# Double and triple pole solutions for the Gerdjikov–Ivanov type of derivative nonlinear Schrödinger equation with zero/ nonzero boundary conditions

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# Double and triple pole solutions for the Gerdjikov-Ivanov type of derivative nonlinear Schrödinger equation with zero/nonzero boundary conditions

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## ABSTRACT

In this work, the double and triple pole soliton solutions for the Gerdjikov–Ivanov type of the derivative nonlinear Schrödinger equation with zero boundary conditions (ZBCs) and nonzero boundary conditions (NZBCs) are studied via the Riemann–Hilbert (RH) method. With spectral problem analysis, we first obtain the Jost function and scattering matrix under ZBCs and NZBCs. Then, according to the analyticity, symmetry, and asymptotic behavior of the Jost function and scattering matrix, the RH problem (RHP) with ZBCs and NZBCs is constructed. Furthermore, the obtained RHP with ZBCs and NZBCs can be solved in the case that reflection coefficients have double or triple poles. Finally, we derive the general precise formulas of N-double and N-triple pole solutions corresponding to ZBCs and NZBCs, respectively. In addition, the asymptotic states of the one-double pole soliton solution and the one-triple pole soliton solution are analyzed when t tends to infinity. The dynamical behaviors for these solutions are further discussed by image simulation.

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## I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is well known to be one of the most vital integrable systems, which has been constantly reported in various fields such as nonlinear optics and water waves.<sup>1,2</sup> To describe more abundant physical effects and research the effects of high-order perturbations, the Gerdjikov–Ivanov (GI) type of the derivative NLS equation was also proposed as follows:<sup>3</sup>

$$iu_t + u_{xx} - iu^2 u_x^* + \frac{1}{2} |u|^4 u = 0,$$
<sup>(1)</sup>

where the asterisk \* denotes the complex conjugation and *u* denotes the transverse magnetic field perturbation function with spatial variable *x* and temporal variable *t*. As an extended form of the NLS equation, the GI equation (1) is used to describe Alfvén waves propagating parallel to the ambient magnetic field in plasma physics. The explicit soliton-like solutions for Eq. (1) were constructed via applying its Darboux transformation,<sup>4</sup> and its algebro-geometric solutions were established according to the Riemann theta functions.<sup>5</sup> By the DT method, the breather wave and rogue wave solutions of the GI equation were presented in Refs. 6 and 7. The soliton molecules and dynamics of the smooth positons for the GI equation were discussed in Ref. 8. In addition, the *N*-soliton for Eq. (1) under zero boundary conditions (ZBCs) was given through using the Riemann-Hilbert (RH) method.<sup>9</sup> By using the nonlinear steepest descent method,

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the long-time asymptotic behavior for Eq. (1) was discussed in Refs. 10-12. Recently, the RH method and Dbar-dressing method were used to construct simple pole solutions of the GI equation with nonzero boundary conditions (NZBCs) in Refs. 13 and 14, respectively. Higher-order pole solutions of the GI equation with ZBCs were derived by the dressing method based on the technique of regularization.<sup>15</sup> Multiple higher-order pole solutions were also obtained by the Laurent's series and generalization of the residue theorem.<sup>16</sup>

In soliton theory, the inverse scattering transform (IST) method is the most powerful tool for analyzing initial value problems of integrable nonlinear evolution equations (NLEEs). The method was raised first by Gardner et al. in 1967 for the KdV equation.<sup>17</sup> After that, the dressing Zakharov-Shabat technique was used to construct the soliton solutions.<sup>18,19</sup> As is well known, the classic IST method was based on the Gel'fand-Levitan-Marchenko (GLM) integral equations. Subsequently, Zakharov et al. developed a RH formulation to replace the GLM equation, which simplifies the IST method.<sup>20</sup> After decades of development, the RH formulation has been successfully applied to numerous integrable equations, and it is still a hot topic today.<sup>21-31</sup> In terms of the direct scattering, the corresponding Riemann-Hilbert problem (RHP) is constructed and solution of the GI equation can be represented by the solution of the RHP. Then, through solving the RHP under a reflection-less case, we can obtain the formulas of multiple solitons. However, since we assume that the reflection coefficient has multiple poles, the PHP will be non-regular, which cannot be solved directly by the Plemelj formula. To solve this obstacle, we need to regularize the RHP via eliminating singularity. Different from the process in Refs. 15, 24, and 32, which transforms the non-regular RHP into the regular problem via multiplying a dressing operator, the method pioneered by Ablowitz, Biondini, Demontis, and Prinary, et al. regularize the RHP by subtracting the asymptotic behavior and the pole contribution.<sup>2,33–39</sup> Inspired by this idea, the research on the soliton solution of the integrable system under ZBCs and NZBCs has been completed in recent years.<sup>40–45</sup> In this work, we apply this idea to derive the double and triple pole solutions for the GI equation with ZBCs and NZBCs, which is different with the simple pole solutions obtained in Refs. 9, 13, and 14. Solitons with higher order of poles have been investigated in the early literature, including the higher-order solitons for the N-wave system, NLS equation, long-wave-short-wave equation, n-component NLS equations, and so on.<sup>46-50</sup> It is worth mentioning that the method used in this paper is different from computing soliton solutions with higher-order poles in Refs. 15 and 16 because the techniques of dealing with severe spectral singularities are disparate. Moreover, for the case of NZBCs, general double poles and triple pole solutions of the GI equation with mixed discrete spectra, i.e.,  $2N_1$ -breather- $2N_2$  soliton solutions and  $3N_1$ -breather- $3N_2$  soliton solutions, are analyzed in this paper, which are not known yet for the GI equation. Therefore, to our knowledge, for the GI equation, the research using this idea for double and triple pole solutions under ZBCs and NZBCs has not been reported yet.

It is well known that the Lax pair of the GI equation (1) can be given by

$$\Phi_x = X\Phi, \qquad \Phi_t = T\Phi, \tag{2}$$

where

and

$$X = -ik^{2}\sigma_{3} + kQ - \frac{i}{2}Q^{2}\sigma_{3},$$
  

$$T = -2ik^{4}\sigma_{3} + 2k^{3}Q - ik^{2}Q^{2}\sigma_{3} - ikQ_{x}\sigma_{3} + \frac{1}{2}(Q_{x}Q - QQ_{x}) + \frac{i}{4}Q^{4}\sigma_{3},$$
(3)

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \tag{4}$$

where  $k \in \mathbb{C}$  is a spectral parameter and  $\Phi = \Phi(x, t, k)$  is the 2 × 2 matrix-valued eigenfunction. Equation (1) can be derived from the compatibility condition  $X_t - T_x + [X, T] = 0$  of system (2).

The outline of this paper is organized as follows: In Sec. II, we establish the RHP for the GI equation under ZBCs by spectral analysis, and the potential function is shown by the solution of the RHP. In Sec. III, through solving the RHP, we obtain the explicit *N*-double pole soliton solutions for the reflection coefficients with double poles under ZBCs. Additionally, the general *N*-triple pole soliton solutions for the reflection coefficients with double poles under ZBCs. IV. In Sec. V, the RHP for the GI equation under NZBCs is presented via more miscellaneous spectral problem analysis. Following that, the accurate *N*-double and *N*-triple pole solutions under NZBCs are given in Secs. VI and VII, respectively. Finally, some summaries are given in Sec. VIII.

### II. THE CONSTRUCTION OF RIEMANN-HILBERT PROBLEM WITH ZBCs

The direct scattering problem for the GI equation (1) with ZBCs has been discussed in Ref. 16. In this section, we would like to recall some results for the direct scattering problem for the targeted GI equation with following ZBCs at infinity:

$$\lim_{x \to \pm \infty} u(x,t) = 0.$$
<sup>(5)</sup>

## A. Spectral analysis

Let  $x \to \pm \infty$ , and the Lax pair (2) under ZBCs (5) changes into

$$\Phi_x^{\infty} = X_0 \Phi^{\infty} = -ik^2 \sigma_3 \Phi^{\infty}, \qquad \Phi_t^{\infty} = T_0 \Phi^{\infty} = 2k^2 X_0 \Phi^{\infty}, \tag{6}$$

which admits the following fundamental matrix solution:

$$\Phi^{\infty}(x,t;k) = e^{-i\theta(x,t;k)\sigma_3}, \qquad \theta(x,t;k) = k^2(x+2k^2t).$$
(7)

Defining  $\Sigma := \mathbb{R} \cup i\mathbb{R}$ , the Jost solutions  $\Phi_{\pm}(x, t; k)$  can be written as

$$\Phi_{\pm}(x,t;k) = e^{-i\theta(x,t;k)\sigma_3} + o(1), \quad k \in \Sigma, \quad \text{as} \quad x \to \pm \infty.$$
(8)

Through the variable transformation

$$\mu_{\pm}(x,t;k) = \Phi_{\pm}(x,t;k)e^{i\theta(x,t;k)\sigma_{3}},$$
(9)

the modified Jost solutions  $\mu_{\pm}(x, t; k)$  tend to *I* as  $x \to \pm \infty$ , and it can be solved as

$$\mu_{\pm}(x,t;k) = I + \int_{\pm\infty}^{x} e^{-ik^{2}(x-y)\hat{\sigma}_{3}} \left[ \left( kQ - \frac{i}{2}Q^{2}\sigma_{3} \right)(y,t)\mu_{\pm}(y,t;k) \right] dy,$$
(10)

where operator  $e^{\vartheta \hat{\sigma}_3} \Delta = e^{\vartheta \sigma_3} \Delta e^{-\vartheta \sigma_3}$  is defined for a matrix  $\Delta$ .

Proposition 2.1. Suppose that  $u \in L^1(\mathbb{R}^{\pm})$ ; then,  $\mu_{\pm}(x,t; k)$  given in Eq. (9) are unique solutions for the Jost integral equation (10) in  $\Sigma$ , and they satisfy the following characteristics:

- $\mu_{-1}(x,t; k)$  and  $\mu_{+2}(x,t; k)$  become analytical for  $D_+$  and continuous in  $D_+ \cup \Sigma$ ;
- $\mu_{+1}(x,t; k)$  and  $\mu_{-2}(x,t; k)$  become analytical for  $D_{-}$  and continuous in  $D_{-} \cup \Sigma$ ;
- $\mu_{\pm}(x,t;k) \rightarrow I \text{ as } k \rightarrow \infty; and$
- det  $\mu_{\pm}(x,t;k) = 1$ ,  $x, t \in \mathbb{R}$ ,  $k \in \Sigma$ .

Since the Jost solutions  $\Phi_{\pm}(x, t; k)$  are the simultaneous solutions of spectral problem (2), which satisfy the following linear relation by the constant scattering matrix  $S(k) = (s_{ij}(k))_{2\times 2}$ :

$$\Phi_+(x,t;k) = \Phi_-(x,t;k)S(k), \quad k \in \Sigma,$$
(11)

where  $S(k) = \sigma_2 S^*(k^*) \sigma_2$ ,  $S(k) = \sigma_1 S^*(-k^*) \sigma_1$ , and  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The scattering coefficients can be shown in what follows by the Wronskian determinant

$$s_{11}(k) = Wr(\Phi_{+,1}, \Phi_{-,2}), \quad s_{12}(k) = Wr(\Phi_{+,2}, \Phi_{-,2}),$$
  

$$s_{21}(k) = Wr(\Phi_{-,1}, \Phi_{+,1}), \quad s_{22}(k) = Wr(\Phi_{-,1}, \Phi_{+,2}),$$
(12)

where  $Wr(\cdot, \cdot)$  denotes the Wronskian determinant. From these representations, it is not hard to get the following proposition.

*Proposition 2.2.* The scattering matrix S(k) satisfies the following:

- det S(k) = 1 for  $k \in \Sigma$ .
- $s_{22}(k)$  becomes analytical for  $D_+$  and continuous in  $D_+ \cup \Sigma$ .
- $s_{11}(k)$  becomes analytical for  $D_-$  and continuous in  $D_- \cup \Sigma$ .
- $S(x,t,k) \rightarrow I \text{ as } k \rightarrow \infty$ .

## B. The Riemann-Hilbert problem

In terms of the analytic properties of Jost solutions  $\mu_{\pm}(x,t; k)$  in Proposition 2.1, we have the following sectionally meromorphic matrices:

$$M_{-}(x,t;k) = \left(\frac{\mu_{+,1}}{s_{11}}, \mu_{-,2}\right), \qquad M_{+}(x,t;k) = \left(\mu_{-,1}, \frac{\mu_{+,2}}{s_{22}}\right), \tag{13}$$

where superscripts  $\pm$  represent analyticity in  $D_+$  and  $D_-$ , respectively. Naturally, a matrix RHP is proposed.

## 1. Riemann-Hilbert problem 1

M(x, t; k) solves the following RHP:

 $\begin{cases} M(x,t;k) \text{ is analytic in } \mathbb{C} \setminus \Sigma, \\ M_{-}(x,t;k) = M_{+}(x,t;k)(I - G(x,t;k)), & k \in \Sigma, \\ M(x,t;k) \to I, & k \to \infty, \end{cases}$ (14)

of which the jump matrix G(x, t; k) is

$$G = \begin{pmatrix} \rho(k)\tilde{\rho}(k) & e^{-2i\theta(x,t;k)}\tilde{\rho}(k) \\ -e^{2i\theta(x,t;k)}\rho(k) & 0 \end{pmatrix},$$
(15)

where  $\rho(k) = \frac{s_{21}(k)}{s_{11}(k)}$ ,  $\tilde{\rho}(k) = \frac{s_{12}(k)}{s_{22}(k)}$ . Taking

$$M(x,t;k) = I + \frac{1}{k}M^{(1)}(x,t;k) + O\left(\frac{1}{k^2}\right), \qquad k \to \infty,$$
(16)

then the potential u(x, t) of the GI equation (1) with ZBCs is denoted by

$$u(x,t) = 2iM_{12}^{(1)}(x,t;k) = 2i\lim_{k \to \infty} kM_{12}(x,t;k).$$
(17)

## **III. THE SOLUTION OF GI EQUATION UNDER ZBCs WITH DOUBLE POLES**

In this section, we will discuss the inverse scattering problem with the double poles discrete spectrum for the GI equation (1) under ZBCs and present the general N-double pole solutions.

#### A. Inverse scattering problem with ZBCs and double poles

We suppose that  $s_{22}(k)$  has N double zeros  $k_n$  (n = 1, 2, ..., N) in  $D_0 = \{k \in \mathbb{C} : \text{Re}k > 0, \text{Im}k > 0\}$ , which means  $s_{22}(k_n) = s'_{22}(k_n) = 0$  and  $s''_{22}(k_n) \neq 0$ . According to the symmetries relation of the scattering matrix, one has

$$\begin{cases} s_{22}(k_n) = s_{22}(-k_n) = s_{11}(k_n^*) = s_{11}(-k_n^*) = 0, \\ s'_{22}(k_n) = s'_{22}(-k_n) = s'_{11}(k_n^*) = s'_{11}(-k_n^*) = 0. \end{cases}$$
(18)

Thus, the corresponding discrete spectrum can be collected as

$$\Gamma = \left\{ k_n, k_n^*, -k_n^*, -k_n \right\}_{n=1}^N,$$
(19)

whose distributions are displayed in Fig. 1.

Since  $s_{22}(k_0) = 0$   $(k_0 \in \Gamma \cap D_+)$ , we easily know that  $\Phi_{-1}(x, t; k_0)$  and  $\Phi_{+2}(x, t; k_0)$  are linearly dependent. Similarly,  $\Phi_{+1}(x, t; k_0)$  and  $\Phi_{-2}(x, t; k_0)$  are linearly dependent due to  $s_{11}(k_0) = 0$  for  $k_0 \in \Gamma \cap D_-$ . That is to say,



FIG. 1. Distribution of the discrete spectrum and jumping curves for the RHP on the complex k-plane. Region  $D_+ = \{k \in \mathbb{C} | \text{RekIm}k > 0\}$  (gray region) and region  $D_- = \{k \in \mathbb{C} | \text{RekIm}k < 0\}$  (white region).

$$\Phi_{+2}(x,t;k_0) = b[k_0]\Phi_{-1}(x,t;k_0), \quad k_0 \in \Gamma \cap D_+, \Phi_{+1}(x,t;k_0) = b[k_0]\Phi_{-2}(x,t;k_0), \quad k_0 \in \Gamma \cap D_-,$$
(20)

where  $b[k_0]$  is a norming constant. Due to  $s'_{22}(k_0) = 0$  in  $k_0 \in \Gamma \cap D_+$ , we find that  $\Phi'_{+2}(x,t;k_0) - b[k_0]\Phi'_{-1}(x,t;k_0)$  and  $\Phi_{-1}(x,t;k_0)$  are linearly dependent. Analogously,  $\Phi'_{+1}(x,t;k_0) - b[k_0]\Phi'_{-2}(x,t;k_0)$  and  $\Phi_{-2}(x,t;k_0)$  are linearly dependent for  $k_0 \in \Gamma \cap D_-$ . Then, we obtain

$$\Phi'_{+2}(x,t;k_0) - b[k_0]\Phi'_{-1}(x,t;k_0) = d[k_0]\Phi_{-1}(x,t;k_0), \quad k_0 \in \Gamma \cap D_+,$$
  
$$\Phi'_{+1}(x,t;k_0) - b[k_0]\Phi'_{-2}(x,t;k_0) = d[k_0]\Phi_{-2}(x,t;k_0), \quad k_0 \in \Gamma \cap D_-,$$
(21)

where  $d[k_0]$  is also a norming constant. Therefore, one has

$$\begin{split} & L_{-2} \Biggl[ \frac{\Phi_{+2}(x,t;k)}{s_{22}(k)} \Biggr] = A[k_0] \Phi_{-1}(x,t;k_0), \quad k_0 \in \Gamma \cap D_+, \\ & L_{-2} \Biggl[ \frac{\Phi_{+1}(x,t;k)}{s_{11}(k)} \Biggr] = A[k_0] \Phi_{-2}(x,t;k_0), \quad k_0 \in \Gamma \cap D_-, \\ & \operatorname{Res} \Biggl[ \frac{\Phi_{+2}(x,t;k)}{s_{22}(k)} \Biggr] = A[k_0] \Biggl[ \Phi_{-1}'(x,t;k_0) + B[k_0] \Phi_{-1}(x,t;k_0) \Biggr], \quad k_0 \in \Gamma \cap D_+, \\ & \operatorname{Res} \Biggl[ \frac{\Phi_{+1}(x,t;k)}{s_{11}(k)} \Biggr] = A[k_0] \Biggl[ \Phi_{-2}'(x,t;k_0) + B[k_0] \Phi_{-2}(x,t;k_0) \Biggr], \quad k_0 \in \Gamma \cap D_-, \end{split}$$

$$(22)$$

where  $L_{-2}[f(x, t; k)]$  means the coefficient of the  $O((k - k_0)^{-2})$  term in the Laurent series expansion of f(x, t; k) at  $k = k_0$  and

$$A[k_0] = \begin{cases} \frac{2b[k_0]}{s_{22}''(k_0)}, & k_0 \in \Gamma \cap D_+, \\ \frac{2b[k_0]}{s_{11}''(k_0)}, & k_0 \in \Gamma \cap D_-, \end{cases}$$
(23)

$$B[k_0] = \begin{cases} \frac{d[k_0]}{b[k_0]} - \frac{s_{22}''(k_0)}{3s_{22}''(k_0)}, & k_0 \in \Gamma \cap D_+, \\ \frac{d[k_0]}{b[k_0]} - \frac{s_{11}''(k_0)}{3s_{11}''(k_0)}, & k_0 \in \Gamma \cap D_-. \end{cases}$$
(24)

*Proposition 3.1.* Let  $k_0 \in \Gamma$ . Then, the following symmetry relations are satisfied:

- The first symmetry relation  $A[k_0] = -A[k_0^*]^*, B[k_0] = B[k_0^*]^*$ .
- The second symmetry relation  $A[k_0] = A[-k_0^*]^*$ ,  $B[k_0] = -B[-k_0^*]^*$ .

In order to solve the RHP conveniently, we take

$$\xi_n = \begin{cases} k_n, & n = 1, 2, \dots N, \\ -k_{n-N}, & n = N+1, N+2, \dots 2N. \end{cases}$$
(25)

Then, the residue and the coefficient  $L_{-2}$  of M(x, t; k) can be expressed as

$$\begin{split} &L_{-2} M_{+} = \left(0, A[\xi_{n}]e^{-2i\theta(x,t;\xi_{n})}\mu_{-1}(x,t;\xi_{n})\right), \\ &L_{-2} M_{-} = \left(A[\xi_{n}^{*}]e^{2i\theta(x,t;\xi_{n}^{*})}\mu_{-2}(x,t;\xi_{n}^{*}),0\right), \\ &Res_{k=\xi_{n}} M_{+} = \left(0, A[\xi_{n}]e^{-2i\theta(x,t;\xi_{n})}\left[\mu_{-1}'(x,t;\xi_{n}) + \left[B[\xi_{n}] - 2i\theta'(x,t;\xi_{n})\right]\mu_{-1}(x,t;\xi_{n})\right]\right), \\ &Res_{k=\xi_{n}} M_{-} = \left(A[\xi_{n}^{*}]e^{2i\theta(x,t;\xi_{n}^{*})}\left[\mu_{-2}'(x,t;\xi_{n}^{*}) + \left[B[\xi_{n}^{*}] + 2i\theta'(x,t;\xi_{n}^{*})\right]\mu_{-2}(x,t;\xi_{n}^{*})\right],0\right). \end{split}$$
(26)

By subtracting out the residue, the coefficient  $L_{-2}$ , and the asymptotic values as  $k \to \infty$  from the original non-regular RHP, one can obtain the following regular RHP:

$$M_{-} - I - \sum_{n=1}^{2N} \left[ \frac{L_{-2} M_{+}}{(k - \xi_{n})^{2}} + \frac{\operatorname{Res} M_{+}}{k - \xi_{n}} + \frac{L_{-2} M_{-}}{(k - \xi_{n}^{*})^{2}} + \frac{\operatorname{Res} M_{-}}{k - \xi_{n}^{*}} \right]$$
$$= M_{+} - I - \sum_{n=1}^{2N} \left[ \frac{L_{-2} M_{+}}{(k - \xi_{n})^{2}} + \frac{\operatorname{Res} M_{+}}{k - \xi_{n}} + \frac{L_{-2} M_{-}}{(k - \xi_{n}^{*})^{2}} + \frac{\operatorname{Res} M_{-}}{k - \xi_{n}^{*}} \right] - M_{+}G,$$
(27)

which can be solved by the Plemelj's formulas, given by

$$M(x,t;k) = I + \sum_{n=1}^{2N} \left[ \frac{\sum_{k=\xi_n}^{2-2} M_+}{(k-\xi_n)^2} + \frac{\operatorname{Res} M_+}{k-\xi_n} + \frac{\sum_{k=\xi_n^*}^{2-2} M_-}{(k-\xi_n^*)^2} + \frac{\operatorname{Res} M_-}{k-\xi_n^*} \right] \\ + \frac{1}{2\pi i} \int_{\Sigma} \frac{M_+(x,t;\zeta) G(x,t;\zeta)}{\zeta-k} d\zeta, \qquad k \in \mathbb{C} \setminus \Sigma,$$
(28)

where

$$\frac{L_{-2}M_{+}}{\frac{k-\xi_{n}}{(k-\xi_{n})^{2}}} + \frac{\operatorname{Res}M_{+}}{k-\xi_{n}} + \frac{L_{-2}M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res}M_{-}}{k-\xi_{n}^{*}}$$
$$= \left(\hat{C}_{n}(k) \left[\mu_{-2}'(\xi_{n}^{*}) + \left(\hat{D}_{n} + \frac{1}{k-\xi_{n}^{*}}\right)\mu_{-2}(\xi_{n}^{*})\right], C_{n}(k) \left[\mu_{-1}'(\xi_{n}) + \left(D_{n} + \frac{1}{k-\xi_{n}}\right)\mu_{-1}(\xi_{n})\right]\right),$$
(29)

and

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$$C_{n}(k) = \frac{A[\xi_{n}]e^{-2i\theta(\xi_{n})}}{k - \xi_{n}}, \quad D_{n} = B[\xi_{n}] - 2i\theta'(\xi_{n}),$$
$$\hat{C}_{n}(k) = \frac{A[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}}{k - \xi_{n}^{*}}, \quad \hat{D}_{n} = B[\xi_{n}^{*}] + 2i\theta'(\xi_{n}^{*}).$$
(30)

Furthermore, according to (16), one has

$$M^{(1)}(x,t;k) = -\frac{1}{2\pi i} \int_{\Sigma} M_{+}(x,t;\zeta) G(x,t;\zeta) d\zeta + \sum_{n=1}^{2N} \left( A[\xi_{n}^{*}] e^{2i\theta(\xi_{n}^{*})} (\mu_{-2}'(\xi_{n}^{*}) + \hat{D}_{n}\mu_{-2}(\xi_{n}^{*})), A[\xi_{n}] e^{-2i\theta(\xi_{n})} (\mu_{-1}'(\xi_{n}) + D_{n}\mu_{-1}(\xi_{n})) \right).$$
(31)

The potential u(x, t) with double poles for the GI equation with ZBCs is redefined into

$$u(x,t) = 2iM_{12}^{(1)} = -\frac{1}{\pi} \int_{\Sigma} (M_{+}(x,t;\zeta)G(x,t;\zeta))_{12}d\zeta + 2i\sum_{n=1}^{2N} A[\xi_{n}]e^{-2i\theta(\xi_{n})} (\mu_{-11}'(\xi_{n}) + D_{n}\mu_{-11}(\xi_{n})).$$
(32)

## **B. Trace formulas**

The discrete spectral points  $\xi_n$  are double zeros of the scattering coefficients  $s_{22}(k)$ , and  $\xi_n^*$  are double zeros of the scattering coefficients  $s_{11}(k)$ . Then, we can define

$$\beta^{+}(k) = s_{22}(k) \prod_{n=1}^{2N} \left( \frac{k - \xi_{n}^{*}}{k - \xi_{n}} \right)^{2}, \ \beta^{-}(k) = s_{11}(k) \prod_{n=1}^{2N} \left( \frac{k - \xi_{n}}{k - \xi_{n}^{*}} \right)^{2}.$$
(33)

Therefore,  $\beta^+(k)$  is analytic and has no zeros in  $D_+$ , and  $\beta^-(k)$  is analytic and has no zeros in  $D_-$ . They both tend to o(1) as  $k \to \infty$ . In terms of Cauchy projectors and the Plemelj's formulas,  $\beta^+(k)$  can be written as

$$\log \beta^{\pm}(k) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho\tilde{\rho})}{s - k} ds, \quad k \in D^{\pm}.$$
(34)

According to Eqs. (33) and (34), we get

$$\log s_{22}(k) = -\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho\tilde{\rho})}{s - k} ds + \sum_{n=1}^{2N} \log\left(\frac{k - \xi_n}{k - \xi_n^*}\right)^2.$$
(35)

Furthermore, let

$$\log s_{22}(k) = \sum_{j=1}^{\infty} \frac{I_j}{k^j},$$
(36)

and using the well-known expansion for  $\log(1 - x) = -\sum_{j=1}^{+\infty} \frac{x^j}{j}$ , then we easily obtain the trace formulas, given by

$$I_{j} = 2\sum_{n=1}^{2N} \left( \frac{(\xi_{n}^{*})^{j} - \xi_{n}^{j}}{j} \right) + \frac{1}{2\pi i} \int_{\Sigma} \log(1 - \rho \tilde{\rho}) s^{j-1} ds.$$
(37)

*Remark.*  $\beta^+(k)$  can also be defined as  $s_{22}(k)\prod_{n=1}^N (\frac{k^2-k_n^{*2}}{k^2-k_n^2})^2$ , which is equivalent to  $\beta^+(k) = s_{22}(k)\prod_{n=1}^{2N} (\frac{k-\xi_n^*}{k-\xi_n})^2$ . For convenience, we choose the definition  $\beta^+(k) = s_{22}(k)\prod_{n=1}^{2N} (\frac{k-\xi_n^*}{k-\xi_n})^2$ .

## C. Double pole soliton solutions with ZBCs

To derive the explicit double pole soliton solutions of the GI equation with ZBCs, we take  $\rho(k) = \tilde{\rho}(k) = 0$  called the reflectionless. Then, the second and first columns of Eq. (28) yield

$$\mu_{-2}(\xi_{j}^{*}) = \begin{pmatrix} 0\\ 1 \end{pmatrix} + \sum_{n=1}^{2N} C_{n}(\xi_{j}^{*}) \left[ \mu_{-1}'(\xi_{n}) + \left( D_{n} + \frac{1}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-1}(\xi_{n}) \right],$$
  

$$\mu_{-2}'(\xi_{j}^{*}) = -\sum_{n=1}^{2N} \frac{C_{n}(\xi_{j}^{*})}{\xi_{j}^{*} - \xi_{n}} \left[ \mu_{-1}'(\xi_{n}) + \left( D_{n} + \frac{2}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-1}(\xi_{n}) \right],$$
  

$$\mu_{-1}(\xi_{j}) = \begin{pmatrix} 1\\ 0 \end{pmatrix} + \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) \left[ \mu_{-2}'(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}(\xi_{n}^{*}) \right],$$
  

$$\mu_{-1}'(\xi_{j}) = -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{\xi_{j} - \xi_{n}^{*}} \left[ \mu_{-2}'(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}(\xi_{n}^{*}) \right].$$
(38)

Then, we have

$$\mu_{-12}(\xi_{j}^{*}) = \sum_{n=1}^{2N} C_{n}(\xi_{j}^{*}) \left[ \mu_{-11}'(\xi_{n}) + \left( D_{n} + \frac{1}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-11}(\xi_{n}) \right],$$
  

$$\mu_{-12}'(\xi_{j}^{*}) = -\sum_{n=1}^{2N} \frac{C_{n}(\xi_{j}^{*})}{\xi_{j}^{*} - \xi_{n}} \left[ \mu_{-11}'(\xi_{n}) + \left( D_{n} + \frac{2}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-11}(\xi_{n}) \right],$$
  

$$\mu_{-11}(\xi_{j}) = 1 + \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) \left[ \mu_{-12}'(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}(\xi_{n}^{*}) \right],$$
  

$$\mu_{-11}'(\xi_{j}) = -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{\xi_{j} - \xi_{n}^{*}} \left[ \mu_{-12}'(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}(\xi_{n}^{*}) \right].$$
  
(39)

**Theorem 3.1.** The explicit formula of the double pole soliton solution for the GI equation (1) with ZBCs (5) is expressed as

-

$$u(x,t) = -2i \frac{\det \begin{pmatrix} I-H & \beta \\ \alpha^T & 0 \end{pmatrix}}{\det(I-H)},$$
(40)

where  $\beta$ , H, and  $\alpha$  are given in (42), (43), and (45).

*Proof.* We can rewrite the linear system (39) in the matrix form,

$$\psi - H\psi = \beta,\tag{41}$$

where

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}, \psi^{(1)} = (\mu_{-11}(\xi_1), \dots, \mu_{-11}(\xi_{2N}))^T, \psi^{(2)} = (\mu'_{-11}(\xi_1), \dots, \mu'_{-11}(\xi_{2N}))^T,$$
  
$$\beta = \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix}, \beta^{(1)} = (1)_{2N \times 1}, \beta^{(2)} = (0)_{2N \times 1}.$$
(42)

The 4N × 4N matrix 
$$H = \begin{pmatrix} H^{(11)} & H^{(12)} \\ H^{(21)} & H^{(22)} \end{pmatrix}$$
 with  $H^{(im)} = \begin{pmatrix} H^{(im)}_{jk} \end{pmatrix}_{2N \times 2N} (i, m = 1, 2)$  is given by

(42)

$$\begin{aligned} H_{jk}^{(11)} &= \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*}) \bigg[ -\frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg( D_{k} + \frac{2}{\xi_{n}^{*} - \xi_{k}} \bigg) + \bigg( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg) \bigg], \\ H_{jk}^{(12)} &= \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*}) \bigg[ -\frac{1}{\xi_{n}^{*} - \xi_{k}} + \bigg( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg], \\ H_{jk}^{(21)} &= \sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*})}{\xi_{j} - \xi_{n}^{*}} \bigg[ \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg( D_{k} + \frac{2}{\xi_{n}^{*} - \xi_{k}} \bigg) - \bigg( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg) \bigg], \\ H_{jk}^{(22)} &= \sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*})}{\xi_{j} - \xi_{n}^{*}} \bigg[ \frac{1}{\xi_{n}^{*} - \xi_{k}} - \bigg( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg]. \end{aligned}$$

$$(43)$$

According to the reflectionless potential, Eq. (32) can be rewritten as

$$u = 2i\alpha^T \psi, \tag{44}$$

where

$$\alpha = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \end{pmatrix}, \alpha^{(2)} = (A[\xi_1]e^{-2i\theta(\xi_1)}, A[\xi_2]e^{-2i\theta(\xi_2)}, \dots, A[\xi_{2N}]e^{-2i\theta(\xi_{2N})})^T,$$
  

$$\alpha^{(1)} = (A[\xi_1]e^{-2i\theta(\xi_1)}D_1, A[\xi_2]e^{-2i\theta(\xi_2)}D_2, \dots, A[\xi_{2N}]e^{-2i\theta(\xi_{2N})}D_{2N})^T.$$
(45)

# Combining Eq. (41), the expression of the double pole soliton solution can be derived.

For example, we have the one-double pole soliton solution of the GI equation with ZBCs: when N = 1, then  $\xi_1 = k_1, \xi_2 = -k_1$ . According to symmetry relations in Proposition 3.1, we have  $A[\xi_1] = A[k_1], A[\xi_2] = -A[k_1], A[\xi_1^*] = -A[k_1]^*, A[\xi_2^*] = A[k_1]^*, B[\xi_1] = B[k_1], B[\xi_2] = -B[k_1], B[\xi_1^*] = B[k_1]^*, B[\xi_2^*] = -B[k_1]^*$ . Substituting above data into formula (40), we get the one-double pole soliton

$$u(x,t) = -2i \frac{\left( \begin{matrix} I - H_{11}^{(11)} & H_{12}^{(11)} & H_{11}^{(12)} & H_{12}^{(12)} & 1 \\ H_{21}^{(11)} & 1 - H_{22}^{(11)} & H_{21}^{(12)} & H_{22}^{(12)} & 1 \\ H_{11}^{(21)} & H_{12}^{(21)} & 1 - H_{11}^{(22)} & H_{12}^{(22)} & 0 \\ H_{21}^{(21)} & H_{22}^{(22)} & H_{21}^{(22)} & 1 - H_{22}^{(22)} & 0 \\ \alpha_{1}^{(1)} & \alpha_{2}^{(1)} & \alpha_{1}^{(2)} & \alpha_{2}^{(2)} & \alpha_{2}^{(2)} & 0 \\ \end{pmatrix}},$$
(46)  
$$\frac{\det \begin{pmatrix} I - H_{11}^{(11)} & H_{12}^{(11)} & H_{11}^{(12)} \\ H_{21}^{(11)} & 1 - H_{22}^{(11)} & H_{21}^{(12)} \\ H_{21}^{(11)} & H_{12}^{(12)} & 1 - H_{11}^{(22)} \\ H_{11}^{(21)} & H_{22}^{(12)} & H_{21}^{(22)} \\ H_{21}^{(21)} & H_{22}^{(22)} & 1 - H_{22}^{(22)} \end{pmatrix}},$$

where

$$\begin{split} H_{jk}^{(11)} &= \sum_{n=1}^{2} \hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*}) \bigg[ -\frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg( D_{k} + \frac{2}{\xi_{n}^{*} - \xi_{k}} \bigg) + \bigg( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg) \bigg], \\ H_{jk}^{(12)} &= \sum_{n=1}^{2} \hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*}) \bigg[ -\frac{1}{\xi_{n}^{*} - \xi_{k}} + \bigg( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg], \\ H_{jk}^{(21)} &= \sum_{n=1}^{2} \frac{\hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*})}{\xi_{j} - \xi_{n}^{*}} \bigg[ \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg( D_{k} + \frac{2}{\xi_{n}^{*} - \xi_{k}} \bigg) - \bigg( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*} - \xi_{k}} \bigg) \bigg], \\ H_{jk}^{(22)} &= \sum_{n=1}^{2} \frac{\hat{C}_{n}(\xi_{j}) C_{k}(\xi_{n}^{*})}{\xi_{j} - \xi_{n}^{*}} \bigg[ \frac{1}{\xi_{n}^{*} - \xi_{k}} - \bigg( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \bigg) \bigg], \end{split}$$

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$$C_{k}(\xi_{n}^{*}) = \frac{A[\xi_{k}]e^{-2i\theta(\xi_{k})}}{\xi_{n}^{*} - \xi_{k}}, \quad D_{n} = B[\xi_{n}] - 2i\theta'(\xi_{n}), \quad \hat{C}_{n}(\xi_{j}) = \frac{A[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}}{\xi_{j} - \xi_{n}^{*}}, \quad \hat{D}_{n} = B[\xi_{n}^{*}] + 2i\theta'(\xi_{n}^{*}),$$

$$\alpha_{j}^{(1)} = A[\xi_{j}]e^{-2i\theta(\xi_{j})}D_{j}, \quad \alpha_{j}^{(2)} = A[\xi_{j}]e^{-2i\theta(\xi_{j})}, \quad \theta(\xi_{j}) = \xi_{j}^{2}(x + 2(\xi_{j})^{2}t), \\ \theta(\xi_{j}) = 2\xi_{j}(2\xi_{j}^{2}t + x) + 4\xi_{j}^{3}t, \quad \theta'(\xi_{j}^{*}) = 2\xi_{j}^{*}(2(\xi_{j}^{*})^{2}t + x) + 4(\xi_{j}^{*})^{3}t, \quad j, \quad k, \quad n = 1, 2.$$

$$(47)$$

In order to compare the one-double pole soliton solution with the standard one soliton solution, we first derive the standard one soliton solution through similar steps as above, given by

$$u(x,t) = \frac{16ie^{i\theta_2}(4Aa^4b^4e^{4ab\theta_1} + A^3(a+ib)^2a^2b^2e^{12ab\theta_1})}{8A^2a^2b^2(a^2 - b^2)e^{8ab\theta_1} + A^4(a^2 + b^2)^2e^{16ab\theta_1} + 16a^4b^4},$$
(48)

where we have set  $k_1 = a + ib$ ,  $\theta_1 = 4a^2t - 4b^2t + x$ ,  $\theta_2 = -4ta^2 + (24b^2t - 2x)a^2 - 4tb^4 + 2xb^2$  and *A*, *a*, *b* are all arbitrary constants. Then, we have

$$|u(x,t)|^{2} = \frac{256A^{2}a^{4}b^{4}e^{8ab\theta_{1}}}{A^{4}(a^{2}+b^{2})^{2}e^{16ab\theta_{1}}+8a^{2}b^{2}(A^{2}(a^{2}-b^{2})e^{8ab\theta_{1}}+2a^{2}b^{2})}.$$
(49)

By taking  $|u(x,t)|_x^2 = |u(x,t)|_t^2 = 0$ , we get that the center trajectory of the standard one soliton solution is

$$x = -4(a^2 - b^2)t + \frac{1}{4ab}\log\frac{2ab}{A\sqrt{a^2 + b^2}}$$

and its velocity is  $-4(a^2 - b^2)$ . Taking parameters A = a = b = 1 in (49), the exact expression of the standard one soliton solution is

$$|u(x,t)|^2 = \frac{64e^{8x}}{e^{16x}+4}.$$
(50)

Since the expression of the one-double pole soliton solution is complicated, we will compare the standard soliton solution and one-double pole soliton solution in terms of same parameters, i.e.,  $A[k_1] = B[k_1] = 1$ ,  $k_1 = a + ib = 1 + i$ . Therefore, according to (46), the exact expression of the one-double pole soliton solution is

$$\begin{aligned} \left|u(x,t)\right|^{2} &= \frac{524\,288e^{8x}\Xi}{\Omega e^{16x} + (16\,384t + 256)e^{8x} - (1024t + 16)e^{24x} + e^{32x} + 256},\\ \Xi &= \left(t^{2} + \frac{1}{64}x^{2} + \frac{1}{32}t - \frac{1}{256}x + \frac{1}{2048}\right)e^{16x} + 2x\left(t + \frac{1}{64}\right)e^{8x} + 16t^{2} + \frac{x^{2}}{4} + \frac{t}{2} + \frac{x}{16} + \frac{1}{128},\\ \Omega &= 268\,435\,456t^{4} + 16\,777\,216t^{3} + (8\,388\,608x^{2} + 786\,432)t^{2} \\ &+ (262\,144x^{2} + 16\,384)t + 65\,536x^{4} + 4096x^{2} + 96. \end{aligned}$$
(51)

Comparing (51) and (50), we find that the one-double pole soliton solution is more complicated than the standard one soliton solution, and the highest power of e-exponential for the denominator of the one-double pole soliton solution is double of the one for the standard one soliton solution. The highest power of e-exponential for the numerator of the one-double pole soliton solution is triple of the one for the standard one soliton solution.

Furthermore, we analyze the asymptotic states of the one-double pole soliton solution as  $t \to \pm \infty$ . Without loss of generality, let  $A[k_1] = 1$ ,  $B[k_1] = 1$ ,  $k_1 = a + bi$ , through maple symbol calculations, and we can derive the asymptotic state of the one-double pole soliton solution, given by

$$|u(x,t)|^{2} \rightarrow \begin{cases} \frac{4\,194\,304a^{10}b^{10}(a^{2}+b^{2})e^{8ab\theta_{3}}}{(a^{2}+b^{2})^{4}e^{16ab\theta_{3}}+131\,072a^{8}b^{8}(a^{4}-b^{4})e^{8ab\theta_{3}}+4\,294\,967\,296a^{16}b^{16}} \text{ as } \theta_{3} = \theta_{1} - \frac{1}{4ab}\log(t), \\ \frac{16\,384a^{2}b^{2}(a^{2}+b^{2})e^{8ab\theta_{3}}}{65\,536(a^{2}+b^{2})^{4}e^{16ab\theta_{3}}+512(a^{4}-b^{4})e^{8ab\theta_{3}}+1} \text{ as } \theta_{3} = \theta_{1} + \frac{1}{4ab}\log(t). \end{cases}$$
(52)



**FIG. 2.** The one-double pole soliton solution for Eq. (1) with ZBCs and N = 1. The parameters are  $A[k_1] = 1$ ,  $B[k_1] = i$ ,  $k_1 = \frac{1}{3} + \frac{1}{2}i$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -5 (long dashed curve), t = 0 (solid curve), and t = 5 (dashed-dotted curve).



**FIG. 3.** The two-double pole soliton solution for Eq. (1) with ZBCs and N = 2. The parameters are  $A[k_1] = 1$ ,  $B[k_1] = 1$ ,  $A[k_2] = 1$ ,  $B[k_2] = 1$ ,  $k_1 = \frac{1}{2} + \frac{1}{2}i$ ,  $k_2 = \frac{1}{3} + \frac{1}{2}i$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -10 (long dashed curve), t = 0 (solid curve), and t = 10 (dashed-dotted curve).

From the above expression, we see that the one-double pole soliton solution degrades into the two one-soliton solution as  $t \to \infty$ . Of which, the center trajectories are  $x = -4(a^2 - b^2)t + \frac{\log t}{4ab}$  and  $x = -4(a^2 - b^2)t - \frac{\log t}{4ab}$ , respectively. When  $t \to \infty$ , the position shift of the two standard one soliton solution is  $\frac{\log t}{2ab}$ , which depends on *t*.

We also analyze the dynamical behaviors of the double pole soliton solution for the GI equation with ZBCs via image simulation. In the case of N = 1, we display in Fig. 2 by choosing suitable parameters. From Figs. 2(a) and 2(b), we easily find that the one-double pole soliton solution is actually a kind of bound-state soliton solution that represents the interaction of the two bright solitons. In addition, the interaction between two bright solitons is an elastic collision with the shape and size of the soliton unchanging. Figure 2(c) represents the wave propagation along the *x* axis at t = -5, 0, 5. Figure 3 displays the interaction of two pairs of one-double soliton solutions, i.e., two-double soliton solutions, in the case of N = 2.

## IV. THE SOLUTION OF GI EQUATION UNDER ZBCs WITH TRIPLE POLES

In this section, we devote to derive the general *N*-triple pole solutions through discussing the inverse scattering problem with the triple poles discrete spectrum for the GI equation (1) under ZBCs.

# A. Inverse scattering problem with ZBCs and triple poles

The triple zeros  $k_n$  (n = 1, 2, ..., N) in  $D_0 = \{k \in \mathbb{C} : \text{Rek} > 0, \text{Imk} > 0\}$  mean  $s_{22}(k_n) = s'_{22}(k_n) = s''_{22}(k_n) = 0$  and  $s'''_{22}(k_n) \neq 0$ . Similar to expressions (20) and (21) and since  $s''_{22}(k_0) = 0$  in  $k_0 \in \Gamma \cap D_+$ ,  $\Phi''_{+2}(x,t;k_0) - b[k_0]\Phi''_{-1}(x,t;k_0) - 2d[k_0]\Phi'_{-1}(x,t;k_0)$  and  $\Phi_{-1}(x,t;k_0)$  are linearly dependent. In addition,  $\Phi''_{+1}(x,t;k_0) - b[k_0]\Phi''_{-2}(x,t;k_0) - 2d[k_0]\Phi'_{-2}(x,t;k_0)$  and  $\Phi_{-2}(x,t;k_0)$  are linearly dependent for  $k_0 \in \Gamma \cap D_-$ . Naturally, we get

$$\Phi_{+2}^{\prime\prime}(x,t;k_0) - b[k_0]\Phi_{-1}^{\prime\prime}(x,t;k_0) - 2d[k_0]\Phi_{-1}^{\prime}(x,t;k_0) = h[k_0]\Phi_{-1}(x,t;k_0), \quad k_0 \in \Gamma \cap D_+,$$

$$\Phi_{+1}^{\prime\prime}(x,t;k_0) - b[k_0]\Phi_{-2}^{\prime\prime}(x,t;k_0) - 2d[k_0]\Phi_{-2}^{\prime}(x,t;k_0) = h[k_0]\Phi_{-2}(x,t;k_0), \quad k_0 \in \Gamma \cap D_-,$$
(53)

where  $h[k_0]$  is also a norming constant. Thus, one has

$$\begin{split} &L_{-3} \left[ \frac{\Phi_{+2}(x,t;k)}{s_{22}(k)} \right] = \tilde{A}[k_0] \Phi_{-1}(x,t;k_0), \quad k_0 \in \Gamma \cap D_+, \\ &L_{-3} \left[ \frac{\Phi_{+1}(x,t;k)}{s_{11}(k)} \right] = \tilde{A}[k_0] \Phi_{-2}(x,t;k_0), \quad k_0 \in \Gamma \cap D_-, \\ &L_{-2} \left[ \frac{\Phi_{+2}(x,t;k)}{s_{22}(k)} \right] = \tilde{A}[k_0] \left[ \Phi_{-1}'(x,t;k_0) + \tilde{B}[k_0] \Phi_{-1}(x,t;k_0) \right], \quad k_0 \in \Gamma \cap D_+, \\ &L_{-2} \left[ \frac{\Phi_{+1}(x,t;k)}{s_{11}(k)} \right] = \tilde{A}[k_0] \left[ \Phi_{-2}'(x,t;k_0) + \tilde{B}[k_0] \Phi_{-2}(x,t;k_0) \right], \quad k_0 \in \Gamma \cap D_-, \\ &R_{ek_0} \left[ \frac{\Phi_{+2}(x,t;k)}{s_{22}(k)} \right] = \tilde{A}[k_0] \left[ \frac{1}{2} \Phi_{-1}''(x,t;k_0) + \tilde{B}[k_0] \Phi_{-2}'(x,t;k_0) + \tilde{C}[k_0] \Phi_{-1}(x,t;k_0) \right], \quad k_0 \in \Gamma \cap D_+, \\ &R_{ek_0} \left[ \frac{\Phi_{+1}(x,t;k)}{s_{22}(k)} \right] = \tilde{A}[k_0] \left[ \frac{1}{2} \Phi_{-1}''(x,t;k_0) + \tilde{B}[k_0] \Phi_{-2}'(x,t;k_0) + \tilde{C}[k_0] \Phi_{-1}(x,t;k_0) \right], \quad k_0 \in \Gamma \cap D_+, \\ &R_{ek_0} \left[ \frac{\Phi_{+1}(x,t;k)}{s_{11}(k)} \right] = \tilde{A}[k_0] \left[ \frac{1}{2} \Phi_{-2}''(x,t;k_0) + \tilde{B}[k_0] \Phi_{-2}'(x,t;k_0) + \tilde{C}[k_0] \Phi_{-2}(x,t;k_0) \right], \quad k_0 \in \Gamma \cap D_-, \end{aligned}$$

where  $L_{-3}[f(x, t; k)]$  denotes the coefficient of  $O((k - k_0)^{-3})$  term in the Laurent series expansion of f(x, t; k) at  $k = k_0$  and

$$\tilde{A}[k_0] = \begin{cases} \frac{6b[k_0]}{s_{22}''(k_0)}, & k_0 \in \Gamma \cap D_+, \\ \frac{6b[k_0]}{s_{11}''(k_0)}, & k_0 \in \Gamma \cap D_-, \end{cases}$$
(55)

$$\tilde{B}[k_0] = \begin{cases} \frac{d[k_0]}{b[k_0]} - \frac{s_{22}^{\prime\prime\prime\prime}(k_0)}{4s_{22}^{\prime\prime\prime}(k_0)}, & k_0 \in \Gamma \cap D_+, \\ \frac{d[k_0]}{b[k_0]} - \frac{s_{11}^{\prime\prime\prime\prime}(k_0)}{4s_{11}^{\prime\prime\prime}(k_0)}, & k_0 \in \Gamma \cap D_-, \end{cases}$$
(56)

$$\tilde{C}[k_0] = \begin{cases} \frac{h[k_0]}{2b[k_0]} - \frac{d[k_0]s_{22}^{\prime\prime\prime\prime}(k_0)}{4b[k_0]s_{22}^{\prime\prime\prime\prime}(k_0)} + \frac{(s_{22}^{\prime\prime\prime\prime})^2(k_0)}{16(s_{22}^{\prime\prime\prime\prime})^2(k_0)}, & k_0 \in \Gamma \cap D_+, \\ \frac{h[k_0]}{2b[k_0]} - \frac{d[k_0]s_{11}^{\prime\prime\prime\prime}(k_0)}{4b[k_0]s_{11}^{\prime\prime\prime\prime}(k_0)} + \frac{(s_{11}^{\prime\prime\prime\prime})^2(k_0)}{16(s_{11}^{\prime\prime\prime\prime})^2(k_0)}, & k_0 \in \Gamma \cap D_-. \end{cases}$$
(57)

*Proposition 4.1.* Let  $k_0 \in \Gamma$ . Then, the following symmetry relations are satisfied:

- The first symmetry relation Ã[k₀] = -Ã[k₀\*]\*, B̃[k₀] = B̃[k₀\*]\*, C̃[k₀] = C̃[k₀\*]\*.
  The second symmetry relation Ã[k₀] = -Ã[-k₀\*]\*, B̃[k₀] = -B̃[-k₀\*]\*, C̃[k₀] = C̃[-k₀\*]\*.

Then, the residue, the coefficient  $L_{-2}$ , and the coefficient  $L_{-3}$  of M(x, t; k) can be written as

$$\begin{split} &L_{-3} M_{+} = \left(0, \tilde{A}[\xi_{n}]e^{-2i\theta(\xi_{n})}\mu_{-1}(\xi_{n})\right), \\ &L_{-3} M_{-} = \left(\tilde{A}[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}\mu_{-2}(\xi_{n}^{*}), 0\right), \\ &L_{-2} M_{+} = \left(0, \tilde{A}[\xi_{n}]e^{-2i\theta(\xi_{n})}\left[\mu_{-1}'(x, t; \xi_{n}) + \left[\tilde{B}[\xi_{n}] - 2i\theta'(\xi_{n})\right]\mu_{-1}(\xi_{n})\right]\right), \\ &L_{-2} M_{-} = \left(\tilde{A}[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}\left[\mu_{-2}'(\xi_{n}^{*}) + \left[\tilde{B}[\xi_{n}^{*}] + 2i\theta'(\xi_{n}^{*})\right]\mu_{-2}(\xi_{n}^{*})\right], 0\right), \\ &L_{-2} M_{-} = \left(\tilde{A}[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}\left[\mu_{-2}'(\xi_{n}^{*}) + \left[\tilde{B}[\xi_{n}] - 2i\theta'(\xi_{n})\right]\mu_{-2}(\xi_{n}^{*})\right], 0\right), \\ &Res_{k=\xi_{n}} M_{+} = \left(0, \tilde{A}[\xi_{n}]e^{-2i\theta(\xi_{n})}\left[\frac{1}{2}\mu_{-1}''(\xi_{n}) + \left[\tilde{B}[\xi_{n}] - 2i\theta'(\xi_{n})\right]\mu_{-1}'(\xi_{n}) + \left[\tilde{C}[\xi_{n}] - \Theta_{1}(\xi_{n})\right]\mu_{-1}(\xi_{n})\right]\right), \\ &Res_{k=\xi_{n}} M_{-} = \left(\tilde{A}[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}\left[\frac{1}{2}\mu_{-2}''(\xi_{n}^{*}) + \left[\tilde{B}[\xi_{n}^{*}] + 2i\theta'(\xi_{n}^{*})\right]\mu_{-2}'(\xi_{n}^{*}) + \left[\tilde{C}[\xi_{n}^{*}] - \Theta_{2}(\xi_{n}^{*})\right]\mu_{-2}(\xi_{n}^{*})\right], 0\right), \end{aligned}$$

$$(58)$$

where

$$\Theta_{1}(\xi_{n}) = 2(\theta'(\xi_{n}))^{2} + i\theta''(\xi_{n}) + 2\tilde{B}[\xi_{n}]i\theta'(\xi_{n}),$$
  

$$\Theta_{2}(\xi_{n}^{*}) = 2(\theta'(\xi_{n}^{*}))^{2} - i\theta''(\xi_{n}^{*}) - 2\tilde{B}[\xi_{n}^{*}]i\theta'(\xi_{n}^{*}).$$
(59)

Subtracting out the residue, the coefficient  $L_{-2}$ ,  $L_{-3}$ , and the asymptotic values as  $k \to \infty$  from the original non-regular RHP, the regular RHP is derived, given by

 $M_{-} - I -$ 

$$\sum_{n=1}^{2N} \left[ \frac{L_{-3} M_{+}}{(k-\xi_{n})^{3}} + \frac{L_{-2} M_{+}}{(k-\xi_{n})^{2}} + \frac{\operatorname{Res} M_{+}}{k-\xi_{n}} + \frac{L_{-3} M_{-}}{(k-\xi_{n}^{*})^{3}} + \frac{L_{-2} M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res} M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res} M_{-}}{(k-\xi_{n}^{*})^{2}} \right] M_{+} - I -$$

$$=\sum_{n=1}^{2N}\left[\frac{L_{-3}}{(k-\xi_n)^3} + \frac{L_{-2}}{(k-\xi_n)^2} + \frac{\operatorname{Res} M_+}{k-\xi_n} + \frac{L_{-3}}{(k-\xi_n^*)^3} + \frac{L_{-2}}{(k-\xi_n^*)^3} + \frac{L_{-2}}{(k-\xi_n^*)^2} + \frac{\operatorname{Res} M_-}{(k-\xi_n^*)^2} + \frac{\operatorname{Res} M_-}{(k-\xi_n^*)^2} \right] - M_+G,$$
(60)

which can be solved as follows by Plemelj's formulas:

+

$$M(x,t;k) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{M_{+}(x,t;\zeta)G(x,t;\zeta)}{\zeta - k} d\zeta$$

$$\sum_{n=1}^{2N} \left[ \frac{L_{-3}M_{+}}{(k-\xi_{n})^{3}} + \frac{L_{-2}M_{+}}{(k-\xi_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{k-\xi_{n}} + \frac{L_{-3}M_{-}}{(k-\xi_{n}^{*})^{3}} + \frac{L_{-2}M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res}M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res}M_{-}}{(k-\xi_{n}^{*})^{2}} \right],$$
(61)

where

$$\frac{L_{-3}M_{+}}{(k-\xi_{n})^{3}} + \frac{L_{-2}M_{+}}{(k-\xi_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{k-\xi_{n}} + \frac{L_{-3}M_{-}}{(k-\xi_{n}^{*})^{3}} + \frac{L_{-2}M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res}M_{-}}{(k-\xi_{n}^{*})^{2}} + \frac{\operatorname{Res}M_$$

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$$C_{n}(k) = \frac{\tilde{A}[\xi_{n}]e^{-2i\theta(\xi_{n})}}{k - \xi_{n}}, D_{n} = \tilde{B}[\xi_{n}] - 2i\theta'(\xi_{n}), F_{n} = \tilde{C}[\xi_{n}] - \Theta_{1}(\xi_{n}),$$
$$\hat{C}_{n}(k) = \frac{\tilde{A}[\xi_{n}^{*}]e^{2i\theta(\xi_{n}^{*})}}{k - \xi_{n}^{*}}, \hat{D}_{n} = \tilde{B}[\xi_{n}^{*}] + 2i\theta'(\xi_{n}^{*}), \hat{F}_{n} = \tilde{C}[\xi_{n}^{*}] - \Theta_{2}(\xi_{n}^{*}).$$
(63)

Furthermore, according to (16), we have

$$M^{(1)}(x,t;k) = -\frac{1}{2\pi i} \int_{\Sigma} M_+(x,t;\zeta) G(x,t;\zeta) d\zeta +$$

$$\sum_{n=1}^{2N} \left( \tilde{A}[\xi_n^*] e^{2i\theta(\xi_n^*)} \left( \frac{1}{2} \mu_{-2}^{\prime\prime}(\xi_n^*) + \hat{D}_n \mu_{-2}^{\prime}(\xi_n^*) + \hat{F}_n \mu_{-2}(\xi_n^*) \right), \tilde{A}[\xi_n] e^{-2i\theta(\xi_n)} \left( \frac{1}{2} \mu_{-1}^{\prime\prime}(\xi_n) + D_n \mu_{-1}^{\prime}(\xi_n) + F_n \mu_1(\xi_n) \right) \right).$$
(64)

The potential u(x, t) with triple poles for the GI equation with ZBCs (5) is reconstructed as

$$u(x,t) = 2iM_{12}^{(1)} = -\frac{1}{\pi}\int_{\Sigma} (M_+(x,t;\zeta)G(x,t;\zeta))_{12}d\zeta$$

$$+2i\sum_{n=1}^{2N}\tilde{A}[\xi_{n}]e^{-2i\theta(\xi_{n})}\left(\frac{1}{2}\mu_{-11}^{\prime\prime\prime}(\xi_{n})+D_{n}\mu_{-11}^{\prime}(\xi_{n})+F_{n}\mu_{11}(\xi_{n})\right).$$
(65)

# **B.** Trace formulas

Since  $\xi_n$  and  $\xi_n^*$  are triple zeros of the scattering coefficients  $s_{22}(k)$  and  $s_{11}(k)$ , respectively, we can take

$$\beta^{+}(k) = s_{22}(k) \prod_{n=1}^{2N} \left( \frac{k - \xi_{n}^{*}}{k - \xi_{n}} \right)^{3}, \ \beta^{-}(k) = s_{11}(k) \prod_{n=1}^{2N} \left( \frac{k - \xi_{n}}{k - \xi_{n}^{*}} \right)^{3}.$$
(66)

Then,  $\beta^+(k)$  is analytic and has no zeros in  $D_+$ , and  $\beta^-(k)$  is analytic and has no zeros in  $D_-$ . They both tend to o(1) as  $k \to \infty$ . According to Cauchy projectors and Plemelj's formulas,  $\beta^{\pm}(k)$  can be expressed as

$$\log \beta^{\pm}(k) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho \tilde{\rho})}{s - k} ds, \quad k \in D^{\pm}.$$
(67)

Combining Eqs. (66) and (67), we get

$$\log s_{22}(k) = -\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s-k} ds + \sum_{n=1}^{2N} \log\left(\frac{k-\xi_n}{k-\xi_n^*}\right)^3.$$
(68)

Furthermore, let

$$\log s_{22}(k) = \sum_{j=1}^{\infty} \frac{I_j}{k^j},$$
(69)

and the following trace formulas can be easily obtained:

$$I_{j} = 3\sum_{n=1}^{2N} \left( \frac{(\xi_{n}^{*})^{j} - \xi_{n}^{j}}{j} \right) + \frac{1}{2\pi i} \int_{\Sigma} \log(1 - \rho \tilde{\rho}) s^{j-1} ds.$$
(70)

# C. Triple pole soliton solutions with ZBCs

Taking  $\rho(k) = \tilde{\rho}(k) = 0$ , we can derive the explicit triple pole soliton solutions of the GI equation with ZBCs. Then, from Eq. (61), we get

(71)

$$\begin{split} \mu_{-2}(\xi_{j}^{*}) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{n=1}^{2N} C_{n}(\xi_{j}^{*}) \left[ \frac{1}{2} \mu_{-1}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{1}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-1}^{\prime}(\xi_{n}) + \left( \frac{1}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-1}(\xi_{n}) \right], \\ \mu_{-2}^{\prime}(\xi_{j}^{*}) &= -\sum_{n=1}^{2N} \frac{C_{n}(\xi_{j}^{*})}{\xi_{j}^{*} - \xi_{n}} \left[ \frac{1}{2} \mu_{-1}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{2}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-1}^{\prime}(\xi_{n}) + \left( \frac{3}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{2D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-1}(\xi_{n}) \right], \\ \mu_{-2}^{\prime\prime}(\xi_{j}^{*}) &= \sum_{n=1}^{2N} \frac{2C_{n}(\xi_{j}^{*})}{(\xi_{j}^{*} - \xi_{n})^{2}} \left[ \frac{1}{2} \mu_{-1}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{3}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-1}^{\prime}(\xi_{n}) + \left( \frac{6}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{3D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-1}(\xi_{n}) \right], \\ \mu_{-1}(\xi_{j}) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) \left[ \frac{1}{2} \mu_{-2}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}^{\prime}(\xi_{n}^{*}) + \left( \frac{1}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-2}(\xi_{n}^{*}) \right], \\ \mu_{-1}^{\prime}(\xi_{j}) &= -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-2}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}^{\prime}(\xi_{n}^{*}) + \left( \frac{3}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-2}(\xi_{n}^{*}) \right], \\ \mu_{-1}^{\prime\prime}(\xi_{j}) &= -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-2}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}^{\prime}(\xi_{n}^{*}) + \left( \frac{3}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-2}(\xi_{n}^{*}) \right], \\ \mu_{-1}^{\prime\prime}(\xi_{j}) &= \sum_{n=1}^{2N} \frac{2\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-2}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{3}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-2}^{\prime}(\xi_{n}^{*}) + \left( \frac{6}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{3\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-2}^{\prime}(\xi_{n}^{*}) \right].$$

Furthermore, we have

$$\mu_{-12}(\xi_{j}^{*}) = \sum_{n=1}^{2N} C_{n}(\xi_{j}^{*}) \left[ \frac{1}{2} \mu_{-11}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{1}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-11}^{\prime}(\xi_{n}) + \left( \frac{1}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-11}(\xi_{n}) \right],$$

$$\mu_{-12}^{\prime}(\xi_{j}^{*}) = -\sum_{n=1}^{2N} \frac{C_{n}(\xi_{j}^{*})}{\xi_{j}^{*} - \xi_{n}} \left[ \frac{1}{2} \mu_{-11}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{2}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-11}^{\prime}(\xi_{n}) + \left( \frac{3}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{2D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-11}(\xi_{n}) \right],$$

$$\mu_{-12}^{\prime\prime}(\xi_{j}^{*}) = \sum_{n=1}^{2N} \frac{2C_{n}(\xi_{j}^{*})}{(\xi_{j}^{*} - \xi_{n})^{2}} \left[ \frac{1}{2} \mu_{-11}^{\prime\prime}(\xi_{n}) + \left( D_{n} + \frac{3}{\xi_{j}^{*} - \xi_{n}} \right) \mu_{-11}^{\prime}(\xi_{n}) + \left( \frac{6}{(\xi_{j}^{*} - \xi_{n})^{2}} + \frac{3D_{n}}{\xi_{j}^{*} - \xi_{n}} + F_{n} \right) \mu_{-11}(\xi_{n}) \right],$$

$$\mu_{-11}(\xi_{j}) = 1 + \sum_{n=1}^{2N} \hat{C}_{n}(\xi_{j}) \left[ \frac{1}{2} \mu_{-12}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{1}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}^{\prime}(\xi_{n}^{*}) + \left( \frac{1}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-12}(\xi_{n}^{*}) \right],$$

$$\mu_{-11}^{\prime}(\xi_{j}) = -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-12}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}^{\prime}(\xi_{n}^{*}) + \left( \frac{3}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-12}(\xi_{n}^{*}) \right],$$

$$\mu_{-11}^{\prime}(\xi_{j}) = \sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-12}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{2}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}^{\prime}(\xi_{n}^{*}) + \left( \frac{3}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-12}(\xi_{n}^{*}) \right],$$

$$\mu_{-11}^{\prime}(\xi_{j}) = \sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})}{(\xi_{j} - \xi_{n}^{*})^{2}} \left[ \frac{1}{2} \mu_{-12}^{\prime\prime}(\xi_{n}^{*}) + \left( \hat{D}_{n} + \frac{3}{\xi_{j} - \xi_{n}^{*}} \right) \mu_{-12}^{\prime\prime}(\xi_{n}^{*}) + \left( \frac{3}{(\xi_{j} - \xi_{n}^{*})^{2}} + \frac{3\hat{D}_{n}}{\xi_{j} - \xi_{n}^{*}} + \hat{F}_{n} \right) \mu_{-12}(\xi_{n}^{*}) \right].$$

$$(72)$$

**Theorem 4.1.** The explicit formula of the triple pole soliton solution for the GI equation (1) with ZBCs (5) is expressed as

$$u(x,t) = -2i \frac{\det \begin{pmatrix} I - \tilde{H} & \tilde{\beta} \\ \tilde{\alpha}^T & 0 \end{pmatrix}}{\det(I - \tilde{H})},$$
(73)

where  $\tilde{\beta}$ ,  $\tilde{H}$ ,  $\tilde{\alpha}$  are given in (75), (76), and (80).

*Proof.* We rewrite the linear system (72) in the following matrix form:

$$\tilde{\psi} - \tilde{H}\tilde{\psi} = \tilde{\beta},$$
(74)

where

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$$\tilde{\psi} = \begin{pmatrix} \tilde{\psi}^{(1)} \\ \tilde{\psi}^{(2)} \\ \tilde{\psi}^{(3)} \end{pmatrix}, \quad \tilde{\psi}^{(1)} = (\mu_{-11}''(\xi_1), \dots, \mu_{-11}''(\xi_{2N}))^T, \quad \tilde{\psi}^{(2)} = (\mu_{-11}'(\xi_1), \dots, \mu_{-11}'(\xi_{2N}))^T,$$

$$\tilde{\psi}^{(3)} = (\mu_{-11}(\xi_1), \dots, \mu_{-11}(\xi_{2N}))^T, \quad \tilde{\beta} = \begin{pmatrix} \tilde{\beta}^{(1)} \\ \tilde{\beta}^{(2)} \\ \tilde{\beta}^{(3)} \end{pmatrix}, \quad \tilde{\beta}^{(1)} = (0)_{2N \times 1}, \quad \tilde{\beta}^{(2)} = (0)_{2N \times 1}, \quad \tilde{\beta}^{(3)} = (1)_{2N \times 1}. \tag{75}$$

The 6N × 6N matrix  $\tilde{H} = \left(\tilde{H}^{(im)}\right)_{3\times3}$  with  $\tilde{H}^{(im)} = \left(\tilde{H}^{(im)}_{jk}\right)_{2N\times2N}(i, m = 1, 2, 3)$  given by

$$\begin{split} \tilde{H}_{jk}^{(11)} &= \sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{(\xi_{j}-\xi_{n}^{*})^{2}} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} - \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{3}{\xi_{j}-\xi_{n}^{*}} \bigg) + \bigg( \frac{6}{(\xi_{j}-\xi_{n}^{*})^{2}} + \frac{3\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg], \\ \tilde{H}_{jk}^{(12)} &= \sum_{n=1}^{2N} \frac{2\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{(\xi_{j}-\xi_{n}^{*})^{2}} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} \bigg( D_{k} + \frac{3}{\xi_{n}^{*}-\xi_{k}} \bigg) - \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{3}{\xi_{j}-\xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{2}{\xi_{n}^{*}-\xi_{k}} \bigg) \\ &+ \bigg( \frac{6}{(\xi_{j}-\xi_{n}^{*})^{2}} + \frac{3\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg) \bigg], \\ \tilde{H}_{jk}^{(13)} &= \sum_{n=1}^{2N} \frac{2\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{(\xi_{j}-\xi_{n}^{*})^{2}} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} \bigg( \frac{6}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{3D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \\ &- \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{3}{\xi_{j}-\xi_{n}^{*}} \bigg) \bigg( \frac{3}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{2D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \\ &+ \bigg( \frac{6}{(\xi_{j}-\xi_{n}^{*})^{2}} + \frac{3\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg( \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \bigg], \end{split}$$
(76)

$$\begin{split} \tilde{H}_{jk}^{(21)} &= -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{2(\xi_{j}-\xi_{n}^{*})} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} - \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{2}{\xi_{j}-\xi_{n}^{*}} \bigg) + \bigg( \frac{3}{(\xi_{j}-\xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg], \\ \tilde{H}_{jk}^{(22)} &= -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{\xi_{j}-\xi_{n}^{*}} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} \bigg( D_{k} + \frac{3}{\xi_{n}^{*}-\xi_{k}} \bigg) - \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{2}{\xi_{j}-\xi_{n}^{*}} \bigg) \bigg( D_{k} + \frac{2}{\xi_{n}^{*}-\xi_{k}} \bigg) \\ &+ \bigg( \frac{3}{(\xi_{j}-\xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg( D_{k} + \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg) \bigg], \\ \tilde{H}_{jk}^{(23)} &= -\sum_{n=1}^{2N} \frac{\hat{C}_{n}(\xi_{j})C_{k}(\xi_{n}^{*})}{\xi_{j}-\xi_{n}^{*}} \bigg[ \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} \bigg( \frac{6}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{3D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \\ &- \frac{1}{\xi_{n}^{*}-\xi_{k}} \bigg( \hat{D}_{n} + \frac{2}{\xi_{j}-\xi_{n}^{*}} \bigg) \bigg( \frac{3}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{2D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \\ &+ \bigg( \frac{3}{(\xi_{j}^{*}-\xi_{n}^{*})^{2}} + \frac{2\hat{D}_{n}}{\xi_{j}-\xi_{n}^{*}} + \hat{F}_{n} \bigg) \bigg( \frac{1}{(\xi_{n}^{*}-\xi_{k})^{2}} + \frac{D_{k}}{\xi_{n}^{*}-\xi_{k}} + F_{k} \bigg) \bigg], \end{split}$$

$$\begin{split} \tilde{H}_{jk}^{(31)} &= \sum_{n=1}^{2N} \frac{\hat{C}_n(\xi_j) C_k(\xi_n^*)}{2} \bigg[ \frac{1}{(\xi_n^* - \xi_k)^2} - \frac{1}{\xi_n^* - \xi_k} \Big( \hat{D}_n + \frac{1}{\xi_j - \xi_n^*} \Big) + \Big( \frac{1}{(\xi_j - \xi_n^*)^2} + \frac{\hat{D}_n}{\xi_j - \xi_n^*} + \hat{F}_n \Big) \bigg], \\ \tilde{H}_{jk}^{(32)} &= \sum_{n=1}^{2N} \hat{C}_n(\xi_j) C_k(\xi_n^*) \bigg[ \frac{1}{(\xi_n^* - \xi_k)^2} \Big( D_k + \frac{3}{\xi_n^* - \xi_k} \Big) - \frac{1}{\xi_n^* - \xi_k} \Big( \hat{D}_n + \frac{1}{\xi_j - \xi_n^*} \Big) \Big( D_k + \frac{2}{\xi_n^* - \xi_k} \Big) \\ &+ \Big( \frac{1}{(\xi_j - \xi_n^*)^2} + \frac{\hat{D}_n}{\xi_j - \xi_n^*} + \hat{F}_n \Big) \Big( D_k + \frac{1}{\xi_n^* - \xi_k} \Big) \bigg], \\ \tilde{H}_{jk}^{(33)} &= \sum_{n=1}^{2N} \hat{C}_n(\xi_j) C_k(\xi_n^*) \bigg[ \frac{1}{(\xi_n^* - \xi_k)^2} \Big( \frac{6}{(\xi_n^* - \xi_k)^2} + \frac{3D_k}{\xi_n^* - \xi_k} + F_k \Big) \\ &- \frac{1}{\xi_n^* - \xi_k} \Big( \hat{D}_n + \frac{1}{\xi_j - \xi_n^*} \Big) \Big( \frac{3}{(\xi_n^* - \xi_k)^2} + \frac{2D_k}{\xi_n^* - \xi_k} + F_k \Big) \\ &+ \Big( \frac{1}{(\xi_j - \xi_n^*)^2} + \frac{\hat{D}_n}{\xi_j - \xi_n^*} + \hat{F}_n \Big) \Big( \frac{1}{(\xi_n^* - \xi_k)^2} + \frac{D_k}{\xi_n^* - \xi_k} + F_k \Big) \bigg]. \end{split}$$

In the case of reflectionless potential, Eq. (65) can be redefined as

$$u = 2i\tilde{\alpha}^T \tilde{\psi},\tag{79}$$

where

$$\tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}^{(1)} \\ \tilde{\alpha}^{(2)} \\ \tilde{\alpha}^{(3)} \end{pmatrix}, \ \tilde{\alpha}^{(1)} = \begin{pmatrix} \frac{1}{2} \tilde{A}[\xi_1] e^{-2i\theta(\xi_1)}, \frac{1}{2} \tilde{A}[\xi_2] e^{-2i\theta(\xi_2)}, \dots, \frac{1}{2} \tilde{A}[\xi_{2N}] e^{-2i\theta(\xi_{2N})} \end{pmatrix}^T,$$

$$\tilde{\alpha}^{(2)} = (\tilde{A}[\xi_1] e^{-2i\theta(\xi_1)} D_1, \tilde{A}[\xi_2] e^{-2i\theta(\xi_2)} D_2, \dots, \tilde{A}[\xi_{2N}] e^{-2i\theta(\xi_{2N})} D_{2N})^T,$$

$$\tilde{\alpha}^{(3)} = (\tilde{A}[\xi_1] e^{-2i\theta(\xi_1)} F_1, \tilde{A}[\xi_2] e^{-2i\theta(\xi_2)} F_2, \dots, \tilde{A}[\xi_{2N}] e^{-2i\theta(\xi_{2N})} F_{2N})^T.$$
(80)

Combining Eq. (74), the triple pole soliton solution (73) can be given out.

Noting that the general expression of the one-triple poles solution for N = 1 in (73) is very complicated, we omit it here. However, by Maple computation, we find that the highest power of e-exponential for the denominator is triple of the one for the standard one soliton solution, and the highest power of e-exponential for the numerator is quintuple of the one for the standard one soliton. Moreover, let  $k_1 = \frac{1}{2} + i$  and  $\tilde{A}[k_1] = \tilde{B}[k_1] = \tilde{C}[k_1] = 1$ , through maple symbol calculations, and we can derive the asymptotic state of the one-triple pole soliton, given by

$$|u(x,t)|^{2} \rightarrow \begin{cases} \frac{1677721600e^{4\theta_{4}}}{15625e^{8\theta_{4}} - 157286400e^{4\theta_{4}} + 1099511627776} \text{as } \theta_{4} = \theta_{1} - \log(t), \\ \frac{102400e^{4\theta_{4}}}{15625e^{8\theta_{4}} - 9600e^{4\theta_{4}} + 4096} \text{as } \theta_{4} = \theta_{1}, \\ \frac{409600e^{4\theta_{4}}}{1024000000e^{8\theta_{4}} - 38400e^{4\theta_{4}} + 1} \text{as } \theta_{4} = \theta_{1} + \log(t). \end{cases}$$

$$(81)$$

(78)



**FIG. 4.** The one-triple pole soliton solution for Eq. (1) with ZBCs and N = 1. The parameters are  $\tilde{A}[k_1] = \tilde{B}[k_1] = \tilde{C}[k_1] = 1$ ,  $k_1 = \frac{1}{2} + \frac{1}{2}i$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -5 (long dashed curve), t = 0-(solid curve), and t = 5-(dashed-dotted curve).

From the above expression, we see that the one-triple pole soliton solution degrades into the three one soliton solutions as  $t \to \infty$ . Of which, the center trajectories are  $x = 3t + \log t$ , x = 3t and  $x = 3t - \log t$ , respectively. When  $t \to \infty$ , the position shift of two adjacent standard one soliton solutions is log *t*, which depends on *t*.

As a matter of convenience, we take  $\tilde{A}[k_1] = \tilde{B}[k_1] = \tilde{C}[k_1] = 1, k_1 = \frac{1}{2} + \frac{1}{2}i$  as an example to illustrate the correlative dynamic behavior for the one-triple pole soliton solution for the GI equation with ZBCs (5) via image simulation. As see in Fig. 4, it displays the bright-bright-bright soliton solutions, which stand for the interaction of three bright soliton waves.

# V. THE CONSTRUCTION OF RIEMANN-HILBERT PROBLEM WITH NZBCs

The direct scattering problem for the GI equation (1) with NZBCs has been studied in Refs. 13 and 16. In this section, we first recall some results for the direct scattering problem for the targeted GI equation with following NZBCs at infinity:

$$\lim_{x \to 1^{-1}} u(x,t) = u_{\pm} e^{-\frac{3}{2}iu_0^4 t + iu_0^2 x},$$
(82)

where  $|u_{\pm}| = u_0 > 0$  and  $u_{\pm}$  are constant.

## A. Spectral analysis

As  $x \to \pm \infty$ , the Lax pair (2) under the boundary (82) becomes

$$\Phi_x^{\infty} = X_{\pm} \Phi^{\infty} = (-ik^2 \sigma_3 + kQ_{\pm}) \Phi^{\infty}, \qquad \Phi_t^{\infty} = T_{\pm} \Phi^{\infty} = (2k^2 - u_0^2) X_{\pm} \Phi^{\infty}, \tag{83}$$

where

$$Q_{\pm} = \begin{pmatrix} 0 & u_{\pm} \\ -u_{\pm}^* & 0 \end{pmatrix}.$$
(84)

The fundamental matrix solution of this lax pair is

$$\Phi^{\infty}(x,t;k) = \begin{cases} Y_{\pm}(k)e^{-i\theta(x,t;k)\sigma_3}, & k \neq \pm iu_0, \\ I + (x - 3u_0^2 t)X_{\pm}(k), & k = \pm iu_0, \end{cases}$$
(85)

where

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$$Y_{\pm} = \begin{pmatrix} 1 & -\frac{iu_{\pm}}{\lambda+k} \\ -\frac{iu_{\pm}^{*}}{\lambda+k} & 1 \end{pmatrix}, \quad \theta(x,t;k) = k\lambda[x + (2k^{2} - u_{0}^{2})t], \quad \lambda^{2} = k^{2} + u_{0}^{2}.$$
(86)

Introducing a uniformization variable  $z = k + \lambda$ , we obtain

$$k = \frac{1}{2} \left( z - \frac{u_0^2}{z} \right), \qquad \lambda = \frac{1}{2} \left( z + \frac{u_0^2}{z} \right), \tag{87}$$

which means that the scattering problem can be analyzed on a standard *z*-plane instead of the two-sheeted Riemann surface. Defining  $D_+$ ,  $D_-$ , and  $\Sigma$  on the *z*-plane as  $\Sigma = \mathbb{R} \cup i\mathbb{R} \setminus \{0\}, D_{\pm} = \{z \in \mathbb{C} | \text{RezIm} z \ge 0\}$ , the Jost solutions  $\Phi_{\pm}(x, t, ; z)$  are given by

$$\Phi_{\pm}(x,t;z) = Y_{\pm}e^{-i\theta(x,t;z)\sigma_3} + o(1), \quad z \in \Sigma, \quad \text{as} \quad x \to \pm \infty.$$
(88)

After the variable transformation

$$\mu_{\pm}(x,t;z) = \Phi_{\pm}(x,t;z)e^{i\theta(x,t;z)\sigma_3},$$
(89)

the modified Jost solutions  $\mu_{\pm}(x,t;z)$  tend to  $Y_{\pm}(z)$  as  $x \to \pm \infty$ , and they also satisfy the following Volterra integral equations:

$$\mu_{\pm}(x,t,z) = \begin{cases} Y_{\pm} + \int_{\pm\infty}^{x} Y_{\pm} e^{-ik\lambda(x-y)\hat{\sigma}_{3}} [Y_{\pm}^{-1}\Delta X_{\pm}(y,t)\mu_{\pm}(y,t;z)] dy, & z \neq \pm iu_{0}, \\ Y_{\pm} + \int_{\pm\infty}^{x} [I + (x-y)X_{\pm}(z)]\Delta X_{\pm}(y,t)\mu_{\pm}(y,t;z) dy, & z = \pm iu_{0}, \end{cases}$$
(90)

where  $\Delta X_{\pm} = X - X_{\pm}$ .

Proposition 5.1. Suppose that  $u - u_{\pm} \in L^1(\mathbb{R}^{\pm})$ . Then,  $\mu_{\pm}(x,t;z)$  given in Eq. (89) are unique solutions for the Jost integral Eq. (90) in  $\Sigma_0 = \Sigma \setminus \{\pm iu_0\}$ , and  $\mu_{\pm}(x,t;z)$  have the following characteristics:

- $\mu_{-1}(x,t; z)$  and  $\mu_{+2}(x,t; z)$  become analytical for  $D_+$  and continuous in  $D_+ \cup \Sigma_0$ .
- $\mu_{+1}(x,t;z)$  and  $\mu_{-2}(x,t;z)$  become analytical for  $D_-$  and continuous in  $D_- \cup \Sigma_0$ .
- $\mu_{\pm}(x,t;z) \rightarrow I \text{ as } z \rightarrow \infty.$
- $\mu_{\pm}(x,t;z) \rightarrow -\frac{i}{z}\sigma_3 Q_{\pm} \text{ as } z \rightarrow 0.$
- det  $\mu_{\pm}(x,t;z) = \det Y_{\pm} = \gamma = 1 + \frac{u_0^2}{z^2}, \quad x,t \in \mathbb{R}, \quad z \in \Sigma_0.$

Since the Jost solutions  $\Phi_{\pm}(x, t; z)$  are the simultaneous solutions of spectral problem (2), which satisfy the following linear relation by the constant scattering matrix  $S(z) = (s_{ij}(z))_{2\times 2}$ :

$$\Phi_{+}(x,t;z) = \Phi_{-}(x,t;z)S(z), \quad z \in \Sigma_{0},$$
(91)

where  $S(z) = \sigma_2 S^*(z^*)\sigma_2$ ,  $S(z) = \sigma_1 S^*(-z^*)\sigma_1$ ,  $S(z) = (\sigma_3 Q_-)^{-1}S(-\frac{u_0^2}{z})\sigma_3 Q_+$  for  $z \in \Sigma$ . The scattering coefficients can be expressed as the Wronskian determinant in the following form:

$$s_{11}(z) = \frac{Wr(\Phi_{+,1}, \Phi_{-,2})}{\gamma(z)}, \quad s_{12}(z) = \frac{Wr(\Phi_{+,2}, \Phi_{-,2})}{\gamma(z)},$$
  

$$s_{21}(z) = \frac{Wr(\Phi_{-,1}, \Phi_{+,1})}{\gamma(z)}, \quad s_{22}(z) = \frac{Wr(\Phi_{-,1}, \Phi_{+,2})}{\gamma(z)}.$$
(92)

*Proposition 5.2.* The scattering matrix S(z) satisfies the following:

- det S(z) = 1 for  $z \in \Sigma_0$ .
- $s_{22}(z)$  becomes analytical  $D_+$  and continuous in  $D_+ \cup \Sigma_0$ .
- $s_{11}(z)$  becomes analytical  $D_-$  and continuous in  $D_- \cup \Sigma_0$ .
- $S(x,t;z) \rightarrow I \text{ as } z \rightarrow \infty.$
- $S(x,t;z) \rightarrow diag\left(\frac{u_-}{u_+},\frac{u_+}{u_-}\right)$  as  $z \rightarrow 0$ .

## B. The Riemann-Hilbert problem

Based on the analytic properties of Jost solutions  $\mu_{\pm}(x; t; z)$  in Proposition 5.1, we get the following sectionally meromorphic matrices:

$$M_{-}(x,t;z) = \left(\frac{\mu_{+,1}}{s_{11}}, \mu_{-,2}\right), \qquad M_{+}(x,t;z) = \left(\mu_{-,1}, \frac{\mu_{+,2}}{s_{22}}\right), \tag{93}$$

where superscripts  $\pm$  imply analyticity in  $D_+$  and  $D_-$ , respectively. Then, a matrix RHP is raised.

#### 1. Riemann-Hilbert problem 2

M(x, t; z) solves the following RHP:

 $\begin{cases}
M(x,t;z) \text{ is analytic in } \mathbb{C} \setminus \Sigma, \\
M_{-}(x,t;z) = M_{+}(x,t;z)(I - G(x,t;z)), \quad z \in \Sigma, \\
M(x,t;z) \to I, \quad z \to \infty, \\
M(x,t;z) \to -\frac{i}{z}\sigma_{3}Q_{-}, \quad z \to 0,
\end{cases}$ (94)

of which the jump matrix G(x, t; z) is

$$G = \begin{pmatrix} \rho(z)\tilde{\rho}(z) & e^{-2i\theta(x,t;z)}\tilde{\rho}(z) \\ -e^{2i\theta(x,t;z)}\rho(z) & 0 \end{pmatrix},$$
(95)

where  $\rho(z) = \frac{s_{21}(z)}{s_{11}(z)}$ ,  $\tilde{\rho}(z) = \frac{s_{12}(z)}{s_{22}(z)}$ . Taking

$$M(x,t;z) = I + \frac{1}{z}M^{(1)}(x,t;z) + O\left(\frac{1}{z^2}\right), \qquad z \to \infty,$$
(96)

then the potential u(x, t) of the GI equation (1) with NZBCs is given by

$$u(x,t) = iM_{12}^{(1)}(x,t;z) = i\lim_{z \to \infty} zM_{12}(x,t;z).$$
(97)

#### VI. THE SOLUTION OF GI EQUATION UNDER NZBCs WITH DOUBLE POLES

In this section, the inverse scattering problem with the double poles discrete spectrum for the GI equation (1) under NZBCs will be considered, and the general *N*-double pole solutions will be given.

#### A. Inverse scattering problem with NZBCs and double poles

We first suppose that  $s_{22}(z)$  has  $N_1$  double zeros  $z_n$   $(n = 1, 2, ..., N_1)$  in  $D_+ \cap \{z \in \mathbb{C} : \text{Re}z > 0$ ,  $|mz > 0, |z| > u_0\}$  and  $N_2$  double zeros  $\omega_m$   $(m = 1, 2, ..., N_2)$  in  $\{z = u_0 e^{i\vartheta} : 0 < \vartheta < \frac{\pi}{2}\}$ , which denotes  $s_{22}(z_0) = s'_{22}(z_0) = 0$  and  $s''_{22}(z_0) \neq 0$  when  $z_0$  is the double zero of  $s_{22}(z)$ . From the symmetries of the scattering matrix, the corresponding discrete spectrum is summed up as (see Fig. 5)

$$\Upsilon = \left\{ \pm z_n, \pm z_n^*, \pm \frac{u_0^2}{z_n}, \pm \frac{u_0^2}{z_n^*} \right\}_{n=1}^{N_1} \cup \left\{ \pm \omega_m, \pm \omega_m^* \right\}_{m=1}^{N_2}.$$
(98)

Due to  $s_{22}(z_0) = 0$  ( $z_0 \in \Upsilon \cap D_+$ ), we can find that  $\Phi_{-1}(x, t; z_0)$  and  $\Phi_{+2}(x, t; z_0)$  are linearly dependent. Analogously, since  $s_{11}(z_0) = 0$  ( $z_0 \in \Upsilon \cap D_-$ ),  $\Phi_{+1}(x, t; z_0)$  and  $\Phi_{-2}(x, t; z_0)$  can be linearly dependent. Then, we have

$$\Phi_{+2}(x,t,z_0) = b[z_0]\Phi_{-1}(x,t,z_0), \quad z_0 \in \Upsilon \cap D_+, \Phi_{+1}(x,t,z_0) = b[z_0]\Phi_{-2}(x,t,z_0), \quad z_0 \in \Upsilon \cap D_-,$$
(99)

where  $b[z_0]$  is a norming constant. Due to  $s'_{22}(z_0) = 0$   $(z_0 \in \Upsilon \cap D_+)$ , we find that  $\Phi'_{+2}(x,t;z_0) - b[z_0]\Phi'_{-1}(x,t;z_0)$  and  $\Phi_{-1}(x,t;z_0)$  are linearly dependent. Similarly, when  $z_0 \in \Upsilon \cap D_-$ ,  $\Phi'_{+1}(x,t;z_0) - b[z_0]\Phi'_{-2}(x,t;z_0)$  and  $\Phi_{-2}(x,t;z_0)$  are linearly dependent. Then, we have



FIG. 5. Distribution of the discrete spectrum and jumping curves for the RHP on the complex z-plane. Region  $D_+ = \{z \in \mathbb{C} | \text{RezIm} z > 0\}$  (gray region) and region  $D_- = \{z \in \mathbb{C} | \text{RezIm} z < 0\}$  (white region).

$$\Phi'_{+2}(x,t;z_0) - b[z_0]\Phi'_{-1}(x,t;z_0) = d[z_0]\Phi_{-1}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_+, \Phi'_{+1}(x,t;z_0) - b[z_0]\Phi'_{-2}(x,t;z_0) = d[z_0]\Phi_{-2}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_-,$$
(100)

where  $d[z_0]$  is a norming constant. Therefore, one obtains

$$\begin{split} &L_{-2} \left[ \frac{\Phi_{+2}(x,t;z)}{s_{22}(z)} \right] = A[z_0] \Phi_{-1}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_+, \\ &L_{-2} \left[ \frac{\Phi_{+1}(x,t;z)}{s_{11}(z)} \right] = A[z_0] \Phi_{-2}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_-, \\ &\operatorname{Res}_{z=z_0} \left[ \frac{\Phi_{+2}(x,t;z)}{s_{22}(z)} \right] = A[z_0] \left[ \Phi_{-1}'(x,t;z_0) + B[z_0] \Phi_{-1}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_+, \\ &\operatorname{Res}_{z=z_0} \left[ \frac{\Phi_{+1}(x,t;z)}{s_{11}(z)} \right] = A[z_0] \left[ \Phi_{-2}'(x,t;z_0) + B[z_0] \Phi_{-2}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_-, \end{split}$$
(101)

where  $L_{-2}[f(x,t;z)]$  is the coefficient of the  $O((z-z_0)^{-2})$  term in the Laurent series expansion of f(x,t;z) at  $z = z_0$  and

$$A[z_{0}] = \begin{cases} \frac{2b[z_{0}]}{s_{22}''(z_{0})}, & z_{0} \in Y \cap D_{+}, \\ \frac{2b[z_{0}]}{s_{11}''(z_{0})}, & z_{0} \in Y \cap D_{-}, \end{cases}$$
$$B[z_{0}] = \begin{cases} \frac{d[z_{0}]}{b[z_{0}]} - \frac{s_{22}''(z_{0})}{3s_{22}''(z_{0})}, & z_{0} \in Y \cap D_{+}, \\ \frac{d[z_{0}]}{b[z_{0}]} - \frac{s_{11}''(z_{0})}{3s_{11}''(z_{0})}, & z_{0} \in Y \cap D_{-}. \end{cases}$$
(102)

*Proposition 6.1.* Let  $z_0 \in \Upsilon$ . Then, the following symmetry relations are satisfied:

- The first symmetry relation A[z<sub>0</sub>] = −A[z<sub>0</sub><sup>\*</sup>]<sup>\*</sup>, B[z<sub>0</sub>] = B[z<sub>0</sub><sup>\*</sup>]<sup>\*</sup>.
  The second symmetry relation A[z<sub>0</sub>] = A[-z<sub>0</sub><sup>\*</sup>]<sup>\*</sup>, B[z<sub>0</sub>] = −B[-z<sub>0</sub><sup>\*</sup>]<sup>\*</sup>.
  The third symmetry relation A[z<sub>0</sub>] = z<sub>0</sub><sup>4</sup>/u<sub>1</sub>/u<sub>0</sub><sup>4</sup>u<sub>1</sub><sup>\*</sup> A[-u<sub>0</sub><sup>2</sup>/z<sub>0</sub>], B[z<sub>0</sub>] = u<sub>0</sub><sup>2</sup>/z<sub>0</sub><sup>2</sup> B[-u<sub>0</sub><sup>2</sup>/z<sub>0</sub>] + z<sub>0</sub>/z<sub>0</sub>.

As a matter of convenience, we set  $\hat{\zeta}_n = -\frac{u_0^2}{\zeta_n}$  and

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$$\zeta_{n} = \begin{cases}
z_{n}, & n = 1, 2, \dots, N_{1}, \\
-z_{n-N_{1}}, & n = N_{1} + 1, N_{1} + 2, \dots, 2N_{1}, \\
\frac{u_{0}^{2}}{z_{n-2N_{1}}^{*}}, & n = 2N_{1} + 1, 2N_{1} + 2, \dots, 3N_{1}, \\
-\frac{u_{0}^{2}}{z_{n-3N_{1}}^{*}}, & n = 3N_{1} + 1, 3N_{1} + 2, \dots, 4N_{1}, \\
\omega_{n-4N_{1}}, & n = 4N_{1} + 1, 4N_{1} + 2, \dots, 4N_{1} + N_{2}, \\
-\omega_{n-4N_{1}-N_{2}}, & n = 4N_{1} + N_{2} + 1, 4N_{1} + N_{2} + 2, \dots, 4N_{1} + 2N_{2}.
\end{cases}$$
(103)

Then, the residue and the coefficient  $L_{-2}$  of M(x, t; z) can be written as

$$\begin{split} &L_{-2} M_{+} = \left(0, A[\zeta_{n}]e^{-2i\theta(x,t;\zeta_{n})}\mu_{-1}(x,t;\zeta_{n})\right), \\ &L_{-2} M_{-} = \left(A[\hat{\zeta}_{n}]e^{2i\theta(x,t;\hat{\zeta}_{n})}\mu_{-2}(x,t;\hat{\zeta}_{n}),0\right), \\ &Res_{z=\hat{\zeta}_{n}} M_{+} = \left(0, A[\zeta_{n}]e^{-2i\theta(x,t;\hat{\zeta}_{n})}[\mu_{-1}'(x,t;\zeta_{n}) + [B[\zeta_{n}] - 2i\theta'(x,t;\zeta_{n})]\mu_{-1}(x,t;\zeta_{n})]\right), \\ &Res_{z=\hat{\zeta}_{n}} M_{-} = \left(A[\hat{\zeta}_{n}]e^{2i\theta(x,t;\hat{\zeta}_{n})}[\mu_{-2}'(x,t;\hat{\zeta}_{n}) + [B[\hat{\zeta}_{n}] + 2i\theta'(x,t;\hat{\zeta}_{n})]\mu_{-2}(x,t;\hat{\zeta}_{n})],0\right). \end{split}$$
(104)

By subtracting out the asymptotic values as  $z \to \infty, z \to 0$ , the residue, and the coefficient  $L_{-2}$  from the original non-regular RHP, one can obtain the following regular RHP:

$$M_{-} + \frac{i}{z}\sigma_3 Q_{-} - I_{-}$$

$$\sum_{n=1}^{4N_1+2N_2} \left[ \frac{L_{-2}}{(z-\zeta_n)^2} + \frac{\operatorname{Res} M_+}{z-\zeta_n} + \frac{L_{-2}}{(z-\hat{\zeta}_n)^2} + \frac{\operatorname{Res} M_-}{(z-\hat{\zeta}_n)^2} + \frac{\operatorname{Res} M_-}{(z-\hat{\zeta}_n)} \right]$$
$$= M_+ + \frac{i}{z}\sigma_3 Q_- - I_-$$

$$\sum_{n=1}^{4N_1+2N_2} \left[ \frac{L_{-2}}{(z-\zeta_n)^2} + \frac{\operatorname{Res} M_+}{z-\zeta_n} + \frac{L_{-2}}{(z-\hat{\zeta}_n)^2} + \frac{\operatorname{Res} M_-}{(z-\hat{\zeta}_n)^2} \right] - M_+ G.$$
(105)

According to Plemelj's formulas, the above RHP can be solved as

$$M(x,t;z) = I - \frac{i}{z}\sigma_{3}Q_{-} + \frac{1}{2\pi i}\int_{\Sigma} \frac{M_{+}(x,t;\zeta)G(x,t;\zeta)}{\zeta - z}d\zeta + \sum_{n=1}^{4N_{1}+2N_{2}} \left[\frac{L_{-2}M_{+}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{z-\zeta_{n}} + \frac{L_{-2}M_{-}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{-}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{-}}{(z-\zeta_{n})}\right],$$
(106)

where

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$$\frac{L_{-2}}{z=\zeta_{n}}\frac{M_{+}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res} M_{+}}{z-\zeta_{n}} + \frac{L_{-2}M_{-}}{(z-\hat{\zeta}_{n})^{2}} + \frac{\operatorname{Res} M_{-}}{(z-\hat{\zeta}_{n})} + \frac{L_{-2}M_{-}}{(z-\hat{\zeta}_{n})} + \frac{L_{-2}M_{-}}{$$

and

$$C_{n}(z) = \frac{A[\zeta_{n}]e^{-2i\theta(\zeta_{n})}}{z-\zeta_{n}}, \quad D_{n} = B[\zeta_{n}] - 2i\theta'(\zeta_{n}),$$
$$\hat{C}_{n}(z) = \frac{A[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}}{z-\hat{\zeta}_{n}}, \quad \hat{D}_{n} = B[\hat{\zeta}_{n}] + 2i\theta'(\hat{\zeta}_{n}).$$
(108)

Moreover, according to (96), we obtain

=

$$M^{(1)}(x,t;z) = -\frac{1}{2\pi i} \int_{\Sigma} M_{+}(x,t;\zeta) G(x,t;\zeta) d\zeta - i\sigma_{3}Q_{-}$$
  
+ 
$$\sum_{n=1}^{4N_{1}+2N_{2}} \left( A[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})} \left( \mu_{-2}'(\hat{\zeta}_{n}) + \hat{D}_{n}\mu_{-2}(\hat{\zeta}_{n}) \right), A[\zeta_{n}]e^{-2i\theta(\zeta_{n})} \left( \mu_{-1}'(\zeta_{n}) + D_{n}\mu_{-1}(\zeta_{n}) \right) \right).$$
(109)

The potential u(x, t) with double poles for the GI equation with NZBCs is given by

$$u(x,t) = iM_{12}^{(1)} = u_{-} - \frac{1}{2\pi} \int_{\Sigma} (M_{+}(x,t;\zeta)G(x,t;\zeta))_{12}d\zeta$$
  
+  $i\sum_{n=1}^{4N_{1}+2N_{2}} A[\zeta_{n}]e^{-2i\theta(\zeta_{n})} (\mu'_{-11}(\zeta_{n}) + D_{n}\mu_{-11}(\zeta_{n})).$  (110)

## B. Trace formulas and theta condition

Since  $\zeta_n$  and  $\hat{\zeta}_n$  are double zeros of the scattering coefficients  $s_{22}(z)$  and  $s_{11}(z)$ , respectively, we can set

$$\beta^{+}(z) = s_{22}(z) \prod_{n=1}^{4N_{1}+2N_{2}} \left(\frac{z-\hat{\zeta}_{n}}{z-\zeta_{n}}\right)^{2}, \ \beta^{-}(z) = s_{11}(z) \prod_{n=1}^{4N_{1}+2N_{2}} \left(\frac{z-\zeta_{n}}{z-\hat{\zeta}_{n}}\right)^{2}$$
(111)

such that  $\beta^+(z)$  is analytic and has no zeros in  $D_+$  and  $\beta^-(z)$  is analytic and has no zeros in  $D_-$ . They both tend to o(1) as  $z \to \infty$ . Through using Cauchy projectors and Plemelj's formulas,  $\beta^{\pm}(z)$  can be expressed as

$$\log \beta^{\pm}(z) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho \tilde{\rho})}{s - z} ds, \quad z \in D^{\pm}.$$
(112)

According to Eq. (111), the trace formulas are given as

$$s_{22}(z) = \exp\left[-\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s-z} ds\right]^{4N_1+2N_2} \left(\frac{z-\zeta_n}{z-\zeta_n}\right)^2,$$
  

$$s_{11}(z) = \exp\left[\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s-z} ds\right]^{4N_1+2N_2} \left(\frac{z-\zeta_n}{z-\zeta_n}\right)^2.$$
(113)

Furthermore, let  $z \rightarrow 0$ , and one has

$$\frac{u_{+}}{u_{-}} = \exp\left[\frac{i}{2\pi} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s} ds\right] \prod_{n=1}^{N_{1}} \left(\frac{z_{n}}{z_{n}^{*}}\right)^{8} \prod_{n=1}^{N_{2}} \left(\frac{\omega_{n}}{u_{0}}\right)^{8}.$$
(114)

Therefore, we easily obtain the theta condition for Eq. (114), given by

$$\arg\left(\frac{u_{+}}{u_{-}}\right) = \frac{1}{2\pi} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s} ds + 16 \sum_{n=1}^{N_{1}} \arg(z_{n}) + 8 \sum_{n=1}^{N_{2}} \arg(\omega_{n}).$$
(115)

## C. Double pole soliton solutions with NZBCs

To get the double pole soliton solution, we let  $\rho(z) = \tilde{\rho}(z) = 0$ . Then, the second column of Eq. (106) yields

$$\mu_{-2}(z) = \begin{pmatrix} -\frac{i}{z} \\ z \\ 1 \end{pmatrix} + \sum_{n=1}^{4N_1 + 2N_2} C_n(z) \Big[ \mu'_{-1}(\zeta_n) + \Big( D_n + \frac{1}{z - \zeta_n} \Big) \mu_{-1}(\zeta_n) \Big],$$
(116)

$$\mu_{-2}'(z) = \begin{pmatrix} \frac{i}{z^2} \mu_{-1} \\ 0 \end{pmatrix} - \sum_{n=1}^{4N_1 + 2N_2} \frac{C_n(z)}{z - \zeta_n} \Big[ \mu_{-1}'(\zeta_n) + \Big( D_n + \frac{2}{z - \zeta_n} \Big) \mu_{-1}(\zeta_n) \Big].$$
(117)

According to the symmetric relation, one can obtain

$$\mu_{-2}(z) = -\frac{iu_{-}}{z}\mu_{-1}\left(-\frac{u_{0}^{2}}{z}\right).$$
(118)

Taking the first-order derivative about  $\boldsymbol{z}$  in the above formula, we get

$$\mu_{-2}'(z) = \frac{iu_{-}}{z^2} \mu_{-1} \left( -\frac{u_0^2}{z} \right) - \frac{iu_{-}u_0^2}{z^3} \mu_{-1}' \left( -\frac{u_0^2}{z} \right).$$
(119)

Putting Eqs. (118) and (119) into Eqs. (116) and (117) and letting  $z = \hat{\zeta}_j$ ,  $j = 1, 2, \dots, 4N_1 + 2N_2$ , we obtain a  $8N_1 + 4N_2$  linear system, given by

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ C_{n}(\hat{\zeta}_{j})\mu_{-1}'(\zeta_{n}) + \left[ C_{n}(\hat{\zeta}_{j})\left(D_{n} + \frac{1}{\hat{\zeta}_{j} - \zeta_{n}}\right) + \frac{iu_{-}}{\hat{\zeta}_{j}}\delta_{j,n}\right]\mu_{-1}(\zeta_{n}) \right\} = \begin{pmatrix} \frac{i}{\hat{\zeta}_{j}}u_{-}\\ -1 \end{pmatrix},$$
(120)  
$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ \left(\frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j} - \zeta_{n}} - \frac{iu_{-}u_{0}^{2}}{\hat{\zeta}_{j}^{3}}\delta_{j,n}\right)\mu_{-1}'(\zeta_{n}) + \left[\frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j} - \zeta_{n}}\left(D_{n} + \frac{2}{\hat{\zeta}_{j} - \zeta_{n}}\right) + \frac{iu_{-}}{\hat{\zeta}_{j}^{2}}\delta_{j,n}\right]\mu_{-1}(\zeta_{n}) \right\} = \begin{pmatrix} \frac{i}{\hat{\zeta}_{j}^{2}}u_{-}\\ 0 \end{pmatrix}.$$
(121)

**Theorem 6.1.** The general formula of the double poles solution for the GI equation (1) with NZBCs (82) is expressed as

$$u(x,t) = u_{-} - i \frac{\det \begin{pmatrix} \mathcal{H} & \varphi \\ \chi^{T} & 0 \end{pmatrix}}{\det(\mathcal{H})},$$
(122)

where  $\varphi$ ,  $\mathcal{H}$ ,  $\chi$  are given by (126), (127), and (129), respectively.

*Proof.* From Eqs. (120) and (121), we get a  $8N_1 + 4N_2$  linear system with respect to  $\mu_{-11}(\zeta_n), \mu'_{-11}(\zeta_n)$ ,

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ C_{n}(\hat{\zeta}_{j})\mu_{-11}'(\zeta_{n}) + \left[ C_{n}(\hat{\zeta}_{j})\left(D_{n} + \frac{1}{\hat{\zeta}_{j} - \zeta_{n}}\right) + \frac{iu_{-}}{\hat{\zeta}_{j}}\delta_{j,n}\right]\mu_{-11}(\zeta_{n}) \right\} = \frac{iu_{-}}{\hat{\zeta}_{j}}$$

$$(123)$$

$$\sum_{j=1}^{4N_{1}+2N_{2}} \left\{ \left(\frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j}} - \frac{iu_{-}u_{0}^{2}}{\hat{\zeta}_{3}}\delta_{j,n}\right)\mu_{-11}'(\zeta_{n}) \right\} = \frac{iu_{-}}{\hat{\zeta}_{j}}$$

$$+ \left[\frac{C_n(\hat{\zeta}_j)}{\hat{\zeta}_j - \zeta_n} \left(D_n + \frac{2}{\hat{\zeta}_j - \zeta_n}\right) + \frac{iu_-}{\hat{\zeta}_j^2}\delta_{j,n}\right]\mu_{-11}(\zeta_n)\right\} = \frac{iu_-}{\hat{\zeta}_j^2},$$
(124)

(125)

which can be rewritten in the following matrix form:

where

$$\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \quad \varphi^{(1)} = \begin{pmatrix} \frac{iu_{-}}{\dot{\zeta}_{1}}, \frac{iu_{-}}{\dot{\zeta}_{2}}, \dots, \frac{iu_{-}}{\dot{\zeta}_{4N_{1}+2N_{2}}} \end{pmatrix}^{T}, \quad \varphi^{(2)} = \begin{pmatrix} \frac{iu_{-}}{\dot{\zeta}_{1}^{2}}, \frac{iu_{-}}{\dot{\zeta}_{2}^{2}}, \dots, \frac{iu_{-}}{\dot{\zeta}_{4N_{1}+2N_{2}}} \end{pmatrix}^{T}, \\ \gamma = \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \end{pmatrix}, \quad \gamma^{(1)} = (\mu_{-11}(\zeta_{1}), \mu_{-11}(\zeta_{2}), \dots, \mu_{-11}(\zeta_{4N_{1}+2N_{2}}))^{T}, \\ \gamma^{(2)} = (\mu'_{-11}(\zeta_{1}), \mu'_{-11}(\zeta_{2}), \dots, \mu'_{-11}(\zeta_{4N_{1}+2N_{2}}))^{T}, \quad \mathcal{H} = \begin{pmatrix} \mathcal{H}^{(11)} & \mathcal{H}^{(12)} \\ \mathcal{H}^{(21)} & \mathcal{H}^{(22)} \end{pmatrix}, \quad (126)$$

 $\mathcal{H}\gamma =$ 

with  $\mathcal{H}^{(im)} = \left(\mathcal{H}_{jn}^{(im)}\right)_{4N_1+2N_2 \times 4N_1+2N_2} (i, m = 1, 2)$  given by

$$\mathcal{H}_{jn}^{(11)} = C_n(\hat{\zeta}_j) \left( D_n + \frac{1}{\hat{\zeta}_j - \zeta_n} \right) + \frac{iu_-}{\hat{\zeta}_j} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(12)} = C_n(\hat{\zeta}_j),$$

$$\mathcal{H}_{jn}^{(21)} = \frac{C_n(\hat{\zeta}_j)}{\hat{\zeta}_j - \zeta_n} \left( D_n + \frac{2}{\hat{\zeta}_j - \zeta_n} \right) + \frac{iu_-}{\hat{\zeta}_j^2} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(22)} = \frac{C_n(\hat{\zeta}_j)}{\hat{\zeta}_j - \zeta_n} - \frac{iu_-u_0^2}{\hat{\zeta}_j^3} \delta_{j,n}.$$
(127)

According to the reflectionless potential, Eq. (110) can be rewritten as

$$u = u_{-} + i\chi^{T}\gamma, \tag{128}$$

where

$$\chi = \begin{pmatrix} \chi^{(1)} \\ \chi^{(2)} \end{pmatrix}, \chi^{(2)} = (A[\zeta_1]e^{-2i\theta(\zeta_1)}, A[\zeta_2]e^{-2i\theta(\zeta_2)}, \dots, A[\zeta_{4N_1+2N_2}]e^{-2i\theta(\zeta_{4N_1+2N_2})})^T,$$
  

$$\chi^{(1)} = (A[\zeta_1]e^{-2i\theta(\zeta_1)}D_1, A[\zeta_2]e^{-2i\theta(\zeta_2)}D_2, \dots, A[\zeta_{4N_1+2N_2}]e^{-2i\theta(\zeta_{4N_1+2N_2})}D_{4N_1+2N_2})^T.$$
(129)

From Eq. (125), the expression of the double pole soliton solution is derived.

For example, we have the one-double pole soliton solution of the GI equation with NZBCs.

• When  $N_1 = 0$ ,  $N_2 = 1$ , we have  $\zeta_1 = \omega_1$ ,  $\zeta_2 = -\omega_1$ ,  $\hat{\zeta}_1 = -\frac{u_0^2}{\omega_1}$ ,  $\hat{\zeta}_2 = \frac{u_0^2}{\omega_1}$ ,  $u_0 = |u_-|$ . Through the theta condition, we can give  $u_+$ . According to symmetry relations in Proposition 6.1, we have  $A[\zeta_1] = A[\omega_1]$ ,  $A[\zeta_2] = -A[\omega_1]$ ,  $B[\zeta_1] = B[\omega_1]$ ,  $B[\zeta_2] = -B[\omega_1]$ . Substituting above data into formula (122), we get the double poles dark-bright soliton solution,

$$u(x,t) = u_{-} - i \frac{\det \begin{pmatrix} \mathcal{H}_{11}^{(11)} & \mathcal{H}_{12}^{(11)} & \mathcal{H}_{11}^{(12)} & \mathcal{H}_{12}^{(12)} & \varphi_{1}^{(1)} \\ \mathcal{H}_{21}^{(11)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{21}^{(12)} & \mathcal{H}_{22}^{(22)} & \varphi_{2}^{(1)} \\ \mathcal{H}_{11}^{(21)} & \mathcal{H}_{12}^{(21)} & \mathcal{H}_{11}^{(22)} & \mathcal{H}_{12}^{(22)} & \varphi_{2}^{(2)} \\ \mathcal{H}_{21}^{(11)} & \mathcal{H}_{22}^{(12)} & \mathcal{H}_{21}^{(22)} & \mathcal{H}_{22}^{(2)} & \varphi_{2}^{(2)} \\ \chi_{1}^{(1)} & \chi_{2}^{(1)} & \chi_{1}^{(2)} & \chi_{2}^{(2)} & \varphi_{2}^{(2)} \\ \chi_{1}^{(1)} & \mathcal{H}_{21}^{(1)} & \mathcal{H}_{11}^{(12)} & \mathcal{H}_{12}^{(2)} \\ \mathcal{H}_{11}^{(11)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{22}^{(2)} & \varphi_{2}^{(2)} \\ \mathcal{H}_{11}^{(11)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{12}^{(2)} \\ \mathcal{H}_{21}^{(11)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{12}^{(2)} \\ \mathcal{H}_{21}^{(21)} & \mathcal{H}_{21}^{(21)} & \mathcal{H}_{22}^{(2)} \\ \mathcal{H}_{21}^{(21)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{22}^{(2)} \end{pmatrix},$$
(130)

where

$$\begin{aligned} \mathcal{H}_{jn}^{(11)} &= C_{n}(\hat{\zeta}_{j}) \left( D_{n} + \frac{1}{\hat{\zeta}_{j} - \zeta_{n}} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(12)} &= C_{n}(\hat{\zeta}_{j}), \\ \mathcal{H}_{jn}^{(21)} &= \frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j} - \zeta_{n}} \left( D_{n} + \frac{2}{\hat{\zeta}_{j} - \zeta_{n}} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}^{2}} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(22)} &= \frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j} - \zeta_{n}} - \frac{iu_{-}u_{0}^{2}}{\hat{\zeta}_{j}^{3}} \delta_{j,n}, \\ C_{n}(\hat{\zeta}_{j}) &= \frac{A[\zeta_{n}]e^{-2i\theta(\zeta_{n})}}{\hat{\zeta}_{j} - \zeta_{n}}, \quad D_{n} = B[\zeta_{n}] - 2i\theta'(\zeta_{n}), \\ \chi_{j}^{(1)} &= A[\zeta_{j}]e^{-2i\theta(\zeta_{j})}D_{j}, \ \chi_{j}^{(2)} &= A[\zeta_{j}]e^{-2i\theta(\zeta_{j})}, \ \varphi_{j}^{(1)} &= \frac{iu_{-}}{\hat{\zeta}_{j}}, \ \varphi_{j}^{(2)} &= \frac{iu_{-}}{\hat{\zeta}_{j}^{2}}, \\ \theta(\zeta_{j}) &= \frac{(\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t)(\zeta_{j}^{4} - 1)}{8\zeta_{j}^{4}}, \ j, k, n = 1, 2, \\ \theta'(\zeta_{j}) &= \frac{(4\zeta_{j}^{3}t - 8\zeta_{j}t + 4\zeta_{j}x)(\zeta_{j}^{4} - 1)}{8\zeta_{j}^{4}} - \frac{(\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t)(\zeta_{j}^{4} - 1)}{2\zeta_{j}^{5}} + \frac{\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t}{2\zeta_{j}}. \end{aligned}$$

$$(131)$$

• When  $N_1 = 1, N_2 = 0$ , we get  $\zeta_1 = z_1, \zeta_2 = -z_1, \zeta_3 = \frac{u_0^2}{z_1^*}, \zeta_4 = -\frac{u_0^2}{z_1^*}, \hat{\zeta}_1 = -\frac{u_0^2}{z_1}, \hat{\zeta}_2 = \frac{u_0^2}{z_1}, \hat{\zeta}_3 = -z_1^*, \hat{\zeta}_4 = z_1^*, u_0 = |u_-|$ . From the theta condition, we can give  $u_+$ . According to symmetry relations in Proposition 6.1, we have  $A[\zeta_1] = A[z_1], A[\zeta_2] = -A[z_1], A[\zeta_3] = -\frac{u_0^4 u_-^*}{z_1^{*4} u_-} A^*[z_1], A[\zeta_4] = \frac{u_0^4 u_-^*}{z_1^{*4} u_-} A^*[z_1], B[\zeta_1] = B[z_1], B[\zeta_2] = -B[z_1], B[\zeta_3] = \frac{2z_1^* - z_1^{*2}B[z_1]^*}{u_0^2}, B[\zeta_4] = \frac{-2z_1^* + z_1^{*2}B[z_1]^*}{u_0^2}$ . Substituting above data into formula (122), we get the double poles two breather solution,

$$u(x,t) = u_{-} - i \frac{\begin{pmatrix} \mathcal{H}_{11}^{(11)} & \mathcal{H}_{12}^{(11)} & \mathcal{H}_{13}^{(11)} & \mathcal{H}_{14}^{(11)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{13}^{(12)} & \mathcal{H}_{14}^{(12)} & \varphi_{1}^{(1)} \\ \mathcal{H}_{21}^{(11)} & \mathcal{H}_{22}^{(11)} & \mathcal{H}_{23}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{22}^{(12)} & \mathcal{H}_{23}^{(12)} & \mathcal{H}_{24}^{(12)} & \varphi_{21}^{(1)} \\ \mathcal{H}_{31}^{(11)} & \mathcal{H}_{32}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{31}^{(12)} & \mathcal{H}_{31}^{(12)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{42}^{(21)} & \mathcal{H}_{41}^{(21)} & \mathcal{H}_{41}^{(21)} & \mathcal{H}_{41}^{(22)} & \mathcal{H}_{42}^{(22)} & \mathcal{H}_{41}^{(22)} & \mathcal{H}_{41}^{(22)} & \varphi_{41}^{(2)} \\ \mathcal{H}_{21}^{(11)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{22}^{(22)} & \mathcal{H}_{23}^{(22)} & \mathcal{H}_{22}^{(22)} & \mathcal{H}_{23}^{(22)} & \mathcal{H}_{22}^{(2)} \\ \mathcal{H}_{21}^{(21)} & \mathcal{H}_{22}^{(21)} & \mathcal{H}_{23}^{(21)} & \mathcal{H}_{21}^{(21)} & \mathcal{H}_{22}^{(22)} & \mathcal{H}_{23}^{(22)} & \mathcal{H}_{24}^{(22)} & \varphi_{2}^{(2)} \\ \mathcal{H}_{31}^{(11)} & \mathcal{H}_{32}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{31}^{(22)} & \mathcal{H}_{32}^{(22)} & \mathcal{H}_{33}^{(22)} & \mathcal{H}_{33}^{(22)} & \mathcal{H}_{24}^{(22)} & \varphi_{2}^{(2)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{42}^{(11)} & \mathcal{H}_{41}^{(11)} & \mathcal{H}_{41}^{(12)} & \mathcal{H}_{42}^{(22)} & \mathcal{H}_{42}^{(22)} & \mathcal{H}_{24}^{(22)} & \varphi_{2}^{(2)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{42}^{(11)} & \mathcal{H}_{31}^{(11)} & \mathcal{H}_{11}^{(11)} & \mathcal{H}_{11}^{(12)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{13}^{(22)} & \mathcal{H}_{24}^{(22)} & \varphi_{4}^{(2)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{11}^{(11)} & \mathcal{H}_{11}^{(11)} & \mathcal{H}_{11}^{(12)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{12}^{(12)} & \mathcal{H}_{12}^{(12)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{41}^{(11)} & \mathcal{H}_{41}^{(11)} & \mathcal{H}_{41}^{(12)} & \mathcal{H}_{41}^{(22)} & \mathcal{H}_{41}^{(22)} & \mathcal{H}_{42}^{(22)} & \mathcal{H}_{42}^{(22)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{12}^{(21)} & \mathcal{H}_{12}^{(21)} & \mathcal{H}_{12}^{(22)} & \mathcal{H}_{12}^{(22)} & \mathcal{H}_{12}^{(22)} \\ \mathcal{H}_{41}^{(11)} & \mathcal{H}_{12}^{(11)} & \mathcal{H}_{11}^{(11)} & \mathcal{H}_{11}^{(12)} & \mathcal{H}_{12}^{(22)} & \mathcal{H}_{12}^{(22)} & \mathcal{H$$

where



**FIG. 6.** The one-double pole soliton solution for Eq. (1) with NZBCs and  $N_1 = 0$ ,  $N_2 = 1$ . The parameters are  $u_{\pm} = 1$ ,  $A[\omega_1] = 1$ ,  $B[\omega_1] = 1$ ,  $\omega_1 = e^{\frac{\pi}{4}i}$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -5 (long dashed curve), t = 0 (solid curve), and t = 5 (dashed-dotted curve).

$$\mathcal{H}_{jn}^{(11)} = C_n(\hat{\zeta}_j) \left( D_n + \frac{1}{\hat{\zeta}_j - \zeta_n} \right) + \frac{iu_-}{\hat{\zeta}_j} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(12)} = C_n(\hat{\zeta}_j),$$

$$\mathcal{H}_{jn}^{(21)} = \frac{C_n(\hat{\zeta}_j)}{\hat{\zeta}_j - \zeta_n} \left( D_n + \frac{2}{\hat{\zeta}_j - \zeta_n} \right) + \frac{iu_-}{\hat{\zeta}_j^2} \delta_{j,n}, \quad \mathcal{H}_{jn}^{(22)} = \frac{C_n(\hat{\zeta}_j)}{\hat{\zeta}_j - \zeta_n} - \frac{iu_-u_0^2}{\hat{\zeta}_j^3} \delta_{j,n},$$

$$C_n(\hat{\zeta}_j) = \frac{A[\zeta_n]e^{-2i\theta(\zeta_n)}}{\hat{\zeta}_j - \zeta_n}, \quad D_n = B[\zeta_n] - 2i\theta'(\zeta_n),$$

$$\chi_j^{(1)} = A[\zeta_j]e^{-2i\theta(\zeta_j)}D_j, \; \chi_j^{(2)} = A[\zeta_j]e^{-2i\theta(\zeta_j)}, \; \varphi_j^{(1)} = \frac{iu_-}{\hat{\zeta}_j}, \; \varphi_j^{(2)} = \frac{iu_-}{\hat{\zeta}_j^2}, \quad (133)$$

$$\theta(\zeta_{j}) = \frac{(\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t)(\zeta_{j}^{4} - 1)}{8\zeta_{j}^{4}}, \ j, k, n = 1, 2, 3, 4,$$
  
$$\theta'(\zeta_{j}) = \frac{(4\zeta_{j}^{3}t - 8\zeta_{j}t + 4\zeta_{j}x)(\zeta_{j}^{4} - 1)}{8\zeta_{j}^{4}} - \frac{(\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t)(\zeta_{j}^{4} - 1)}{2\zeta_{j}^{5}} + \frac{\zeta_{j}^{4}t - 4\zeta_{j}^{2}t + 2\zeta_{j}^{2}x + t}{2\zeta_{j}}.$$
 (134)



**FIG. 7.** The one-double pole soliton solution for Eq. (1) with NZBCs and  $N_1 = 1$ ,  $N_2 = 0$ . The parameters are  $u_- = 1$ ,  $u_+ = e^{\frac{2}{3}\pi i}$ ,  $A[z_1] = 1$ ,  $B[z_1] = i$ ,  $z_1 = 2e^{\frac{\pi}{6}i}$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -15 (long dashed curve), t = 0 (solid curve), and t = 15 (dashed-dotted curve).

As an example, through choosing some appropriate parameters, we discuss the dynamical behaviors for double poles dark-bright and double poles two breather solutions in the case of  $\{N_1 = 0, N_2 = 1\}$ ,  $\{N_1 = 1, N_2 = 0\}$ , and  $\{N_1 = 1, N_2 = 1\}$ , respectively. For  $N_1 = 0, N_2 = 1$ , we choose the parameters  $u_- = 1, A[\omega_1] = 1, B[\omega_1] = 1, \omega_1 = e^{\frac{\pi}{4}i}$  in which the theta condition is  $\arg(\frac{u_+}{u_-}) = 2\pi$  such that  $u_+ = u_-$ . For this case, the one-double pole soliton solution exhibits the interaction of dark and bright solitons, which can be verified in Fig. 6. Additionally, from Fig. 6(c), it is easily to find that the collision is an elastic collision since the shape and size of the dark and bright solitons remain unchanged after the collision. On the other hand, when  $N_1 = 1, N_2 = 0$ , we select the parameters  $u_- = 1, A[z_1] = 1, B[z_1] = i, z_1 = 2e^{\frac{\pi}{6}i}$  and the theta condition is  $\arg(\frac{u_+}{u_-}) = \frac{8}{3}\pi$ , which leads to  $u_+ = e^{\frac{2}{3}\pi i}$ . As shown in Fig. 7, the one-double pole soliton solutions can be obtained, which display the interaction of two breathers and two solitons.

## VII. THE SOLUTION OF GI EQUATION UNDER NZBCs WITH TRIPLE POLES

In this section, we aim to derive the general *N*-triple pole solutions via analyzing the inverse scattering problem with the triple poles discrete spectrum for the GI equation (1) under NZBCs.

## A. Inverse scattering problem with NZBCs and triple poles

Similar to expressions (99) and (100) and since  $s_{22}''(z_0) = 0$  in  $z_0 \in Y \cap D_+$  for the case of triple poles, it is not hard to find that  $\Phi_{+2}''(x,t;z_0) - b[z_0]\Phi_{-1}''(x,t;z_0) - 2d[z_0]\Phi_{-1}'(x,t;z_0)$  and  $\Phi_{-1}(x,t;z_0)$  are linearly dependent, and  $\Phi_{+1}''(x,t;z_0) - b[z_0]\Phi_{-2}''(x,t;z_0) - 2d[z_0]\Phi_{-2}'(x,t;z_0)$  are linearly dependent for  $z_0 \in Y \cap D_-$ . That is to say,

$$\Phi_{+2}''(x,t;z_0) - b[z_0]\Phi_{-1}''(x,t;z_0) - 2d[z_0]\Phi_{-1}'(x,t;z_0) = h[z_0]\Phi_{-1}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_+, \Phi_{+1}''(x,t;z_0) - b[z_0]\Phi_{-2}''(x,t;z_0) - 2d[z_0]\Phi_{-2}'(x,t;z_0) = h[z_0]\Phi_{-2}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_-,$$

$$(135)$$

where  $h[z_0]$  is also a norming constant. Therefore, one has

$$\begin{split} & L_{-3} \left[ \frac{\Phi_{+2}(x,t;z)}{s_{22}(z)} \right] = \tilde{A}[z_0] \Phi_{-1}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_+, \\ & L_{-3} \left[ \frac{\Phi_{+1}(x,t;z)}{s_{11}(z)} \right] = \tilde{A}[z_0] \Phi_{-2}(x,t;z_0), \quad z_0 \in \Upsilon \cap D_-, \\ & L_{-2} \left[ \frac{\Phi_{+2}(x,t;z)}{s_{22}(z)} \right] = \tilde{A}[z_0] \left[ \Phi_{-1}'(x,t;z_0) + \tilde{B}[z_0] \Phi_{-1}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_+, \\ & L_{-2} \left[ \frac{\Phi_{+1}(x,t;z)}{s_{11}(z)} \right] = \tilde{A}[z_0] \left[ \Phi_{-2}'(x,t;z_0) + \tilde{B}[z_0] \Phi_{-2}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_-, \\ & \text{Res} \left[ \frac{\Phi_{+2}(x,t;z)}{s_{22}(z)} \right] = \tilde{A}[z_0] \left[ \frac{1}{2} \Phi_{-1}''(x,t;z_0) + \tilde{B}[z_0] \Phi_{-1}'(x,t;z_0) + \tilde{C}[z_0] \Phi_{-1}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_+, \\ & \text{Res} \left[ \frac{\Phi_{+1}(x,t;z)}{s_{11}(z)} \right] = \tilde{A}[z_0] \left[ \frac{1}{2} \Phi_{-2}''(x,t;z_0) + \tilde{B}[z_0] \Phi_{-2}'(x,t;z_0) + \tilde{C}[z_0] \Phi_{-2}(x,t;z_0) \right], \quad z_0 \in \Upsilon \cap D_-, \end{split}$$

where  $L_{-3}[f(x, t; z)]$  means the coefficient of  $O((z - z_0)^{-3})$  term in the Laurent series expansion of f(x, t; z) at  $z = z_0$  and

$$\tilde{A}[z_0] = \begin{cases} \frac{6b[z_0]}{s_{22}^{\prime\prime\prime}(z_0)}, & z_0 \in \Upsilon \cap D_+, \\ \frac{6b[z_0]}{s_{11}^{\prime\prime\prime}(z_0)}, & z_0 \in \Upsilon \cap D_-, \end{cases}$$
(137)

$$\tilde{B}[z_0] = \begin{cases} \frac{d[z_0]}{b[z_0]} - \frac{s_{22}'''(z_0)}{4s_{22}''(z_0)}, & z_0 \in \Upsilon \cap D_+, \\ \frac{d[z_0]}{b[z_0]} - \frac{s_{11}'''(z_0)}{4s_{11}''(z_0)}, & z_0 \in \Upsilon \cap D_-, \end{cases}$$
(138)

$$\tilde{C}[z_0] = \begin{cases} \frac{h[z_0]}{2b[z_0]} - \frac{d[z_0]s_{22}^{'''}(z_0)}{4b[z_0]s_{22}^{'''}(z_0)} + \frac{(s_{22}^{'''})^2(z_0)}{16(s_{22}^{'''})^2(z_0)}, & z_0 \in \Upsilon \cap D_+, \\ \frac{h[z_0]}{2b[z_0]} - \frac{d[z_0]s_{11}^{''''}(z_0)}{4b[z_0]s_{11}^{'''}(z_0)} + \frac{(s_{11}^{'''})^2(z_0)}{16(s_{11}^{'''})^2(z_0)}, & z_0 \in \Gamma \cap D_-. \end{cases}$$
(139)

*Proposition 7.1.* Let  $z_0 \in Y$ . Then, the following symmetry relations are satisfied:

- The first symmetry relation \$\tilde{A}[z\_0] = -\tilde{A}[z\_0^\*]^\*\$, \$\tilde{B}[z\_0] = \tilde{B}[z\_0^\*]^\*\$, \$\tilde{C}[z\_0] = \tilde{C}[z\_0^\*]^\*\$.
  The second symmetry relation \$\tilde{A}[z\_0] = -\tilde{A}[-z\_0^\*]^\*\$, \$\tilde{B}[z\_0] = -\tilde{B}[-z\_0^\*]^\*\$, \$\tilde{C}[z\_0] = \tilde{C}[-z\_0^\*]^\*\$.
  The third symmetry relation \$\tilde{A}[z\_0] = \frac{z\_0^5u\_-}{u\_0^5u\_-^\*} \tilde{A}[-\frac{u\_0^2}{z\_0}]\$, \$\tilde{B}[z\_0] = \frac{u\_0^2}{z\_0^2} \tilde{B}[-\frac{u\_0^2}{z\_0}]\$, \$\tilde{C}[z\_0] = \tilde{C}[-z\_0^\*]^\*\$.

The residue and the coefficient  $L_{-2}$ ,  $L_{-3}$  of M(x, t; z) are

$$\begin{split} &L_{-3} M_{+} = \left(0, \tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})}\mu_{-1}(\zeta_{n})\right), \\ &L_{-3} M_{-} = \left(\tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}\mu_{-2}(\hat{\zeta}_{n}), 0\right), \\ &L_{-3} M_{-} = \left(\tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}[\mu_{-2}(\hat{\zeta}_{n}), 0] + \left[\tilde{B}[\zeta_{n}] - 2i\theta'(\zeta_{n})\right]\mu_{-1}(\zeta_{n})\right]\right), \\ &L_{-2} M_{+} = \left(0, \tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})}\left[\mu_{-2}'(\hat{\zeta}_{n}) + \left[\tilde{B}[\hat{\zeta}_{n}] + 2i\theta'(\hat{\zeta}_{n})\right]\mu_{-2}(\hat{\zeta}_{n})\right], 0\right), \\ &L_{-2} M_{-} = \left(\tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}\left[\mu_{-2}'(\hat{\zeta}_{n}) + \left[\tilde{B}[\hat{\zeta}_{n}] + 2i\theta'(\hat{\zeta}_{n})\right]\mu_{-2}(\hat{\zeta}_{n})\right], 0\right), \\ &R_{es} M_{+} = \left(0, \tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})}\left[\frac{1}{2}\mu_{-1}''(\zeta_{n}) + \left[\tilde{B}[\zeta_{n}] - 2i\theta'(\zeta_{n})\right]\mu_{-1}'(\zeta_{n}) + \left[\tilde{C}[\zeta_{n}] - \Theta_{1}(\zeta_{n})\right]\mu_{-1}(\zeta_{n})\right]\right), \\ &R_{es} M_{-} = \left(\tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}\left[\frac{1}{2}\mu_{-2}''(\hat{\zeta}_{n}) + \left[\tilde{B}[\hat{\zeta}_{n}] + 2i\theta'(\hat{\zeta}_{n})\right]\mu_{-2}'(\hat{\zeta}_{n}) + \left[\tilde{C}[\hat{\zeta}_{n}] - \Theta_{2}(\hat{\zeta}_{n})\right]\mu_{-2}(\hat{\zeta}_{n})\right], 0\right), \end{aligned}$$

$$(140)$$

where

$$\Theta_{1}(\zeta_{n}) = 2(\theta'(\zeta_{n}))^{2} + i\theta''(\zeta_{n}) + 2\tilde{B}[\zeta_{n}]i\theta'(\zeta_{n}),$$
  

$$\Theta_{2}(\hat{\zeta}_{n}) = 2(\theta'(\hat{\zeta}_{n}))^{2} - i\theta''(\hat{\zeta}_{n}) - 2\tilde{B}[\hat{\zeta}_{n}]i\theta'(\hat{\zeta}_{n}).$$
(141)

By subtracting out the residue, the coefficient  $L_{-2}, L_{-3}$ , and the asymptotic values as  $z \to \infty$  from the original non-regular RHP, the following regular RHP is derived:  $M + i \sigma O I$ 

$$M_{-} + \frac{-\sigma_{3}Q_{-} - I_{-}}{z\sigma_{3}Q_{-} - I_{-}}$$

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left[ \frac{L_{-3}M_{+}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{+}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{z-\zeta_{n}} + \frac{L_{-3}M_{-}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{-}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{-}}{(z-\zeta_{n})} \right]$$

$$= M_{+} + \frac{i}{z}\sigma_{3}Q_{-} - I_{-}$$

$$\sum_{n=1}^{4N_1+2N_2} \left[ \frac{L_{-3}}{(z-\zeta_n)^3} M_+ + \frac{L_{-2}}{(z-\zeta_n)^2} M_+ + \frac{\text{Res}\,M_+}{z-\zeta_n} + \frac{L_{-3}M_-}{(z-\hat{\zeta}_n)^3} + \frac{L_{-2}M_-}{(z-\hat{\zeta}_n)^2} + \frac{\text{Res}\,M_-}{(z-\hat{\zeta}_n)} \right] - M_+G, \tag{142}$$

which can be solved by the Plemelj's formulas, given by

$$M(x,t;z) = I - \frac{i}{z}\sigma_{3}Q_{-} + \frac{1}{2\pi i}\int_{\Sigma} \frac{M_{+}(x,t;\zeta)G(x,t;\zeta)}{\zeta - z}d\zeta$$

$$\sum_{n=1}^{N_{1}+2N_{2}} \left[ \frac{L_{-3}M_{+}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{+}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{z-\zeta_{n}} + \frac{L_{-3}M_{-}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{-}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{-}}{(z-\zeta_{n})^{2}} \right],$$
(143)

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where

$$\frac{L_{-3}M_{+}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{+}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{+}}{z-\zeta_{n}} + \frac{L_{-3}M_{-}}{(z-\zeta_{n})^{3}} + \frac{L_{-2}M_{-}}{(z-\zeta_{n})^{2}} + \frac{\operatorname{Res}M_{-}}{z-\zeta_{n}}$$

$$= \left(\hat{C}_{n}(z)\left[\frac{1}{2}\mu_{-2}^{\prime\prime}(\hat{\zeta}_{n}) + \left(\hat{D}_{n} + \frac{1}{z-\zeta_{n}}\right)\mu_{-2}^{\prime}(\hat{\zeta}_{n}) + \left(\frac{1}{(z-\zeta_{n})^{2}} + \frac{\hat{D}_{n}}{z-\zeta_{n}} + \hat{F}_{n}\right)\mu_{-2}(\hat{\zeta}_{n})\right],$$

$$C_{n}(z)\left[\frac{1}{2}\mu_{-1}^{\prime\prime}(\zeta_{n}) + \left(D_{n} + \frac{1}{z-\zeta_{n}}\right)\mu_{-1}^{\prime}(\zeta_{n}) + \left(\frac{1}{(z-\zeta_{n})^{2}} + \frac{D_{n}}{z-\zeta_{n}} + F_{n}\right)\mu_{-1}(\zeta_{n})\right],$$
(144)

and

$$C_{n}(z) = \frac{\tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})}}{z-\zeta_{n}}, D_{n} = \tilde{B}[\zeta_{n}] - 2i\theta'(\zeta_{n}), F_{n} = \tilde{C}[\zeta_{n}] - \Theta_{1}(\zeta_{n}),$$
$$\hat{C}_{n}(z) = \frac{\tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})}}{z-\hat{\zeta}_{n}}, \hat{D}_{n} = \tilde{B}[\hat{\zeta}_{n}] + 2i\theta'(\hat{\zeta}_{n}), \hat{F}_{n} = \tilde{C}[\hat{\zeta}_{n}] - \Theta_{2}(\hat{\zeta}_{n}).$$
(145)

Furthermore, according to (96), we get

$$M^{(1)}(x,t;z) = -\frac{1}{2\pi i} \int_{\Sigma} M_{+}(x,t;\zeta) G(x,t;\zeta) d\zeta - i\sigma_{3}Q_{-}$$

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left( \tilde{A}[\hat{\zeta}_{n}]e^{2i\theta(\hat{\zeta}_{n})} \left( \frac{1}{2}\mu_{-2}^{\prime\prime}(\hat{\zeta}_{n}) + \hat{D}_{n}\mu_{-2}^{\prime}(\hat{\zeta}_{n}) + \hat{F}_{n}\mu_{-2}(\hat{\zeta}_{n}) \right),$$

$$\tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})} \left( \frac{1}{2}\mu_{-1}^{\prime\prime}(\zeta_{n}) + D_{n}\mu_{-1}^{\prime}(\zeta_{n}) + F_{n}\mu_{1}(\zeta_{n}) \right) \right).$$
(146)

The potential u(x, t) with triple poles for the GI equation with NZBCs turns into

$$u(x,t) = iM_{12}^{(1)} = u_{-} - \frac{1}{2\pi} \int_{\Sigma} (M_{+}(x,t;\zeta)G(x,t;\zeta))_{12}d\zeta$$
  
+  $i\sum_{n=1}^{4N_{1}+2N_{2}} \tilde{A}[\zeta_{n}]e^{-2i\theta(\zeta_{n})} \Big(\frac{1}{2}\mu_{-11}''(\zeta_{n}) + D_{n}\mu_{-11}'(\zeta_{n}) + F_{n}\mu_{11}(\zeta_{n})\Big).$  (147)

# B. Trace formulas and theta condition

In consideration of  $\zeta_n$  and  $\hat{\zeta}_n$  being trace zeros of the scattering coefficients  $s_{22}(z)$  and  $s_{11}(z)$ , respectively, we introduce following functions:

$$\beta^{+}(z) = s_{22}(z) \prod_{n=1}^{4N_{1}+2N_{2}} \left(\frac{z-\hat{\zeta}_{n}}{z-\zeta_{n}}\right)^{3}, \ \beta^{-}(z) = s_{11}(z) \prod_{n=1}^{4N_{1}+2N_{2}} \left(\frac{z-\zeta_{n}}{z-\hat{\zeta}_{n}}\right)^{3},$$
(148)

of which  $\beta^+(z)$  is analytic and has no zeros in  $D_+$  and  $\beta^-(z)$  is analytic and has no zeros in  $D_-$ . As  $z \to \infty$ , they both tend to o(1). Furthermore,  $\beta^{\pm}(z)$  can be shown as

$$\log \beta^{\pm}(z) = \mp \frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1 - \rho \tilde{\rho})}{s - z} ds, \quad z \in D^{\pm},$$
(149)

and the trace formulas are

$$s_{22}(z) = \exp\left[-\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s-z} ds\right]^{4N_1+2N_2} \left(\frac{z-\zeta_n}{z-\hat{\zeta}_n}\right)^3,$$
  

$$s_{11}(z) = \exp\left[\frac{1}{2\pi i} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s-z} ds\right]^{4N_1+2N_2} \left(\frac{z-\hat{\zeta}_n}{z-\zeta_n}\right)^3.$$
(150)

Let  $z \rightarrow 0$ , and it arrives in

$$\frac{u_{+}}{u_{-}} = \exp\left[\frac{i}{2\pi} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s} ds\right] \prod_{n=1}^{N_{1}} \left(\frac{z_{n}}{z_{n}^{*}}\right)^{12} \prod_{n=1}^{N_{2}} \left(\frac{\omega_{n}}{u_{0}}\right)^{12}.$$
(151)

Ultimately, the theta condition for Eq. (151) is given by

$$\arg\left(\frac{u_{+}}{u_{-}}\right) = \frac{1}{2\pi} \int_{\Sigma} \frac{\log(1-\rho\tilde{\rho})}{s} ds + 24 \sum_{n=1}^{N_{1}} \arg(z_{n}) + 12 \sum_{n=1}^{N_{2}} \arg(\omega_{n}).$$
(152)

# C. Triple pole soliton solutions with NZBCs

We take  $\rho(z) = \tilde{\rho}(z) = 0$  to generate the explicit triple pole soliton solutions of the GI equation with NZBCs. Then, the second column of Eq. (143) yields

$$\mu_{-2}(z) = \begin{pmatrix} -\frac{i}{z} \mu_{-1} \\ 1 \end{pmatrix}$$

$$+ \sum_{n=1}^{4N_1 + 2N_2} C_n(z) \left[ \frac{1}{2} \mu_{-1}''(\zeta_n) + \left( D_n + \frac{1}{z - \zeta_n} \right) \mu_{-1}'(\zeta_n) + \left( \frac{1}{(z - \zeta_n)^2} + \frac{D_n}{z - \zeta_n} + F_n \right) \mu_{-1}(\zeta_n) \right],$$

$$\mu_{-2}'(z) = \begin{pmatrix} \frac{i}{z^2} \mu_{-1} \\ 0 \end{pmatrix}$$

$$- \sum_{n=1}^{4N_1 + 2N_2} \frac{C_n(z)}{z - \zeta_n} \left[ \frac{1}{2} \mu_{-1}''(\zeta_n) + \left( D_n + \frac{2}{z - \zeta_n} \right) \mu_{-1}'(\zeta_n) + \left( \frac{3}{(z - \zeta_n)^2} + \frac{2D_n}{z - \zeta_n} + F_n \right) \mu_{-1}(\zeta_n) \right]$$

$$\mu_{-2}''(z) = \begin{pmatrix} -\frac{2i}{z^3} \mu_{-1} \\ 0 \end{pmatrix}$$

$$+\sum_{n=1}^{2N} \frac{2C_n(z)}{(z-\xi_n)^2} \left[ \frac{1}{2} \mu_{-1}^{\prime\prime}(\zeta_n) + \left( D_n + \frac{3}{z-\zeta_n} \right) \mu_{-1}^{\prime}(\zeta_n) + \left( \frac{6}{(z-\zeta_n)^2} + \frac{3D_n}{z-\zeta_n} + F_n \right) \mu_{-1}(\zeta_n) \right].$$
(153)

Taking the second-order derivative about z in the above formula (118), we get

$$\mu_{-2}^{\prime\prime}(z) = -\frac{2iu_{-}}{z^{3}}\mu_{-1}\left(-\frac{u_{0}^{2}}{z}\right) + \frac{4iu_{-}u_{0}^{2}}{z^{4}}\mu_{-1}^{\prime}\left(-\frac{u_{0}^{2}}{z}\right) - \frac{iu_{-}u_{0}^{4}}{z^{5}}\mu_{-1}^{\prime\prime}\left(-\frac{u_{0}^{2}}{z}\right).$$
(154)

Putting Eqs. (118), (119), and (154) into Eq. (153) and letting  $z = \hat{\zeta}_j, j = 1, 2, \dots, 4N_1 + 2N_2$ , we obtain the  $12N_1 + 6N_2$  following linear system:

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$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ C_{n}(\hat{\zeta}_{j}) \left( \frac{1}{2} \mu_{-1}^{\prime\prime}(\zeta_{n}) + \left( D_{n} + \frac{1}{\hat{\zeta}_{j} - \zeta_{n}} \right) \mu_{-1}^{\prime}(\zeta_{n}) \right) + \left[ C_{n}(\hat{\zeta}_{j}) \left( \frac{1}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{D_{n}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}} \delta_{j,n} \right] \mu_{-1}(\zeta_{n}) \right\} = \begin{pmatrix} \frac{i}{\hat{\zeta}_{j}} \mu_{-} \\ -1 \end{pmatrix},$$

$$(155)$$

$$= \frac{1}{2(\zeta_{j} - \zeta_{n})^{i}} \left( \frac{2}{\zeta_{j} - \zeta_{n}} \left( \frac{1}{\zeta_{j} - \zeta_{n}} + F_{n} \right) + \frac{iu_{-}}{\zeta_{j}^{2}} \delta_{j,n} \right) \mu_{-1}(\zeta_{n}) \right\} = \begin{pmatrix} \frac{i}{\zeta_{j}^{2}} \mu_{-} \\ 0 \end{pmatrix},$$
(156)

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ \left[ \frac{C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{iu_{0}^{4}u_{-}}{\hat{\zeta}_{j}^{5}} \delta_{j,n} \right] \mu_{-1}^{\prime\prime}(\zeta_{n}) + \left[ \frac{2C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} \left( D_{n} + \frac{3}{\hat{\zeta}_{j}-\zeta_{n}} \right) - \frac{4iu_{0}^{2}u_{-}}{\hat{\zeta}_{j}^{4}} \delta_{j,n} \right] \mu_{-1}^{\prime}(\zeta_{n}) + \left[ \frac{2C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} \left( \frac{6}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{3D_{n}}{\hat{\zeta}_{j}-\zeta_{n}} + F_{n} \right) + \frac{2iu_{-}}{\hat{\zeta}_{j}^{3}} \delta_{j,n} \right] \mu_{-1}(\zeta_{n}) \right\} = \left( \frac{2i}{\hat{\zeta}_{j}^{3}} u_{-} \\ 0 \right).$$

$$(157)$$

**Theorem 7.1.** The general formula of the triple poles solution for the GI equation (1) with NZBCs (82) is expressed as

$$u(x,t) = u_{-} - i \frac{\det \begin{pmatrix} \tilde{\mathcal{H}} & \tilde{\varphi} \\ \tilde{\chi}^{T} & 0 \end{pmatrix}}{\det(\tilde{\mathcal{H}})},$$
(158)

where  $\tilde{\varphi}$ ,  $\tilde{\mathcal{H}}$ ,  $\tilde{\chi}$  are given by (163), (164), and (166), respectively.

+

*Proof.* From Eqs. (155)–(157), we get a  $12N_1 + 6N_2$  linear system with respect to  $\mu_{-11}(\zeta_n), \mu'_{-11}(\zeta_n)$ ,

$$\sum_{n=1}^{4N_{1}+2N_{2}} \left\{ C_{n}(\hat{\zeta}_{j}) \left( \frac{1}{2} \mu_{-11}^{\prime\prime}(\zeta_{n}) + \left( D_{n} + \frac{1}{\hat{\zeta}_{j} - \zeta_{n}} \right) \mu_{-11}^{\prime}(\zeta_{n}) \right) + \left[ C_{n}(\hat{\zeta}_{j}) \left( \frac{1}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{D_{n}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}} \delta_{j,n} \right] \mu_{-11}(\zeta_{n}) \right\} = \frac{i}{\hat{\zeta}_{j}} u_{-}, \quad (159)$$

$$\frac{4N_{1}+2N_{2}}{\sum_{n=1}^{2}} \left\{ \frac{C_{n}(\hat{\zeta}_{j})}{2(\hat{\zeta}_{j} - \zeta_{n})} \mu_{-11}^{\prime\prime}(\zeta_{n}) + \left[ \frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j} - \zeta_{n}} \left( D_{n} + \frac{2}{\hat{\zeta}_{j} - \zeta_{n}} \right) - \frac{iu_{0}^{2}u_{-}}{\hat{\zeta}_{j}^{3}} \delta_{j,n} \right] \mu_{-11}^{\prime}(\zeta_{n}) + \left[ \frac{C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{2D_{n}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}^{2}} \delta_{j,n} \right] \mu_{-11}(\zeta_{n}) \right\} = \frac{i}{\hat{\zeta}_{j}^{2}} u_{-}, \quad (160)$$

$$\frac{4N_{1}+2N_{2}}{\sum_{n=1}^{2}} \left\{ \left[ \frac{C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{iu_{0}^{4}u_{-}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{iu_{-}}{\hat{\zeta}_{j}^{2}} \delta_{j,n} \right] \mu_{-11}(\zeta_{n}) \right\} = \frac{i}{\hat{\zeta}_{j}^{2}} u_{-}, \quad (160)$$

$$\frac{4N_{1}+2N_{2}}{\sum_{n=1}^{2}} \left\{ \left[ \frac{C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{iu_{0}^{4}u_{-}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{2iu_{-}}{\hat{\zeta}_{j}^{2}} \delta_{j,n} \right] - \frac{4iu_{0}^{2}u_{-}}{\hat{\zeta}_{j}^{4}} \delta_{j,n} \right] \mu_{-11}^{\prime}(\zeta_{n}) + \left[ \frac{2C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} \left( \frac{6}{(\hat{\zeta}_{j} - \zeta_{n})^{2}} + \frac{3D_{n}}{\hat{\zeta}_{j} - \zeta_{n}} + F_{n} \right) + \frac{2iu_{-}}{\hat{\zeta}_{j}^{3}} \delta_{j,n} \right] \mu_{-11}(\zeta_{n}) \right\} = \frac{2i}{\hat{\zeta}_{j}^{3}} u_{-}, \quad (161)$$

 $\tilde{\varphi}$ 

(162)

which can be rewritten in the following matrix form:

where

$$= \begin{pmatrix} \tilde{\varphi}^{(1)} \\ \tilde{\varphi}^{(2)} \\ \tilde{\varphi}^{(3)} \end{pmatrix}, \quad \tilde{\varphi}^{(1)} = \begin{pmatrix} iu_{-} \\ \dot{\zeta}_{1}^{-}, \dots, \frac{iu_{-}}{\dot{\zeta}_{4N_{1}+2N_{2}}} \end{pmatrix}^{T}, \quad \tilde{\varphi}^{(2)} = \begin{pmatrix} iu_{-} \\ \dot{\zeta}_{1}^{2} \\ \dot{\zeta}_{1}^{2} \end{pmatrix}, \quad \vdots, \quad \tilde{\varphi}^{(3)} = \begin{pmatrix} 2iu_{-} \\ \dot{\zeta}_{1}^{3} \\ \dot{\zeta}_{1}^{3} \\ \dot{\zeta}_{4N_{1}+2N_{2}} \end{pmatrix}^{T}, \quad \tilde{\varphi}^{(3)} = \begin{pmatrix} 2iu_{-} \\ \dot{\zeta}_{1}^{3} \\ \dot{\zeta}_{1}^{3} \\ \dot{\zeta}_{4N_{1}+2N_{2}} \end{pmatrix}^{T}, \quad \tilde{\varphi}^{(2)} = \begin{pmatrix} \tilde{\psi}^{(1)} \\ \tilde{\psi}^{(2)} \\ \tilde{\psi}^{(3)} \end{pmatrix}, \quad \tilde{\psi}^{(1)} = (\mu_{-11}(\zeta_{1}), \dots, \mu_{-11}(\zeta_{4N_{1}+2N_{2}}))^{T}, \quad \tilde{\psi}^{(2)} = (\mu_{-11}(\zeta_{1}), \dots, \mu_{-11}'(\zeta_{4N_{1}+2N_{2}}))^{T},$$

 $\tilde{\mathcal{H}}\tilde{\gamma}=\tilde{\varphi},$ 

$$\tilde{\gamma}^{(3)} = (\mu_{-11}^{\prime\prime}(\zeta_1), \dots, \mu_{-11}^{\prime\prime}(\zeta_{4N_1+2N_2}))^T, \tilde{\mathcal{H}} = \begin{pmatrix} \tilde{\mathcal{H}}^{(11)} & \tilde{\mathcal{H}}^{(12)} & \tilde{\mathcal{H}}^{(13)} \\ \tilde{\mathcal{H}}^{(21)} & \tilde{\mathcal{H}}^{(22)} & \tilde{\mathcal{H}}^{(23)} \\ \tilde{\mathcal{H}}^{(31)} & \tilde{\mathcal{H}}^{(32)} & \tilde{\mathcal{H}}^{(33)} \end{pmatrix},$$
(163)

with 
$$\tilde{\mathcal{H}}^{(im)} = \left(\tilde{\mathcal{H}}_{jn}^{(im)}\right)_{4N_{1}+2N_{2}\times 4N_{1}+2N_{2}}(i,m=1,2,3)$$
 given by  

$$\tilde{\mathcal{H}}_{jn}^{(11)} = C_{n}(\hat{\zeta}_{j})\left(\frac{1}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{D_{n}}{\hat{\zeta}_{j}-\zeta_{n}} + F_{n}\right) + \frac{iu_{-}}{\hat{\zeta}_{j}}\delta_{j,n}, \quad \tilde{\mathcal{H}}_{jn}^{(12)} = C_{n}(\hat{\zeta}_{j})\left(D_{n} + \frac{1}{\hat{\zeta}_{j}-\zeta_{n}}\right), \quad \tilde{\mathcal{H}}_{jn}^{(13)} = \frac{1}{2}C_{n}(\hat{\zeta}_{j}), \quad \tilde{\mathcal{H}}_{jn}^{(13)} = \frac{1}{2}C_{n}(\hat{\zeta}_{j}), \quad \tilde{\mathcal{H}}_{jn}^{(21)} = \frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j}-\zeta_{n}}\left(\frac{3}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{2D_{n}}{\hat{\zeta}_{j}-\zeta_{n}} + F_{n}\right) + \frac{iu_{-}}{\hat{\zeta}_{j}^{2}}\delta_{j,n}, \quad \tilde{\mathcal{H}}_{jn}^{(22)} = \frac{C_{n}(\hat{\zeta}_{j})}{\hat{\zeta}_{j}-\zeta_{n}}\left(D_{n} + \frac{2}{\hat{\zeta}_{j}-\zeta_{n}}\right) - \frac{iu_{0}^{2}u_{-}}{\hat{\zeta}_{j}^{3}}\delta_{j,n}, \quad \tilde{\mathcal{H}}_{jn}^{(23)} = \frac{C_{n}(\hat{\zeta}_{j})}{2(\hat{\zeta}_{j}-\zeta_{n})^{2}}, \quad \tilde{\mathcal{H}}_{jn}^{(31)} = \frac{2C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}}\left(\frac{6}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{3D_{n}}{\hat{\zeta}_{j}-\zeta_{n}} + F_{n}\right) + \frac{2iu_{-}}{\hat{\zeta}_{j}^{3}}\delta_{j,n}, \quad \tilde{\mathcal{H}}_{jn}^{(32)} = \frac{2C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}}\left(D_{n} + \frac{3}{\hat{\zeta}_{j}-\zeta_{n}}\right) - \frac{4iu_{0}^{2}u_{-}}{\hat{\zeta}_{j}^{4}}\delta_{j,n}, \quad \tilde{\mathcal{H}}_{jn}^{(33)} = \frac{C_{n}(\hat{\zeta}_{j})}{(\hat{\zeta}_{j}-\zeta_{n})^{2}} + \frac{iu_{0}^{4}u_{-}}{\hat{\zeta}_{j}^{5}}\delta_{j,n}. \quad (164)$$

For the case of reflectionless potential, Eq. (147) is denoted as

$$u = u_{-} + i\tilde{\chi}^{T}\tilde{\gamma}, \tag{165}$$

where

$$\begin{split} \tilde{\chi} &= \begin{pmatrix} \tilde{\chi}^{(1)} \\ \tilde{\chi}^{(2)} \\ \tilde{\chi}^{(3)} \end{pmatrix}, \ \tilde{\chi}^{(1)} &= (A[\zeta_1]e^{-2i\theta(\zeta_1)}F_1, \dots, A[\zeta_{4N_1+2N_2}]e^{-2i\theta(\zeta_{4N_1+2N_2})}F_{4N_1+2N_2})^T, \\ \tilde{\chi}^{(2)} &= (A[\zeta_1]e^{-2i\theta(\zeta_1)}D_1, \dots, A[\zeta_{4N_1+2N_2}]e^{-2i\theta(\zeta_{4N_1+2N_2})}D_{4N_1+2N_2})^T, \\ \tilde{\chi}^{(3)} &= \left(\frac{1}{2}A[\zeta_1]e^{-2i\theta(\zeta_1)}, \dots, \frac{1}{2}A[\zeta_{4N_1+2N_2}]e^{-2i\theta(\zeta_{4N_1+2N_2})}\right)^T. \end{split}$$
(166)

Using Eq. (162), the expression of the triple pole soliton solution can be derived finally.



**FIG. 8.** The one-triple pole soliton solution for Eq. (1) with NZBCs and  $N_1 = 0$ ,  $N_2 = 1$ . The parameters are  $u_{\pm} = 1$ ,  $\tilde{A}[\omega_1] = \tilde{B}[\omega_1] = \tilde{C}[\omega_1] = 1$ ,  $\omega_1 = e^{\frac{d}{6}}$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -5 (long dashed curve), t = 0 (solid curve), and t = 5 (dashed-dotted curve).



**FIG. 9.** The one-triple pole soliton solution for Eq. (1) with NZBCs and  $N_1 = 1$ ,  $N_2 = 0$ . The parameters are  $u_{\pm} = 1$ ,  $\tilde{A}[z_1] = \tilde{B}[z_1] = 1$ ,  $z_1 = 2e^{\frac{\pi}{6}}$ . (a) Three-dimensional plot. (b) The density plot. (c) The wave propagation along the *x* axis at t = -10 (long dashed cruve), t = 0 (solid curve), and t = 10 (dashed-dotted curve).

Due to the fact that the general expressions of the one-triple pole solutions for  $N_1 = 0$ ,  $N_2 = 1$  and  $N_1 = 1$ ,  $N_2 = 0$  in (158) are very complicated, we omit them here. We just exhibit some pictures by selecting appropriate parameters. In the case of  $N_1 = 0$ ,  $N_2 = 1$ , the one-triple pole soliton become dark-bright-dark solutions with the parameters  $u_- = 1$ ,  $\tilde{A}[\omega_1] = \tilde{B}[\omega_1] = \tilde{C}[\omega_1] = 1$ ,  $\omega_1 = e^{\frac{\pi i}{6}}$  in which the theta condition becomes  $\arg(\frac{u_+}{u_-}) = 2\pi$  such that  $u_+ = u_-$  (see Fig. 8). On the other hand, when selecting  $N_1 = 1$ ,  $N_2 = 0$ , the one-triple pole soliton displays breather-breather-breather shown in Fig. 9 with the parameters  $u_- = 1$ ,  $\tilde{A}[z_1] = \tilde{B}[z_1] = \tilde{C}[z_1] = 1$ ,  $z_1 = 2e^{\frac{\pi i}{6}}$  in which the theta condition is  $\arg(\frac{u_+}{u_-}) = 4\pi$  such that  $u_+ = u_-$ . In addition, when  $N_1 = N_2 = 1$ , i.e., mixed discrete spectra, it turns into the three-breather-three-soliton solutions, meaning the interaction of three breathers and three solitons.

## **VIII. CONCLUSION**

In this paper, we have applied the RH method to discuss the GI type of the derivative NLS equation with ZBCs and NZBCs. Through solving the RHP in the case of double and triple poles, we have given out the *N*-double and *N*-triple pole soliton solutions under ZBCs and NZBCs. The critical technique shown in this work is to eliminate the properties of singularities via subtracting the residue and the coefficient  $L_{-2}$  from the original non-regular RHP when reflection coefficients have double poles. For the case of triple poles, we have to subtract another coefficient  $L_{-3}$ . Additionally, the asymptotic value of the jump matrix is subtracted from the original non-regular RHP. Then, the regular RHP can be displayed, which can be solved by the Plemelj formula. Finally, the *N*-double and *N*-triple pole soliton solutions can be derived by using the solution of RHP to reconstruct the potential function. Furthermore, we analyze the asymptotic state of the one-double pole soliton solution and the one-triple pole soliton solution as *t* tends to infinity. When  $t \rightarrow \infty$ , the one-double pole soliton solution degrades into the two one soliton solutions and the one-triple pole soliton solution turns into three one soliton solutions. In addition, through choosing suitable parameters, the dynamic behaviors of one-double pole soliton, two-double pole soliton, one-triple pole soliton corresponding to ZBCs, one-double pole soliton, and one-triple pole soliton corresponding to NZBCs are analyzed. In the near future, more works remain to be solved for other integrable systems via the technique shown in this paper.

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### AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

## DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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