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Long-time Asymptotics for the Reverse Space-time Nonlocal Hirota Equation with Decaying Initial Value Problem: without Solitons

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Abstract In this work, we mainly consider the Cauchy problem for the reverse space-time nonlocal Hirota equation with the initial data rapidly decaying in the solitonless sector. Start from the Lax pair, we first construct the basis Riemann-Hilbert problem for the reverse space-time nonlocal Hirota equation. Furthermore, using the approach of Deift-Zhou nonlinear steepest descent, the explicit long-time asymptotics for the reverse space-time nonlocal Hirota is derived. For the reverse space-time nonlocal Hirota equation, since the symmetries of its scattering matrix are different with the local Hirota equation, the $\vartheta(\lambda_i)$ $(i = 0, 1)$ would like to be imaginary, which results in the $\delta^0_{\lambda_i}$ contains an increasing $t^{\frac{\pm Im\vartheta(\lambda_i)}{2}}$, and then the asymptotic behavior for nonlocal Hirota equation becomes differently.

Keywords Riemann-Hilbert problem; reverse space-time nonlocal Hirota equation; long-time asymptotics; nonlinear steepest descent method
2020 MR Subject Classification 35C15; 35Q51

1 Introduction

In recent years, more and more scholars pay attention to nonlocal integrable equations in the area of integrable systems. The nonlocal nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + 2q^2 q^*(-x,t) = 0 \tag{1.1}$$

was first introduced by Ablowitz and Musslimani, and they derived its soliton solutions through the method of inverse scattering transform(IST)[2, 3]. The nonlocal NLS equation(1.1) contains the PT symmetric potential which is invariant under the transformation $x \to -x$ and complex conjugation. The IST for the nonlocal NLS equation with nonzero boundary conditions at infinity was studied in Ref.[1]. It is worth mentioning that the long-time asymptotics for the integrable nonlocal NLS equation with decaying boundary conditions has been presented in Ref.[23]. Moreover, other nonlocal integrable equations were also investigated including nonlocal Davey-Stewartson equations, nonlocal modified KdV equation, nonlocal sine-Gordon equation, nonlocal derivative NLS equation, etc.[17, 21, 27, 28, 33, 35].

Recently, the reverse space-time nonlocal Hirota equation

$$iq_t + \alpha\left[q_{xx} - 2q^2 q(-x,-t)\right] + i\beta\left[q_{xxx} - 6qq(-x,-t)q_x\right] = 0 \tag{1.2}$$

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was introduced by Cen, Correa and Fring in 2019[6] as a PT symmetric reduction ($r = q(-x,-t)$) of the following system:

$$\begin{aligned} &iq_t + \alpha[q_{xx} - 2q^2 r] + i\beta[q_{xxx} - 6qrq_x] = 0, \\ &ir_t - \alpha[r_{xx} - 2qr^2] + i\beta[r_{xxx} - 6qrr_x] = 0, \end{aligned} \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$. These two equations are transformed into each other by means of a PT-symmetry transformation. This reduction leads to (1.2) and is consistent with the PT-symmetry condition: if $q(x,t)$ is a solution of (1.4), then $q(-x,-t)$ is a solution as well. The other types of integrable nonlocal Hirota equation are also introduced in Ref. [6] including the reverse space nonlocal Hirota equation, reverse time nonlocal Hirota equation and the conjugate reverse space-time nonlocal Hirota equation. The multi-soliton solutions of these nonlocal Hirota equations have been generated by DarbouxCrum transformations and Hirota bilinear method[6]. The conjugate reverse space-time nonlocal Hirota equation with nonzero boundary conditions was investigated via Riemann-Hilbert(RH) method[19]. Soliton solutions of the conjugate reverse space nonlocal Hirota equation were obtained by the IST method and Darboux transformation method in Ref. [15] and Ref. [16], respectively. Using the Darboux transformation, some types of exact solutions of the reverse space-time nonlocal Hirota equation were found in Ref. [29].

In 1967, the IST was used to solve the KdV equation with Lax pairs by Gardner et al. for the first time[9]. Since then, it played an increasingly important role in finding the exact solutions for integrable systems. Later on, a modern version of IST method, named RH formulation, was developed by Zakharov et al.[32], and then the exact solutions and long-time asymptotics of various integrable equations were investigated using RH formulation[5, 20, 25, 31, 34]. It is worth mentioning that Pelinovsky and Shimabukuro proved the existence of global solutions to the derivative NLS equation on the line from the perspective of inverse scattering transform based on the representation of a RH problem, which is a milestone in the development of IST[18]. And what's more, in 1993, Deift-Zhou put forward the nonlinear steepest descent method for the first time to solve the oscillatory RH problem and derive the long-time asymptotics of solutions for the modified KdV equation[8]. After that, this method has been employed to discuss the asymptotic analysis in a variety of integrable models[4, 7, 10–12, 14, 26, 30]. In 2019, Dmitry Shepelsky et al. applied this method to study the long-time behavior of solutions to the initial boundary value problem of nonlocal NLS equations[23]. Recently, they have extended the Deift-Zhou method to study the long-time asymptotic behavior of nonlocal integrable NLS solutions with nonzero boundary conditions and step-like initial data, respectively[22, 24]. Besides, the Deift-Zhou nonlinear steepest-descent method was used to analyze the long-time asymptotics for the solution of the nonlocal mKdV equation[13].

As we know, long-time asymptotics for the reverse space-time nonlocal Hirota equation (1.2) has not been reported. In this paper, we are committed to the Cauchy problem for the so-called defocusing reverse space-time nonlocal Hirota equation

$$\begin{aligned} &iq_t + \alpha\left[q_{xx} - 2q^2 q(-x,-t)\right] + i\beta\left[q_{xxx} - 6qq(-x,-t)q_x\right] = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &q(x,0) = q_0(x), \end{aligned} \tag{1.4}$$

with the initial data $q_0(x)$ belonging to the Schwartz space and rapidly decaying to 0 as $|x| \to \infty$. For the nonlocal Hirota equation, except to the symmetries of its Lax pair are different with the local Hirota equation, another major difference is that $\vartheta(\lambda_i)(i = 0,1)$ are imaginary in our case, which leads to the $\delta^0_{\lambda_i}$ contains an increasing $t^{\frac{\pm Im\vartheta(\lambda_i)}{2}}$, and then the asymptotic behavior for nonlocal Hirota equation will behave differently. The pivotal result of this paper is generalized in what follows:

Theorem 1.1. *Let $q(x,t)$ be the solution of the Cauchy problem of the reverse space-time nonlocal Hirota equation(1.4) with $q_0(x)$ lying in the Schwartz space. As $t \to \infty$, for $\alpha^2 - 3\beta\frac{t}{x} >$*

0, *the leading asymptotics of the solution* $q(x,t)$ *is*

$$\begin{aligned} q(x,t) =& \frac{\sqrt{\pi} t^{-\frac{1}{2}-Im\vartheta(\lambda_0)} e^{4i\lambda_0^2 t(4\beta\lambda_0+\alpha)+2\chi_0(\lambda_0)+\frac{\pi\vartheta(\lambda_0)}{2}+\frac{\pi i}{4}+iRe\vartheta(\lambda_0)\ln t+i\vartheta(\lambda_0)\ln(32\lambda_0^2(\alpha+6\beta\lambda_0))}}{\sqrt{\alpha+6\beta\lambda_0} r_1(\lambda_0)\Gamma(i\vartheta(\lambda_0))} \\ &+ \frac{\sqrt{\pi} t^{-\frac{1}{2}+Im\vartheta(\lambda_1)} e^{4i\lambda_1^2 t(4\beta\lambda_1+\alpha)+2\chi_1(\lambda_1)-\frac{\pi}{2}\vartheta(\lambda_1)+\frac{\pi i}{4}-iRe\vartheta(\lambda_1)\ln t-i\vartheta(\lambda_1)\ln(32\lambda_1^2(\alpha+6\beta\lambda_1))}}{\sqrt{\alpha+6\beta\lambda_1} r_1(\lambda_1)\Gamma(-i\vartheta(\lambda_1))} \\ &+ O(t^{-\frac{1}{2}-\max\{|\mathrm{Im}\vartheta(\lambda_0)|,|\mathrm{Im}\vartheta(\lambda_1)|\}}), \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} &\lambda_0 = \frac{-\alpha-\sqrt{\alpha^2-3\beta\frac{x}{t}}}{6\beta}, \quad \lambda_1 = \frac{-\alpha+\sqrt{\alpha^2-3\beta\frac{x}{t}}}{6\beta}, \\ &\chi_0(\lambda) = \frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} \ln\left(\frac{1-r_1(s)r_2(s)}{1-r_1(\lambda_0)r_2(\lambda_0)}\right)\frac{\mathrm{d}s}{s-\lambda}, \\ &\chi_1(\lambda) = \frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} \ln\left(\frac{1-r_1(s)r_2(s)}{1-r_1(\lambda_1)r_2(\lambda_1)}\right)\frac{\mathrm{d}s}{s-\lambda}, \\ &\vartheta(\lambda_0) = -\frac{1}{2\pi}\ln(1-r_1(\lambda_0)r_2(\lambda_0)), \\ &\vartheta(\lambda_1) = -\frac{1}{2\pi}\ln(1-r_1(\lambda_1)r_2(\lambda_1)), \end{aligned} \tag{1.6}$$

where Γ *is Gamma function, and* $r_1(\lambda), r_2(\lambda)$ *are the reflection coefficients.*

This paper is organized as follows. In section 2, we perform the direct scattering theory to generate the associated RH problem, further the phase analysis is discussed in detail. In section 3, the nonlinear steepest descent method is used to analyse the long-time asymptotics of the solution for the reverse space-time nonlocal Hirota equation.

2 Inverse Scattering Transform and the Riemann-Hilbert Problem

At the very start, we should carry out the direct scattering analysis to construct the basis RH problem for the nonlocal Hirota equation (1.4), which is a compatibility condition of the following Lax pair[6]

$$\begin{aligned} &\phi_x = X\phi, \qquad X \equiv \begin{pmatrix} -i\lambda & q(x,t) \\ q(-x,-t) & i\lambda \end{pmatrix}, \qquad \phi_t = T\phi, \qquad T \equiv \begin{pmatrix} Q & B \\ C & -Q \end{pmatrix}, \\ &Q = -i\alpha q q(-x,-t) - 2i\alpha\lambda^2 + \beta\left[q(-x,-t)q_x + q q_x(-x,-t) - 4i\lambda^3 - 2i\lambda q q(-x,-t)\right], \\ &B = i\alpha q_x + 2\alpha\lambda q + \beta\left[2q^2 q(-x,-t) - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right], \\ &C = i\alpha q_x(-x,-t) + 2\alpha\lambda q(-x,-t) \\ &\quad + \beta\left[2q q(-x,-t)^2 - q_{xx}(-x,-t) + 2i\lambda q_x(-x,-t) + 4\lambda^2 q(-x,-t)\right], \end{aligned} \tag{2.1}$$

where λ means the spectral parameter, $\phi = \phi(x,t;\lambda)$ is the eigenfunction.

As $x \to \pm\infty$, due to the initial data rapidly decaying, the Lax pair (2.1) turns into

$$\phi_x^\infty = X_0\phi^\infty = -i\lambda\sigma_3\phi^\infty, \qquad \phi_t^\infty = T_0\phi^\infty = (2\alpha\lambda + 4\beta\lambda^2)X_0\phi^\infty, \tag{2.2}$$

where σ_3 represents one of the following Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

We can easily find that the Eq.(2.2) arrives at the following fundamental matrix solution

$$\phi^{\infty}(x,t;\lambda)=e^{-i\theta(x,t;\lambda)\sigma_3},\qquad \theta(x,t;\lambda)=\lambda(x+\left[2\alpha\lambda+4\beta\lambda^2\right]t). \tag{2.4}$$

Taking D_+, D_- and Σ on λ-plane as $D_\pm=\{\lambda\in\mathbb{C}|\mathrm{Im}\lambda\gtrless 0\},\Sigma=\mathbb{R}$, the Jost solutions $\phi_\pm(x,t;\lambda)$ are

$$\phi_\pm(x,t;\lambda)=e^{-i\theta(x,t;\lambda)\sigma_3}+o(1),\qquad \lambda\in\Sigma,\quad \text{as}\quad x\to\pm\infty. \tag{2.5}$$

Through the variable transformation

$$\mu_\pm(x,t;\lambda)=\phi_\pm(x,t;\lambda)e^{i\theta(x,t;\lambda)\sigma_3}, \tag{2.6}$$

the spectral problem (2.1) can be solved as the following Jost solutions $\mu_\pm$, given by

$$\begin{cases}\mu_-=I+\int_{-\infty}^{x}\exp\left[-i\lambda\sigma_3\left(x-y\right)\right](X-X_0)\mu_-\exp\left[i\lambda\sigma_3\left(x-y\right)\right]\mathrm{d}y,\\ \mu_+=I-\int_{x}^{+\infty}\exp\left[-i\lambda\sigma_3\left(x-y\right)\right](X-X_0)\mu_+\exp\left[i\lambda\sigma_3\left(x-y\right)\right]\mathrm{d}y.\end{cases} \tag{2.7}$$

Proposition 2.1. *Suppose $q\in L^1(\mathbb{R}^\pm)$, then $\mu_\pm(x,t,\lambda)$ given in Eq.(2.7) satisfy the following properties:*
- $\mu_{-1}(x,t,\lambda)$ *and* $\mu_{+2}(x,t,\lambda)$ *is analytical in* D_+ *and continuous in* $D_+\cup\Sigma$*;*
- $\mu_{+1}(x,t,\lambda)$ *and* $\mu_{-2}(x,t,\lambda)$ *is analytical in* D_- *and continuous in* $D_-\cup\Sigma$*;*
- $\mu_\pm(x,t,\lambda)\to I$ as $\lambda\to\infty$*;*
- $\det\mu_\pm(x,t,\lambda)=1,\quad x,t\in\mathbb{R},\quad \lambda\in\Sigma.$

Since the Jost solutions $\phi_\pm(x,t,\lambda)$ are the simultaneous solutions of Lax pair (2.1), they can meet the following linear relation contacted by a scattering matrix $S(\lambda)=(s_{ij}(\lambda))_{2\times2}$, given by

$$\phi_-(x,t,\lambda)=\phi_+(x,t,\lambda)S(\lambda),\qquad \lambda\in\Sigma, \tag{2.8}$$

of which the scattering coefficients can be written into what follows

$$\begin{aligned}s_{11}(\lambda)&=Wr(\phi_{-1},\phi_{+2}), &\qquad s_{12}(\lambda)&=Wr(\phi_{-2},\phi_{+2}),\\ s_{21}(\lambda)&=Wr(\phi_{+1},\phi_{-1}), &\qquad s_{22}(\lambda)&=Wr(\phi_{+1},\phi_{-2}).\end{aligned} \tag{2.9}$$

Proposition 2.2. *Suppose $q\in L^1(\mathbb{R}^\pm)$, then the scattering matrix $S(\lambda)$ has the following properties:*
- $\det S(\lambda)=1$ *for* $\lambda\in\Sigma$*;*
- $s_{11}(\lambda)$ *is analytical in* D_+ *and continuous in* $D_+\cup\Sigma$*;*
- $s_{22}(\lambda)$ *is analytical in* D_- *and continuous in* $D_-\cup\Sigma$*;*
- $S(x,t,\lambda)\to I$ as $\lambda\to\infty$.

Furthermore, we need to study the symmetries of the Jost solutions $\phi(x,t,\lambda)$ and scattering matrix $S(\lambda)$ for the nonlocal Hirota equation. The detail reduction conditions for $X(x,t,\lambda)$ and $T(x,t,\lambda)$ in the Lax pair (2.1) on λ-plane are as follows:

$$X(x,t,\lambda)=-\sigma_2X(-x,-t,\lambda)\sigma_2,\qquad T(x,t,\lambda)=-\sigma_2T(-x,-t,\lambda)\sigma_2, \tag{2.10}$$

which results in the Jost solutions $\Psi(x,t,\lambda)$, and scattering matrix $S(\lambda)$ has the following reduction conditions on λ-plane:

$$\phi_\pm(x,t,\lambda)=\sigma_2\phi_\mp(-x,-t,\lambda)\sigma_2,\qquad S(\lambda)=\sigma_2S^{-1}(\lambda)\sigma_2, \tag{2.11}$$

which means $s_{12}(\lambda)=s_{21}(\lambda)$, and $s_{11}(\lambda), s_{22}(\lambda)$ are not directly related. This is different from the case of local Hirota equation.

According to the analyticity of Jost solutions $\mu_{\pm}(x,t,\lambda)$ in Proposition 2.1, we can define the following sectionally meromorphic matrices

$$M_{+}(x,t,\lambda)=\Big(\frac{\mu_{-1}}{s_{11}},\mu_{+2}\Big),\qquad M_{-}(x,t,\lambda)=\Big(\mu_{+1},\frac{\mu_{-2}}{s_{22}}\Big),\tag{2.12}$$

where $\pm$ denote analyticity in D_{+} and D_{-}, respectively. Then, a matrix RH problem is generated:

Riemann-Hilbert Problem. $M(x,t,\lambda)$ solves the following RH problem:

$$\begin{cases} M(x,t,\lambda) \text{ is analytic in } \mathbb{C}\setminus\Sigma, & \\ M_{+}(x,t,\lambda)=M_{-}(x,t,\lambda)J(x,t,\lambda), & \lambda\in\Sigma, \\ M(x,t,\lambda)\to I, & \lambda\to\infty, \end{cases}\tag{2.13}$$

of which the jump matrix $J(x,t,\lambda)$ is

$$J(x,t,\lambda)=\begin{pmatrix} 1-r_1(\lambda)r_2(\lambda) & -r_2(\lambda)e^{-2i\theta(x,t,\lambda)} \\ r_1(\lambda)e^{2i\theta(x,t,\lambda)} & 1 \end{pmatrix},\tag{2.14}$$

where $r_1(\lambda)=\frac{s_{21}(\lambda)}{s_{11}(\lambda)},\ r_2(\lambda)=\frac{s_{12}(\lambda)}{s_{22}(\lambda)}$.

Let

$$M(x,t,\lambda)=I+\frac{1}{\lambda}M_1(x,t;\lambda)+O\Big(\frac{1}{\lambda^2}\Big),\qquad \lambda\to\infty,\tag{2.15}$$

then the potential $q(x,t)$ of the nonlocal Hirota equation (1.4) is given by

$$q(x,t)=2i\left[M_1\right]_{12}(x,t,\lambda)=2i\lim_{\lambda\to\infty}\lambda\left[M\right]_{12}(x,t,\lambda).\tag{2.16}$$

3 The Long-time Behavior for the Nonlocal Hirota Equation

In this section, we primarily devote to discuss the long-time behavior for the nonlocal Hirota equation(1.4). Let's start with phase analysis, in terms of the works of Deift and Zhou[8], we take $\frac{df}{d\lambda}=0$, and then the stationary points of the function f are $\lambda_0=\frac{-\alpha-\sqrt{\alpha^2-3\beta\xi}}{6\beta}$, $\lambda_1=\frac{-\alpha+\sqrt{\alpha^2-3\beta\xi}}{6\beta}$ for $\alpha^2-3\beta\xi>0$, there we have defined $f=\lambda(\xi+2\alpha\lambda+4\beta\lambda^2)$, $\xi=\frac{x}{t}$, and the signature distribution for $\mathrm{Re}\,(if)$ is shown in Figure 3.1. The steepest decent contours are

$$\begin{aligned} &L:\{\lambda=\lambda_1+\lambda_1\rho e^{\frac{3\pi i}{4}}:-\infty<\rho\le\sqrt{2}\}\cup\{\lambda=\lambda_0-\lambda_0\rho e^{\frac{\pi i}{4}}:-\infty<\rho\le\sqrt{2}\},\\ &L^*:\{\lambda=\lambda_1+\lambda_1\rho e^{-\frac{3\pi i}{4}}:-\infty<\rho\le\sqrt{2}\}\cup\{\lambda=\lambda_0-\lambda_0\rho e^{-\frac{\pi i}{4}}:-\infty<\rho\le\sqrt{2}\}.\end{aligned}\tag{3.1}$$

3.1 Factorization of the Jump Matrix and Contour Deformation

We decompose the jump matrix $J(x,t,\lambda)$ into following two cases:

$$J=\begin{cases}\begin{pmatrix}1 & -r_2e^{-2ift}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ r_1e^{2ift} & 1\end{pmatrix},\\[2ex] \begin{pmatrix}1 & 0\\ \dfrac{r_1e^{2ift}}{1-r_1r_2} & 1\end{pmatrix}\begin{pmatrix}1-r_1r_2 & 0\\ 0 & \dfrac{1}{1-r_1r_2}\end{pmatrix}\begin{pmatrix}1 & -\dfrac{r_2e^{-2ift}}{1-r_1r_2}\\ 0 & 1\end{pmatrix}.\end{cases}\tag{3.2}$$

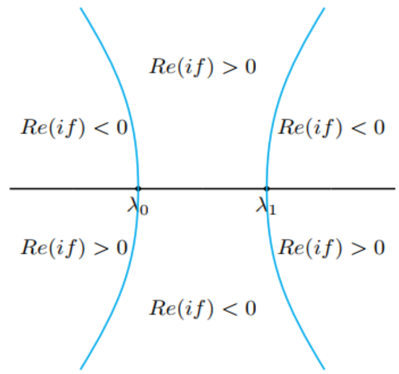

Figure 3.1. The signature table for $Re(if)$ in the complex λ-plane.

Then, we define a RH problem for the function $\delta(\lambda)$

$$\begin{cases} \delta_+(\lambda) = (1 - r_1(\lambda)r_2(\lambda))\delta_-(\lambda), & \lambda \in (\lambda_1, \lambda_2), \\ \delta(\lambda) \to 1, \quad \lambda \to \infty, \end{cases} \tag{3.3}$$

which can be solved by the Plemelj formula as

$$\begin{aligned} \delta(\lambda) &= \exp\left\{\frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} \frac{\ln(1 - r_1(s)r_2(s))}{s - \lambda} \mathrm{d}s\right\} \\ &= \left(\frac{\lambda - \lambda_1}{\lambda - \lambda_0}\right)^{i\vartheta(\lambda_0)} e^{\chi_0(\lambda)} = \left(\frac{\lambda - \lambda_1}{\lambda - \lambda_0}\right)^{i\vartheta(\lambda_1)} e^{\chi_1(\lambda)}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \chi_0(\lambda) &= \frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} \ln\left(\frac{1 - r_1(s)r_2(s)}{1 - r_1(\lambda_0)r_2(\lambda_0)}\right) \frac{\mathrm{d}s}{s - \lambda}, \\ \chi_1(\lambda) &= \frac{1}{2\pi i}\int_{\lambda_0}^{\lambda_1} \ln\left(\frac{1 - r_1(s)r_2(s)}{1 - r_1(\lambda_1)r_2(\lambda_1)}\right) \frac{\mathrm{d}s}{s - \lambda}, \\ \vartheta(\lambda_0) &= -\frac{1}{2\pi}\ln(1 - r_1(\lambda_0)r_2(\lambda_0)), \\ \vartheta(\lambda_1) &= -\frac{1}{2\pi}\ln(1 - r_1(\lambda_1)r_2(\lambda_1)), \end{aligned} \tag{3.5}$$

so that

$$\mathrm{Im}\vartheta(\lambda_i) = -\frac{1}{2\pi}\int_{-\infty}^{\lambda_i} \mathrm{d}\ \arg(1 - r_1(s)r_2(s)), \qquad i = 0, 1. \tag{3.6}$$

Assuming that $\int_{-\infty}^{\lambda_i} \mathrm{d}\ \arg(1 - r_1(s)r_2(s)) \in (-\pi, \pi)$, we have

$$|\mathrm{Im}\vartheta(\lambda)| < \frac{1}{2}, \qquad \lambda \in \mathbb{R}, \tag{3.7}$$

then we get that $\ln(1 - r_1(\lambda)r_2(\lambda))$ is single-valued, and the singularity of $\delta(\lambda, \xi)$ at $\lambda = \lambda_0$ and $\lambda = \lambda_1$ is square integrable.

Let

$$M^{(1)}(x, t; \lambda) = M(x, t; \lambda)\delta^{-\sigma_3}(\lambda), \tag{3.8}$$

then $M^{(1)}$ solves the RH problem on the jump contour $\mathbb{R}$ shown in Figure 3.2,

$$\begin{cases} M_+^{(1)}(x,t;\lambda) = M_-^{(1)}(x,t;\lambda)J^{(1)}(x,t;\lambda), & \lambda \in \mathbb{R} = \Sigma^{(1)}, \\ M^{(1)}(x,t;\lambda) \to I, & \lambda \to \infty, \end{cases} \tag{3.9}$$

where

$$J^{(1)} = \begin{pmatrix} 1 & 0 \\ \gamma_1(\lambda)e^{2ift}\delta_-^{-2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma_2(\lambda)e^{-2ift}\delta_+^2 \\ 0 & 1 \end{pmatrix}, \tag{3.10}$$

the functions $\gamma_1(\lambda), \gamma_2(\lambda)$ are defined as

$$\gamma_1(\lambda) = \begin{cases} \dfrac{r_1(\lambda)}{1 - r_1(\lambda)r_2(\lambda)}, & \lambda_0 < \lambda < \lambda_1, \\ -r_1(\lambda), & \lambda < \lambda_0 \cup \lambda > \lambda_1, \end{cases}$$
$$\gamma_2(\lambda) = \begin{cases} \dfrac{r_2(\lambda)}{1 - r_1(\lambda)r_2(\lambda)}, & \lambda_0 < \lambda < \lambda_1, \\ -r_2(\lambda), & \lambda < \lambda_0 \cup \lambda > \lambda_1. \end{cases} \tag{3.11}$$

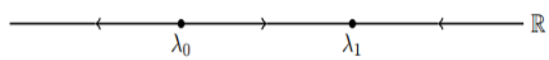

Figure 3.2. The jump contour $\mathbb{R} = \Sigma^{(1)}$.

Performing the decomposition $J^{(1)} = (b_-)^{-1}b_+$, where

$$b_- = \begin{pmatrix} 1 & 0 \\ -\gamma_1(\lambda)e^{2ift}\delta_-^{-2} & 1 \end{pmatrix}, \qquad b_+ = \begin{pmatrix} 1 & -\gamma_2(\lambda)e^{-2ift}\delta_+^2 \\ 0 & 1 \end{pmatrix}, \tag{3.12}$$

and taking

$$M^{(2)} = \begin{cases} M^{(1)}(\lambda), & \lambda \in \Omega_1 \cup \Omega_2, \\ M^{(1)}(\lambda)(b_-)^{-1}, & \lambda \in \Omega_3 \cup \Omega_4 \cup \Omega_5, \\ M^{(1)}(\lambda)(b_+)^{-1}, & \lambda \in \Omega_6 \cup \Omega_7 \cup \Omega_8, \end{cases} \tag{3.13}$$

we can deform the contour $\Sigma^{(1)}$ into the contour $\Sigma^{(2)} = L \cup L^*$ as displayed in Figure 3.3 and derive the following RH problem on the contour $\Sigma^{(2)} = L \cup L^* \cup \mathbb{R}$

$$\begin{cases} M_+^{(2)}(x,t;\lambda) = M_-^{(2)}(x,t;\lambda)J^{(2)}(x,t;\lambda), & \lambda \in \Sigma^{(2)}, \\ M^{(2)}(x,t;\lambda) \to I, & \lambda \to \infty, \end{cases} \tag{3.14}$$

where the jump matrix is

$$J^{(2)} = \begin{cases} I, & \lambda \in \mathbb{R}, \\ b_+, & \lambda \in L, \\ (b_-)^{-1}, & \lambda \in L^*. \end{cases} \tag{3.15}$$

Considering the jump matrix $J^{(2)}$ decaying exponentially to identity away from the stationary phase point λ_0, λ_1 as $t \to \infty$, we need take $D_{\lambda_0}^{\epsilon}$ and $D_{\lambda_1}^{\epsilon}$ be a disk of radius ϵ centered at λ_0 and λ_1, with ϵ sufficiently small. Thus, we can change the contour $\Sigma^{(2)}$ into the contours $\Sigma^{(\mathrm{app})}$ and $\Sigma^{(\mathrm{err})}$ (see Figure 3.4).

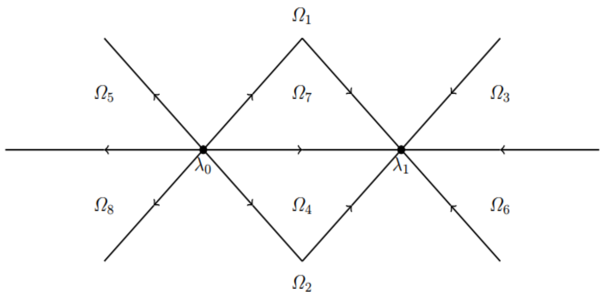

Figure 3.3. The jump contour $\Sigma^{(2)}$ and domains Ω_j $(j = 1, \cdots, 8)$.

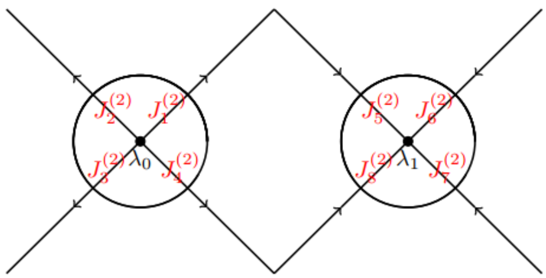

Figure 3.4. The jump contour $\Sigma^{(2)}$.

Define

$$M^{(\mathrm{app})} = M_{\lambda_0}^{(\mathrm{app})} M_{\lambda_1}^{(\mathrm{app})} = \begin{cases} I, & \text{outside } D_{\lambda_0}^{\epsilon} \cup D_{\lambda_1}^{\epsilon}, \\ \text{parametrixof } M^{(2)}, & \text{inside } D_{\lambda_0}^{\epsilon} \cup D_{\lambda_1}^{\epsilon}, \end{cases} \tag{3.16}$$

which means $M^{(\mathrm{app})}$ has the same jump conditions as $M^{(2)}$ inside $D_{\lambda_0}^{\epsilon} \cup D_{\lambda_1}^{\epsilon}$. $M_{\lambda_0}^{(\mathrm{app})}$ should possess a jump $J_{\lambda_0}^{(\mathrm{app})}$ across the circle $D_{\lambda_0}^{\epsilon}$, $M_{\lambda_1}^{(\mathrm{app})}$ should possess a jump $J_{\lambda_1}^{(\mathrm{app})}$ across the circle $D_{\lambda_1}^{\epsilon}$. Besides, we obtain (see Appendix A)

$$\begin{aligned} J_{\lambda_0}^{(\mathrm{app})} - I &= \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2} - \mathrm{Im}(\vartheta(\lambda_0))}) \\ O(t^{-\frac{1}{2} + \mathrm{Im}(\vartheta(\lambda_0))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}, \\ J_{\lambda_1}^{(\mathrm{app})} - I &= \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2} + \mathrm{Im}(\vartheta(\lambda_1))}) \\ O(t^{-\frac{1}{2} - \mathrm{Im}(\vartheta(\lambda_1))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}. \end{aligned} \tag{3.17}$$

So the following RH problem is given for matrix $M^{(\mathrm{app})(x,t,\lambda)}$

$$\begin{cases} M^{(\mathrm{app})}(x,t,\lambda) \text{ is analytic in } \mathbb{C} \setminus \Sigma^{(\mathrm{app})}, \\ M_+^{(\mathrm{app})}(x,t,\lambda) = M_-^{(\mathrm{app})}(x,t,\lambda) J^{(\mathrm{app})}(x,t,\lambda), & \lambda \in \Sigma^{(\mathrm{app})}, \\ M^{(\mathrm{app})}(x,t,\lambda) \to I, & \lambda \to \infty, \end{cases} \tag{3.18}$$

of which the jump matrix $J^{(\mathrm{app})}(x,t,\lambda)$ is

$$J^{(\mathrm{app})}(x,t,\lambda) = \begin{cases} J_i^{(\mathrm{app})} = J_i^{(2)} \ (i = 1,2,3,4), & \text{inside } D_{\lambda_0}^{\epsilon}, \\ J_i^{(\mathrm{app})} = J_i^{(2)} \ (i = 5,6,7,8), & \text{inside } D_{\lambda_1}^{\epsilon}, \\ J_{\lambda_0}^{(\mathrm{app})} \ \text{on} \ \ D_{\lambda_0}^{\epsilon}, \\ J_{\lambda_1}^{(\mathrm{app})} \ \text{on} \ \ D_{\lambda_1}^{\epsilon}, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} J_1^{(2)} &= J_5^{(2)} = \begin{pmatrix} 1 & -\dfrac{r_2(\lambda)}{1-r_1(\lambda)r_2(\lambda)}e^{-2ift}\delta_+^2 \\ 0 & 1 \end{pmatrix}, \\ J_2^{(2)} &= J_6^{(2)} = \begin{pmatrix} 1 & 0 \\ -r_1(\lambda)e^{2ift}\delta_-^{-2} & 1 \end{pmatrix}, \\ J_3^{(2)} &= J_7^{(2)} = \begin{pmatrix} 1 & r_2(\lambda)e^{-2ift}\delta_+^2 \\ 0 & 1 \end{pmatrix}, \\ J_4^{(2)} &= J_8^{(2)} = \begin{pmatrix} 1 & 0 \\ \dfrac{r_1(\lambda)}{1-r_1(\lambda)r_2(\lambda)}e^{2ift}\delta_-^{-2} & 1 \end{pmatrix}. \end{aligned} \tag{3.20}$$

For large λ, we define the factorization

$$M^{(2)} = M^{(\mathrm{err})}M^{(\mathrm{app})}, \tag{3.21}$$

where the error term contains higher-order contribution from the contour $\Sigma^{(2)}$. Then matrix $M^{(\mathrm{err})}(x,t,\lambda)$ meets the following RH problem:

$$\begin{cases} M^{(\mathrm{err})}(x,t,\lambda) \text{ is analytic in } \mathbb{C}\setminus\Sigma^{(\mathrm{err})}, \\ M_+^{(\mathrm{err})}(x,t,\lambda) = M_-^{(\mathrm{err})}(x,t,\lambda)J^{(\mathrm{err})}(x,t,\lambda), & \lambda\in\Sigma^{(\mathrm{err})}, \\ M^{(\mathrm{err})}(x,t,\lambda)\to I, & \lambda\to\infty, \end{cases} \tag{3.22}$$

of which the jump matrix $J^{(\mathrm{err})}(x,t,\lambda)$ is (see Appendix B)

$$J^{(\mathrm{err})}(x,t,\lambda) = \begin{cases} J_i^{(\mathrm{err})} = J_i^{(2)} = I + O(e^{-\tilde{C}t}),\ (i=1,2,\cdots,8), & \text{outside } D_{\lambda_0}^{\epsilon}\cup D_{\lambda_1}^{\epsilon}, \\ J_{\lambda_0}^{(\mathrm{err})} = (J_{\lambda_0}^{(\mathrm{app})})^{-1} \text{ on } D_{\lambda_0}^{\epsilon}, \\ J_{\lambda_1}^{(\mathrm{err})} = (J_{\lambda_1}^{(\mathrm{app})})^{-1} \text{ on } D_{\lambda_1}^{\epsilon}. \end{cases} \tag{3.23}$$

Let's expand the matrices $M^{(2)}, M_{\lambda_0}^{(\mathrm{app})}, M_{\lambda_1}^{(\mathrm{app})}, M^{(\mathrm{err})}$ at infinity into the Laurent series

$$\begin{aligned} M^{(2)} &= I + \frac{M_1^{(2)}}{\lambda} + \frac{M_2^{(2)}}{\lambda^2} + \cdots, && \lambda\to\infty, \\ M_{\lambda_0}^{(\mathrm{app})} &= I + \frac{(M_{\lambda_0}^{(\mathrm{app})})_1}{\lambda} + \frac{(M_{\lambda_0}^{(\mathrm{app})})_2}{\lambda^2} + \cdots, && \lambda\to\infty, \\ M_{\lambda_1}^{(\mathrm{app})} &= I + \frac{(M_{\lambda_1}^{(\mathrm{app})})_1}{\lambda} + \frac{(M_{\lambda_1}^{(\mathrm{app})})_2}{\lambda^2} + \cdots, && \lambda\to\infty, \\ M^{(\mathrm{err})} &= I + \frac{M_1^{(\mathrm{err})}}{\lambda} + \frac{M_2^{(\mathrm{err})}}{\lambda^2} + \cdots, && \lambda\to\infty. \end{aligned} \tag{3.24}$$

According to the factorization (3.21), comparing the coefficients of $\frac{1}{\lambda}$, we find

$$M_1^{(2)} = (M_{\lambda_0}^{(\mathrm{app})})_1 + (M_{\lambda_1}^{(\mathrm{app})})_1 + M_1^{(\mathrm{err})}. \tag{3.25}$$

Thus the solution of the nonlocal Hirota equation (1.4) is

$$q(x,t) = 2i[M_1^{(2)}]_{12} = 2i[(M_{\lambda_0}^{(\mathrm{app})})_1]_{12} + 2i[(M_{\lambda_1}^{(\mathrm{app})})_1]_{12} + 2i[M_1^{(\mathrm{err})}]_{12}. \tag{3.26}$$

Similar to Refs.[5, 26], the absolution of matrix $M_1^{(\mathrm{err})}(x,t,\lambda)$ in Eq.(3.22) satisfies (see Appendix C)

$$|M_1^{(\mathrm{err})}(x,t,\lambda)| = O(t^{-\frac{1}{2}-\max\{|\mathrm{Im}\vartheta(\lambda_0)|,|\mathrm{Im}\vartheta(\lambda_1)|\}}). \tag{3.27}$$

3.2 Reduction to a Model Riemann-Hilbert Problem

In this subsection, we will define a scaling transformation to separate the time t from the jump matrix, given by

$$\begin{aligned} \tilde{\lambda} = T_0(\lambda) = \sqrt{-8t(\alpha+6\beta\lambda_0)}(\lambda-\lambda_0), \qquad \lambda \in \Sigma_{\lambda_0}^{(\mathrm{app})}, \\ \tilde{\lambda} = T_1(\lambda) = \sqrt{8t(\alpha+6\beta\lambda_1)}(\lambda-\lambda_1), \qquad \lambda \in \Sigma_{\lambda_1}^{(\mathrm{app})}. \end{aligned} \tag{3.28}$$

For a given function $\varphi(\zeta)$, one has

$$T_0(\varphi(\lambda)) = \varphi\Big(\frac{\tilde{\lambda}}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big), \qquad T_1(\varphi(\lambda)) = \varphi\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1\Big).$$

Hence, we have

$$T_0(e^{-itf}\delta(\lambda)) = \delta_{\lambda_0}^0\delta_{\lambda_0}^1, \qquad T_1(e^{-itf}\delta(\lambda)) = \delta_{\lambda_1}^0\delta_{\lambda_1}^1, \tag{3.29}$$

where

$$\begin{aligned} \delta_{\lambda_0}^0 =& \big[-32\lambda_0^2 t(\alpha+6\beta\lambda_0)\big]^{\frac{i\vartheta(\lambda_0)}{2}} e^{2i\lambda_0^2 t(4\beta\lambda_0+\alpha)+\chi_0(\lambda_0)}, \\ \delta_{\lambda_0}^1 =& \tilde{\lambda}^{-i\vartheta(\lambda_0)}\Big(\frac{2\lambda_0}{\tilde{\lambda}/\sqrt{-8t(\alpha+6\beta\lambda_0)}+\lambda_0-\lambda_1}\Big)^{-i\vartheta(\lambda_0)} \\ &\times e^{\frac{i}{4}\tilde{\lambda}^2(1-\frac{i\beta\tilde{\lambda}}{\sqrt{2t}(6\beta\lambda_0+\alpha)^{\frac{3}{2}}})} e^{\chi_0(\frac{\tilde{\lambda}}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0)-\chi_0(\lambda_0)}, \\ \delta_{\lambda_1}^0 =& \big[32\lambda_1^2 t(\alpha+6\beta\lambda_1)\big]^{-\frac{i\vartheta(\lambda_1)}{2}} e^{2i\lambda_1^2 t(4\beta\lambda_1+\alpha)+\chi_1(\lambda_1)}, \\ \delta_{\lambda_1}^1 =& \tilde{\lambda}^{i\vartheta(\lambda_1)}\Big(\frac{2\lambda_1}{\tilde{\lambda}/\sqrt{8t(\alpha+6\beta\lambda_1)}+\lambda_1-\lambda_0}\Big)^{i\vartheta(\lambda_1)} \\ &\times e^{-\frac{i}{4}\tilde{\lambda}^2(1+\frac{\beta\tilde{\lambda}}{\sqrt{2t}(6\beta\lambda_1+\alpha)^{\frac{3}{2}}})} e^{\chi_1(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1)-\chi_1(\lambda_1)}. \end{aligned} \tag{3.30}$$

3.2.1 The T_0 Scaling Transformation

Here, we first consider the scaling transformation of T_0, it is easy to obtain the following RH problem

$$M_+^{(3)}(x,t;\tilde{\lambda}) = M_-^{(3)}(x,t;\tilde{\lambda})J^{(3)}, \qquad \tilde{\lambda} \in \Sigma^{(3)}, \tag{3.31}$$

where we have defined

$$M^{(3)}(x,t;\tilde{\lambda}) = T_0(M_{\lambda_0}^{(\mathrm{app})}(x,t;\lambda)), \qquad J^{(3)}(x,t;\tilde{\lambda}) = T_0(J_i^{(2)}(x,t;\lambda)), \qquad i=1,2,3,4.$$

In terms of the above analysis, one gets the new jump matrix $J^{(3)}$, given by (see Figure 3.5)

$$\begin{aligned} J_1^{(3)} &= \begin{pmatrix} 1 & -(\delta_{\lambda_0}^0\delta_{\lambda_0}^1)^2\frac{r_2}{1-r_1r_2}\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) \\ 0 & 1 \end{pmatrix}, \\ J_2^{(3)} &= \begin{pmatrix} 1 & 0 \\ -(\delta_{\lambda_0}^0\delta_{\lambda_0}^1)^{-2}r_1\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) & 1 \end{pmatrix}, \\ J_3^{(3)} &= \begin{pmatrix} 1 & (\delta_{\lambda_0}^0\delta_{\lambda_0}^1)^2 r_2\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) \\ 0 & 1 \end{pmatrix}, \\ J_4^{(3)} &= \begin{pmatrix} 1 & 0 \\ (\delta_{\lambda_0}^0\delta_{\lambda_0}^1)^{-2}\frac{r_1}{1-r_1r_2}\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) & 1 \end{pmatrix}. \end{aligned} \tag{3.32}$$

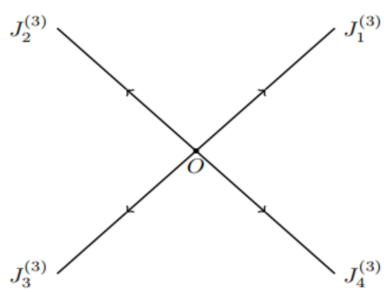

Figure 3.5. The jump contour $\Sigma^{(3)}$.

Since

$$\begin{aligned} M^{(3)} &= T_0(M_{\lambda_0}^{(\mathrm{app})}(\lambda)) = M_{\lambda_0}^{(\mathrm{app})}\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) \\ &= I + \frac{(M_{\lambda_0}^{(\mathrm{app})})_1}{\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0} + \cdots = I + \frac{M_1^{(3)}}{\tilde\lambda} + \cdots . \end{aligned} \tag{3.33}$$

Comparing the coefficient of $\tilde\lambda$ in above formulas, we have

$$M_1^{(3)} = \sqrt{-8t(\alpha+6\beta\lambda_0)}(M_{\lambda_0}^{(\mathrm{app})})_1. \tag{3.34}$$

Moreover, as $t\to\infty$, one obtains

$$\begin{aligned} &\lim_{t\to\infty}\Big(\frac{\tilde\lambda}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}+\lambda_0\Big) = \lambda_0, \\ &\lim_{t\to\infty}\delta_{\lambda_0}^1 = \tilde\lambda^{-i\vartheta(\lambda_0)}e^{\frac{1}{4}i\tilde\lambda^2}. \end{aligned} \tag{3.35}$$

To separate the time t completely, we perform the following limiting operation

$$M^{(\infty)} = \lim_{t\to\infty}(\delta_{\lambda_0}^0)^{-\hat\sigma_3}M^{(3)}, \tag{3.36}$$

which changes the jumping curve $\Sigma^{(3)}$ into $\Sigma^{(\infty)}$, and leads to the following RH problem:

$$\begin{cases} M^{(\infty)}(x,t;\tilde{\lambda}) \text{ is analytic in } \mathbb{C}\backslash\Sigma^{(\infty)}, \\ M_+^{(\infty)}(x,t;\tilde{\lambda}) = M_-^{(\infty)}(x,t;\tilde{\lambda})J^{(\infty)}(x,t;\tilde{\lambda}), & \tilde{\lambda}\in\Sigma^{(\infty)}, \\ M^{(\infty)}(x,t;\tilde{\lambda}) \to I, & \tilde{\lambda}\to\infty, \end{cases} \tag{3.37}$$

where

$$\begin{aligned} J_1^{(\infty)} &= \begin{pmatrix} 1 & -\tilde{\lambda}^{-2i\vartheta(\lambda_0)}e^{\frac{1}{2}i\tilde{\lambda}^2}\dfrac{r_2}{1-r_1r_2}(\lambda_0) \\ 0 & 1 \end{pmatrix}, \\ J_2^{(\infty)} &= \begin{pmatrix} 1 & 0 \\ -\tilde{\lambda}^{2i\vartheta(\lambda_0)}e^{-\frac{1}{2}i\tilde{\lambda}^2}r_1(\lambda_0) & 1 \end{pmatrix}, \\ J_3^{(\infty)} &= \begin{pmatrix} 1 & \tilde{\lambda}^{-2i\vartheta(\lambda_0)}e^{\frac{1}{2}i\tilde{\lambda}^2}r_2(\lambda_0) \\ 0 & 1 \end{pmatrix}, \\ J_4^{(\infty)} &= \begin{pmatrix} 1 & 0 \\ \tilde{\lambda}^{2i\vartheta(\lambda_0)}e^{-\frac{1}{2}i\tilde{\lambda}^2}\dfrac{r_1}{1-r_1r_2}(\lambda_0) & 1 \end{pmatrix}. \end{aligned} \tag{3.38}$$

To obtain the model RH problem, we define the following transformation

$$M^{(\mathrm{mod})} = M^{(\infty)}G_j, \qquad \tilde{\lambda}\in\Omega_j, \qquad j=0,\cdots,4, \tag{3.39}$$

where

$$\begin{aligned} &G_0 = e^{\frac{1}{4}i\tilde{\lambda}^2\sigma_3}\tilde{\lambda}^{-i\vartheta(\lambda_0)\sigma_3}, \\ &G_1 = G_0\begin{pmatrix} 1 & -\dfrac{r_2}{1-r_1r_2}(\lambda_0) \\ 0 & 1 \end{pmatrix}, \qquad G_2 = G_0\begin{pmatrix} 1 & 0 \\ r_1(\lambda_0) & 1 \end{pmatrix}, \\ &G_3 = G_0\begin{pmatrix} 1 & r_2(\lambda_0) \\ 0 & 1 \end{pmatrix}, \qquad G_4 = G_0\begin{pmatrix} 1 & 0 \\ -\dfrac{r_1}{1-r_1r_2}(\lambda_0) & 1 \end{pmatrix}. \end{aligned} \tag{3.40}$$

Through this transformation, we obtain a model RH problem for $M^{(\mathrm{mod})}$ with a constant jump matrix, given by

$$\begin{cases} M^{(\mathrm{mod})}(x,t;\tilde{\lambda}) \text{ is analytic in } \mathbb{C}\backslash\mathbb{R}, \\ M_+^{(\mathrm{mod})}(x,t;\tilde{\lambda}) = M_-^{(\mathrm{mod})}(x,t;\tilde{\lambda})J^{(\mathrm{mod})}(x,t;\tilde{\lambda}), & \tilde{\lambda}\in\mathbb{R}, \\ M^{(\mathrm{mod})}(x,t;\tilde{\lambda}) \to e^{\frac{1}{4}i\tilde{\lambda}^2\sigma_3}\tilde{\lambda}^{-i\vartheta(\lambda_0)\sigma_3}, & \tilde{\lambda}\to\infty, \end{cases} \tag{3.41}$$

where

$$J^{(\mathrm{mod})} = \begin{pmatrix} 1-r_1(\lambda_0)r_2(\lambda_0) & -r_2(\lambda_0) \\ r_1(\lambda_0) & 1 \end{pmatrix}. \tag{3.42}$$

The solution $M^{(\mathrm{mod})}(\tilde{\lambda})$ of this RH problem can be given explicitly via using the parabolic cylinder functions.

In order to derive the asymptotic formulas in Theorem 1.1, we give the large-$\tilde{\lambda}$ behavior of $M^{(\infty)}(\tilde{\lambda})$(see Appendix D)

$$M^{(\infty)}(\tilde{\lambda}) = I + \frac{M_1^{(\infty)}}{\tilde{\lambda}} + O\Big(\frac{1}{\tilde{\lambda}}\Big), \qquad \tilde{\lambda}\to\infty, \tag{3.43}$$

where

$$
\begin{aligned}
&[M_1^{(\infty)}]_{12} = \frac{\sqrt{2\pi} i (-1)^{i\vartheta(\lambda_0)-\frac{1}{2}} e^{-\frac{3\pi i}{4}+\frac{\pi\vartheta(\lambda_0)}{2}}}{r_1(\lambda_0)\Gamma(i\vartheta(\lambda_0))}, \\
&[M_1^{(\infty)}]_{21} = \frac{\sqrt{2\pi} i (-1)^{-i\vartheta(\lambda_0)-\frac{1}{2}} e^{-\frac{\pi i}{4}+\frac{\pi\vartheta(\lambda_0)}{2}}}{r_2(\lambda_0)\Gamma(-i\vartheta(\lambda_0))}.
\end{aligned} \tag{3.44}
$$

Therefore, combining (3.34), (3.36), (3.43) and (3.44), we have

$$
\begin{aligned}
[(M_{\lambda_0}^{(\mathrm{app})})_1]_{12} &= \frac{1}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}[M_1^{(3)}]_{12} = \frac{(\delta_{\lambda_0}^0)^2}{\sqrt{-8t(\alpha+6\beta\lambda_0)}}[M_1^{(\infty)}]_{12} \\
&= \frac{\sqrt{2\pi}[32\lambda_0^2 t(\alpha+6\beta\lambda_0)]^{i\vartheta(\lambda_0)} e^{4i\lambda_0^2 t(4\beta\lambda_0+\alpha)+2\chi_0(\lambda_0)-\frac{3\pi i}{4}+\frac{\pi\vartheta(\lambda_0)}{2}}}{\sqrt{-8t(\alpha+6\beta\lambda_0)} r_1(\lambda_0)\Gamma(i\vartheta(\lambda_0))}.
\end{aligned} \tag{3.45}
$$

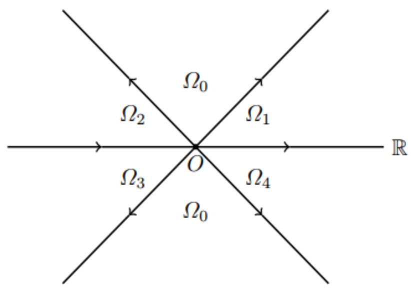

Figure 3.6. The jump contour $\Sigma^{(3)} \cup \Sigma^{(\mathrm{mod})}$.

3.2.2 The T_1 Scaling Transformation

Here, we perform the T_1 scaling transformation, and obtain the following RH problem

$$
M_+^{(3)}(x,t;\tilde{\lambda}) = M_-^{(3)}(x,t;\tilde{\lambda}) J^{(3)}, \qquad \tilde{\lambda} \in \Sigma^{(3)}, \tag{3.46}
$$

where

$$
M^{(3)}(x,t;\tilde{\lambda}) = T_1(M_{\lambda_1}^{(\mathrm{app})}(x,t;\lambda)), \qquad J^{(3)}(x,t;\tilde{\lambda}) = T_1(J_i^{(2)}(x,t;\lambda)), \qquad i = 5,6,7,8.
$$

and the new jump matrix $J^{(3)}$ is (see Figure 3.7)

$$
\begin{aligned}
J_5^{(3)} &= \begin{pmatrix} 1 & -(\delta_{\lambda_1}^0 \delta_{\lambda_1}^1)^2 \frac{r_2}{1-r_1 r_2}\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1\Big) \\ 0 & 1 \end{pmatrix}, \\
J_6^{(3)} &= \begin{pmatrix} 1 & 0 \\ -(\delta_{\lambda_1}^0 \delta_{\lambda_1}^1)^{-2} r_1\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1\Big) & 1 \end{pmatrix}, \\
J_7^{(3)} &= \begin{pmatrix} 1 & (\delta_{\lambda_1}^0 \delta_{\lambda_1}^1)^2 r_2\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1\Big) \\ 0 & 1 \end{pmatrix}, \\
J_8^{(3)} &= \begin{pmatrix} 1 & 0 \\ (\delta_{\lambda_1}^0 \delta_{\lambda_1}^1)^{-2} \frac{r_1}{1-r_1 r_2}\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}}+\lambda_1\Big) & 1 \end{pmatrix}.
\end{aligned} \tag{3.47}
$$

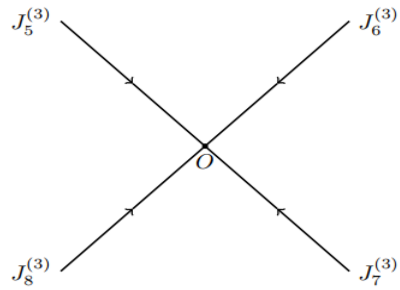

Figure 3.7. The jump contour $\Sigma^{(3)}$.

Due to

$$\begin{aligned} M^{(3)} &= T_1(M_{\lambda_1}^{(\text{app})}(\lambda)) = M_{\lambda_1}^{(\text{app})}\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}} + \lambda_1\Big) \\ &= I + \frac{(M_{\lambda_1}^{(\text{app})})_1}{\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}} + \lambda_1} + \cdots = I + \frac{M_1^{(3)}}{\tilde{\lambda}} + \cdots , \end{aligned} \tag{3.48}$$

we have

$$M_1^{(3)} = \sqrt{8t(\alpha+6\beta\lambda_1)}(M_{\lambda_1}^{(\text{app})})_1. \tag{3.49}$$

Besides, we arrive at

$$\lim_{t\to\infty}\Big(\frac{\tilde{\lambda}}{\sqrt{8t(\alpha+6\beta\lambda_1)}} + \lambda_1\Big) = \lambda_1, \qquad \lim_{t\to\infty}\delta_{\lambda_1}^1 = \tilde{\lambda}^{i\vartheta(\lambda_1)}e^{-\frac{1}{4}i\tilde{\lambda}^2}.$$

Similarly, we carry out the following limitation

$$M^{(\infty)} = \lim_{t\to\infty}(\delta_{\lambda_1}^0)^{-\hat{\sigma}_3}M^{(3)}, \tag{3.50}$$

and we obtain the following new RH problem:

$$\begin{cases} M^{(\infty)}(x,t;\tilde{\lambda}) \text{ is analytic in } \mathbb{C}\backslash\Sigma^{(\infty)}, \\ M_+^{(\infty)}(x,t;\tilde{\lambda}) = M_-^{(\infty)}(x,t;\tilde{\lambda})J^{(\infty)}(x,t;\tilde{\lambda}), & \tilde{\lambda}\in\Sigma^{(\infty)}, \\ M^{(\infty)}(x,t;\tilde{\lambda}) \to I, & \tilde{\lambda}\to\infty, \end{cases} \tag{3.51}$$

where

$$\begin{aligned} J_5^{(\infty)} &= \begin{pmatrix} 1 & -\tilde{\lambda}^{2i\vartheta(\lambda_1)}e^{-\frac{1}{2}i\tilde{\lambda}^2}\dfrac{r_2}{1-r_1r_2}(\lambda_1) \\ 0 & 1 \end{pmatrix}, \\ J_6^{(\infty)} &= \begin{pmatrix} 1 & 0 \\ -\tilde{\lambda}^{-2i\vartheta(\lambda_1)}e^{\frac{1}{2}i\tilde{\lambda}^2}r_1(\lambda_1) & 1 \end{pmatrix}, \\ J_7^{(\infty)} &= \begin{pmatrix} 1 & \tilde{\lambda}^{2i\vartheta(\lambda_1)}e^{-\frac{1}{2}i\tilde{\lambda}^2}r_2(\lambda_1) \\ 0 & 1 \end{pmatrix}, \\ J_8^{(\infty)} &= \begin{pmatrix} 1 & 0 \\ \tilde{\lambda}^{-2i\vartheta(\lambda_1)}e^{\frac{1}{2}i\tilde{\lambda}^2}\dfrac{r_1}{1-r_1r_2}(\lambda_1) & 1 \end{pmatrix}. \end{aligned} \tag{3.52}$$

Next, we will aim to obtain a model RH problem by defining

$$M^{(\mathrm{mod})} = M^{(\infty)} G_j, \qquad \tilde{\lambda} \in \Omega_j, \qquad j = 0, \cdots, 4, \tag{3.53}$$

where

$$\begin{aligned} &G_0 = e^{-\frac{1}{4} i \tilde{\lambda}^2 \sigma_3} \tilde{\lambda}^{i\vartheta(\lambda_1)\sigma_3}, \\ &G_1 = G_0 \begin{pmatrix} 1 & -\dfrac{r_2}{1 - r_1 r_2}(\lambda_1) \\ 0 & 1 \end{pmatrix}, \qquad G_2 = G_0 \begin{pmatrix} 1 & 0 \\ r_1(\lambda_1) & 1 \end{pmatrix}, \\ &G_3 = G_0 \begin{pmatrix} 1 & r_2(\lambda_1) \\ 0 & 1 \end{pmatrix}, \qquad G_4 = G_0 \begin{pmatrix} 1 & 0 \\ -\dfrac{r_1}{1 - r_1 r_2}(\lambda_1) & 1 \end{pmatrix}, \end{aligned} \tag{3.54}$$

then we obtain the following model RH problem:

$$\begin{cases} M^{(\mathrm{mod})}(x, t; \tilde{\lambda}) \text{ is analytic in } \mathbb{C} \backslash \mathbb{R}, \\ M_+^{(\mathrm{mod})}(x, t; \tilde{\lambda}) = M_-^{(\mathrm{mod})}(x, t; \tilde{\lambda}) J^{(\mathrm{mod})}(x, t; \tilde{\lambda}), & \tilde{\lambda} \in \mathbb{R}, \\ M^{(\mathrm{mod})}(x, t; \tilde{\lambda}) \to e^{-\frac{1}{4} i \tilde{\lambda}^2 \sigma_3} \tilde{\lambda}^{i\vartheta(\lambda_1)\sigma_3}, & \tilde{\lambda} \to \infty, \end{cases} \tag{3.55}$$

where

$$J^{(\mathrm{mod})} = \begin{pmatrix} 1 - r_1(\lambda_1) r_2(\lambda_1) & -r_2(\lambda_1) \\ r_1(\lambda_1) & 1 \end{pmatrix}. \tag{3.56}$$

Performing the same procedure in Appendix D, we get the large-$\tilde{\lambda}$ behavior of $M^{(\infty)}(\tilde{\lambda})$:

$$M^{(\infty)}(\tilde{\lambda}) = I + \frac{M_1^{(\infty)}}{\tilde{\lambda}} + O\Big(\frac{1}{\tilde{\lambda}}\Big), \qquad \tilde{\lambda} \to \infty, \tag{3.57}$$

where

$$[M_1^{(\infty)}]_{12} = \frac{\sqrt{2\pi} i e^{-\frac{\pi}{2}\vartheta(\lambda_1)} e^{-\frac{3\pi i}{4}}}{r_1(\lambda_1)\Gamma(-i\vartheta(\lambda_1))}, \qquad [M_1^{(\infty)}]_{21} = \frac{\sqrt{2\pi} i e^{-\frac{\pi}{2}\vartheta(\lambda_1)} e^{-\frac{\pi i}{4}}}{r_2(\lambda_1)\Gamma(i\vartheta(\lambda_1))}. \tag{3.58}$$

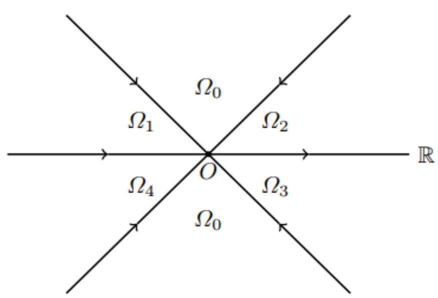

Figure 3.8. The jump contour $\Sigma^{(3)} \cup \Sigma^{(\mathrm{mod})}$.

Therefore, combining (3.49), (3.50), (3.57) and (3.58), we have

$$\begin{aligned} [(M_{\lambda_1}^{(\mathrm{app})})_1]_{12} &= \frac{1}{\sqrt{8t(\alpha + 6\beta\lambda_1)}} [M_1^{(3)}]_{12} = \frac{(\delta_{\lambda_1}^0)^2}{\sqrt{8t(\alpha + 6\beta\lambda_1)}} [M_1^{(\infty)}]_{12} \\ &= \frac{\sqrt{2\pi} i [32\lambda_1^2 t(\alpha + 6\beta\lambda_1)]^{-i\vartheta(\lambda_1)} e^{4i\lambda_1^2 t(4\beta\lambda_1 + \alpha) + 2\chi_1(\lambda_1) - \frac{\pi}{2}\vartheta(\lambda_1) - \frac{3\pi i}{4}}}{\sqrt{8t(\alpha + 6\beta\lambda_1)} r_1(\lambda_1) \Gamma(-i\vartheta(\lambda_1))}. \end{aligned} \tag{3.59}$$

Finally, combining (3.26), (3.27), (3.45) and (3.59), we can achieve the result of Theorem 1.1.

Appendix A

From Ref.[8], we have

$$M^{(\infty)} = (\delta^0_{\lambda_0})^{-\hat{\sigma}_3} M^{(3)} + O(t^{-\frac{1}{2}} \ln t). \tag{A.1}$$

For $\tilde{\lambda} \to \infty$,

$$M^{(3)} = I + M^{(3)}_1 \tilde{\lambda}^{-1} + O(\tilde{\lambda}^{-2}) = I + (\delta^0_{\lambda_0})^{\hat{\sigma}_3}(M^{(\infty)}_1 + O(t^{-\frac{1}{2}} \ln t))\tilde{\lambda}^{-1} + O(\tilde{\lambda}^{-2}). \tag{A.2}$$

On $D^{\epsilon}_{\lambda_0}$, $M^{(\text{app})}_{\lambda_0 -} = I$, one has

$$\begin{aligned} M^{(\text{app})}_{\lambda_0 +} = T_0^{-1}(M^{(3)}) =& I + (\delta^0_{\lambda_0})^{\hat{\sigma}_3}(T_0^{-1}(M^{(\infty)}_1) + O(t^{-\frac{1}{2}} \ln t))(\sqrt{8t(\alpha + 6\beta|\lambda_0|)}(\lambda - \lambda_0))^{-1} \\ &+ O((\sqrt{8t(\alpha + 6\beta|\lambda_0|)}(\lambda - \lambda_0))^{-2}). \end{aligned} \tag{A.3}$$

Therefore, we get

$$J^{(\text{app})}_{\lambda_0} - I = (M^{(\text{app})}_{\lambda_0 -})^{-1} M^{(\text{app})}_{\lambda_0 +} - I = \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2} - \text{Im}(\vartheta(\lambda_0))}) \\ O(t^{-\frac{1}{2} + \text{Im}(\vartheta(\lambda_0))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}. \tag{A.4}$$

In a similar way, we have

$$J^{(\text{app})}_{\lambda_1} - I = \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2} + \text{Im}(\vartheta(\lambda_1))}) \\ O(t^{-\frac{1}{2} - \text{Im}(\vartheta(\lambda_1))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}. \tag{A.5}$$

Appendix B

According to Eq.(3.21), we obtain

$$J^{(\text{err})} = (M^{(\text{err})}_{-})^{-1} M^{(\text{err})}_{+} = M^{(\text{app})}_{-} J^{(2)} (J^{(\text{app})})^{-1} (M^{(\text{app})}_{-})^{-1}. \tag{B.1}$$

Since $M^{(\text{app})}_{-} = I$ on $\Sigma^{(\text{err})}$, one has

$$J^{(\text{err})} = J^{(2)} (J^{(\text{app})})^{-1}. \tag{B.2}$$

Furthermore, we get

$$\begin{aligned} &J^{(\text{err})}_i = J^{(2)}_i \ (i = 1, 2, \cdots, 8), \quad \text{outside } D^{\epsilon}_{\lambda_0} \cup D^{\epsilon}_{\lambda_1}, \\ &J^{(\text{err})}_{\lambda_0} = (J^{(\text{app})}_{\lambda_0})^{-1} \qquad \text{on } D^{\epsilon}_{\lambda_0}, \\ &J^{(\text{err})}_{\lambda_1} = (J^{(\text{app})}_{\lambda_1})^{-1} \qquad \text{on } D^{\epsilon}_{\lambda_1}. \end{aligned} \tag{B.3}$$

Now, we will estimate the error of $J^{(\text{err})}_7$ outside $D^{\epsilon}_{\lambda_1}$. On the jump contour $\lambda_1 + \lambda_1 \rho e^{\frac{3\pi i}{4}} (\rho > \epsilon)$, the jump matrix $J^{(\text{err})}_7$ is

$$J^{(\text{err})}_7 = J^{(2)}_7 = \begin{pmatrix} 1 & r_2(\lambda) e^{-2ift} \delta^2_+ \\ 0 & 1 \end{pmatrix}. \tag{B.4}$$

Observing

$$|e^{-2ift}| = e^{4\lambda_1^2 \rho^2 (\sqrt{2}\beta\lambda_1\rho - 6\beta\lambda_1 - \alpha)t} \le e^{-\tilde{C}t}, \qquad \tilde{C} > 0. \tag{B.5}$$

Thus, we have

$$J_7^{(\mathrm{err})} - I = J_7^{(2)} - I = O(e^{-\tilde{C}t}). \tag{B.6}$$

Similarly, we have

$$J_i^{(\mathrm{err})} - I = J_i^{(2)} - I = O(e^{-\tilde{C}t}), \qquad i = 1, 2 \cdots, 8. \tag{B.7}$$

Moreover, from Eq.(A.4) and Eq.(A.5), it is easy to find

$$\begin{aligned} J_{\lambda_0}^{(\mathrm{err})} - I = (J_{\lambda_0}^{(\mathrm{app})})^{-1} - I = \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2}-\mathrm{Im}(\vartheta(\lambda_0))}) \\ O(t^{-\frac{1}{2}+\mathrm{Im}(\vartheta(\lambda_0))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}, \\ J_{\lambda_1}^{(\mathrm{err})} - I = (J_{\lambda_1}^{(\mathrm{app})})^{-1} - I = \begin{pmatrix} O(t^{-\frac{1}{2}}) & O(t^{-\frac{1}{2}+\mathrm{Im}(\vartheta(\lambda_1))}) \\ O(t^{-\frac{1}{2}-\mathrm{Im}(\vartheta(\lambda_1))}) & O(t^{-\frac{1}{2}}) \end{pmatrix}. \end{aligned} \tag{B.8}$$

Appendix C

The Cauchy integral formula on contour Σ can be defined as

$$(C_\Sigma(f))(\lambda) = \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-\lambda} ds. \tag{C.1}$$

Let

$$C_V^-(f) = C_\Sigma^-(f(V-I)), \tag{C.2}$$

where V is a matrix given in Σ, and C_Σ^+, C_Σ^+ denote the nontangential limits of the bounded operator C_Σ approaching Σ from left and right, respectively.

According to the RH problem (3.22), we can obtain

$$\begin{aligned} M^{(\mathrm{err})} - I &= C_{\Sigma^{(\mathrm{err})}} M_-^{(\mathrm{err})}(J^{(\mathrm{err})} - I) \\ &= -\frac{1}{2\pi i\lambda}\int_{\Sigma^{(\mathrm{err})}} M_-^{(\mathrm{err})}(J^{(\mathrm{err})} - I) ds + O(\lambda^{-2}), \end{aligned} \tag{C.3}$$

which indicates

$$M_1^{(\mathrm{err})} = -\frac{1}{2\pi i}\int_{\Sigma^{(\mathrm{err})}} M_-^{(\mathrm{err})}(J^{(\mathrm{err})} - I) ds. \tag{C.4}$$

Using Holder inequality, we have

$$| M_1^{(\mathrm{err})} | \leq C_1 ||M_-^{(\mathrm{err})} - I||_{L^2} ||J^{(\mathrm{err})} - I||_{L^2} + C_2 ||J^{(\mathrm{err})} - I||_{L^1}, \qquad C_1, C_2 > 0. \tag{C.5}$$

Beside, it is not hard to get $||M_-^{(\mathrm{err})} - I||_{L^2} \leq C_3 ||J^{(\mathrm{err})} - I||_{L^2}, C_3 > 0$, then we finally obtain

$$| M_1^{(\mathrm{err})} | \leq C_1 C_3 ||J^{(\mathrm{err})} - I||_{L^2} + C_2 ||J^{(\mathrm{err})} - I||_{L^1}. \tag{C.6}$$

Combining Eq.(B.7) and Eq.(B.8), we arrive at

$$|M_1^{(\mathrm{err})}(x,t,\lambda)| = O(t^{-\frac{1}{2}-\max\{|\mathrm{Im}\vartheta(\lambda_0)|, |\mathrm{Im}\vartheta(\lambda_1)|\}}). \tag{C.7}$$

Appendix D

The solution $M^{(\mathrm{mod})}(\tilde\lambda)$ of the model RH problem (3.41) can be given explicitly via using the Liouville's theorem and parabolic cylinder functions. Since the jump matrix $J^{(\mathrm{mod})}$ is constant, the logarithmic derivative $\frac{d}{d\tilde\lambda}M^{(\mathrm{mod})}(M^{(\mathrm{mod})})^{-1}$ possesses continuous jump along any of the rays, which indicates that $M^{(\mathrm{mod})}$ solves the following ordinary differential equation

$$\frac{d}{d\tilde\lambda}M^{(\mathrm{mod})}+\begin{pmatrix}-\dfrac{i}{2}\tilde\lambda & \Psi\\ \Phi & \dfrac{i}{2}\tilde\lambda\end{pmatrix}M^{(\mathrm{mod})}=0, \tag{D.1}$$

where $\Psi=i[M_1^{(\infty)}]_{12}, \Phi=-i[M_1^{(\infty)}]_{21}$. The solution of (D.1) can be written as

$$M^{(\mathrm{mod})}=\begin{pmatrix} M_{11}^{(\mathrm{mod})} & \dfrac{-\frac{i}{2}\tilde\lambda M_{22}^{(\mathrm{mod})}-\frac{dM_{22}^{(\mathrm{mod})}}{d\tilde\lambda}}{\Phi}\\ \dfrac{\frac{i}{2}\tilde\lambda M_{11}^{(\mathrm{mod})}-\frac{dM_{11}^{(\mathrm{mod})}}{d\tilde\lambda}}{\Psi} & M_{22}^{(\mathrm{mod})}\end{pmatrix}, \tag{D.2}$$

where the functions $M_{jj}^{(\mathrm{mod})},\ j=1,2$, satisfy the parabolic cylinder equations

$$\begin{aligned}&\frac{d^2}{d\tilde\lambda^2}M_{11}^{(\mathrm{mod})}+\Big(-\frac{i}{2}-\Phi\Psi+\frac{\tilde\lambda^2}{4}\Big)M_{11}^{(\mathrm{mod})}=0,\\ &\frac{d^2}{d\tilde\lambda^2}M_{22}^{(\mathrm{mod})}-\Big(-\frac{i}{2}+\Phi\Psi-\frac{\tilde\lambda^2}{4}\Big)M_{22}^{(\mathrm{mod})}=0.\end{aligned} \tag{D.3}$$

According to the property of standard parabolic cylinder equation and $M_{11}^{(\mathrm{mod})}\to e^{\frac14 i\tilde\lambda^2}\tilde\lambda^{-i\vartheta}$, $M_{22}^{(\mathrm{mod})}\to e^{-\frac14 i\tilde\lambda^2}\tilde\lambda^{i\vartheta}, \tilde\lambda\to\infty$, we obtain

$$M_{11}^{(\mathrm{mod})}=\begin{cases}(ie^{-\frac{3\pi}{4}i})^{i\vartheta}D_{-i\vartheta}(ie^{-\frac{3\pi}{4}i}\tilde\lambda) & \mathrm{Im}\ (\tilde\lambda)>0,\\ (ie^{\frac{\pi}{4}i})^{i\vartheta}D_{-i\vartheta}(ie^{\frac{\pi}{4}i}\tilde\lambda) & \mathrm{Im}\ (\tilde\lambda)<0,\end{cases} \tag{D.4}$$

$$M_{22}^{(\mathrm{mod})}=\begin{cases}(ie^{-\frac{\pi}{4}i})^{-i\vartheta}D_{i\vartheta}(ie^{-\frac{\pi}{4}i}\tilde\lambda) & \mathrm{Im}\ (\tilde\lambda)>0,\\ (ie^{\frac{3\pi}{4}i})^{-i\vartheta}D_{i\vartheta}(ie^{\frac{3\pi}{4}i}\tilde\lambda) & \mathrm{Im}\ (\tilde\lambda)<0.\end{cases} \tag{D.5}$$

Then, we can get

$$\begin{aligned}M_{-}^{(\mathrm{mod})}(\tilde\lambda)^{-1}M_{+}^{(\mathrm{mod})}(\tilde\lambda)=&M_{-}^{(\mathrm{mod})}(0)^{-1}M_{+}^{(\mathrm{mod})}(0)\\ =&\begin{pmatrix}(ie^{\frac{\pi}{4}i})^{i\vartheta}\dfrac{2^{\frac{-i\vartheta}{2}}\sqrt{\pi}}{\Gamma(\frac{1+i\vartheta}{2})} & (ie^{\frac{3\pi}{4}i})^{1-i\vartheta}\dfrac{2^{\frac{1+i\vartheta}{2}}\sqrt{\pi}}{\Phi\Gamma(\frac{-i\vartheta}{2})}\\ (ie^{\frac{\pi}{4}i})^{1+i\vartheta}\dfrac{2^{\frac{1-i\vartheta}{2}}\sqrt{\pi}}{\Psi\Gamma(\frac{i\vartheta}{2})} & (ie^{\frac{3\pi}{4}i})^{-i\vartheta}\dfrac{2^{\frac{i\vartheta}{2}}\sqrt{\pi}}{\Gamma(\frac{1-i\vartheta}{2})}\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}(ie^{-\frac{3\pi}{4}i})^{i\vartheta}\dfrac{2^{\frac{-i\vartheta}{2}}\sqrt{\pi}}{\Gamma(\frac{1+i\vartheta}{2})} & (ie^{-\frac{\pi}{4}i})^{1-i\vartheta}\dfrac{2^{\frac{1+i\vartheta}{2}}\sqrt{\pi}}{\Phi\Gamma(\frac{-i\vartheta}{2})}\\ (ie^{-\frac{3\pi}{4}i})^{1+i\vartheta}\dfrac{2^{\frac{1-i\vartheta}{2}}\sqrt{\pi}}{\Psi\Gamma(\frac{i\vartheta}{2})} & (ie^{-\frac{\pi}{4}i})^{-i\vartheta}\dfrac{2^{\frac{i\vartheta}{2}}\sqrt{\pi}}{\Gamma(\frac{1-i\vartheta}{2})}\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} 1 - r_1(\lambda_1) r_2(\lambda_1) & -r_2(\lambda_1) \\ r_1(\lambda_1) & 1 \end{pmatrix}, \tag{D.6}$$

which leads to

$$\Psi = \frac{\sqrt{2\pi}(-1)^{i\vartheta(\lambda_0)+\frac{1}{2}} e^{-\frac{3\pi i}{4}+\frac{\pi\vartheta(\lambda_0)}{2}}}{r_1(\lambda_0)\Gamma(i\vartheta(\lambda_0))}, \qquad \Phi = -\frac{\sqrt{2\pi}(-1)^{-i\vartheta(\lambda_0)+\frac{1}{2}} e^{-\frac{\pi i}{4}+\frac{\pi\vartheta(\lambda_0)}{2}}}{r_2(\lambda_0)\Gamma(-i\vartheta(\lambda_0))}. \tag{D.7}$$

Conflict of Interest

The authors declare no conflict of interest.

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