*(-x,-t)\right]_x \\ \left[\alpha q(x,t) - \beta^* q^*(-x,-t)\right] = 0.$$

The real shifted reverse-space-time nonlocal generalized SS equation

$$q_t(x,t) + q_{xxx}(x,t) + 6q(x,t)q_x(x,t) \\ \left[\beta q(x,t) - \alpha q(x_0 - x, t_0 - t)\right] \\ +3\left[q(x,t)q(x_0 - x, t_0 - t)\right]_x \\ \left[\alpha q(x,t) - \beta^* q(x_0 - x, t_0 - t)\right] = 0.$$

All the integrability of the above nonlocal equations can be proved for the corresponding compatibility conditions, which can be obtained with a reduction from the Lax pair (3) and (4).

3 Infinitely many conserved quantities and conservation laws

The nonlocal generalized SS Eq. (1) possesses many important properties such as reverse-space-time symmetry and gauge invariant. Besides, due to the integrability of the nonlocal Eq. (1), the corresponding infinitely many conserved quantities and conservation laws can be gotten. Wadati et al. [46] proposed an efficient algebraic method to generalize the higher quantities. This section will focus on the method to develop the infinite number of conserved quantities for the nonlocal Eq. (1). Consider the solution $Y = (y_1, y_2, y_3)^T$ of the Lax pair (11). Introduce two new functions defined as

$$\mu_1 = \frac{y_1}{y_3}, \quad \mu_2 = \frac{y_2}{y_3}. \tag{7}$$

Then, substituting Eq. (7) into (11), the following equations can be gotten:

$$\begin{aligned} (\ln y_3)_x &= (\alpha r + \beta q)\mu_1 + (\alpha q + \beta^* r)\mu_2 - i\lambda, \\ (\ln y_3)_t &= [-\beta q_{xx} - \alpha r_{xx} + 2i\lambda(\alpha r_x + \beta q_x) \\ &\quad +2(\beta q^2 + 2\alpha rq + \beta^* r^2 + 2\lambda^2)(\alpha r + \beta q)]\,\mu_1 \\ &\quad +\left[-\alpha q_{xx} - \beta^* r_{xx} + 2i\lambda(\alpha q_x + \beta^* r_x)\right. \\ &\quad \left.+2(\beta q^2 + 2\alpha rq + \beta^* r^2 + 2\lambda^2)(\alpha q + \beta^* r)\right]\mu_2 \\ &\quad -2i(\beta q^2 + 2\alpha rq + \beta^* r^2)\lambda - 4i\lambda^3. \end{aligned} \tag{8}$$

Cross-differentiating these two equations with respect to t and x, respectively, we have

$$\begin{aligned}&\left[(\alpha r+\beta q)\mu_1+(\alpha q+\beta^* r)\mu_2\right]_t\\&\quad=[[-\beta q_{xx}-\alpha r_{xx}+2i\lambda(\alpha r_x+\beta q_x)\\&\qquad+2(\beta q^2+2\alpha rq+\beta^* r^2+2\lambda^2)(\alpha r+\beta q)]\,\mu_1\\&\qquad-2i(\beta q^2+2\alpha rq+\beta^* r^2)\lambda\\&\qquad+\left[-\alpha q_{xx}-\beta^* r_{xx}+2i\lambda(\alpha q_x+\beta^* r_x)\right.\\&\qquad\left.+2(\beta q^2+2\alpha rq+\beta^* r^2+2\lambda^2)(\alpha q+\beta^* r)\right]]_x,\end{aligned}$$

then expand μ_1 and μ_2 into the following series

$$\mu_k(x,t,\lambda)=\sum_{n=1}^{\infty}\frac{\mu_k^{(n)}(x,t,\lambda)}{(2i\lambda)^n},\ k=1,2. \tag{9}$$

Equating the same powers of λ, the density of the infinitely conservation law is $(\alpha r+\beta q)\mu_1^n+(\alpha q+\beta^* r)\mu_2^n$. Furthermore, the infinitely conserved quantities can also be concluded that:

$$I_n=\int_{-\infty}^{\infty}\left[(\alpha r+\beta q)\mu_1^{(n)}+(\alpha q+\beta^* r)\mu_2^{(n)}\right]dx,\quad n=1,2,\ldots.$$

From the space part of Lax pair (11), after eliminating $y_k,\ k=1,2,3$, we can also get that μ_1 and μ_2 satisfy the following Riccati equations

$$\begin{cases}\mu_{1,x}+(\beta q+\alpha r)\mu_1^2+(\alpha q+\beta^* r)\mu_1\mu_2=2i\lambda\mu_1+q,\\ \mu_{2,x}+(\alpha q+\beta^* r)\mu_2^2+(\beta q+\alpha r)\mu_1\mu_2=2i\lambda\mu_2+r.\end{cases} \tag{10}$$

Similarly, after making the asymptotic expansions of μ_k as in (9) and collecting the same powers of λ, we can get the expansions of $\mu_k^n,\ k=1,2,\ n=1,2,\ldots$

$$\begin{aligned}&\mu_1^{(1)}=q,\quad \mu_2^{(1)}=r,\\&\mu_1^{(2)}=q_x,\quad \mu_2^{(2)}=r_x,\\&\mu_1^{(3)}=q_{xx}+(\beta q+\alpha r)q^2+(\alpha q+\beta^* r)qr,\\&\mu_2^{(3)}=r_{xx}+(\alpha q+\beta^* r)r^2+(\beta q+\alpha r)qr,\\&\qquad\vdots\\&\mu_1^{(n)}=\mu_{1,x}^{(n-1)}+(\beta q+\alpha r)\sum_{k,j=1..(n-2)}^{k+j=n-1}\mu_1^{(k)}\mu_1^{(j)}\\&\qquad+(\alpha q+\beta^* r)\sum_{k,j=1..(n-2)}^{k+j=n-1}\mu_1^{(k)}\mu_2^{(j)},\\&\mu_2^{(n)}=\mu_{2,x}^{(n-1)}+(\alpha q+\beta^* r)\sum_{k,j=1..(n-2)}^{k+j=n-1}\mu_2^{(k)}\mu_2^{(j)}\\&\qquad+(\beta q+\alpha r)\sum_{k,j=1..(n-2)}^{k+j=n-1}\mu_1^{(k)}\mu_2^{(j)},\\&\qquad\vdots\end{aligned}$$

Under the reverse-space-time reduction $r(x,t)=-q(-x,-t)$, the first three conserved quantities of the nonlocal Eq. (1) are:

$$\begin{aligned}I_1&=\int_{-\infty}^{\infty}[(\beta q(x,t)-\alpha q(-x,-t))q(x,t)\\&\quad-(\alpha q(x,t)-\beta^* q(-x,-t))q(-x,-t)]\,dx,\\I_2&=\int_{-\infty}^{\infty}[(\beta q(x,t)-\alpha q(-x,-t))q_x(x,t)\\&\quad-(\alpha q(x,t)-\beta^* q(-x,-t))q_x(x,-t)]\,dx,\\I_3&=\int_{-\infty}^{\infty}[[q_{xx}(x,t)+(\beta q(x,t)-\alpha q(-x,-t))q(x,t)^2\\&\quad-(\alpha q(x,t)-\beta^* q(-x,-t))q(x,t)q(-x,-t)]\\&\quad(\beta q(x,t)-\alpha q(-x,-t))-(\alpha q(x,t)\\&\quad-\beta^* q(-x,-t))\,[-q_{xx}(x,-t)+(\alpha q(x,t)\\&\quad-\beta^* q(-x,-t))q(-x,-t)^2\\&\quad-(\beta q(x,t)-\alpha q(-x,-t))q(x,t)q(-x,-t)]]\,dx.\end{aligned}$$

The other higher-order conserved quantities can also be obtained by continuous iteration. In addition, starting from the temporal part of Lax pair (11) and repeating similar steps of the spatial part, the expansions of $\mu_k^n,\ k=1,2,\ n=1,2,\ldots$ and conserved quantities which are associated with t can be present.

4 Inverse scattering theory and N soliton solutions

To establish the inverse scattering theory for the initial problem of the nonlocal Eq. (1) in the framework of the RH problem, it is required to construct the inverse scattering theory and analyze the properties of the eigenvectors and scattering data.

4.1 The framework of the Riemann–Hilbert problem

Review the generic process framework of the RH method. We start from the following linear matrix equations

$$\begin{cases}Y_x=i\lambda\Lambda Y+QY,\\ Y_t=4i\lambda^3\Lambda Y+(4\lambda^2 Q+2i\lambda(Q^2+Q_x)\Lambda\\ \qquad-[Q,Q_x]-Q_{xx}+2Q^3)Y,\end{cases} \tag{11}$$

where $[Q,Q_x]=Q_xQ-QQ_x$ with

$$Q=\begin{pmatrix}0&0&q\\0&0&r\\\beta q+\alpha r&\alpha q+\beta^* r&0\end{pmatrix}.$$

By the initial value condition (2), the potential function decays rapidly to zero at infinity. Thus, the asymptotic properties of the solution of the Lax pair are $Y\propto \mathrm{e}^{i\lambda\Lambda(x+4\lambda^2 t)}$, $x\to\infty$. In what follows, we introduce the matrix spectral function transform:

$$Y=J\mathrm{e}^{-i\lambda\Lambda\left(x+4\lambda^2 t\right)}. \tag{12}$$

where $J = J(x, t; \lambda)$, so J satisfies the asymptotic properties

$$J \to I, \ x \to \infty \text{ or } t \to \infty \tag{13}$$

and by Lax pair (11), and the transformation (12) J satisfies the following linear equations

$$\begin{cases} J_x = i\lambda[\Lambda, J] + QJ, \\ T_t = 4i\lambda^3[\Lambda, J] + (4\lambda^2 Q + 2i\lambda(Q^2 + Q_x)\Lambda \\ \quad -[Q, Q_x] - Q_{xx} + 2Q^3)J. \end{cases} \tag{14}$$

with $[\Lambda, J] = \Lambda J - J\Lambda$. It is obvious that the Lax pair (14) is equivalent to the Lax pair (11) through a transformation. Besides, on account of $tr(Q) = 0$ and $tr(\tilde{Q}) = 0$, by Liouville's formula and the asymptotic properties (13), we have $det(J) = 1$. In order to study the solution of (1) with the help of RH problem, we treat the Lax pair (14) as the scattering problem with time t fixed. Thus, we consider the RH problem of the following scattering equation:

$$J_x(x; \lambda) = i\lambda[\Lambda, J(x; \lambda)] + Q(x)J(x; \lambda) \tag{15}$$

We introduce the adjoint problem of the scattering problem (15):

$$\tilde{J}_x(x, \lambda) = i\lambda[\Lambda, \tilde{J}] - \tilde{J}Q. \tag{16}$$

We introduce $J_{1,2}$ as the Jost solutions of Eq. (15)

$$J_1 \to I, x \to -\infty, \quad J_2 \to I, x \to +\infty \tag{17}$$

and sign

$$\begin{aligned} J_1 E &= \Phi = (\phi_1, \phi_2, \phi_3), \\ J_2 E &= \Psi = (\psi_1, \psi_2, \psi_3) \end{aligned} \tag{18}$$

with $E = e^{i\lambda\Lambda x}$. It is easy to verify that J^{-1} solves Eq. (16). So $\tilde{J}_1 = J_1^{-1}$ and $\tilde{J}_2 = J_2^{-1}$ solve Eq. (15). Since Φ and Ψ are solutions of the linear ordinary differential Eq. (11), there should be a matrix $S(\lambda) = (s_{ij})_{3\times 3}$, which is independent of x that satisfies

$$\Phi = \Psi S(\lambda), \ \lambda \in \mathbb{R}. \tag{19}$$

Furthermore, we can get $J_1 = J_2 ESE^{-1}$, $\tilde{J}_1 = ESE^{-1}\tilde{J}_2$, by the Abel's identity and $tr(Q) = 0$, we have $\det(\Phi) = \det(\Psi) = 1$, then

$$\det S(\lambda) = 1, \lambda \in \mathbb{R}.$$

Imposing the boundary conditions (17) and by (11), $J_{1,2}$ can be resigned as Volterra-type integral equations

$$\begin{aligned} J_1 &= I - \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)} Q(y) J_1(y, \lambda) e^{-i\lambda\Lambda(x-y)} dy, \\ J_2 &= I + \int_{x}^{+\infty} e^{i\lambda\Lambda(x-y)} Q(y) J_2(y, \lambda) e^{-i\lambda\Lambda(x-y)} dy. \end{aligned} \tag{20}$$

Then, expressions (20) imply the analytical properties of $J_{1,2}$, $J_1 = \left[J_{1,1}^-, J_{1,2}^-, J_{1,3}^+\right]$, $J_2 = \left[J_{2,1}^+, J_{2,2}^+, J_{2,3}^-\right]$, where the superscript "±" represents that the functions can be analytically extended to on the upper half or lower half complex λ-plane, respectively. Notice that $J_j^{-1} (j = 1, 2)$ satisfy Eq. (16) and the analytical properties

$$\begin{aligned} J_1^{-1} &= E\hat{\Phi} = \hat{J}_1 = (\hat{J}_{1,1}^+, \hat{J}_{1,2}^+, \hat{J}_{1,3}^-)^T, \\ J_2^{-1} &= E\hat{\Psi} = \hat{J}_2 = (\hat{J}_{2,2}^{\ -}, \hat{J}_{2,2}^-, \hat{J}_{2,3}^+)^T. \end{aligned}$$

Introducing matrix $H_1 = \text{diag}(1, 1, 0)$ and $H_2 = \text{diag}(0, 0, 1)$, we denote

$$\begin{aligned} P_+ &= \left[J_{2,1}^+, J_{2,2}^+, J_{1,3}^+\right] = J_1 H_2 + J_2 H_1, \\ P_-^{-1} &= \left[\hat{J}_{1,3}^-, \hat{J}_{2,2}^-, \hat{J}_{2,2}^-\right]^T = H_2 J_1^{-1} + H_1 J_2^{-1}. \end{aligned} \tag{21}$$

So P_+ can be analytically extended to $\lambda \in \mathbb{C}_+$ and P_-^{-1} and can be analytically extended to $\lambda \in \mathbb{C}_-$. The large λ-asymptotic behavior

$$\begin{aligned} &P_+(x, \lambda) \to I, \ \lambda \in \mathbb{C}_+ \to \infty, \\ &P_-^{-1} \to I, \lambda \in \mathbb{C}_- \to \infty. \end{aligned} \tag{22}$$

Implementing the direct calculation on (21), we take the notation $S^{-1}(\lambda) = (\hat{s}_{ij})_{3\times 3}$, and we have

$$\det(P_+) = \hat{s}_{33}, \quad \det(P_-^{-1}) = s_{33}.$$

By (18) and the analytical properties of $J_{1,2}$ and combined with Eq. (19), the analytical properties of the scattering matrix can be obtained as:

$$\begin{aligned} S &= \Phi^{-1}\Psi = \begin{pmatrix} s_{11}^+ & s_{12}^+ & s_{13} \\ s_{21}^+ & s_{22}^+ & s_{23} \\ s_{31} & s_{32} & s_{33}^- \end{pmatrix}, \\ S^{-1} &= \Psi^{-1}\Phi = \begin{pmatrix} \hat{s}_{11}^- & \hat{s}_{12}^- & \hat{s}_{13} \\ \hat{s}_{21}^- & \hat{s}_{22}^- & \hat{s}_{23} \\ \hat{s}_{31} & \hat{s}_{32} & \hat{s}_{33}^+ \end{pmatrix}, \end{aligned} \tag{23}$$

where the absence of a superscript means that the function cannot be analytically extended to the upper or lower half complex plane.

4.2 Symmetry relations of scattering data

To investigate the symmetry relation of the eigenvalues and scattering data for the coupled generalized SS Eq. (5) of q and r, the symmetry of the potential matrix

$$Q = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & r \\ \beta q + \alpha r & \alpha q + \beta^* r & 0 \end{pmatrix}$$

can be given. For the potential matrix Q, we have

$$Q = -B_0^{-1} Q^T B_0, \tag{24}$$

where

$$B_0 = \begin{pmatrix} \beta & \alpha & 0 \\ \alpha & \beta^* & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The establishment of Eq. (24) can be checked by direct matrix calculation. This special symmetry constraints (24) of Q will add new constraints for the eigenvectors, eigenvalues and scattering data.

Proposition 1 *For the coupled generalized SS Eq. (5), if J is a solution of scattering Eq. (15), then $B_0^{-1} J^T B_0$ is also a solution of the adjoint scattering Eq. (16) when $\bar{\lambda} = -\lambda$. The Jost function J, the scattering matrix S and the Jost solution P satisfy the following relation:*

$$J^{-1}(-\lambda) = B_0^{-1} J^T(-\lambda) B_0, \tag{25}$$

$$S^{-1}(\lambda) = -B_0^{-1} S^T(-\lambda) B_0, \tag{26}$$

$$P_-^{-1}(-\lambda) = B_0^{-1} {P_+}^T(\lambda) B_0. \tag{27}$$

Proof For the scattering Eq. (15),

$$J_x(x;\lambda) = i\lambda \Lambda J(x;\lambda) - i\lambda J(x;\lambda)\Lambda + Q(x)J(x;\lambda).$$

Taking transpose firstly and then multiplying B_0^{-1} from the left side on the above equation, we have

$$\begin{aligned} B_0^{-1} J_x^T(x;\lambda) B_0 = &\, i\lambda B_0^{-1} J^T(x;\lambda) \Lambda B_0 \\ & - i\lambda B_0^{-1} \Lambda J^T(x;\lambda) B_0 \\ & + B_0^{-1} J^T(x;\lambda) Q^T(x) B_0. \end{aligned}$$

By the symmetry relation (24), we can get that the following equation

$$\begin{aligned} B_0^{-1} J_x^T(x;\lambda) B_0 = &\, i\lambda B_0^{-1} J^T(x;\lambda) \Lambda B_0 \\ & - i\lambda B_0^{-1} \Lambda J^T(x;\lambda) B_0 \\ & - B_0^{-1} J^T(x;\lambda) B_0 Q. \end{aligned}$$

i.e.,

$$\begin{aligned} B_0^{-1} J_x^T(x;\lambda) B_0 = &\, -i\lambda [\Lambda, B_0^{-1} J^T(x;\lambda) B_0] \\ & + B_0^{-1} J^T(x;\lambda) B_0 Q. \end{aligned}$$

So if $J(x;\lambda)$ is a solution of scattering Eq. (15), then $B_0^{-1} J^T(x;\lambda) B_0$ is also a solution of the adjoint scattering Eq. (16) when $\bar{\lambda} = -\lambda$. Due to the asymptotic behaviors (17) with symmetry relation of Jost function J,

$$J^{-1}(-\lambda) = B_0^{-1} J^T(-\lambda) B_0. \tag{28}$$

Moreover, by Eq. (19) and Proposition 1, the symmetry property of the scattering matrix S and the relation between P_-^{-1} and P_+ can also be obtained. Then, we can conclude the following proposition. The details of the proof are omitted here. By Eq. (19) and the notations of (18), we have

$$J_1(x,\lambda) E = J_2(x,\lambda) E S(\lambda) \tag{29}$$

Taking transpose on the above equation and then multiplying B_0^{-1} from left side and B_0 from right side, we have

$$B_0^{-1} E J_1^T(x,\lambda) B_0 = B_0^{-1} S^T(\lambda) E J_2^T(x,\lambda) B_0.$$

Applying Eq. (25) to the above equation, the following equation can be gotten:

$$E J_1^{-1}(x,-\lambda) = B_0^{-1} S^T(\lambda) B_0 E J_2^{-1}(x,-\lambda).$$

For Eq. (29), taking the inverse of the matrix equation and making some deformations, then

$$E J_1^{-1}(x,-\lambda) = S^{-1}(-\lambda) E J_2^{-1}(x,-\lambda)$$

So $S^{-1}(-\lambda) = B_0^{-1} S^T(\lambda) B_0$. Besides, by the definition of P_-^{-1} and P_+ and Eq. (25), the symmetry relation can be obtained. □

Lemma 1 *For the couple generalized SS Eq. (5), if $\lambda_j \in \mathbb{C}_+$ is a eigenvalue of the spectral Eq. (15), the $\bar{\lambda}_j = -\lambda_j \in \mathbb{C}_-$ is eigenvalue of the adjoint spectral Eq. (16). And the scattering eigenvectors satisfy: $\bar{v}_{k_0} = v_{k_0}^T B_0^{-1}$, and the scattering data are connected by the relations*

$$\bar{a}_k = \frac{b_k \alpha - a_k \beta^*}{\alpha^2 - |\beta|^2},\ \bar{b}_k = \frac{a_k \alpha - b_k \beta}{\alpha^2 - |\beta|^2},\ \bar{c}_k = -c_k.$$

Proof By Eq. (26) in Proposition 1, it is natural that the symmetry relations of the elements for S and S^{-1} can be concluded.

$$\begin{aligned} \hat{s}_{31}(\lambda) &= -a s_{23}(-\lambda) - b s_{13}(-\lambda), \\ \hat{s}_{32}(\lambda) &= -a s_{13}(-\lambda) - b^* s_{23}(-\lambda), \\ \hat{s}_{33}(\lambda) &= s_{33}(-\lambda). \end{aligned} \tag{30}$$

Recall the determinant of P_+, P_-^{-1}

$$\det(P_+) = \hat{s}_{33},\ \det(P_-^{-1}) = s_{33},$$

so by (30), if $\lambda_k \in \mathbb{C}^+$ is the zero of $\hat{s}_{33}$, then $-\lambda_k$ is the zero of s_{33}, i.e., the zero of $\det(P_-^{-1})$.

$$P_+(\lambda_k) v_k = 0,\ \bar{v}_k P_-(\bar{\lambda}_k) = 0 \tag{31}$$

The eigenvector solutions of the (31) can be expressed as $v_k = e^{i\lambda_k \sigma x} v_{k_0} = e^{i\lambda_k \sigma x}(a_k, b_k, c_k)^T$ and $\bar{v}_k = (\bar{a}_k, \bar{b}_k, \bar{c}_k)^T e^{i\bar{\lambda}_k \sigma x} = \bar{v}_{k_0} e^{-i\bar{\lambda}_k \sigma x}$.

Take transpose on the equation $P_+(\lambda_k)v_k = 0$, and by symmetry constraints (27), then

$$v_k^T B_0^{-1} P_-(-\lambda_k) = 0.$$

Obviously, $v_k^T B_0^{-1}$ is the eigenvector of P_-, i.e., $\bar{v}_k = v_k^T B_0^{-1}$. We expand the matrix and the symmetry relations of the scattering data

$$\bar{a}_k = \frac{b_k\alpha - a_k\beta^*}{\alpha^2 - |\beta|^2},\ \bar{b}_k = \frac{a_k\alpha - b_k\beta}{\alpha^2 - |\beta|^2},\ \bar{c}_k = -c_k.$$

This completes the proof of the lemma. □

4.3 Riemann–Hilbert problem and the N-soliton solutions

We take the real axis as the closed contour which pass through the infinity, and the matrix function $P_\pm$ analytical inside and outside of the contour, respectively. Thus, the RH problem of the spectral problem (15) can be constructed as follows. *Riemann–Hilbert Problem 1* For $(x, t) \in R^2$, solving a 3×3 matrix-value function $P(x, t, \lambda)$ in the complex $\lambda-$plane such that

- The matrix function $P_\pm$ is analytic in $\mathbb{C}_\pm$.
- The canonical normalization condition is P_+, $P_-^{-1} \to I$ as $\lambda \to \infty$.
- The RH problem is well defined on the real line

  $$P_-^{-1}(x, \lambda)P_+(x, \lambda) = G(x, \lambda),\ \lambda \in \mathbb{R} \tag{32}$$

  with the jump matrix

  $$G(x, \lambda) = E\begin{bmatrix} 1 & 0 & s_{13} \\ 0 & 1 & s_{23} \\ \hat{s}_{31} & \hat{s}_{32} & 1 \end{bmatrix} E^{-1}.$$

Expand P_+, P_-^{-1} at $\lambda = \infty$, then substitute $P_+ = I + \lambda^{-1}P_+^{[1]} + O(\lambda^{-2})$, $P_-^{-1} = I + \lambda^{-1}P_-^{[1]} + O(\lambda^{-2})$ into (15) and (16), respectively, and collect the same power of λ, we can get $Q = -i[\Lambda, P_+^{[1]}]$, $Q = i[\Lambda, P_-^{[1]}]$. So the potentials can be recovered

$$u = -2i(P_+^{[1]})_{13},\quad v = -2i(P_+^{[1]})_{23}. \tag{33}$$

4.4 N-soliton formulas with N simple zeros

In most cases, the RH problem 1 is nonregular, i.e., $\det(P_+)$ and $\det(P_-^{-1})$ have zeros in λ-plane. Here we begin with the case of simple zeros. If $\det(P_+)$ and $\det(P_-^{-1})$ only have simple zeros in complex λ-plane. By the symmetry relations of the scattering data in Lemma 1, suppose the simple zeros are $(\lambda_k, -\lambda_k)(k = 1, 2, \ldots, N)$, respectively. Thus, for each pair of the eigenvalues $(\lambda_k, -\lambda_k)$, there exist only one pair linearly independent eigenvectors $(v_k, \bar{v}_k^T)$. By Lemma 1, the eigenvectors are

$$\begin{aligned} v_k &= e^{i\theta(\lambda_k)\Lambda}(a_k, b_k, c_k)^T, \\ \bar{v}_k^T &= \left(\frac{b_k\alpha - a_k\beta^*}{\alpha^2 - |\beta|^2}, \frac{a_k\alpha - b_k\beta}{\alpha^2 - |\beta|^2}, -c_k\right) e^{i\theta(\lambda_k)\Lambda} \end{aligned} \tag{34}$$

Then, when $G = I$ and the RH problem is reflectionless, similar to the process in [45], the N-soliton formulas can be derived in the following theorem.

Theorem 1 *The N-soliton formulas for the nonlocal generalized SS Eq. (1) can be represented as*

$$q(x, t) = 2i\frac{\begin{vmatrix} M & Y_3 \\ \bar{Y}_1^T & 0 \end{vmatrix}}{|M|},\quad r(x, t) = 2i\frac{\begin{vmatrix} M & Y_3 \\ \bar{Y}_2^T & 0 \end{vmatrix}}{|M|} \tag{35}$$

where $Y = [v_1, \ldots, v_N]$, $\bar{Y} = [\bar{v}_1, \ldots, \bar{v}_N]$ and Y_k $\bar{Y}_k$ are the k-th row of the matrix Y, $\bar{Y}$ with vectors $(v_j, \bar{v}_j)$ are given in (34) and $M = (m_{jk})_{N\times N}$ with
$$m_{jk} = \frac{\bar{v}_j^T v_k}{\bar{\lambda}_j - \lambda_k},\quad 1 \le j, k \le N.$$

Remark Compared to [45], it is obvious that the different symmetry relations of scattering data lead to quite different forms of the solution.

5 The nonlocal constraints of the nonlocal generalized SS Eq. (1)

This section will focus on the nonlocal constraints of the eigenvectors and scattering data for Eq. (1). The symmetry relations for Eq. (1), after a lot of complicated calculations and analysis, the symmetry relations for Eq. (1) cannot be obtained directly. The symmetry constraints for the coupled generalized SS Eq. (5) and the corresponding symmetry relations for the eigenvectors and scattering data have been imposed in the last section. Based on the previous solution for Eq. (5), let $r(x, t) = -q(-x, -t)$, and then, the corresponding nonlocal constraints for Eq. (1) can be obtained.

Theorem 2 *For the nonlocal generalized SS Eq. (1), if $\lambda_k \in \mathbb{C}_+$ is a discrete eigenvalue, then the parameters of the corresponding eigenvectors are linearly dependent with each other. The parameters in the single- and* *two-soliton solutions (35) meet the following relationships. For the eigenvalues $\lambda_k \in \mathbb{C}_+$, the corresponding eigenvector satisfies the following constraints*

$$b_k = \frac{(\alpha+\beta^*)\, a_k}{\alpha+\beta},\quad c_k = \frac{\sqrt{2\alpha+\beta+\beta^*}\, a_k}{\alpha+\beta},\ k=1,2. \tag{36}$$

or

$$b_k = -\frac{(\alpha-\beta^*)\, a_k}{\alpha-\beta},\quad c_k = \frac{\sqrt{2\alpha-\beta-\beta^*}\, a_k}{\alpha-\beta},\ k=1,2. \tag{37}$$

Proof The nonlocal Eq. (1) is reduced from the coupled Eq. (5); the eigenvalues for Eq. (5) in $\mathbb{C}_+$ and $\mathbb{C}_-$ are exhibited in pairs $(\lambda_k, -\lambda_k)$, so the eigenvalues for the nonlocal Eq. (1) are also in pairs. And according to Theorem 1, the corresponding eigenvectors are connected in $\bar{v} = v^T B_0^{-1}$. However, the nonlocal Eq. (1) possess more symmetry relations due to the nonlocal reduction, which cannot be obtained directly. Next, we will consider how to get the special nonlocal constraints for the soliton solutions of the nonlocal Eq. (1). We begin from the single soliton solution. When $N=1$ in (35), for the eigenvalues $\lambda_1 \in \mathbb{C}_+$, the single soliton solution for the coupled Eq. (5) can be obtained as:

$$\begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} \alpha b_1 - \beta^* a_1 \\ \alpha a_1 - \beta b_1 \end{bmatrix} \frac{4\mathrm{i} c_1 \lambda_1}{(-\alpha^2 c_1^2 + |\beta|^2 c_1^2)\, \mathrm{e}^{-2\mathrm{i}\lambda_1(4t\lambda_1^2+x)} + (2\alpha a_1 b_1 - \beta b_1^2 - \beta^* a_1^2)\, \mathrm{e}^{2\mathrm{i}\lambda_1(4t\lambda_1^2+x)}} \tag{38}$$

Replacing (x,t) with $(-x,-t)$ in q and taking the opposite sign in (38), by the reduction $r(x,t) = -q(-x,-t)$, then the nonlocal constraints for the single-soliton solution of the nonlocal Eq. (1) are

$$b_k = \frac{(\alpha+\beta^*)}{\alpha+\beta} a_k,\quad c_k = \pm\frac{\sqrt{-2\alpha+\beta+\beta^*}}{\alpha+\beta} a_k,$$

or

$$b_k = -\frac{(\alpha-\beta^*)}{\alpha-\beta} a_k,\quad c_k = \pm\frac{\sqrt{-2\alpha-\beta-\beta^*}}{\alpha-\beta} a_k.$$

When $N=2$, for the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}_+$, we introduce the notations $\theta_1 = 2\mathrm{i}\lambda_1(4t\lambda_1^2+x), \theta_2 = 2\mathrm{i}\lambda_2(4t\lambda_2^2+x\ )$; the two-soliton solutions for the coupled Eq. (5) can be written as:

$$q = \frac{4\mathrm{i}(\lambda_1+\lambda_2)^2\big[A_{1,1}e^{\theta_1} + A_{1,2}e^{\theta_2}\big] - \big[c_1\lambda_2(\alpha b_2 - \beta^* b_2)e^{-\theta_1} - c_2\lambda_1(\alpha b_1 - \beta^* a_1)e^{-\theta_2}\big]A_0}{\big[c_2^2(\beta b_1^2 + \beta^* a_1^2 - 2\alpha a_1 b_1)e^{\theta_1-\theta_2} + c_1^2(\beta b_2^2 + \beta^* a_2^2 - 2\alpha a_2 b_2)e^{\theta_2-\theta_1}\big]B_0 + B_1 e^{\theta_1+\theta_2} + B_2 e^{-\theta_1-\theta_2} + C_0}$$

$$r = \frac{4\mathrm{i}(\lambda_1+\lambda_2)^2\big[A_{1,1}e^{\theta_1} + A_{1,2}e^{\theta_2}\big] - \big[c_1\lambda_2(\alpha a_2 - \beta b_2)e^{-\theta_1} - c_2\lambda_1(\alpha a_1 - \beta b_1)e^{-\theta_2}\big]A_0}{\big[c_2^2(\beta b_1^2 + \beta^* a_1^2 - 2\alpha a_1 b_1)e^{\theta_1-\theta_2} + c_1^2(\beta b_2^2 + \beta^* a_2^2 - 2\alpha a_2 b_2)e^{\theta_2-\theta_1}\big]B_0 + B_1 e^{\theta_1+\theta_2} + B_2 e^{-\theta_1-\theta_2} + C_0} \tag{39}$$

with

$$\begin{aligned}
A_0 &= 4\mathrm{i}c_1c_2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)\left(|\beta|^2-\alpha^2\right),\\
B_0 &= \left(|\beta|^2-\alpha^2\right)(\lambda_1+\lambda_2)^2\\
B_1 &= \left(2\alpha a_2 b_2 - \beta {b_2}^2 - \beta^* {a_2}^2\right)\left(2\alpha a_1 b_1 - \beta {b_1}^2 - \beta^* {a_1}^2\right){\lambda_1}^2\\
&\quad -2\big[\left(2{a_1}^2{b_2}^2 + 2{a_2}^2{b_1}^2\right)\alpha^2\\
&\quad -2\,(a_1b_2+a_2b_1)\,(\beta b_1 b_2 + a_1 a_2 \beta^*)\,\alpha\\
&\quad +\beta^{*2}{a_1}^2{a_2}^2 - \beta\left({a_1}^2{b_2}^2 - 4a_1a_2b_1b_2 + {a_2}^2{b_1}^2\right)\beta^*\\
&\quad +\beta^2{b_1}^2{b_2}^2\big]\,\lambda_2\lambda_1\\
&\quad +{\lambda_2}^2\left(2\alpha a_2 b_2 - \beta {b_2}^2 - \beta^* {a_2}^2\right)\left(2\alpha a_1 b_1 - \beta {b_1}^2 - \beta^* {a_1}^2\right),\\
B_2 &= -\left(|\beta|^2-\alpha^2\right)^2 c_1^2 c_2^2 \left(\lambda_1-\lambda_2\right)^2,\\
C_0 &= 8\left(|\beta|^2-\alpha^2\right)\lambda_1\left(\alpha a_1 b_2 + \alpha a_2 b_1 - \beta b_1 b_2 - a_1 a_2 \beta^*\right)\lambda_2 c_1 c_2,\\
A_{1,1} &= c_2\lambda_2\big[\big(-\beta^{*2}{a_1}^2 a_2 + \big(\left({a_1}^2 b_2 + 2a_1a_2b_1\right)\alpha - 2\beta a_1 b_1 b_2\\
&\quad +\beta a_2 {b_1}^2\big)\beta^* + \alpha {b_1}^2\left(-2\alpha a_2 + \beta b_2\right)\big)\lambda_1\\
&\quad -\lambda_2\left(\alpha b_2 - \beta^* a_2\right)\left(-2\alpha a_1 b_1 + \beta {b_1}^2 + \beta^* {a_1}^2\right)\big],\\
A_{1,2} &= -c_1\lambda_1\big[\left(\alpha b_1 - \beta^* a_1\right)\left(-2\alpha a_2 b_2 + \beta {b_2}^2 + \beta^* {a_2}^2\right)\lambda_1\\
&\quad -\big(-\beta^{*2}a_1{a_2}^2 + \big(\left(2a_1a_2b_2 + {a_2}^2 b_1\right)\alpha\\
&\quad +\beta b_2\left(a_1b_2 - 2a_2b_1\right)\big)\beta^* + \alpha {b_2}^2\left(-2\alpha a_1 + \beta b_1\right)\big)\lambda_2\big],\\
A_{2,1} &= c_2\lambda_2\big[-2b_1\big(\big(b_2\left(\lambda_2-\lambda_1\right)\alpha + \beta^* a_2\lambda_1\big)\beta - \lambda_2\alpha^2 a_2\big)a_1\\
&\quad +\beta {b_1}^2\left(\lambda_2-\lambda_1\right)\left(-\alpha a_2 + \beta b_2\right)\\
&\qquad \big(\beta^* b_2\left(\lambda_1+\lambda_2\right)\beta - \alpha\left(2\alpha b_2\lambda_1 + \beta^* a_2\left(\lambda_2-\lambda_1\right)\right)\big)a_1^2\big],\\
A_{2,2} &= c_1\lambda_1\big[-\big(\beta^2 b_1 {b_2}^2 + \big(2\beta^* a_1 a_2 b_2 - \beta^* {a_2}^2 b_1\\
&\quad -\left(a_1{b_2}^2 + 2a_2b_1b_2\right)\alpha\big)\beta + 2\alpha^2{a_2}^2 b_1 - \alpha\beta^* a_1 {a_2}^2\big)\lambda_2\\
&\qquad \left(-\alpha a_1 + \beta b_1\right)\left(-2\alpha a_2 b_2 + \beta {b_2}^2 + \beta^* {a_2}^2\right)\lambda_1\big].
\end{aligned}$$

Replacing (x,t) with $(-x,-t)$ of q in Eq. (39), then we have

$$q(-x,-t) = \frac{4\mathrm{i}\left(\lambda_1+\lambda_2\right)^2\big[A_{1,1}e^{-\theta_1} + A_{1,2}e^{-\theta_2}\big] - \big[c_1\lambda_2\left(\alpha b_2 - \beta^* b_2\right)e^{\theta_1} - c_2\lambda_1\left(\alpha b_1 - \beta^* a_1\right)e^{\theta_2}\big]A_0}{\big[c_2^2\left(\beta b_1^2 + \beta^* a_1^2 - 2\alpha a_1 b_1\right)e^{\theta_2-\theta_1} + c_1^2\left(\beta b_2^2 + \beta^* a_2^2 - 2\alpha a_2 b_2\right)e^{\theta_1-\theta_2}\big]B_0 + B_1 e^{-\theta_1-\theta_2} + B_2 e^{\theta_1+\theta_2} + C_0}$$

when $r(x,t) = -q(-x,-t)$, i.e., $r(x,t) + q(-x,-t) = 0$, one can conclude two sets of nonzero solution:

$$b_k = \frac{(\alpha+\beta^*)\, a_k}{\alpha+\beta},\ c_k = \frac{\sqrt{2\alpha+\beta+\beta^*}\, a_k}{\alpha+\beta},\ k=1,2,$$

or

$$b_k = -\frac{(\alpha-\beta^*)\, a_k}{\alpha-\beta},\ c_k = \frac{\sqrt{2\alpha-\beta-\beta^*}\, a_k}{\alpha-\beta},\ k=1,2.$$

6 Dynamic behaviors of the solutions with simple zeros and N multiple zeros

In this section, we will study the dynamic behaviors of the soliton solutions with simple zeros and N multiple zeros. Firstly, we will derive the exact expressions for single- and two-soliton solutions and the higher-order soliton solutions with double and triple zero for the nonlocal Eq. (1).

6.1 Single-soliton solutions

Rewrite $\beta = \kappa_1 + i\kappa_2$ and $\lambda_k = \xi_k + i\eta_k$ with κ_1, κ_2 are real number and k is a positive integer. When $N = 1$ in Eq. (35), by Theorem 2 and Eq. (36), the single-soliton solutions for the nonlocal Eq. (1) can be obtained.

$$q_1 = \frac{2\mathrm{i}\lambda_1}{\sqrt{2\alpha - 2\kappa_1}\cos\left(8t\lambda_1^3 + 2\lambda_1 x\right)} \tag{40}$$

or by (37), the single-soliton solutions are

$$q_1 = \frac{2\lambda_1}{\sqrt{2\alpha + 2\kappa_1}\sin\left(8\lambda_1^3 t + 2\lambda_1 x\right)}. \tag{41}$$

The two different expressions (40) and (41) are similar in properties with the parameter condition $|\alpha| \neq |\kappa_1|$; we will only analyze the dynamic behaviors of (40). The expression of single soliton (40) contains four real parameters α, κ_1, ξ_1 and η_1. From expression (40), α and κ_1 only influence the amplitude of the general single-soliton solution. With different values of α and κ_1, the amplitude of soliton changes. It is shown in Fig. 1e with different α. In order to investigate the influence of ξ_1 and η_1 and illustrate easily, $\alpha = 1,\ \beta = \frac{1}{2}$ is adopted in the following analysis of this section.

(1) When λ_1 is not purely imaginary, the solution is a singular soliton that collapses periodically which is shown in Fig. 1a, b, where the real part of λ_1 and the imaginary part change the period of collapse and the direction of propagation. Figure 1a shows the collapsed general single-soliton solution with $\lambda_1 = \frac{1}{3} + \frac{i}{2}$. And when ξ_1 increases to $\frac{1}{3}$, the propagation direction of the single soliton closes to $t = 0$. But the soliton stays singular with different λ_1.

(2) When λ_1 is purely imaginary, i.e., $\xi_1 = 0$, the solution does not behave as the fundamental general single-soliton solution because the denominator of the general single-soliton solution is a trigonometric function which zeros are periodically appear in x-axis. Here we give two examples, which are shown in Fig. 1c, d, where $\eta_1 = 2 \times 10^{-2}\mathrm{i}$ in (c) and when $\eta_1 \to 0$, the collapse periodic of the singularities tends to infinity. When $\eta_1 = 2 \times 10^{-3}\mathrm{i}$ in (d), the general soliton solution behaves like a fundamental soliton due to its big collapse periodic.

6.2 General two-soliton solutions

By Theorem 2, the general two-soliton solution can be obtained from Eq. (39) when we implement the nonlocal constraints (36) in Theorem 2; then, cumbersome expression is reduced to the following equation.

$$q = 4\frac{({\lambda_1}^2 - {\lambda_2}^2)\left[\lambda_1 \sin(8{\lambda_2}^3 t + 2\lambda_2 x) - \lambda_2 \sin(8{\lambda_1}^3 t + 2\lambda_1 x)\right]}{\sqrt{2\alpha+\beta+\beta^*}\left[(\lambda_1+\lambda_2)^2\cos(\Theta_2) - (\lambda_1-\lambda_2)^2\cos(\Theta_1) - 4\lambda_2\lambda_1\right]}, \tag{42}$$

with

$$\Theta_1 = 2(\lambda_1 + \lambda_2)\left[4({\lambda_1}^2 - \lambda_2\lambda_1 + {\lambda_2}^2)t + x\right],$$
$$\Theta_2 = 2(\lambda_1 - \lambda_2)\left[4({\lambda_1}^2 + \lambda_2\lambda_1 + {\lambda_2}^2)t + x\right].$$

Under the nonlocal constraints (36), the general two-soliton solution (42) is an equation with only four parameters α, β, λ_1 and λ_2. Next, we will analyze the influence of these four parameters. By Eq. (42), we can find that α and β are independent of the singularities of the general two-soliton solution. We will analyze the dynamical behaviors for different eigenvalue patterns for $\lambda_1, \lambda_2 \in \mathbb{C}_+$. (1) When λ_1 and λ_2 are both pure imaginary, both of the branches of the two-soliton solution are singular, through simple comparison of the corresponding single soliton with $\lambda_1 = \mathrm{i}$ and $\lambda_1 = \frac{1}{3}\mathrm{i}$; we find that in this case, the general two-soliton solution is a nonlinear superposition of two single soliton with puerly imaginary eigenvalues. An example of $\lambda_1 = \mathrm{i},\ \lambda_2 = \frac{1}{3}\mathrm{i}$ is plotted in Fig. 2a. (2) When λ_1 is pure imaginary and $\Re(\lambda_2) \neq 0$, it is worth mentioning that one of the branches of the two-soliton solution is no longer singular and stays bounded, which is quite different from the above case. It is obvious that in this case, the two-soliton solution is not a nonlinear superposition of two single soliton with a purely imaginary eigenvalue and an eigenvalue $\Re(\lambda_2) \neq 0$. This is an interesting situation. Here we take $\lambda_1 = \mathrm{i},\ \lambda_2 = \frac{1}{4} + \frac{1}{3}\mathrm{i}$ to illustrate in Fig. 2b. (3) When $\Re(\lambda_1)\Re(\lambda_2) \neq 0$, under normal parameter values, the two-soliton solution behaves as a wave, which is composed by two

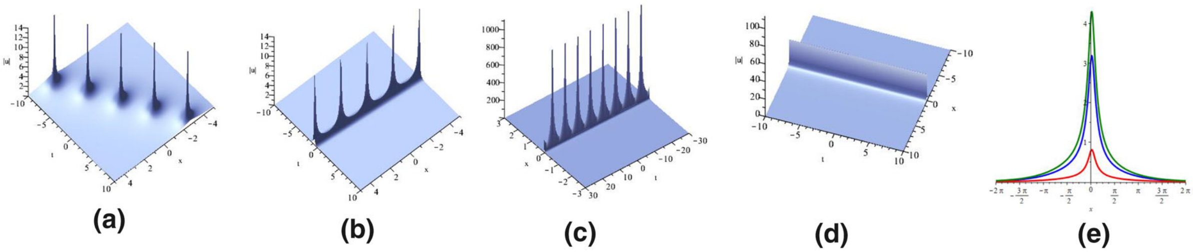

Fig. 1 (Color online) **a–d** are the plots of general solutions for Eq. (1) with the same parameters $\alpha = 1, \beta = \frac{1}{2}$ except for **a** $\lambda_1 = \frac{1}{4} + \frac{1}{3}i$; **b** $\lambda_1 = \frac{3}{4} + \frac{1}{3}i$; **c** $\lambda_1 = 2 \times 10^{-2}i$; **d** $\lambda_1 = 2 \times 10^{-3}i$; **e** is the 2D plot of general single-soliton solution for Eq. (1) with same $\beta = \frac{1}{2}, \lambda_1 = \frac{1}{4} + \frac{1}{3}i$ and different α , $\left[\alpha = \frac{1}{20}, \alpha = \frac{1}{2}, \alpha = 20\right]$=[“Blue”, “Red”, “Green”]

periodically singular single soliton with different propagation directions. But when we let $\Re(\lambda_1) = \Re(\lambda_2)$ and $\Im(\lambda_1) = \Im(\lambda_2)$, the two singular branches began to parallel to each other. We take $\lambda_1 = 1 + \frac{1}{2}i,\ \lambda_2 = 1 + \frac{1}{3}i$ to illustrate in Fig. 2c. In this case, a singular solution in periodic background can also be obtained. In fact, in order to consider the influence of the imaginary part of the eigenvalues, we take $\Im(\lambda_1) = 10^{-3}$ and other parameters are all 1; then, we found that the solution behaves as a singular soliton in a periodic background, and the plot is shown in Fig. 2d. The singular points emerge in pairs except when $x = t = 0$. Except for this case, we found that the value of the $\Re(\lambda_k)$ does not influence the dynamic behaviors in some range. It is natural that how about the real part of the eigenvalues. Taking $\Re(\lambda_1) = 10^{-3}$, we also find that one of the branches is not singular any more, which can be seen in Fig. 2e. The case is similar to the case of Fig. 2b, but where $\Re(\lambda_1) = 0$. So the dynamic feature exists in a small range of $\Re(\lambda_1) = 0$. For the nonlocal equation, in most cases, the singular soliton will be bounded when $\lambda_1 = -\lambda_2^*$. We also try to select such spectral parameters in this way in order to get a bounded solution, but it does not turn out as expected. No matter how the parameters change, when $\lambda_1 = -\lambda_2^*$, the two branches of the soliton solution are twisted together and remain singular over time. In Fig. 2e, we take $\lambda_1 = 1 + \frac{1}{3}i$ to illustrate. Consequently, the two-soliton solution composed with two collapsed general single-soliton solutions can be easily gotten with two complex spectral parameters. But this periodic collapse phenomenon of the two-soliton solutions with two impure imaginary complex spectral parameters is not all the case. In some special cases, the singularity of one of the branches will disappear and the two-soliton solution becomes a wave which is composed by a general singular soliton and a nonsingular one. Besides, an interesting phenomenon is also discovered, i.e., when one of imaginary parts of the parameter is 10^{-3}, the solution behaves as a singular wave in the periodical background. It is worth mentioning that when $\lambda_2 = -\lambda_1^*$, the two-soliton solution did not degenerate into a bound state breathing soliton instead of a breathing singular wave.

6.3 Higher-order soliton solutions for Eq. (1) with N multiple zeros

If $\det(P_+)$ and $\det(P_-^{-1})$ have N multiple zeros in complex λ-plane, respectively. By the symmetry relations of spectral parameters in Lemma 1, suppose $(\zeta_k, -\zeta_k)(k = 1, 2, \ldots, N)$ are N pairs of multiple zeros of $\det(P_+)$ and $\det(P_-^{-1})$, where the geometric dimensions of the multiple zeros are n_k, respectively. In this case, the normal case that P_+ and P_-^{-1} possess the same number of zeros will be considered; then, the determinants of matrix can be rewritten as:

$$\det(P_+) = \prod_{k=1}^{N} (\lambda - \zeta_k)^{n_k} \tau_+,\ \det\left(P_-^{-1}\right) = \prod_{k=1}^{N} (\lambda + \zeta_k)^{n_k} \tau_-,$$

where $\tau_+(\zeta_k)\tau_-(-\zeta_k) \neq 0,\ k = 1, 2, \ldots, N$ and $\sum_{k=1}^{N} n_k = N_0$. In order to convert the multiple zeros into the limit of the simple zeros, implement a perturbational modification on the scattering data

$$\left\{\lambda_{k,j}, a_{k,j}, b_{k,j}, c_{k,j}\right\} \mapsto \Big\{\lambda_{k,j}(\epsilon_{k,j}), a_{k,j}(\epsilon_{k,j-1}), b_{k,j}(\epsilon_{k,j-1}), c_{k,j}(\epsilon_{k,j-1})\Big\},$$

where $\lambda_{k,j}(\epsilon_{k,j}) = \zeta_k + \epsilon_{k,j-1}\ k = 1, 2, \cdots, N,\ j = 1, 2, \cdots, n_k$ with $\lambda_{k,1} = \zeta_k$ and $\epsilon_{k,0} = 0(k = 1, 2, \ldots, N)$. Recall the symmetry properties in Lemma

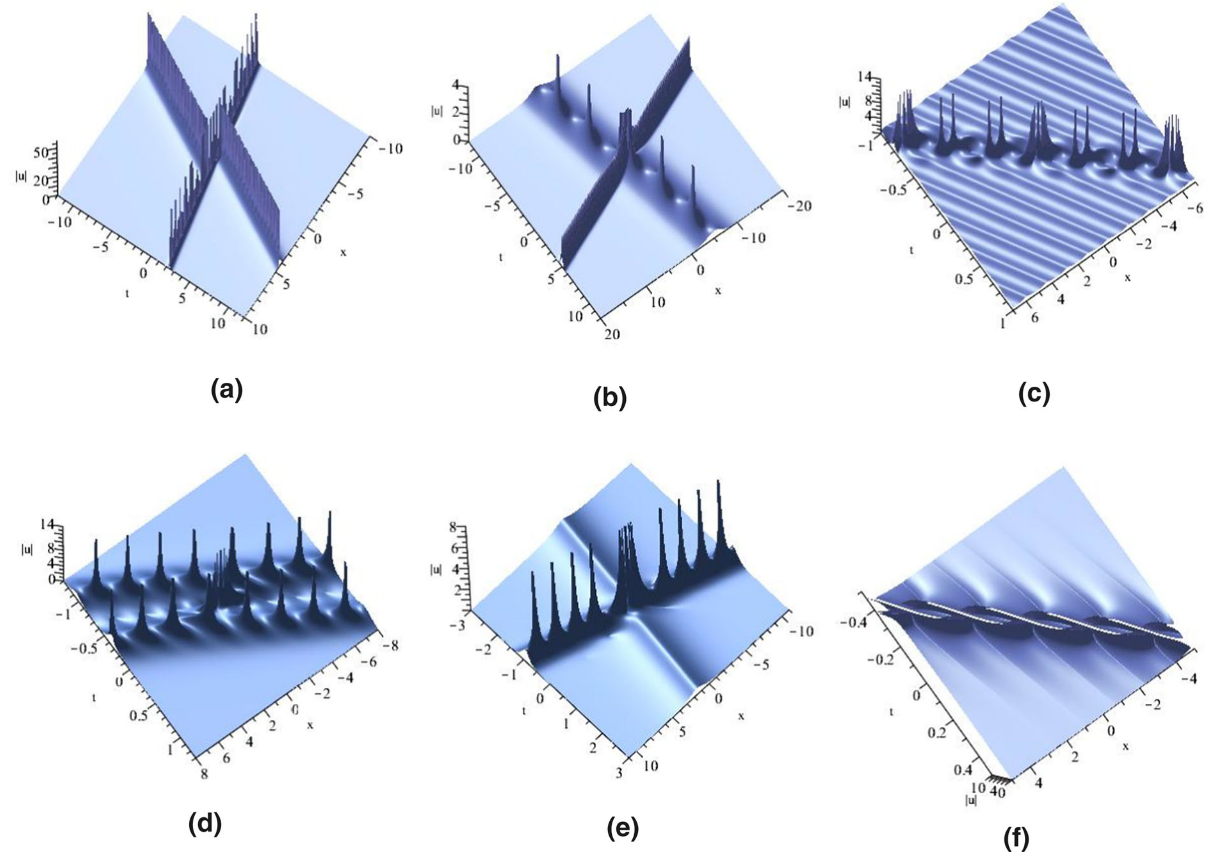

Fig. 2 (Color online) **a**–**f** are the plots of general single-soliton solutions for Eq. (1) by choosing the same parameters $\alpha = 2, \beta = -4$ except for $[\Re(\lambda_1), \Im(\lambda_1), \Re(\lambda_2), \Im(\lambda_2)]$ take the following values: **a** $\left[0, 1, 0, \frac{1}{3}\right]$; **b** $\left[0, 1, \frac{1}{4}, \frac{1}{3}\right]$; **c** $\left[1, 10^{-3}, 1, 1\right]$; **d** $\left[1, \frac{1}{2}, 1, \frac{1}{3}\right]$; **e** $\left[1, \frac{1}{3}, 10^{-3}, \frac{1}{3}\right]$; **f** $\left[1, \frac{1}{3}, -1, \frac{1}{3}\right]$

1 and the nonlocal constraints in Theorem 2, the symmetry relations of perturbed scattering data for the nonlocal generalized SS Eq. (1) can be obtained.

Lemma 2 *For a pair of perturbed eigenvalues* $\left[\lambda_{k,j}(\epsilon_{k,j}), -\lambda_{k,j}(\epsilon_{k,j})\right]$ *of the reverse-space-time nonlocal generalized SS Eq.* (1)*, where* $\lambda_{k,j}(\epsilon_{k,j}) \in \mathbb{C}_+$*. The perturbed eigenvectors* $v_k(\epsilon_{k,j-1})$ *and* $\bar{v}_k^T(\bar{\epsilon}_{k,j-1})$ *with*

$$v_k(\epsilon_{k,j-1}) = e^{i\theta(\zeta_k+\epsilon_{k,j-1})\Lambda} \left[\sum_{j=0}^{\infty} a_{k,j}\epsilon^{k,j-1}, \sum_{j=0}^{\infty} b_{k,j}\epsilon^{j}, \sum_{j=0}^{\infty} c_{k,j}\epsilon^{k,j-1}\right]^T \tag{43}$$

and

$$\bar{v}_k^T(\bar{\epsilon}_{k,j-1}) = \left[\bar{a}_1, \bar{b}_1, \bar{c}_1\right] e^{i\theta\left(\zeta_k-\bar{\epsilon}_{k,j-1}\right)\Lambda} \tag{44}$$

with

$$\bar{a}_1 = \frac{1}{\alpha+\beta} e^{\sum\limits_{l=0}^{\infty} a_{k,j}(-\bar{\epsilon}_{k,j-1})^l}, \quad \bar{b}_1 = \frac{1}{\alpha+\beta} e^{\sum\limits_{l=0}^{\infty} a_{k,j}(-\bar{\epsilon}_{k,j-1})^l},$$

$$\bar{c}_1 = -\frac{\sqrt{2\alpha+\beta+\beta^*}}{\alpha+\beta} e^{\sum\limits_{l=0}^{\infty} a_{k,j}(-\bar{\epsilon}_{k,j-1})^l}, \tag{45}$$

where $\bar{\epsilon}_{k,j} = -\epsilon_{k,j} (k = 1, 2, \ldots, N,\ j = 1, 2, \ldots, n_k)$ *and* $a_{k,l},\ b_{k,l},\ c_{k,l}\ (l = 0, 1, 2, \cdots)$ *are arbitrary complex numbers.*

Proof If $\lambda_{k,j}(\epsilon_{k,j}) = \zeta_k + \epsilon_{k,j-1}$ is the spectral parameter in $\mathbb{C}_+$, by Lemma 1, then the corresponding spectral parameter in $\mathbb{C}_-$ is $-\lambda_{k,j}(\epsilon_{k,j}) = -\zeta_k - \epsilon_{k,j-1} = -\zeta_k + \bar{\epsilon}_{k,j-1}$, so $\bar{\epsilon}_{k,j-1} = -\epsilon_{k,j-1}$. Then, for a pair of zeros $\left[\lambda_{k,j}, -\lambda_{k,j}\right]$, the corresponding perturbation parameters are $\left[\epsilon_{k,j}, -\epsilon_{k,j}\right]$. So by Lemma 1, the perturbed scattering data $\bar{v}_{k_0,j}(-\epsilon_{k,j}) = v_{k_0,j}(\epsilon_{k,j})^T B_0$ with the elements of $v_{k_0,j}(\epsilon_{k,j})$, are taken with the following expansions $a_k(\epsilon_{k,j-1}) = \sum_{j=0}^{\infty} a_{k,j}\epsilon^{k,j-1}$, $b_k(\epsilon_{k,j-1}) = \sum_{j=0}^{\infty} b_{k,j}\epsilon^{j}$, $c_k(\epsilon_{k,j-1}) = \sum_{j=0}^{\infty} c_{k,j}\epsilon^{k,j-1}$ with $a_{k,l},\ b_{k,l},\ c_{k,l}\ (l = 0, 1, 2, \cdots)$ are arbitrary complex numbers. By the N-soliton formula in Theorem 1 and the nonlocal constraints in Theroem 2, we implement a limiting process and a series of determinant transformation; the higher-order soliton solutions for the nonlocal generalized SS Eq. (1) can be obtained. □

Theorem 3 *For the nonlocal generalized SS Eq. (1), the higher-order soliton formula can be represented as the following forms*

$$q(x,t) = 2\mathrm{i}\frac{\det(\tau_{13})}{\det(\tau)} \tag{46}$$

where

$$\tau_{kj} = \begin{bmatrix} \tau & y_j \\ \bar{y}_k^T & 0 \end{bmatrix},$$
$$\tau = \left(M_{h_1,h_2}\right)_{1\le h_1,h_2\le N}, \quad M_{h_1,h_2} = (m_{k,l}^{[l_1,l_2]})_{0\le l_1\le n_{h_1}-1,\ 0\le l_2\le n_{h_2}-1}$$

with y_k and $\bar{y}_j$ are jth-row and kth-row of y and $\bar{y}$, respectively. The perturbed eigenvalues are

$$y = \left[v_1^{(0)}, v_1^{(1)}, \ldots, v_1^{(n_1-1)}, \ldots, v_N^{(0)}, v_N^{(1)}, \ldots, v_N^{(n_N-1)}\right]_{3\times N_0},$$
$$\bar{y} = \left[\bar{v}_1^{(0)}, \bar{v}_1^{(1)}, \ldots, \bar{v}_1^{(n_1-1)}, \ldots, \bar{v}_N^{(0)}, \bar{v}_N^{(0)}, \ldots, \bar{v}_N^{(n_N-1)}\right]_{3\times N_0}, \tag{47}$$

with

$$v_k^{(l)} = \lim_{\epsilon_{k,j-1}\to 0} \frac{\partial^l v_k(\epsilon_{k,j-1})}{l!\,\partial\epsilon_{k,j-1}^l}, \quad \bar{v}_k^{(l)} = \lim_{\bar{\epsilon}_{k,j-1}\to 0} \frac{\partial^l \bar{v}_k^T(\bar{\epsilon}_{k,j-1})}{l!\,\bar{\epsilon}_{k,j-1}^l},$$
$$m_{k,l}^{[l_1,l_2]} = \lim_{\bar{\epsilon}_{k,j_1-1},\epsilon_{l,j_2-1}\to 0} \frac{1}{(l_1-1)!(l_2-1)!} \frac{\partial^{l_1+l_2-2}}{\partial\bar{\epsilon}_{k,j_1-1}^{l_1-1}\partial\epsilon_{l,j_2-1}^{l_2-1}} \left[\frac{\bar{v}_k^T(\bar{\epsilon}_{k,j_1-1})v_l(\epsilon_{l,j_2-1})}{\bar{\zeta}_k-\zeta_l+\bar{\epsilon}_{k,j_1-1}-\epsilon_{l,j_2-1}}\right]. \tag{48}$$

Remark The similar higher-order soliton formulas have been explored in [37,40], where the formula has been proofed from different perspectives. Later, in [43], the higher-order soliton formula for the nonlocal NLS equations was given. But both of the corresponding spectral problems are 2×2. Here the higher-order soliton formula for a 3×3 matrix problem is given; the details of the proof can be seen in [40,43]. The higher-order solitons with one multiple pole can be obtained for $N = 1$ in Eq. (46). When $n_1 = 2$ and $n_1 = 3$, the soliton-like solution with double zeros and triple zeros will be imposed. Here we will analyze the corresponding dynamic behaviors in detail. Taking $N = 1$ and $n_N = 2$ in Eq. (46), the higher-order soliton solution with double zeros of Eq. (1) is

$$q = \frac{4\lambda_1\left[(-24t\lambda_1^3 - 2x\lambda_1 + \mathrm{i})\mathrm{e}^{-2\mathrm{i}\lambda_1(4\lambda_1^2t+x)} - (24t\lambda_1^3 + 2x\lambda_1 + \mathrm{i})\mathrm{e}^{2\mathrm{i}\lambda_1(4\lambda_1^2t+x)}\right]}{\sqrt{2}\sqrt{\kappa_1-\alpha}[1152t^2\lambda_1^6 + 192\lambda_1^4t + 8\lambda_1^2x^2 + \cos(16t\lambda_1^3 + 4x\lambda_1) - 1]}. \tag{49}$$

When $n_1 = 3$ in Eq. (46), the higher-order soliton solution with triple zeros of Eq. (1) is

$$q = \frac{6\lambda_1\sqrt{2}\sqrt{\alpha-\kappa_1}(\alpha^2-\kappa_1^2)f_{41}\mathrm{e}^{-4\mathrm{i}\lambda_1(4\lambda_1^2t+x)} - 6\lambda_1\sqrt{2}\sqrt{\alpha-\kappa_1}\kappa_2^2f_{42}\mathrm{e}^{4\mathrm{i}\lambda_1(4\lambda_1^2t+x)} - 4\lambda_1\sqrt{2}\kappa_2\sqrt{\alpha+\kappa_1}(\alpha-\kappa_1)f_0}{\mathrm{i}\kappa_2(\alpha^2-\kappa_1^2)f_{21}e^{-2\mathrm{i}\lambda_1(4\lambda_1^2t+x)} + \mathrm{i}\kappa_2^2\sqrt{\alpha^2-\kappa_1^2}f_{22}e^{2\mathrm{i}\lambda_1(4\lambda_1^2t+x)} + \mathrm{i}(\alpha^2-\kappa_1^2)^{3/2}\mathrm{e}^{-6\mathrm{i}\lambda_1(4\lambda_1^2t+x)} + \mathrm{i}\kappa_2^3\mathrm{e}^{6\mathrm{i}\lambda_1(4\lambda_1^2t+x)}} \tag{50}$$

with

$$f_{21} = C_{2,1} + C_{2,2}, \quad f_{22} = C_{2,1} - C_{2,2},$$
$$f_{41} = C_{1,1} + C_{1,2}, \quad f_{42} = C_{1,1} - C_{1,2},$$
$$C_{1,2} = -384t^2\lambda_1^6 - 64\lambda_1^4xt + 1 - \frac{8}{3}\lambda_1^2x^2,$$
$$C_{1,1} = 4\mathrm{i}\lambda_1(20\lambda_1^2t + x),$$
$$C_{2,1} = 1327104t^4\lambda_1^{12} + 442368t^3x\lambda_1^{10} + 55296t^2x^2\lambda_1^8 + 3072tx^3\lambda_1^6 - 25344t^2\lambda_1^6 + 64x^4\lambda_1^4 - 1920tx\lambda_1^4 - 48x^2\lambda_1^2 + 3,$$
$$C_{2,2} = -64\mathrm{i}\lambda_1^3(36t\lambda_1^2 + x)(12t\lambda_1^2 + x)^2,$$
$$f_0 = 663552t^4\lambda_1^{12} + 221184t^3x\lambda_1^{10} + 27648t^2x^2\lambda_1^8 + 1536tx^3\lambda_1^6 - 2304t^2\lambda_1^6 + 32x^4\lambda_1^4 - 768tx\lambda_1^4 - 16x^2\lambda_1^2 - 3.$$

The plots for the higher-order solitons with one double zero (49) are exhibited in Fig. 3a, b and c. Similar to the regulations of N-soliton, for λ_1 purely imaginary, the higher-order solitons are not common solutions without singularities. When λ_1 is a purely imaginary complex number, each branch of the higher-order solitons collapsed periodically along the directions of propagations. Figure 3b is the plot of $\lambda_1 = \frac{1}{10}\mathrm{i}$, where we can see that the two branches have a displacement at the intersection. And when $\lambda_1 = \frac{1}{10^3}\mathrm{i}$, one of the branches becomes a normal one without singular, the corresponding plot is shown in Fig. 3a. Another case is that the eigenvalue λ_1 is not purely imaginary, where the higher-order soliton is a collapsed periodically one without displacement at the intersection. It can be seen in Fig. 3c. As for the higher-order soliton with one triple zero (50), we show three cases in the plots Fig. 3d, e

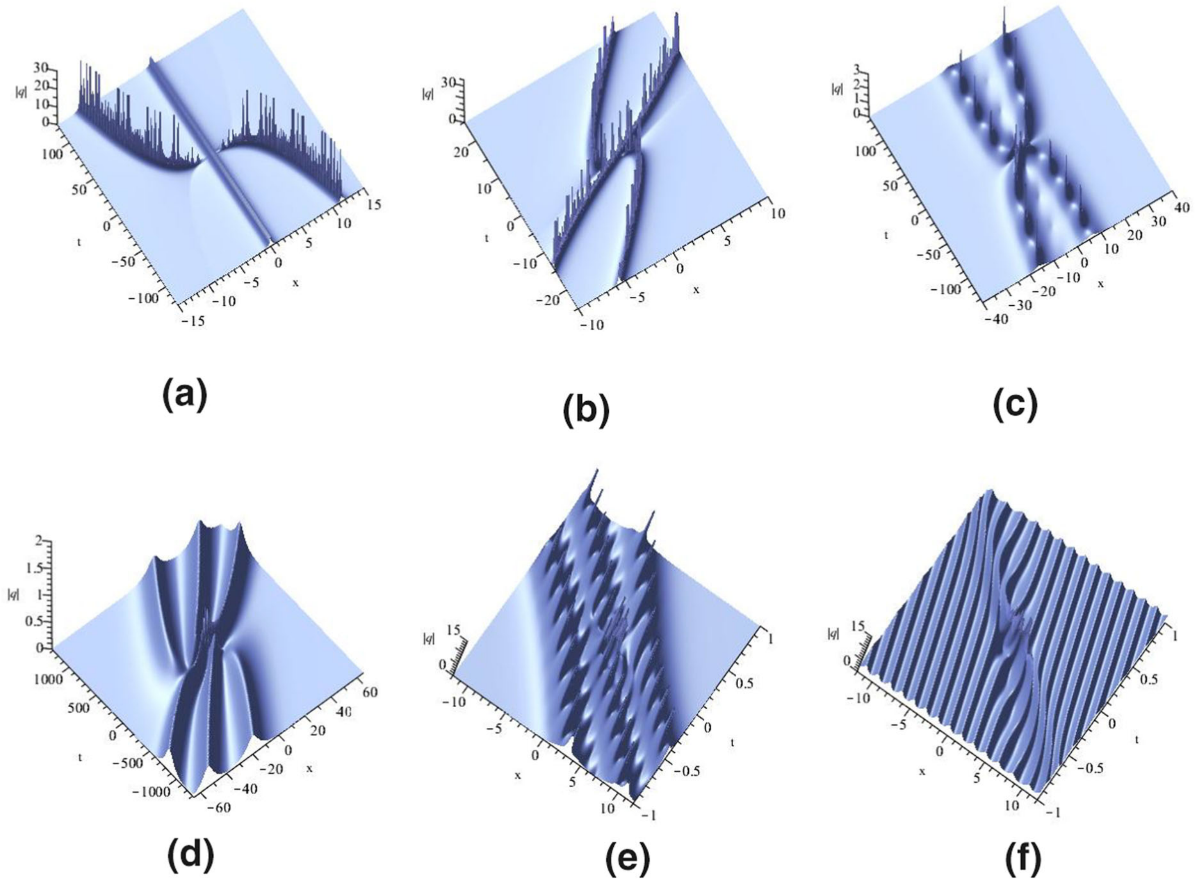

Fig. 3 (Color online) The higher-order solitons with one double zero and triple zero for Eq. (1) by choosing the same parameters $\alpha = -1, \beta = 2 + \frac{i}{2}$ except for λ_1: **a** 10^{-3}i; **b** 10^{-1}i; **c** $\frac{1}{10} + \frac{1}{5}$i; **d** 10^{-1}i; **e** $1 + \frac{1}{2}$i; **f** $1 + 10^{-3}$i

and f. Different from the higher-order solitons with one double zeros, when the eigenvalue λ_1 is purely imaginary, the soliton is bounded all the time and without singularities. Figure 3d is an example when $\lambda_1 = \frac{1}{10}$i. And when eigenvalue λ_1 is not purely imaginary, the soliton with three branches collapsed periodically, and Fig. 3d is a singular higher-order soliton with triple zero when $\lambda_1 = 1 + \frac{1}{2}$i. Finally, we show a special case which is a wave with a periodical background when $\lambda_1 = 1 + \frac{1}{10^3}$i. The corresponding figures are shown in Fig. 3f. In general, for the higher-order soliton, different from the higher-order solitons with one double zeros, when the eigenvalue is purely imaginary, the soliton with one triple zero is bounded all the time and without singularities. But for the local ones, the higher-order solitons with purely imaginary eigenvalues are always nonsingular.

7 Conclusions and discussions

The novel integrable nonlocal generalized SS Eq. (1), which can be reduced to the nonlocal SS equation and nonlocal modified KdV equation is proposed, is the compatibility condition of a 3×3 spectral problem. Due to the integrability of the nonlocal generalized SS Eq. (1), the corresponding infinitely many conserved quantities and conservation laws are developed. Then, the framework of the IST based on the RH problem for the coupled generalized SS Eq. (5) is established. Due to the distinctive structure of the 3×3 potential matrix, a special symmetry property of the potential matrix is found. Moreover, the symmetry properties of the scattering matrix, eigenvalues, and eigenvectors are obtained. If λ_j is the eigenvalue of the original spectral problem, then the eigenvalue of the adjoint equation is $-\lambda_j$. The relation between the corresponding eigenvectors is $\bar{v}_{0k} = v_{0k}^T B_0^{-1}$. Furthermore, due to the reverse-space-time reduction, the symmetry relations cannot be obtained directly, and the nonlocal constraints are obtained by implementing a series of complex computations. The symmetry relations of the scattering data are more complicated than the local ones, which leads to the difference of the formulation for the soliton solutions. The exact expressions for the single-soliton, two-soliton and higher-order soliton with double and triple zero and the corresponding dynamical behaviors were obtained. Some new and interesting properties were gotten. The N-soliton solutions for the coupled generalized SS Eq. (5) have been exhibited by solving the reflectionless RH problem with simple zeros. Then, combining the nonlocal constraints in Theorem 2, the single- and two-soliton solutions are obtained. Some new dynamical behaviors of the general single and two-solitons for the nonlocal generalized SS Eq. (1) were explored. For the dynamical behaviors for the general single-soliton and two-soliton solutions. Firstly, for the single soliton, the spectral configurations did not influence the singularity of the soliton. The single soliton is not a fundamental soliton solution without singularity; the soliton collapses periodically no matter whether the eigenvalue is purely imaginary or not. Different from the dynamics of two-soliton solutions of the reverse-space-time nonlocal equation in [36] and [35], the two-soliton is a nonlinear superposition of two general single soliton when two spectral parameters are both purely imaginary. In some special cases, such as the plots shown in Fig. 2b, d, the singularity of one of the branches will disappear and the two-soliton solution becomes a wave which is composed of a general singular soliton and a nonsingular one. Besides, an interesting phenomenon is also discovered; when one of imaginary parts of the parameter tends to zero, the solution behaves as a singular wave in the periodical background and the plot is shown in Fig. 2d. The periodic collapse phenomenon of the two-soliton solutions with two impure imaginary complex spectral parameters is all the case even when $\lambda_2 = -\lambda_1^*$, this singularity did not disappear and the two-soliton solution become a twisted wave and remain singular over time in Fig. 2f. By the symmetry relations of spectral parameters in Lemma 1 and the nonlocal constraints in Theorem 2, the symmetry relations of perturbed scattering data for the nonlocal generalized SS Eq. (1) are obtained. After implementing a limiting process and a series of determinant transformation, the higher-order soliton formula for the nonlocal generalized SS Eq. (1) is obtained. Compared to the exploration of the higher-order soliton for the local equations in [37,40], the formula has been proofed from different perspectives and the symmetry relations of perturbed scattering data are quite different. The higher-order soliton formula for the nonlocal NLS equations [43] was given. But both of the corresponding spectral problems are 2×2. It is the first time to study the high-order solitons of nonlocal equations related to the 3×3 matrix spectral problem. Under different spectral configurations, the singular and nonsingular higher-order soliton were obtained with the zeros double and triple, respectively. It is mentioned that for the eigenvalue that is purely imaginary, the higher-order solitons with double zero are not common solutions without singularities; it is a singular one which is shown in Fig. 3a and b, but the soliton with one triple zero is bounded all the time and without singularities. When the eigenvalue is a not purely imaginary complex number, each branch of the higher-order solitons collapsed periodically along the directions of propagations in Fig. 3c, e.

However, for system (5) with complex reverse-space-time reduction, through we have implemented many attempts, the symmetry properties of scattering data have not been solved. Besides, the shifted nonlocal reduction was proposed [47], and it will be interesting to construct the solutions and inverse scattering transforms for the shifted nonlocal generalized SS equation. Thirdly, as for the nonlocal equation which is associated with the 3×3 spectral problem with nonzero boundary conditions, the nonlocal integrable equation corresponding to the 3×3 Lax pair will be considered continuously in the future.

Author contributions All authors contributed to the study conception and theory. Calculation and analysis were performed by Wang Minmin. The first draft of the manuscript was written by Wang Minmin, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Data availability The datasets generated and analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or nonfinancial interests to disclose. The authors declare that there is no conflict of interests regarding the publication of this paper.

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