Journal of Mathematical Physics

A new form of general soliton solutions and multiple zeros solutions for a higher-order Kaup–Newell equation

Cite as: J. Math. Phys. **62**, 123501 (2021); https://doi.org/10.1063/5.0064411
Submitted: 22 July 2021 • Accepted: 15 November 2021 • Published Online: 02 December 2021

Jin-Yan Zhu and Yong Chen

AIP Publishing

Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

A new form of general soliton solutions and multiple zeros solutions for a higher-order Kaup–Newell equation

Cite as: J. Math. Phys. 62, 123501 (2021); doi: 10.1063/5.0064411
Submitted: 22 July 2021 • Accepted: 15 November 2021 •
Published Online: 2 December 2021

View Online Export Citation CrossMark

Jin-Yan Zhu[1] and Yong Chen[1,2,a)]

AFFILIATIONS

[1] Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, School of Mathematical Sciences, East China Normal University, Shanghai 200241, China
[2] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

[a)] **Author to whom correspondence should be addressed:** ychen@sei.ecnu.edu.cn

ABSTRACT

Due to the fact that the higher-order Kaup–Newell (KN) system has more complex and diverse solutions than the classical second-order flow KN system, the research on it has attracted much attention. In this paper, we consider a higher-order KN equation with third-order dispersion and fifth-order nonlinearity. Based on the theory of the inverse scattering, the matrix Riemann–Hilbert problem is established. Through the dressing method, the solution matrix with simple zeros without reflection is constructed. In particular, a new form of solution is given which is more direct and simpler than previous methods. In addition, through the determinant solution matrix, the vivid diagrams and dynamic analysis of single-soliton solution and two-soliton solution are given in detail. Finally, by using the technique of limit, we construct the general solution matrix in the case of multiple zeros, and the examples of solutions for the cases of double zeros, triple zeros, single–double zeros, and double–double zeros are especially shown.

I. INTRODUCTION

In Ref. 1, Abhinav *et al.* gave the coupled equations

$$\begin{aligned} q_t &= iq_{xx} - (4\beta+1)q^2 r_x - 4\beta q q_x r + \frac{i}{2}(1+2\beta)(4\beta+1)q^3 r^2,\\ r_t &= -ir_{xx} - (4\beta+1)r^2 q_x - 4\beta r r_x q - \frac{i}{2}(1+2\beta)(4\beta+1)q^2 r^3. \end{aligned} \tag{1}$$

System (1) has three famous Schrödinger-type reductions and these three reductions had been widely studied in recent years.

When $\beta = -\frac{1}{2}$ and $r = -q^*$, system (1) reduces to derivative nonlinear Schrödinger (DNLS) I,

$$iq_t + q_{xx} - i\left(q^2 q^*\right)_x = 0, \tag{2}$$

where the symbol "$*$" represents the complex conjugate and the subscript of x (or t) represents the partial derivative with respect to x (or t). Equation (2) is also called the Kaup–Newell (KN) equation.[2] In recent years, the KN equation related to spectral problems, exact solutions, Hamilton structure, Painléve properties, and other properties have been in-depth research.[2–8] Equation (2) is a typical dispersion equation, which is derived from the magnetohydrodynamic equation with Hall effect, especially describing the nonlinear Alfvén waves in plasma physics.[9–11]

When $\beta = -\frac{1}{4}$ and $r = -q^*$, system (1) reduces to DNLS II,

$$iq_t + q_{xx} - iqq^* q_x = 0, \tag{3}$$

which appears in optical models of ultrashort pulses and is also referred to as the Chen–Lee–Liu (CLL) equation.[12]

When $\beta = 0$ and $r = -q^*$, system (1) reduces to DNLS III,

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 q^{*2} = 0. \tag{4}$$

This equation was first discovered by Gerdjikov and Ivanov in Ref. 13, also known as the GI equation.

In Ref. 14, Fan gave the higher-order generalization of Eq. (1),

$$\begin{aligned} &q_t - \frac{1}{4}\big[2q_{xxx} - 6(2\beta - 1)rq_x^2 - 6(4\beta - 1)qq_x r_x - 6(2\beta - 1)qrq_{xx} + 6(2\beta - 1)(4\beta - 1)q^3 rr_x \\ &\quad + 3(8\beta^2 - 12\beta + 3)q^2 r^2 q_x + 4\beta(2\beta - 1)(4\beta - 1)q^4 r^3\big] = 0, \\ &r_t - \frac{1}{4}\big[2r_{xxx} + 6(2\beta - 1)qr_x^2 - 6(4\beta - 1)rq_x r_x + 6(2\beta - 1)qrr_{xx} + 6(2\beta - 1)(4\beta - 1)qr^3 q_x \\ &\quad + 3(8\beta^2 - 12\beta + 3)q^2 r^2 r_x - 4\beta(2\beta - 1)(4\beta - 1)q^3 r^4\big] = 0. \end{aligned} \tag{5}$$

System (5) can be used to describe the higher-order nonlinear effects in nonlinear optics and other fields. System (5) also has three important Schrödinger-type reductions.

First, when $\beta = 0, x \to ix, t \to it$, and $r = -q^*$, system (5) becomes

$$q_t = -\frac{1}{2} q_{xxx} + \left(\frac{3i}{2}|q|^2 q_x\right)_x + \left(\frac{3}{4}|q|^4 q\right)_x, \tag{6}$$

which can be viewed as the higher-order DNLS I or higher-order KN equation. Equation (6) also can be derived from the generalized KN hierarchy[15] under $n = 3$ and proper parameter.

Second, when $\beta = \frac{1}{4}$ and $x \to ix, t \to it, r = -q^*$, system (5) becomes

$$q_t = -\frac{1}{2} q_{xxx} - \frac{3}{4} i|q|^2 q_{xx} - \frac{3}{4} iq^* q_x^2 + \frac{3}{8}|q|^4 q_x, \tag{7}$$

which can be viewed as the higher-order DNLS II or higher-order CLL equation.

Third, when $\beta = \frac{1}{2}$ and $x \to ix, t \to it$ and $r = -q^*$, system (5) becomes

$$q_t = -\frac{1}{2} q_{xxx} + \frac{3}{2} iqq_x q_x^* - \frac{3}{4}|q|^4 q_x, \tag{8}$$

which can be regarded as the higher-order DNLS III or higher-order GI equation.

It has been proved in Ref. 14 that Eqs. (6)–(8) have multiple Hamiltonian structures and are Liouville integrable. The N-soliton solutions of Eqs. (7) and (8) have been studied in Refs. 16 and 17. In this paper, we mainly consider the soliton solutions and higher-order soliton solution of Eq. (6). In fact, there are several classical methods to obtain the soliton solutions, such as the inverse scattering (IST) method, Hirota bilinear method, Darboux/Bäcklund transform, and Riemann–Hilbert (RH) method.[18–23,26] Here, we will use the RH method to derive the soliton solutions of Eq. (6) since it is more convenient to study the exact long-time asymptotic and large $-n$ asymptotic.[24]

The high-order soliton solution of the nonlinear Schrödinger (NLS) type has been widely concerned by many scholars in recent years. It can be used to describe the weak bound states of solitons, which may appear in the study of soliton train transmission with specific chirp and almost equal velocity and amplitude.[25] There are not many studies on DNLS type higher-order soliton solutions. Recently, Chen's team studied the double and triple zeros of the GI equation[27] and the double zeros of higher-order KN.[28] Here, we study more extensive cases and give the general form of the solutions with multiple zeros.

The main content of this paper is to construct the general soliton solution matrix of the higher-order KN equation by using RH method. It is worth noting that we recover the potential $q(x,t)$ as the spectral parameter $\zeta \to 0$, it effectively reduces the operation process and avoids the interference of implicit function, and the matrix form of the soliton solution is more direct. Taking the single soliton solution and the two-soliton solution as examples, the properties of the soliton are studied. Then, on the basis of the soliton solution, through a certain limit technique, the solution matrix of the high-order soliton solution of the multiple zeros is obtained.

The organization of this paper is as follows: In Sec. II, the inverse scattering theory of the spectrum problem and the corresponding matrix Riemann–Hilbert problem (RHP) are established. The N-soliton formula for the higher-order KN equation is derived by considering the simple zeros in the RHP in Sec. III. In Sec. IV, we construct the higher-order soliton matrix and obtain the general expression of the higher-order soliton, which corresponds to the multiple zeros in the RHP. Section V is devoted to conclusion and discussion.

II. INVERSE SCATTERING THEORY OF (6)

The main work of this part is to study the inverse scattering problem of Eq. (6) and construct the corresponding RHP.

Equation (6) is Lax integrable with the linear spectral problem

$$Y_x = MY, \qquad M = -i\zeta^2\sigma_3 + \zeta Q, \tag{9}$$

$$Y_t = NY, \qquad N = -2i\zeta^6\sigma_3 + N_1, \tag{10}$$

where

$$\begin{aligned} N_1 &= 2Q\zeta^5 - iQ^2\sigma_3\zeta^4 + i\sigma_3 Q_x\zeta^3 + Q^3\zeta^3 - \frac{1}{2}(QQ_x - Q_xQ)\zeta^2 \\ &\quad - \frac{3}{4}iQ^4\sigma_3\zeta^2 - \frac{1}{2}Q_{xx}\zeta + \frac{3i}{2}\sigma_3 Q^2 Q_x\zeta + \frac{3}{4}Q^5\zeta, \end{aligned} \tag{11}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \tag{12}$$

and it is easy to verify that

$$Q^\dagger = -Q, \qquad \sigma_3 Q\sigma_3 = -Q,$$

which plays an important role in symmetry research later, and the symbol "†" represents the conjugate transpose of a matrix. In the following analysis, we assume that the potential function q, q^* rapidly tends to zero as $x \to \pm\infty$. In this case, the solution of the boundary form can be clearly obtained,

$$Y \sim e^{(-i\zeta^2 x - 2i\zeta^6 t)\sigma_3} \quad \text{as} \quad x \to \infty. \tag{13}$$

We make the following transformation:

$$Y = Je^{(-i\zeta^2 x - 2i\zeta^6 t)\sigma_3}. \tag{14}$$

The Lax pair of Eqs. (9) and (10) becomes

$$J_x + i\zeta^2[\sigma_3, J] = \zeta QJ, \tag{15}$$

$$J_t + 2i\zeta^6[\sigma_3, J] = N_1 J, \tag{16}$$

where Q, N_1 have been given by Eqs. (11) and (12).

In the scattering problem, the Lax equation (16) of time t is ignored temporarily. By solving Eq. (9) with the constant variation method and using transformation (14), the solution of Eq. (15) can be obtained, which satisfies the following integral equations:

$$J_M = I + \zeta\int_{-\infty}^{x} e^{i\zeta^2\sigma_3(y-x)}Q(y)J_M e^{i\zeta^2\sigma_3(x-y)}\mathrm{d}y, \tag{17}$$

$$J_P = I - \zeta\int_{x}^{+\infty} e^{i\zeta^2\sigma_3(y-x)}Q(y)J_P e^{i\zeta^2\sigma_3(x-y)}\mathrm{d}y, \tag{18}$$

and these two Jost solutions satisfy the following asymptotics at large distances:

$$J \sim I \quad as\ |x| \sim \infty. \tag{19}$$

In order to analyze the analytical properties of Jost solutions in the ζ plane, we divide the entire ζ plane into two regions, as shown in Fig. 1,

$$\mathbb{C}_{13} = \left\{\zeta \middle| \arg\zeta \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)\right\}, \quad \mathbb{C}_{24} = \left\{\zeta \middle| \arg\zeta \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)\right\}.$$

Dividing J into columns as $J = \left(J^{(1)}, J^{(2)}\right)$, due to the structure (12) of the potential Q and Volterra integral equations (17) and (18), we have the following proposition:

Proposition II.1. The above Volterra integral equations exist and are unique and have the following properties:

- *The column vectors $J_M^{(1)}$ and $J_P^{(2)}$ are analytic for $\zeta \in \mathbb{C}_{13}$ and continuous for $\zeta \in \mathbb{C}_{13} \cup \mathbb{R} \cup i\mathbb{R}$.*
- *The column vectors $J_P^{(1)}$ and $J_M^{(2)}$ are analytical for $\zeta \in \mathbb{C}_{24}$ and continuous for $\zeta \in \mathbb{C}_{24} \cup \mathbb{R} \cup i\mathbb{R}$.*

Through Eq. (14), we know that $J_P E$ and $J_M E$ are both solutions of the linear equation (9), so they are linearly related by a matrix $S(\zeta)$,

$$J_M E = J_P E S(\zeta), \quad \zeta \in \mathbb{R} \cup i\mathbb{R}, \tag{20}$$

where $E = e^{-i\zeta^2 x\sigma_3}$ and $S(\zeta) = (s_{ij})_{2\times 2}$. It should be noted that

$$\mathrm{tr}(-i\zeta^2\sigma_3 + \zeta Q) = 0,$$

and using Abel's formula, we can get that

$$(\det Y)_x = 0, \tag{21}$$

considering that transformation (14) has

$$\det J = \det Y \det(e^{i\zeta^2 x\sigma_3}) = \det Y.$$

Reusing Eq. (21) has

$$(\det J)_x = 0,$$

which means that the $\det J$ is independent of x, and then from the asymptotic (19), we know

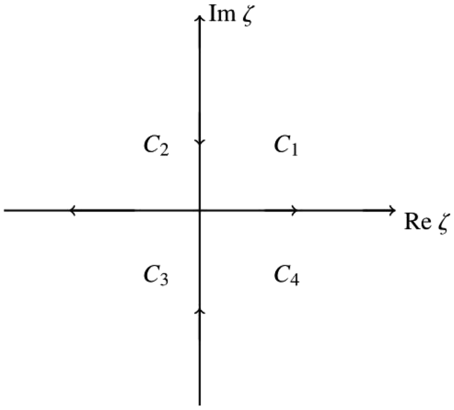

FIG. 1. Definition of the $\mathbb{C}_{13} = C_1 \cup C_3$ and $\mathbb{C}_{24} = C_2 \cup C_4$.

$$\det J = \lim_{|x|\to\infty} \det J = \det\left(\lim_{|x|\to\infty} J\right) = 1.$$

We take the determinant on both sides of relation (20) to get $\det S(\zeta) = 1$.

In order to construct the RHP, we consider the adjoint scattering equation of (15),

$$\Phi_x = -i\zeta^2[\sigma_3, \Phi] - \zeta \Phi Q, \tag{22}$$

and it is easy to see that J^{-1} is the solution of the adjoint equation (22) and satisfies the boundary condition $J^{-1} \to I$ as $x \to \pm\infty$, where the inverse matrices J^{-1} as a collection of rows,

$$(J_P)^{-1} = ((J_P^{-1})^{(1)}, (J_P^{-1})^{(2)})^T, \quad (J_M)^{-1} = ((J_M^{-1})^{(1)}, (J_M^{-1})^{(2)})^T. \tag{23}$$

Due to the structure (12) of the potential Q, we also have the following proposition:

Proposition II.2. According to the properties of the Jost solution, we can deduce that the inverse matrix J^{-1} has the following properties:

- *The row vectors $(J_P^{-1})^{(1)}$ and $(J_M^{-1})^{(2)}$ are analytic for $\zeta \in \mathbb{C}_{13}$ and continuous for $\zeta \in \mathbb{C}_{13} \cup \mathbb{R} \cup i\mathbb{R}$.*
- *The rows $(J_M^{-1})^{(1)}$ and $(J_P^{-1})^{(2)}$ are analytical for $\zeta \in \mathbb{C}_{24}$ and continuous for $\zeta \in \mathbb{C}_{24} \cup \mathbb{R} \cup i\mathbb{R}$.*

Furthermore, the analytical properties of the scattering data can be obtained as follows:

Proposition II.3. Suppose that $q(x,t) \in L^1(\mathbb{R})$, then s_{11} is analytic on $\mathbb{C}_{13}$, s_{22} is analytic on $\mathbb{C}_{24}$, and s_{12} and s_{22} are not analytic in $\mathbb{C}_{13}$ and $\mathbb{C}_{24}$ but are continuous to the real axis $\mathbb{R}$ and imaginary axis $i\mathbb{R}$.

Proof. The scattering matrix can be rewritten as

$$e^{-i\zeta^2 x\sigma_3} S(\zeta) e^{i\zeta^2 x\sigma_3} = J_P^{-1} J_M = \begin{pmatrix} (J_P^{-1})^{(1)} \\ (J_P^{-1})^{(2)} \end{pmatrix} \left(J_M^{(1)}, J_M^{(2)}\right) = \begin{pmatrix} (J_P^{-1})^{(1)} J_M^{(1)} & (J_P^{-1})^{(1)} J_M^{(2)} \\ (J_P^{-1})^{(2)} J_M^{(1)} & (J_P^{-1})^{(2)} J_M^{(2)} \end{pmatrix}, \quad \zeta \in \mathbb{R} \cup i\mathbb{R}. \tag{24}$$

The elements corresponding to the matrices on both sides can be written clearly as

$$s_{11} = \left(J_P^{-1}\right)^{(1)} J_M^{(1)}, \quad s_{12} = \left(J_P^{-1}\right)^{(1)} J_M^{(2)} e^{2i\zeta^2 x},$$

$$s_{21} = \left(J_P^{-1}\right)^{(2)} J_M^{(1)} e^{-2i\zeta^2 x}, \quad s_{12} = \left(J_P^{-1}\right)^{(2)} J_M^{(2)}.$$

According to Propositions 1 and 2, it is easy to know that s_{11} is analytic on $\mathbb{C}_{13}$, s_{22} is analytic on $\mathbb{C}_{24}$, and s_{12} and s_{22} are not analytic in $\mathbb{C}_{13}$ and $\mathbb{C}_{24}$ but are continuous to the real axis $\mathbb{R}$ and imaginary axis $i\mathbb{R}$. □

Hence, we can construct two matrix functions $\mathbf{P}(\zeta, x)$ that are analytic for $\zeta \in \mathbb{C}_{13} \cup \mathbb{C}_{24}$,

$$\mathbf{P}(\zeta, x) := \begin{cases} \left[(J_M^{(1)}(\zeta,x), J_P^{(2)}(\zeta,x)\right], & \zeta \in \mathbb{C}_{13}, \\ \left[(J_M^{-1})^{(1)}(\zeta,x), (J_P^{-1})^{(2)}(\zeta,x)\right], & \zeta \in \mathbb{C}_{24}, \end{cases} \tag{25}$$

and $\det\mathbf{P} = s_{11}$ when $\zeta \in \mathbb{C}_{13}$, $\det\mathbf{P} = \hat{s}_{11}$ and when $\zeta \in \mathbb{C}_{24}$. $\hat{s}_{11}$ is the first element of S^{-1}.

To find the boundary condition of $\mathbf{P}$, we consider the following asymptotic expansion as $\zeta \to 0$:

$$\mathbf{P} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)}\zeta + \mathbf{P}^{(2)}\zeta^2 + O(\zeta^3). \tag{26}$$

Substituting (26) into (15) and equating terms with like powers of ζ, we find

$$\mathbf{P}_x^{(0)} = 0.$$

It can be seen from (17) and (18) that

$$J|_{(\zeta=0)} = I,$$

so we have

$$\mathbf{P} \to I, \quad \zeta \to 0. \tag{27}$$

Then, the RHP of the higher-order KN equation is as follows:

Riemann–Hilbert Problem II.4. The matrix function $\mathbf{P}(\zeta; x)$ has the following properties:

- **Analyticity** : $\mathbf{P}(\zeta; x, t)$ is an analytic function in $\zeta \in \mathbb{C}_{13} \cup \mathbb{C}_{24}$.
- **Jump condition**:

$$\mathbf{P}_+(\zeta; x) = \mathbf{P}_-(\zeta; x) G(\zeta), \qquad \zeta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}. \tag{28}$$

- **Normalization** : $\mathbf{P}(\zeta; x) = I + O(\zeta)$ as $\zeta \to 0$.

Here,

$$G = E \begin{pmatrix} 1 & \hat{s}_{12} \\ s_{21} & 1 \end{pmatrix} E^{-1}. \tag{29}$$

Next, we consider the symmetric properties of Jost solutions and scattering data so that we can consider interesting reduction.

Proposition II.5. There are two symmetries of the Jost solutions and scattering data:

- *The first symmetry reduction:*

$$(J(x, \zeta^*))^\dagger = J^{-1}(x, \zeta), \tag{30}$$

$$(\mathbf{P}(\zeta^*))^\dagger = \mathbf{P}^{-1}(\zeta), \tag{31}$$

$$S^\dagger(\zeta^*) = S^{-1}(\zeta). \tag{32}$$

- *The second symmetry reduction:*

$$J(\zeta) = \sigma_3 J(-\zeta) \sigma_3, \tag{33}$$

$$\mathbf{P}(-\zeta) = \sigma_3 \mathbf{P}(\zeta) \sigma_3, \tag{34}$$

$$S(-\zeta) = \sigma_3 S(\zeta) \sigma_3. \tag{35}$$

Proof. For the first symmetric case, we replace ζ with ζ^* and then take the conjugate transpose of Eq. (15) to get

$$(J^\dagger(x, \zeta^*))_x = -i\zeta^2[\sigma_3, J^\dagger(x, \zeta^*)] + \zeta J^\dagger(x, \zeta^*) Q^\dagger \tag{36}$$

owing to $Q^\dagger = -Q$, so the above equation is

$$(J^\dagger(x, \zeta^*))_x = -i\zeta^2[\sigma_3, J^\dagger(x, \zeta^*)] - \zeta J^\dagger(x, \zeta^*) Q. \tag{37}$$

Comparing with Eq. (22), it is found that $J^{-1}(x, \zeta)$ and $J^\dagger(x, \zeta^*)$ satisfy the same equation form, and then according to the boundary conditions at $x \to \pm\infty$, we know that

$$(J(x, \zeta^*))^\dagger = J^{-1}(x, \zeta).$$

Note that the $\mathbf{P}$ we constructed are part of the Jost solutions and therefore have

$$(\mathbf{P}(\zeta^*))^\dagger = \mathbf{P}^{-1}(\zeta).$$

In addition, in view of the scattering relation (20) between J_M and J_P, we find that S also satisfies the involution property

$$S^\dagger(\zeta^*) = S^{-1}(\zeta).$$

For the second symmetry, replacing ζ with $-\zeta$ and both sides of the equation being multiplied by σ_3,

$$\sigma_3 J_x(-\zeta)\sigma_3 + \mathrm{i}\zeta^2[\sigma_3, \sigma_3 J(-\zeta)\sigma_3] = -\zeta\sigma_3 Q J(-\zeta)\sigma_3,$$

due to $\sigma_3 Q \sigma_3 = -Q$, the above equation can be reduced to

$$\sigma_3 J_x(-\zeta)\sigma_3 + \mathrm{i}\zeta^2[\sigma_3, \sigma_3 J(-\zeta)\sigma_3] = \zeta Q \sigma_3 J(-\zeta)\sigma_3.$$

It is easy to find that $J(\zeta)$ and $J(-\zeta)$ satisfy the same equation, so there is

$$J(\zeta) = \sigma_3 J(-\zeta)\sigma_3, \tag{38}$$

and it follows that

$$\mathbf{P}(-\zeta) = \sigma_3 \mathbf{P}(\zeta)\sigma_3 \tag{39}$$

and

$$S(-\zeta) = \sigma_3 S(\zeta)\sigma_3. \tag{40}$$

□

From Eqs. (32) and (35), we obtain the relations

$$s_{11}^*(\zeta^*) = \hat{s}_{11}(\zeta), \quad s_{21}^*(\zeta^*) = \hat{s}_{12}(\zeta), \quad s_{12}^*(\zeta^*) = \hat{s}_{21}(\zeta), \quad \zeta \in \mathbb{R} \cup i\mathbb{R} \tag{41}$$

and

$$s_{11}(\zeta) = s_{11}(-\zeta), \quad s_{22}(\zeta) = s_{22}(-\zeta), \quad s_{12}(-\zeta) = -s_{12}(\zeta), \quad s_{21}(-\zeta) = -s_{21}(\zeta), \quad \zeta \in \mathbb{R} \cup i\mathbb{R}. \tag{42}$$

Thus, $s_{11}(\lambda)$ is an even function, and each zero ζ_k of s_{11} is accompanied with zero $-\zeta_k$. Similarly, $\hat{s}_{11}$ has two zeros $\pm\zeta_k^*$.

A. Solvability of RHP

In general, if the det $\mathbf{P}(\zeta) \neq 0$ of the RHP, the RHP is considered to be regular, its solution is unique, and can be given by using the Plemelj formula.[29] However, more often than not they are non-regular, where det $\mathbf{P}(\zeta) = 0$, i. e., $s_{11}(\pm\zeta_k) = 0$ and $\hat{s}_{11}(\pm\bar{\zeta}_k) = 0$ at certain discrete locations, $\pm\zeta_k$ and $\pm\bar{\zeta}_k$ are called zeros. Here, we first consider the case of simple zeros $\{\pm\zeta_k \in \mathbb{C}_{13}, 1 \le k \le N\}$ and $\{\pm\bar{\zeta}_k \in \mathbb{C}_{24}, 1 \le k \le N\}$, where N is the number of these zeros. These zeros are known from relation (41),

$$s_{11}(\zeta_k) = \hat{s}_{11}^*(\zeta_k^*) = 0, \quad \hat{s}_{11}(\bar{\zeta}_k) = 0,$$

so

$$\bar{\zeta}_k = \zeta_k^*. \tag{43}$$

In this case, both $\ker(\mathbf{P}(\pm\zeta_k))$ are one-dimensional and spanned by single column vector $|v_k\rangle$ and single row vector $\langle v_k|$, respectively, and thus,

$$\mathbf{P}(\zeta_k)|v_k\rangle = 0, \quad \langle v_k|\mathbf{P}(\zeta_k^*) = 0, \quad \zeta_k \in \mathbb{C}_{13}, \quad 1 \le k \le N. \tag{44}$$

By the symmetry relation (31), it is easy to get

$$|v_k\rangle = \langle v_k|^{\dagger}. \tag{45}$$

Regarding this non-regular RHP (28) under the canonical normalization condition, its solution is also unique. Here, we mainly use the dressing method to turn the non-regular problem into the regular problem.[30] Next, we construct a matrix function $\Gamma(x,t,\zeta)$ that could cancel all the zeros of $\mathbf{P}$. From relations (41) and (42), we should construct a matrix Γ_k whose determinant is

$$\det\Gamma_k(\zeta) = \frac{\zeta^2 - \zeta_k^2}{\zeta^2 - \zeta_k^{*2}}. \tag{46}$$

From the above properties [(31), (34) and (46)], we could construct the form for the matrix

$$\Gamma_k(\zeta) = I + \frac{A_k}{\zeta - \zeta_k^*} - \frac{\sigma_3 A_k \sigma_3}{\zeta + \zeta_k^*}, \quad \Gamma_k^{-1}(\zeta) = I + \frac{A_k^{\dagger}}{\zeta - \zeta_k} - \frac{\sigma_3 A_k^{\dagger}\sigma_3}{\zeta + \zeta_k}, \quad \zeta_k \in \mathbb{C}_{13}, \quad k = 1,2,\ldots,N, \tag{47}$$

where

$$A_k = \frac{\zeta_k^{*2} - \zeta_k^2}{2}\begin{pmatrix} \alpha_k^* & 0 \\ 0 & \alpha_k \end{pmatrix}|w_k\rangle\langle w_k|, \quad \alpha_k^{-1} = \langle w_k|\begin{pmatrix} \zeta_k & 0 \\ 0 & \zeta_k^* \end{pmatrix}|w_k\rangle, \tag{48}$$

$$|w_k\rangle = \Gamma_{k-1}(\zeta_k)\ldots\Gamma_1(\zeta_k)|v_k\rangle, \quad \langle w_k| = |w_k\rangle^{\dagger}, \tag{49}$$

and then $\det \mathbf{P}\Gamma_k^{-1} \neq 0$ at points $\pm\zeta_k$ and $\det\Gamma_k^{-1}\mathbf{P} \neq 0$ at points $\pm\zeta_k^*$. Introducing

$$\Gamma(\zeta) = \Gamma_N(\zeta)\Gamma_{N-1}(\zeta)\ldots\Gamma_1(\zeta), \tag{50}$$

$$\Gamma^{-1}(\zeta) = \Gamma_1^{-1}(\zeta)\Gamma_2^{-1}(\zeta)\ldots\Gamma_N^{-1}(\zeta), \tag{51}$$

the analytic solutions may be represented as

$$\mathbf{P} = \tilde{\mathbf{P}}\Gamma. \tag{52}$$

Then, the RHP of the higher-order KN equation without singularity is as follows:

Riemann–Hilbert Problem II.6. The matrix function $\tilde{\mathbf{P}}(\zeta;x)$ has the following properties:

- **Analyticity**: $\tilde{\mathbf{P}}(\zeta;x,t)$ is the analytic function in $\zeta \in \mathbb{C}_{13} \cup \mathbb{C}_{24}$.
- **Jump condition**:

$$\tilde{\mathbf{P}}_+(\zeta;x) = \tilde{\mathbf{P}}_-(\zeta;x)\Gamma G\Gamma^{-1}(\zeta), \qquad \zeta \in \mathbb{R}\cup i\mathbb{R}. \tag{53}$$

- **Asymptotic behaviors**: $\tilde{\mathbf{P}}(\zeta;x) = \tilde{\mathbf{P}}_0 + O(\zeta), \quad \text{as} \quad \zeta \to 0.$

The form of G has been given by Eq. (29). From Eq. (52), we have

$$\tilde{\mathbf{P}}_0 = \left(\Gamma|_{\zeta=0}\right)^{-1}. \tag{54}$$

B. Scattering data evolution

From the solution of RHP, it can be seen that the minimum scattering data needed for solving RHP and reconstructing potential is

$$\{s_{21}, \hat{s}_{12}, \zeta \in \mathbb{R}\cup i\mathbb{R}; \pm\zeta_k, \pm\zeta_k^*, |v_k\rangle, \langle v_k|, 1 \le k \le N\}.$$

Since J satisfies the temporal equation (16) of the Lax pair and relation (20), then according to the evolution property (20) and $Q \to 0, V \to 0$ as $|x| \to \infty$, we have

$$S_t + 2i\zeta^6[\sigma_3, S] = 0,$$

and the evolutions of the entries of the scattering matrix S satisfy

$$s_{11,t} = s_{22,t} = 0, \quad s_{12}(t;\zeta) = s_{12}(0;\zeta)\exp\left(-4i\zeta^6 t\right), \quad s_{21}(t;\zeta) = s_{21}(0;\zeta)\exp\left(4i\zeta^6 t\right). \tag{55}$$

Differentiating both sides of the first equation of (44) with respect to x and t and recalling the Lax (15) and (16), we have

$$\mathbf{P}(\zeta_k;x)\left(\frac{d|v_k\rangle}{dx} + i\zeta^2\sigma_3|v_k\rangle\right) = 0, \quad \mathbf{P}(\zeta_k;x)\left(\frac{d|v_k\rangle}{dt} + 2i\zeta^6\sigma_3|v_k\rangle\right) = 0, \quad \zeta_k \in \mathbb{C}_{13}.$$

It concludes that

$$|v_k\rangle = e^{-i\zeta_k^2\sigma_3 x - 2i\zeta_k^6\sigma_3 t}|v_{k0}\rangle,$$

where $v_{k0} = v_k|_{x=0}$.

III. *N* SOLITON SOLUTIONS

In this part, we mainly obtain the potential q. The expansion of $\mathbf{P}(\zeta)$ with $\zeta \to 0$ is

$$\mathbf{P}(\zeta) = I + \mathbf{P}^{(1)}\zeta + \mathbf{P}^{(2)}\zeta^2 + O\left(\zeta^2\right). \tag{56}$$

Substituting the expansion into Eq. (15), the potential matrix function can be obtained by comparing the coefficients,

$$Q = \mathbf{P}_x^{(1)}, \tag{57}$$

from this formula, and we can get the potential $q(x,t)$. As we all know, the soliton solutions correspond to the disappearance of the scattering coefficient, $G = I, \hat{G} = 0$. Therefore, we intend to solve the corresponding RHP (53).

According to Eqs. (52) and (54), we can consider the following expansion form:

$$\mathbf{P}(x,t;\zeta) = (\Gamma|_{\zeta=0})^{-1}(\Gamma|_{\zeta=0} + \Gamma^{(1)}(x,t)\zeta + O(\zeta)), \tag{58}$$

which gives $\mathbf{P}^{(1)} = (\Gamma|_{\zeta=0})^{-1}\Gamma^{(1)}(x,t)$. Below, the main effort is to find an explicit expression for $(\Gamma|_{\zeta=0})^{-1}\Gamma^{1}(x,t)$. In fact, the form of Γ from Eqs. (50) and (51) have a more compact form

$$\Gamma(\zeta) = I + \sum_{k=1}^{N}\left[\frac{B_k}{\zeta - \zeta_k^*} - \frac{\sigma_3 B_k \sigma_3}{\zeta + \zeta_k^*}\right] \tag{59}$$

and

$$\Gamma^{-1}(\zeta) = I + \sum_{k=1}^{N}\left[\frac{B_k^\dagger}{\zeta - \zeta_k} - \frac{\sigma_3 B_k^\dagger \sigma_3}{\zeta + \zeta_k}\right], \tag{60}$$

with $B_k = |z_k\rangle\langle v_k|$. To determine the form of matrix B_k, we consider $\Gamma(\zeta)\Gamma^{-1}(\zeta) = I$, we have

$$\text{Res}_{\zeta=\zeta_j}\Gamma(\zeta)\Gamma^{-1}(\zeta) = \Gamma(\zeta_j)B_j^\dagger = 0,$$

and it yields

$$\left[I + \sum_{k=1}^{N}\left(\frac{|z_k\rangle\langle v_k|}{\zeta_j - \zeta_k^*} - \frac{\sigma_3|z_k\rangle\langle v_k|\sigma_3}{\zeta_j + \zeta_k^*}\right)\right]|v_j\rangle = 0, \quad j = 1, 2, \ldots, N, \tag{61}$$

it is easy to figure out

$$|z_k\rangle_1 = \sum_{j=1}^{N}(M^{-1})_{jk}|v_j\rangle_1, \tag{62}$$

where $|z_k\rangle_l$ denotes the lth element of $|z_k\rangle$ and matrix M is defined as

$$M_{jk} = \frac{\langle v_k|\sigma_3|v_j\rangle}{\zeta_j + \zeta_k^*} - \frac{\langle v_k|v_j\rangle}{\zeta_j - \zeta_k^*}. \tag{63}$$

Then, we have

$$(\Gamma|_{\zeta=0}) = I - \sum_{j=1}^{N}\left[\frac{B_j + \sigma_3 B_j \sigma_3}{\zeta_j^*}\right],$$

$$\Gamma^{(1)}(x,t) = -\sum_{j=1}^{N}\frac{B_j - \sigma_3 B_j \sigma_3}{\zeta_j^{*2}}.$$

These equations enable us to have

$$\mathbf{P}^{(1)} = (\Gamma|_{\zeta=0})^{-1}\Gamma^{(1)}(x,t) = \left(I - \sum_{j=1}^{N}\left[\frac{B_j + \sigma_3 B_j\sigma_3}{\zeta_j^*}\right]\right)^{-1}\sum_{j=1}^{N}\frac{\sigma_3 B_j \sigma_3 - B_j}{\zeta_j^{*2}},$$

and by Eq. (57), we can obtain that the potential function $q(x,t)$ is

$$q(x,t) = \left(\left(1 - \sum_{j,k=1}^{N}\frac{2(M^{-1})_{jk}|v_k\rangle_1\langle v_j|_1}{\zeta_j^*}\right)^{-1}\left(\sum_{j,k=1}^{N}\frac{-2(M^{-1})_{jk}|v_k\rangle_1\langle v_j|_2}{\zeta_j^{*2}}\right)\right)_x, \tag{64}$$

where M has been given by Eq. (63). Note that M^{-1} can be expressed as the transpose of M's cofactor matrix divided by $\det M$. Hence, the solution (64) can be rewritten as

$$q(x,t) = \left(\frac{2\frac{\det F}{\det M}}{1 + 2\frac{\det G}{\det M}}\right)_x = \left(\frac{2\det F}{\det M + 2\det G}\right)_x, \tag{65}$$

where

$$F = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} & |v_1\rangle_1 \\ M_{21} & M_{22} & \cdots & M_{2n} & |v_2\rangle_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} & |v_n\rangle_1 \\ \frac{\langle v_1|_2}{\zeta_1{}^{*2}} & \frac{\langle v_2|_2}{\zeta_2{}^{*2}} & \cdots & \frac{\langle v_n|_2}{\zeta_n{}^{*2}} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} & |v_1\rangle_1 \\ M_{21} & M_{22} & \cdots & M_{2n} & |v_2\rangle_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} & |v_n\rangle_1 \\ \frac{\langle v_1|_1}{\zeta_1{}^{*}} & \frac{\langle v_2|_1}{\zeta_2{}^{*}} & \cdots & \frac{\langle v_n|_1}{\zeta_n{}^{*}} & 0 \end{pmatrix}.$$

To get the explicit N-soliton solutions, we may take $v_{k0} = (a_k, b_k)^T$, and then

$$|v_k\rangle = \begin{pmatrix} a_k e^{\theta_k} \\ b_k e^{-\theta_k}\end{pmatrix}, \qquad \langle v_k| = (a_k^* e^{\theta_k^*} \quad b_k^* e^{-\theta_k^*}),$$

where $\theta_k = -i\zeta_k^2 x - 2i\zeta_k^6 t$.

In what follows, we will take single soliton and two-soliton solution as examples to study the properties of solitons in more detail. For convenience, let $\zeta_j = \zeta_{jR} + i\zeta_{jI}$,

$$\theta_{jR} = 2m_j(x - (8m_j^2 - 6\beta_j^2)t), \qquad \theta_{jI} = -\beta_j x - 2(\beta_j^3 - 12m_j^2\beta_j)t,$$
$$m_j = \zeta_{jR}\zeta_{jI}, \quad \beta_j = \zeta_{jR}^2 - \zeta_{jI}^2,$$

where ζ_{jR}, ζ_{jI} are the real and imaginary parts of ζ_j.

A. Single-soliton solution

For $N = 1$, taking the discrete spectrum point $\pm\zeta_1$ and $\pm\zeta_1^*$, then using formula (64) to directly calculate, it can be seen that

$$q(x,t) = \frac{(\zeta_1^2 - \zeta_1^{*2})}{|\zeta_1|^2}\left(\frac{a_1 b_1^* e^{\theta_1-\theta_1^*}}{\zeta_1^*|b_1|^2 e^{-(\theta_1+\theta_1^*)} + \zeta_1|a_1|^2 e^{\theta_1+\theta_1^*}}\right)_x \tag{66}$$

or equal to

$$q(x,t) = 8a_1 b_1^* \zeta_{1R}\zeta_{1I}\frac{\zeta_1|b_1|^2 e^{-2\theta_{1R}} + \zeta_1^*|a_1|^2 e^{2\theta_{1R}}}{(\zeta_1^*|b_1|^2 e^{-2\theta_{1R}} + \zeta_1|a_1|^2 e^{2\theta_{1R}})^2} e^{2i\theta_{1I}}. \tag{67}$$

The velocity for the single soliton is $v = 8\zeta_{1R}^2\zeta_{1I}^2 - 6(\zeta_{1R}^2 - \zeta_{1I}^2)^2$. The center position for $|q|$ locates on the line

$$x - vt - \frac{1}{4\zeta_{1R}^2\zeta_{1I}^2}\ln\frac{|b_1|}{|a_1|} = 0.$$

The amplitudes associated with $|q|^2$ are given by

$$A(q) = \frac{64\zeta_{1R}^2\zeta_{1I}^2}{2|\zeta_1^2| + \zeta_1^2 + \zeta_1^{*2}}.$$

In Fig. 2(a), we give the 3D graph of the single-soliton solution.

B. Two-soliton solution

When $N = 2$, the two-soliton solution of the higher-order KN equation has the form of $q(x,t) = \Delta_1/\Delta_0$ with

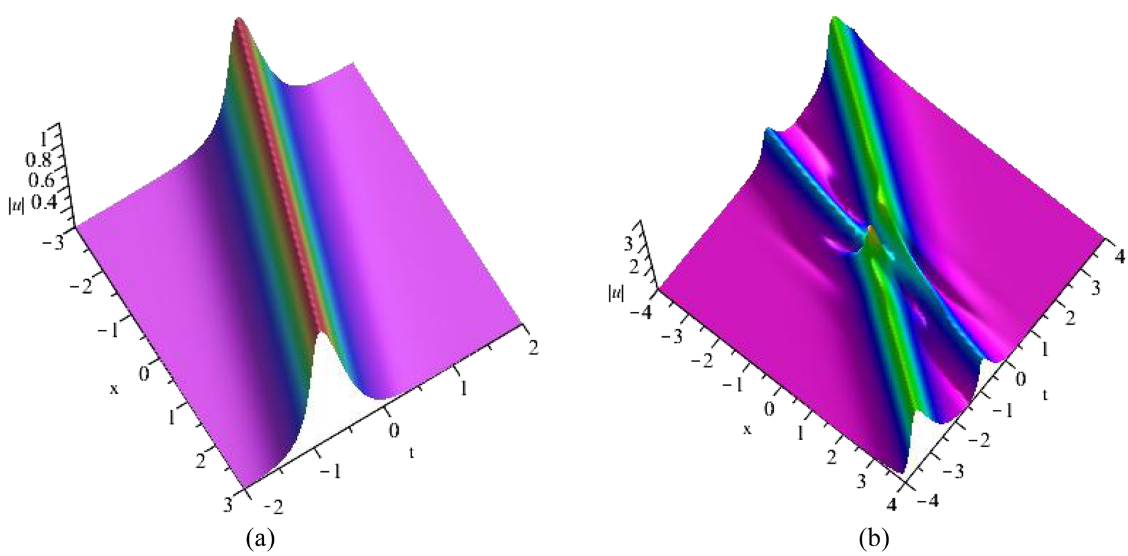

FIG. 2. Soliton solutions for $|q|$. (a) Single-soliton solution in the three-dimensional plot, where $\zeta_1 = 1 + 0.25i, a_1 = 1, b_1 = 0.1 + 0.7i$. (b) Two-soliton solution in the three-dimensional plot, where $\zeta_1 = 1 + 0.25i, a_1 = 1, b_1 = 0.1 + 0.7i, \zeta_2 = 1 + 0.5i, a_2 = 1, b_2 = -0.1 + 0.7i$. Single-soliton solution for $|u|$, where $\xi_1 = 1, \eta_1 = 0.5, c_1 = 1$.

$$
\begin{aligned}
\Delta_1 = {} & \delta_1 e^{-2\theta_{1R}+2i\theta_{1I}-4\theta_{2R}} + \delta_2 e^{2\theta_{1R}+2i\theta_{1I}} + \delta_3 e^{2\theta_{1R}+2i\theta_{1I}-4\theta_{2R}} + \delta_4 e^{-2\theta_{1R}+2i\theta_{1I}} + \delta_5 e^{-2\theta_{2R}+2i\theta_{2I}} + \delta_6 e^{4i\theta_{1I}-2\theta_{2R}-2i\theta_{2I}} \\
& + \delta_7 e^{4\theta_{1R}+2\theta_{2R}+2i\theta_{2I}} + \delta_8 e^{4\theta_{1R}-2\theta_{2R}+2i\theta_{2I}} + \delta_9 e^{2\theta_{2R}+2i\theta_{2I}} + \delta_{10} e^{2\theta_{1R}-2i\theta_{1I}+4i\theta_{2I}} + \delta_{11} e^{2\theta_{1R}+2i\theta_{1I}+4\theta_{2R}} + \delta_{12} e^{-2\theta_{1R}+2i\theta_{1I}+4\theta_{2R}} \\
& + \delta_{13} e^{4i\theta_{1I}+2\theta_{2R}-2i\theta_{2I}} + \delta_{14} e^{-4\theta_{1R}-2\theta_{2R}+2i\theta_{2I}} + \delta_{15} e^{-4\theta_{1R}+2\theta_{2R}+2i\theta_{2I}} + \delta_{16} e^{-2\theta_{1R}-2i\theta_{1I}+4i\theta_{2I}},
\end{aligned}
$$

$$
\begin{aligned}
\Delta_0 = {} & \rho_0 + \rho_1 e^{-4\theta_{1R}-4\theta_{2R}} + \rho_2 e^{-4\theta_{1R}} + \rho_3 e^{-4\theta_{2R}} + \rho_4 e^{-2\theta_{1R}-2i\theta_{1I}-2\theta_{2R}+2i\theta_{2I}} + \rho_5 e^{-2\theta_{1R}+2i\theta_{1I}-2\theta_{2R}-2i\theta_{2I}} + \rho_6 e^{4\theta_{1R}+4\theta_{2R}} \\
& + \rho_7 e^{4\theta_{1R}} + \rho_8 e^{4\theta_{2R}} + \rho_9 e^{2\theta_{1R}-2i\theta_{1I}-2\theta_{2R}+2i\theta_{2I}} + \rho_{10} e^{2\theta_{1R}+2i\theta_{1I}+2\theta_{2R}-2i\theta_{2I}} + \rho_{11} e^{4\theta_{1R}-4\theta_{2R}} + \rho_{12} e^{2\theta_{1R}-2i\theta_{1I}+2\theta_{2R}+2i\theta_{2I}} \\
& + \rho_{13} e^{2\theta_{1R}+2i\theta_{1I}-2\theta_{2R}-2i\theta_{2I}} + \rho_{14} e^{-4\theta_{1R}+4\theta_{2R}} + \rho_{15} e^{-2\theta_{1R}-2i\theta_{1I}+2\theta_{2R}+2i\theta_{2I}} + \rho_{16} e^{-2\theta_{1R}+2i\theta_{1I}+2\theta_{2R}-2i\theta_{2I}} + \rho_{17} e^{-4i\theta_{1I}+4i\theta_{2I}} \\
& + \rho_{18} e^{4i\theta_{1I}-4i\theta_{2I}}.
\end{aligned}
$$

The coefficients of these exponential terms constituted of $a_1, a_1^*, a_2, a_2^*, b_1, b_1^*, b_2, b_2^*$ and $\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*$. However, it is tedious to write them all out, and they can be calculated directly via the computer. Instead of presenting the complex expression, we show the typical solution behavior in Fig. 2(b). It can be seen from Fig. 2(b) that when $t \to -\infty$, the solution consists of two single solitons that are far apart and travel opposite each other. When they collide together, the interaction weakens. When $t \to +\infty$, these are separated into two single solitons, and there is no change in shape and velocity and no energy radiating to the far field. Therefore, the interaction of these solitons is elastic. However, it can be observed from the graph that after the interaction, each soliton has a phase shift and a position shift.

Next, we verify the rationality of the above analysis through the expression of the soliton solution. In general, we make the assumption $\xi_i\eta_i > 0$ and $v_1 < v_2$. This means that at $t \to -\infty$, soliton-2 is on the left side of soliton-1 and moves faster, and the two solitons are in the moving frame with velocity $v_i = 8\zeta_{iR}^2\zeta_{iI}^2 - 6(\zeta_{iR}^2 - \zeta_{iI}^2)^2$; noting that $\theta_{1R} = 2m_1(x - v_1 t), \theta_{2R} = 2m_2(x - v_2 t)$, it yields

$$m_2\theta_{1R} - m_1\theta_{2R} = 2m_1m_2(v_2 - v_1)t.$$

Next, we study the collision dynamics of the two soliton solutions by using the asymptotic analysis technique of Ref. 31. Under different asymptotic states of θ_{1R} and θ_{2R}, the asymptotic expression of $q(x,t)$ is obtained.

(i) Before collision (as $t \to -\infty$).
  (a) If $|\theta_{1R}| < \infty$, then $\theta_{2R} \to \infty$:

$$q(x,t) \sim 8\tilde{a}_1^M\tilde{b}_1^{*M}\zeta_{1R}\zeta_{1I}\frac{\zeta_1|\tilde{b}_1^M|^2e^{-2\theta_{1R}} + \zeta_1^*|\tilde{a}_1^M|^2e^{2\theta_{1R}}}{(\zeta_1^*|\tilde{b}_1^M|^2e^{-2\theta_{1R}} + \zeta_1|\tilde{a}_1^M|^2e^{2\theta_{1R}})^2}e^{2i\theta_{1I}}, \tag{68}$$

  where $\tilde{a}_1^M = a_1(\zeta_2^2 - \zeta_1^2), \tilde{b}_1^M = b_1(\zeta_2^{*2} - \zeta_1^2)$.
  (b) If $|\theta_{2R}| < \infty$, then $\theta_{1R} \to -\infty$:

$$q(x,t) \sim 8\tilde{a}_2^M\tilde{b}_2^{*M}\zeta_{2R}\zeta_{2I}\frac{\zeta_2|\tilde{b}_2^M|^2e^{-2\theta_{2R}} + \zeta_2^*|\tilde{a}_2^M|^2e^{2\theta_{2R}}}{(\zeta_2^*|\tilde{b}_2^M|^2e^{-2\theta_{2R}} + \zeta_2|\tilde{a}_2^M|^2e^{2\theta_{2R}})^2}e^{2i\theta_{2I}}, \tag{69}$$

  where $\tilde{a}_2^M = a_2(\zeta_2^2 - \zeta_1^{*2}), \tilde{b}_2^M = b_2(\zeta_2^2 - \zeta_1^2)$.
(ii) After collision (as $t \to \infty$).
  (a) If $|\theta_{1R}| < \infty$, then $\theta_{2R} \to -\infty$:

$$q(x,t) \sim 8\tilde{a}_1^P\tilde{b}_1^{*P}\zeta_{1R}\zeta_{1I}\frac{\zeta_1|\tilde{b}_1^P|^2e^{-2\theta_{1R}} + \zeta_1^*|\tilde{a}_1^P|^2e^{2\theta_{1R}}}{(\zeta_1^*|\tilde{b}_1^P|^2e^{-2\theta_{1R}} + \zeta_1|\tilde{a}_1^P|^2e^{2\theta_{1R}})^2}e^{2i\theta_{1I}}, \tag{70}$$

  where $\tilde{a}_1^P = a_1(\zeta_2^{*2} - \zeta_1^2), \tilde{b}_1^P = b_1(\zeta_2^2 - \zeta_1^2)$.
  (b) If $|\theta_{2R}| < \infty$, then $\theta_{1R} \to \infty$:

$$q(x,t) \sim 8\tilde{a}_2^P\tilde{b}_2^{*P}\zeta_{2R}\zeta_{2I}\frac{\zeta_2|\tilde{b}_2^P|^2e^{-2\theta_{2R}} + \zeta_2^*|\tilde{a}_2^P|^2e^{2\theta_{2R}}}{(\zeta_2^*|\tilde{b}_2^P|^2e^{-2\theta_{2R}} + \zeta_1|\tilde{a}_2^P|^2e^{2\theta_{2R}})^2}e^{2i\theta_{2I}}, \tag{71}$$

  where $\tilde{a}_2^P = a_2(\zeta_2^2 - \zeta_1^2), \tilde{b}_2^P = b_2(\zeta_2^2 - \zeta_1^{*2})$.

It is pointed out that the asymptotic solutions can also be written as the function of solitary waves, and the respective velocities are v_1 and v_2, which remain unchanged before and after the collision. This elastic interaction is a remarkable property, which shows that the higher-order KN equation (6) is integrable. From the above asymptotic solutions, we can get the phase difference of soliton-1 solution,

$$\Delta\theta_{01} = \frac{1}{2}\left(\ln\frac{|\tilde{b}_1^P|}{|\tilde{a}_1^P|} - \ln\frac{|\tilde{b}_1^M|}{|\tilde{a}_1^M|}\right) = \ln\left|\frac{\zeta_2^2-\zeta_1^2}{\zeta_2^{*2}-\zeta_1^2}\right|.$$

Following similar calculations, we can get the phase difference of soliton-2 solution,

$$\Delta\theta_{02} = \frac{1}{2}\left(\ln\frac{|\tilde{b}_2^P|}{|\tilde{a}_2^P|} - \ln\frac{|\tilde{b}_2^M|}{|\tilde{a}_2^M|}\right) = \ln\left|\frac{\zeta_2^2-\zeta_1^{*2}}{\zeta_2^2-\zeta_1^2}\right| = -\Delta\theta_{01}.$$

IV. SOLITON MATRIX FOR MULTIPLE ZEROS

In this section, we will further consider the case of multiple zeros, where the multiplicity of $\{\pm\zeta_i, \pm\zeta_i^*\}$ is greater than 1, and then the determinant of $\mathbf{P}$ can be written in the following form:

$$\det\mathbf{P}(\zeta) = \left(\zeta^2-\zeta_1^2\right)^{n_1}\left(\zeta^2-\zeta_2^2\right)^{n_2}\ldots\left(\zeta^2-\zeta_r^2\right)^{n_r}\rho(\zeta), \quad \zeta_i\in\mathbb{C}_{13},$$

$$\det\mathbf{P}(\zeta) = \left(\zeta^2-\zeta_1^{*2}\right)^{n_1}\left(\zeta^2-\zeta_2^{*2}\right)^{n_2}\ldots\left(\zeta^2-\zeta_r^{*2}\right)^{n_r}\hat{\rho}(\zeta), \quad \zeta_i^*\in\mathbb{C}_{24},$$

where $\rho(\zeta_i)\neq 0 (i = 1, \ldots, r)$ for all $\zeta\in\mathbb{C}_{13}$ and $\hat{\rho}(\zeta_i)\neq 0\ (i=1,\ \ldots,\ r)$ for all $\zeta\in\mathbb{C}_{24}$.

Compared with the case of simple zeros, the number of kernel functions with multiple zeros is related to the multiplicity of zeros. For example, for the discrete spectral point $\{\zeta_1, \zeta_1^*\}$, its kernel function is

$$\mathbf{P}(\zeta_1)|v_j\rangle = 0, \quad \langle v_j|\mathbf{P}\left(\zeta_1^*\right) = 0, \quad \zeta_1\in\mathbb{C}_{13}, \quad 1\le j\le n_1, \tag{72}$$

$|v_j\rangle$ is linearly independent. For the case of multiple zeros, the corresponding Γ and Γ^{-1} can be given by using the following theorem:

Theorem IV.1 *(Ref. 32, Lemma 3). Consider a pair of higher order zeros of order $n_j(j = 1, \ldots, r)$: $\{\zeta_j, -\zeta_j\}$ in $\mathbb{C}_{13}$ and $\{\zeta_j^*, -\zeta_j^*\}$ in $\mathbb{C}_{24}$. Then, the corresponding soliton matrix $\Gamma_j(\zeta)$ and its inverse can be cast in the following form:*

$$\begin{aligned}\Gamma_j^{-1}(\zeta) &= I + \left(|\phi_{j,1}\rangle, \ldots, |\tilde{\phi}_{j,n_j}\rangle\right)\Xi_j(\zeta)\begin{pmatrix}\langle\tilde{\varphi}_{j,n_j}|\\ \vdots\\ \langle\varphi_{j,1}|\end{pmatrix},\\ \Gamma_j(\zeta) &= I + \left(|\bar{\varphi}_{j,n_j}\rangle, \ldots, |\bar{\tilde{\varphi}}_{j,1}\rangle\right)\overline{\Xi}_j(\zeta)\begin{pmatrix}\langle\bar{\phi}_{j,1}|\\ \vdots\\ \langle\bar{\tilde{\phi}}_{j,n_j}|\end{pmatrix},\end{aligned} \tag{73}$$

where the matrices $\Xi_j(\zeta)$ and $\overline{\Xi}_j(\zeta)$ are defined as

$$\Xi_j(\zeta) = \begin{pmatrix}\mathcal{D}^+(\zeta-\zeta_j) & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathcal{D}^+(\zeta+\zeta_j)\end{pmatrix}, \quad \overline{\Xi}_j(\zeta) = \begin{pmatrix}\mathcal{D}^-\left(\zeta-\zeta_j^*\right) & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathcal{D}^-\left(\zeta+\zeta_j^*\right)\end{pmatrix},$$

$\mathcal{D}^+(\gamma), \mathcal{D}^-(\gamma)$ are upper-triangular and lower-triangular Toeplitz matrices defined as

$$\mathcal{D}^+(\gamma) = \begin{pmatrix}\gamma^{-1} & \gamma^{-2} & \cdots & \gamma^{-n}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \gamma^{-1} & \gamma^{-2}\\ 0 & \cdots & 0 & \gamma^{-1}\end{pmatrix}, \qquad \mathcal{D}^-(\gamma) = \begin{pmatrix}\gamma^{-1} & 0 & \cdots & 0\\ \gamma^{-2} & \gamma^{-1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ \gamma^{-n} & \cdots & \gamma^{-2} & \gamma^{-1}\end{pmatrix},$$

and vectors $|\phi_{j,i}\rangle, |\tilde{\phi}_{j,i}\rangle, \langle\bar{\tilde{\phi}}_{j,i}|, \langle\varphi_{j,i}|, \langle\tilde{\varphi}_{j,i}|, |\bar{\tilde{\varphi}}_{j,i}\rangle (i = 1,\ \ldots,\ n_r)$ are independent of ζ.

Hence,

$$\Gamma(\zeta) = \Gamma_r(\zeta)\Gamma_{r-1}(\zeta)\ldots\Gamma_1(\zeta), \tag{74}$$

$$\Gamma^{-1}(\zeta) = \Gamma_1^{-1}(\zeta)\Gamma_2^{-1}(\zeta)\ldots\Gamma_r^{-1}(\zeta). \tag{75}$$

The rest of the vector parameters in (73) can be derived by calculating the residue of each order in the identity $\tilde{\Gamma}(\zeta)\tilde{\Gamma}^{-1}(\zeta) = I$ at $\zeta = \zeta_j$ and $\zeta = -\zeta_j$,

$$\tilde{\Gamma}(\zeta_j)\begin{pmatrix} |\phi_{j,1}\rangle \\ \vdots \\ |\phi_{j,n_r}\rangle \end{pmatrix} = 0, \quad \tilde{\Gamma}(-\zeta_j)\begin{pmatrix} |\tilde{\phi}_{j,1}\rangle \\ \vdots \\ |\tilde{\phi}_{j,n_r}\rangle \end{pmatrix} = 0, \tag{76}$$

where

$$\tilde{\Gamma}(\zeta) = \begin{pmatrix} \Gamma(\zeta) & 0 & \cdots & 0 \\ \frac{d}{d\zeta}\Gamma(\zeta) & \Gamma(\zeta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(n_r-1)!}\frac{d^{n_r-1}}{d\zeta^{n_r-1}}\Gamma(\zeta) & \cdots & \frac{d}{d\zeta}\Gamma(\zeta) & \Gamma(\zeta) \end{pmatrix}. \tag{77}$$

Using this method, the process of solving the soliton solution is very complex. Next, the corresponding Γ can be constructed by using the method of Ref. 33, and the dressing matrix of multiple zeros is derived by the unipolar limit method. The results are given by the following theorem:

Theorem IV.2. *Suppose $\zeta = \zeta_i$ is the zero of geometric multiplicity $n_j(j = 1, \ldots, r)$ and $\sum_{j=1}^{r} n_j = N$; then, the modified matrix can be expressed as*

$$\Gamma(\zeta) = \Gamma_r^{[n_r-1]}\ldots\Gamma_r^{[0]}\ldots\Gamma_1^{[n_1-1]}\ldots\Gamma_1^{[0]}, \quad \Gamma^{-1}(\zeta) = (\Gamma_1^{[0]})^{-1}\ldots(\Gamma_1^{[n_1-1]})^{-1}\ldots(\Gamma_r^{[0]})^{-1}\ldots(\Gamma_r^{[n_r-1]})^{-1},$$

where

$$\Gamma_i^{[j]}(\zeta) = I + \frac{A_i^{[j]}}{\zeta-\zeta_i^*} - \frac{\sigma_3 A_i^{[j]}\sigma_3}{\zeta+\zeta_i^*}, \quad (\Gamma_i^{[j]})^{-1}(\zeta) = I + \frac{A_i^{\dagger[j]}}{\zeta-\zeta_i} - \frac{\sigma_3 A_i^{\dagger[j]}\sigma_3}{\zeta+\zeta_i},$$

$$A_i^{[j]} = \frac{\zeta_i^{*2}-\zeta_i^2}{2}\begin{pmatrix} \alpha_i^{*[j]} & 0 \\ 0 & \alpha_i^{[j]} \end{pmatrix}|v_i^{[j]}\rangle\langle v_i^{[j]}|, \quad (\alpha_i^{[j]})^{-1} = \langle v_i^{[j]}|\begin{pmatrix} \zeta_i & 0 \\ 0 & \zeta_i^* \end{pmatrix}|v_i^{[j]}\rangle,$$

$$|v_i^{[j]}\rangle = \lim_{\delta\to 0}\frac{(\Gamma_i^{[n_j-1]}\ldots\Gamma_i^{[0]}\ldots\Gamma_1^{[n_1-1]}\ldots\Gamma_1^{[0]})|_{\zeta=\zeta_i+\delta}}{\delta^j}|v_i\rangle(\zeta_i+\delta),$$

$$\langle v_i^{[j]}| = \lim_{\delta\to 0}\langle v_1|(\zeta_i^*+\delta)\frac{(\Gamma_1^{[0]-1}\ldots\Gamma_1^{[n_1-1]-1}\ldots\Gamma_i^{[0]-1}\ldots\Gamma_i^{[n_j-1]-1})|_{\zeta=\zeta_i^*+\delta}}{\delta^j}.$$

Then, we can get

$$P^{(1)} = \left(I - \sum_{i=1}^{r}\sum_{j=0}^{n_i-1}\left[\frac{B_i^{[j]}-\sigma_3 B_i^{[j]}\sigma_3}{\zeta_i^*}\right]\right)^{-1}\sum_{i=1}^{r}\sum_{j=0}^{n_i-1}\frac{\sigma_3 B_i^{[j]}\sigma_3 - B_i^{[j]}}{\zeta_i^{*2}},$$

which leads to

$$q(x,t)=\left(\frac{2\frac{\det\tilde{F}}{\det\tilde{M}}}{1+2\frac{\det\tilde{G}}{\det\tilde{M}}}\right)_x=\left(\frac{2\det\tilde{F}}{\det\tilde{M}+2\det\tilde{G}}\right)_x, \tag{78}$$

where

$$\tilde{F}=\begin{pmatrix}\tilde{M}^{[11]} & \tilde{M}^{[12]} & \cdots & \tilde{M}^{[1r]} & \tilde{\chi}_1\\ \tilde{M}^{[21]} & \tilde{M}^{[22]} & \cdots & \tilde{M}^{[2r]} & \tilde{\chi}_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \tilde{M}^{[r1]} & \tilde{M}^{[r2]} & \cdots & \tilde{M}^{[rr]} & \tilde{\chi}_r\\ \tilde{\psi}_1 & \tilde{\psi}_2 & \cdots & \tilde{\psi}_r & 0\end{pmatrix},\quad \tilde{M}=\begin{pmatrix}\tilde{M}^{[11]} & \tilde{M}^{[12]} & \cdots & \tilde{M}^{[1r]}\\ \tilde{M}^{[21]} & \tilde{M}^{[22]} & \cdots & \tilde{M}^{[2r]}\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{M}^{[r1]} & \tilde{M}^{[r2]} & \cdots & \tilde{M}^{[rr]}\end{pmatrix},$$

$$\tilde{G}=\begin{pmatrix}\tilde{M}^{[11]} & \tilde{M}^{[12]} & \cdots & \tilde{M}^{[1r]} & \tilde{\chi}_1\\ \tilde{M}^{[21]} & \tilde{M}^{[22]} & \cdots & \tilde{M}^{[2r]} & \tilde{\chi}_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \tilde{M}^{[r1]} & \tilde{M}^{[r2]} & \cdots & \tilde{M}^{[rr]} & \tilde{\chi}_r\\ \tilde{\tau}_1 & \tilde{\tau}_2 & \cdots & \tilde{\tau}_r & 0\end{pmatrix},$$

with

$$\tilde{M}^{[ij]}_{kl}=\frac{1}{(k-1)!(l-1)!}\frac{\partial^{k+l-2}}{\partial\zeta^{*k-1}\partial\zeta^{l-1}}\frac{\langle\nu_j|\sigma_3|\nu_i\rangle}{\zeta+\zeta^*}-\frac{\langle\nu_j|\nu_i\rangle}{\zeta-\zeta^*}|_{\zeta=\zeta_i,\zeta^*=\zeta_j^*},$$

$$\tilde{\chi}_i=\left(|\nu_i\rangle_1^{[0]},|\nu_i\rangle_1^{[1]},\quad\ldots,\quad|\nu_i\rangle_1^{[n_i-1]}\right)^T,\qquad |\nu_i\rangle^{[j]}=\frac{1}{(j)!}\frac{\partial^j}{\partial(\zeta)^j}|\nu_i\rangle|_{\zeta=\zeta_i},$$

$$\tilde{\psi}_i=\left(\left(\frac{\langle\nu_i|_2}{\zeta_i^{*2}}\right)^{[0]},\left(\frac{\langle\nu_i|_2}{\zeta_i^{*2}}\right)^{[1]},\quad\ldots,\quad\left(\frac{\langle\nu_i|_2}{\zeta_1^{*2}}\right)^{[n_i-1]}\right),\qquad\left(\frac{\langle\nu_i|}{\zeta_i^{*2}}\right)^{[j]}=\left(\frac{1}{(j)!}\frac{\partial^j}{\partial(\zeta)^j}\left(\frac{\langle\nu_i|}{\zeta^{*2}}\right)|_{\zeta=\zeta_i^*}\right),$$

$$\tilde{\tau}_i=\left(\left(\frac{\langle\nu_i|_1}{\zeta_i^{*}}\right)^{[0]},\left(\frac{\langle\nu_i|_1}{\zeta_i^{*}}\right)^{[1]},\quad\ldots,\quad\left(\frac{\langle\nu_i|_1}{\zeta_i^{*}}\right)^{[n_i-1]}\right),\qquad\left(\frac{\langle\nu_i|}{\zeta_i^{*}}\right)^{[j]}=\left(\frac{1}{(j)!}\frac{\partial^j}{\partial(\zeta)^j}\left(\frac{\langle\nu_i|}{\zeta^{*}}\right)|_{\zeta=\zeta_i^*}\right).$$

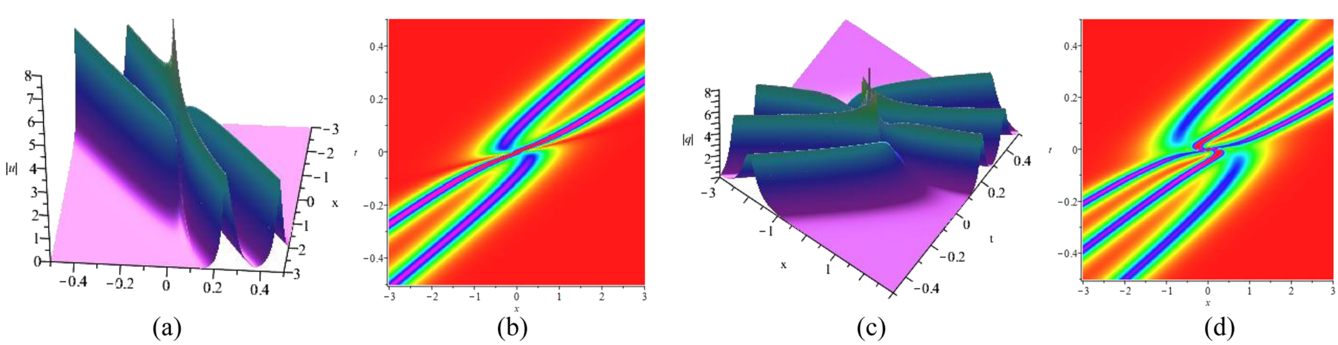

FIG. 3. (a) The double-zero soliton solution for $|q|$ with $n_1=2,\zeta=1+i,a_1=b_1=1$. (b) Density plot of double zeros. (c) The triple-zero soliton solution for $|q|$ and $n_1=3,\zeta=1+i,a_1=b_1=1$. (d) Density plot of triple zeros.

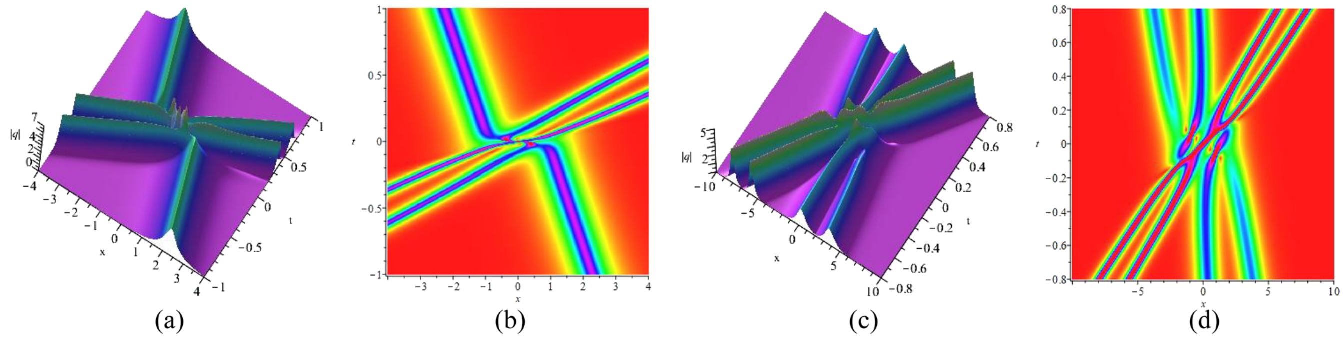

FIG. 4. (a) Mixed solution of double zeros and single zero for $|q|$ with $n_1 = 2,\ n_2 = 1,\ \zeta_1 = 1 + i,\ a_1 = b_1 = 1,\ \zeta_2 = \frac{1}{2} + i,\ a_2 = b_2 = 1$. (b) Density plot of single–double zeros. (c) Mixed solution of double zeros and double zeros for $|q|$ with $n_1 = 2,\ n_2 = 2,\ \zeta_1 = 1 + i,\ a_1 = b_1 = 1,\ \zeta_2 = 1 + \frac{1}{2}i,\ a_2 = b_2 = 1$. (d) Density plot of double–double zeros.

Hence, formula (78) gives the general expression of high-order solitons with multiple zeros. Because the spectral parameters here cannot be pure real or pure virtual, the expression of the high-order soliton is relatively complex, but different n_j and appropriate parameters can be selected, and the graphics of mixed high-order solitons solution can be given by using mathematical software such as Maple and Mathematica. Here, we give several representative mixed solutions. In Fig. 3, let $n_1 = 2, n_j = 0\ (j = 2, \ldots, r)$, in Eq. (78), which represents the double zeros case, and $n_1 = 3, n_j = 0(j = 2, \ldots, r)$, in Eq. (78) is the triple zeros case. In Fig. 4, take $n_1 = 2, n_2 = 1, n_j = 0(j = 3, \ldots, r)$, that is, a mixed solution of a double zeros and a single zero, and take $n_1 = 2, n_2 = 2, n_j = 0(j = 3, \ldots, r)$, which means a mixed solution of two double zeros.

V. CONCLUSION AND DISCUSSION

In a word, the inverse scattering method is applied to the higher-order KN equation with vanishing boundary at infinity, and the soliton matrix is constructed by studying the corresponding RHP. Using RHP regularization of finite simple zeros, the determinant form of general N-solitons of the higher-order KN equation without reflection is obtained, which is different from the soliton solution form of the previous KN system. In the process of inverse scattering, the potential function is recovered when the spectral parameter tends to zero, which effectively avoids the appearance of the implicit function.[28] At the same time, the properties of the single-soliton solution and the collision dynamics and asymptotic behavior of the two-soliton solution are investigated.

In addition, the multiple zeros of RHP are considered, and the higher-order soliton matrix of the higher-order KN equation is obtained by the limit technique. Several typical graphs are given, including the graphs of the double zero soliton solution, triple zero soliton solution, single–double zero soliton solution, and double–double zero soliton solution. It provides a good basis for future experimental observation.

In this context, we merely consider the solutions of the zero boundary condition at infinity. For solutions with non-zero boundary conditions at infinity, the long-term behavior and asymptotic stability need to be further studied.

ACKNOWLEDGMENTS

This project was supported by the National Natural Science Foundation of China (Grant No. 12175069) and the Science and Technology Commission of Shanghai Municipality (Grant Nos. 21JC1402500 and 18dz2271000).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] K. Abhinav, P. Guha, and I. Mukherjee, "Study of quasi-integrable and non-holonomic deformation of equations in the NLS and DNLS hierarchy," J. Math. Phys. **59**, 101507 (2018).

[2] D. J. Kaup and A. C. Newell, "An exact solution for a derivative nonlinear Schrödinger equation," J. Math. Phys. **19**, 798–801 (1978).

[3] E. Fan, "A Liouville integrable Hamiltonian system associated with a generalized Kaup–Newell spectral problem," Physica A **301**, 105–113 (2001).

[4] H. M. Ahmed, W. B. Rabie, and M. A. Ragusa, "Optical solitons and other solutions to Kaup–Newell equation with Jacobi elliptic function expansion method," Anal. Math. Phys. **11**, 23 (2021).
[5] Y. Zeng, "New factorization of the Kaup-Newell hierarchy," Physica D **73**, 171–188 (1994).
[6] W. Liu, Y. S. Zhang, and J. S. He, "Rogue wave on a periodic background for Kaup-Newell equation," Rom. Rep. Phys. **70**, 106 (2018).
[7] J. Lenells, "The solution of the global relation for the derivative nonlinear Schrödinger equation on the half-line," Physica D **240**, 512–525 (2011).
[8] H. Steudel, "The hierarchy of multi-soliton solutions of the derivative nonlinear Schrödinger equation," J. Phys. A: Math. Gen. **36**, 1931–1946 (2003).
[9] K. Mio, T. Ogino, K. Minami, and S. Takeda, "Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas," J. Phys. Soc. Jpn. **41**(1), 265–271 (1976).
[10] E. Mojølhus, "Nonlinear Alfvén waves and the DNLS equation: Oblique aspects," Phys. Scr. **40**(2), 227–237 (1989).
[11] A. Biswas, Y. Yıldırım, E. Yaşar, Q. Zhou, S. P. Moshokoa, and M. Belic, "Sub pico-second pulses in mono-mode optical fibers with Kaup–Newell equation by a couple of integration schemes," Optik **167**, 121–128 (2018).
[12] H. H. Chen, Y. C. Lee, and C. S. Liu, "Integrability of nonlinear Hamiltonian systems by inverse scattering method," Phys. Scr. **20**, 490–492 (1979).
[13] V. S. Gerdjikov and M. I. Ivanov, "A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures," Bulg. J. Phys. **10**, 130–143 (1983).
[14] E. Fan, "Integrable systems of derivative nonlinear Schrödinger type and their multi-Hamiltonian structure," J. Phys. A: Math. Gen. **34**, 513–519 (2001).
[15] G. S. França, J. F. Gomes, and A. H. Zimerman, "The algebraic structure behind the derivative nonlinear Schrödinger equation," J. Phys. A: Math. Theor. **46**(30), 305201 (2013).
[16] J. Hu, J. Xu, and G. F. Yu, "Riemann-Hilbert approach and *N*-soliton formula for a higher-order Chen-Lee-Liu equation," J. Nonlinear Math. Phys. **25**(4), 633–649 (2018).
[17] J. Y. Zhu and Y. Chen, "High-order soliton matrix for the third-order flow equation of the Gerdjikov-Ivanov hierarchy through the Riemann-Hilbert method," arXiv:2105.08412.
[18] A. Kundu, "Exact solutions to higher-order nonlinear equations through gauge transformation," Physica D **25**, 399–406 (1987).
[19] X. J. Chen and W. K. Lam, "Inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions," Phys. Rev. E **69**, 066604 (2004).
[20] K. Imai, "Generalization of the Kaup-Newell inverse scattering formulation and Darboux transformation," J. Phys. Soc. Jpn. **68**, 355–359 (1999).
[21] B. Guo and L. Ling, "Riemann-Hilbert approach and N-soliton formula for coupled derivative Schrödinger equation," J. Math. Phys. **53**, 073506 (2012).
[22] L. Ling and Q. P. Liu, "Darboux transformation for a two-component derivative nonlinear Schrödinger equation," J. Phys. A: Math. Theor. **43**, 434023 (2010).
[23] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, "The inverse scattering transform-Fourier analysis for nonlinear problems," Stud. Appl. Math. **53**, 249–315 (1974).
[24] D. Bilman, R. Buckingham, and D.-S. Wang, "Far-field asymptotics for multiple-pole solitons in the large-order limit," J. Differ. Equations **297**, 320–369 (2021).
[25] L. Gagnon and N. Stiévenart, "*N*-soliton interaction in optical fibers: The multiple-pole case," Opt. Lett. **19**, 619–621 (1994).
[26] A. B. Shabat, "One dimensional perturbations of a differential operator and the inverse scattering problem," in *Problems in Mechanics and Mathematical Physics* (Nauka, Moscow,, 1976), pp. 279–296.
[27] W. Q. Peng and Y. Chen, "Double and triple poles solutions for the Gerdjikov-Ivanov type of derivative nonlinear Schrödinger equation with zero/nonzero boundary conditions," arXiv:2104.12073.
[28] J. C. Pu and Y. Chen, "Riemann-Hilbert approach and double-pole solutions for the third-order flow equation of the Kaup-Newell system with zero/nonzero boundary conditions," arXiv:2105.06098.
[29] J. K. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2010).
[30] V. E. Zakharov and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II," Funct. Anal. Appl. **13**, 166–174 (1979).
[31] T. Xu and B. Tian, "Bright N-soliton solutions in terms of the triple Wronskia for the coupled nonlinear Schrödinger equations in optical fibers," J. Phys. A: Math. Gen. **43**, 245205 (2010).
[32] V. S. Shchesnovich and J. Yang, "Higher-order solitons in the *N*-wave system," Stud. Appl. Math. **110**, 297–332 (2003).
[33] B. Guo, L. Ling, and Q. P. Liu, "High-order solutions and generalized Darboux transformations of derivative nonlinear Schrödinger equations," Stud. Appl. Math. **130**, 317–344 (2012).