The Sasa-Satsuma equation on a non-zero background: Inverse scattering transform and multi-soliton solutions

Li-Li Wen[1], En-Gui Fan[2] and Yong Chen[1,3*]

Abstract

We concentrate on inverse scattering transformation for the Sasa-Satsuma equation with 3×3 matrix spectral and nonzero boundary condition in this article. To circumvent multi-value of eigenvalues, we introduce a suitable two-sheet Riemann surface to map the original spectral parameter k into a single-valued parameter z. The analyticity of the Jost eigenfunctions and scattering coefficients of Lax pair for the SS equation are analyzed in details. According to the analyticity of eigenfunctions and scattering coefficients, the z-complex plane is divided into four analytic regions $D_j,\ j=1,2,3,4$. Since the second column of Jost eigenfunctions is analytic in $D_j,\ j=1,2,3,4$, but in upper-half or lower-half plane, we introduce certain auxiliary eigenfunctions which are necessary for deriving the analytic eigenfunctions in D_j. We find that for the eigenfunctions, scattering coefficients and the auxiliary eigenfunctions all possess three kinds of symmetries, which characterize the distribution of discrete spectrum. The asymptotic behaviors of eigenfunctions, auxiliary eigenfunctions and scattering coefficients are also systematically derived. Then

[1] School of Mathematical Sciences, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, 200241, China.
[2] School of Mathematical Sciences and Key Laboratory for Nonlinear Science, Fudan University, Shanghai 200433, China.
[3] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, 266590, China.
[*] Corresponding author and email address: ychen@sei.ecnu.edu.cn.

a matrix Riemann-Hilbert problem with four kind jump conditions associated with the problem of nonzero asymptotic boundary conditions is established, from which N-soliton solutions is obtained via the corresponding reconstruction formulae. The reflectionless soliton solutions are explicitly given. As application of the N-soliton formula, we present three kinds of single-soliton solutions according to the distribution of discrete spectrum.

Keywords: the Sasa-Satsuma equation; nonzero boundary condition; auxiliary eigenfunctions; Riemann-Hilbert problem; soliton solution.

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Contents

1 Introduction

The nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0 \tag{1.1}$$

is one of the most important integrable equations appearing in various physical systems such as plasma physics, solid-state physics nonlinear optics and so on. The initial value problem of the NLS equation (1.1) was solved by the inverse scattering transformation (IST) method [1]. Hasegawa and Tappert found the possibility of soliton propagation in optical fibers and showed the stability by numerical computations [2]. In 1980, Mollenauer, Stolen and Gordon observing the soliton propagation experimentally [3]. However, by the advancement of experiment accuracy, several phenomena which can not be explained by the classical equation (1.1) have been observed in experimentally works. In order to explain these phenomena, Kodama and Hasegawa proposed a higher-order nonlinear Schrödinger equation [4, 5]

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\{\beta_1 q_{xxx} + \beta_2 |q|^2 q_x + \beta_3 q(|q|^2)_x\} = 0, \quad \beta_j \in \mathbb{R}, \tag{1.2}$$

which is completely integrable for special parameters β_1, β_2 and β_3 [6–9]. For the choice $\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$, the equation (1.2) reduced to [10]

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon\{q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x\} = 0. \tag{1.3}$$

If making a transformation [10]

$$q(x,t) \to q(x,t)e^{\frac{i}{6\varepsilon}(x - \frac{t}{18\varepsilon})}, \quad t \to t, \quad x \to x + \frac{t}{12\varepsilon}.$$

the equation (1.3) reduce to the well-known Sasa-Satsuma (SS) equation

$$q_t + \varepsilon\{q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x\} = 0, \tag{1.5}$$

which can be used to describe the nonlinear phenomena in many situations, such as pulse propagation in optical fibers and deep ocean waves [11–15]. It was shown that the SS equation (1.5) is completely integrable and possesses 3×3 matrix spectral problem [10]. In recent years, there has been much work on the Sasa-Satsuma equation. For example, bilinearization and multisoliton solutions for the SS equation [15], N-soliton solutions for the SS equation with zero boundary condition by IST method [16], initial-boundary problems of the SS equation on the half-line by Fokas uniformed method [17], N-soliton solutions for the SS equation with zero boundary condition by Riemann-Hilbert approach [18], long-time asymptotic of the SS equation by Deift-Zhou steepest descent method [19], high-order soliton solutions for the SS equation with zero boundary condition by Riemann-Hilbert approach [20], binary Darboux transformation of the SS equation [21], Bright-soliton and optical soliton etc [22, 23].

The inverse scattering method, first discovered by Gardner, Green, Kruskal and Miura, is one of the most powerful tool to investigate solitons of nonlinear models [24, 25]. The IST for the focusing Schrödinger equation with zero boundary conditions was first developed by Zakharov and Shabat [1], later for the defocusing case with nonzero boundary conditions [26]. The next important steps of the development of IST method is the Riemann-Hilbert (RH) method which as the modern version of IST was established by Zakharov and Shabat [27]. It has since become clear that the RH method is applicable to investigate exact solutions and asymptotic analysis of solutions for a wide class of integrable systems [28–34]. In recent years, Biondini, Prinari, Manakov et al have made much excellent work on the IST method of integrable systems with nonvanishing boundary conditions [35–42].

In this article we are interested in the inverse scattering method for the SS equation ($\varepsilon = 1$) by means of RH method with the following nonzero boundary condition

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm = \mathbf{q}_0 e^{i\theta_\pm} \tag{1.6}$$

where $\mathbf{q}$ and $\mathbf{q}_0$ are the two-component vectors, $\|\cdot\|$ is the standard Euclidean norm, $\theta_\pm$ are real numbers and $\| \mathbf{q}_0 \|^2 = q_0^2$. Because the SS equation admits a 3×3 matrix spectral problem, which causes the analyticity, symmetries and asymptotic to be more complicated than those of 2×2 matrix spectral problem. The auxiliary eigenfunctions are also necessary to construct a matrix Riemann-Hilbert problem.

The structure of this work is as follows. In section 2, we investigate the spectral problem of SS equation (1.5) with the nonzero boundary condition (1.6) by introducing a Riemann surface. To construct the desired RH problem, we investigate the analyticity of eigenfunctions, auxiliary eigenfunctions and scattering coefficients. In section 3, we show that all eigenfunctions, scattering matrix and reflection coefficient possess three types symmetries. In section 4, we consider the asymptotic of eigenfunctions, auxiliary eigenfunction and scattering coefficient. In section 5, discuss the discrete spectrum and the residue conditions to analyze poles for meromorphic matrices appearing in the RH problem. In section 6, we present a generalized RH problem with four jump matrixes, from which we find a reconstruction formula between solution of the SS equation and the RH problem. In section 7, in reflectionless case, we discuss solvability of the RH problem, from which the N-soliton solutions of the SS equation are obtained. In section 8, as an illustrate examples, we explicitly obtain three kinds of one-soliton solutions according to different distribution of the spectrum, and their dynamical features are analyzed.

2 Eigenfunctions

2.1 Lax Pair

The Sasa-Satsuma equation (1.5) admits 3×3 matrix spectral problem [10]

$$\Phi_x = X\Phi, \quad \Phi_t = T\Phi, \tag{2.1}$$

where

$$X = -ik\Lambda + Q,$$
$$T = -4ik^3\Lambda + 4k^2Q + 2ik\Lambda(Q_x - Q^2) + 2Q^3 - Q_{xx} + [Q_x, Q],$$
$$Q = \begin{pmatrix} \mathbf{0} & \mathbf{q} \\ -\mathbf{q}^\dagger & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\dagger & -1 \end{pmatrix},$$

in which some notations in (2.1) are as follows:

▲ $k \in \mathbb{C}$ is the spectral parameter.

▲ vector function $\mathbf{q} = (q, \bar{q})^T$, the superscript "$T$" denote the matrix transpose and $\bar{q}$ denotes the complex conjugation of q.

▲ the superscript "†" represents the Hermitian of matrix.

▲ $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the zero matrix or vector.

▲ eigenfunction $\Phi = (\Phi_{ij})_{3\times 3}, \quad (i, j = 1, 2, 3)$.

Under the nonzero boundary condition (1.6), we obtain the following asymptotic matrix spectral problem

$$\Phi_x = X_\pm \Phi, \quad \Phi_t = T_\pm \Phi, \tag{2.2}$$

where

$$X_\pm = \lim_{x\to\pm\infty} X = -ik\Lambda + Q_\pm,$$
$$T_\pm = \lim_{x\to\pm\infty} T = -4ik^3\Lambda + 4k^2Q_\pm - 2ik\Lambda Q_\pm^2 + 2Q_\pm^3,$$
$$Q_\pm = \lim_{x\to\pm\infty} Q = \begin{pmatrix} \mathbf{0} & \mathbf{q}_\pm \\ -\mathbf{q}_\pm^\dagger & 0 \end{pmatrix}.$$

The eigenvalues of matrices $X_\pm$ and $T_\pm$ are respectively

$$\begin{matrix} -ik, & i\lambda, & -i\lambda; \\ -4ik^3, & 2i\lambda(2k^2 - q_0^2), & -2i\lambda(2k^2 - q_0^2), \end{matrix}$$

where

$$\lambda^2 = k^2 + q_0^2, \tag{2.4}$$

which is a double-valued function for complex variable k.

Next, we derive the eigenvector of $X_\pm$ and $T_\pm$. For the convenient of expression, we firstly introduce the definition of orthogonal vector in the following Lemma.

Lemma 1. *For any two-component complex-valued vector* $\mathbf{l} = (l_1, l_2)^T$*, one can define its orthogonal vector as* $\mathbf{l}^\perp = (l_2, -l_1)$*, and holds the property* $\mathbf{l}^\perp \mathbf{l} = \mathbf{l}^T \mathbf{l}^\perp = 0$.

It is obvious that $X_\pm$ and $T_\pm$ possess the relation $[X_\pm, T_\pm] = 0$. So $X_\pm$ and $T_\pm$ holds the common eigenvectors. Through some calculations, one can obtain an invertible matrix $\Gamma_\pm$ satisfying the following equation

$$X_\pm \Gamma_\pm = i\Gamma_\pm J, \quad T_\pm \Gamma_\pm = i\Gamma_\pm \Omega,$$

where

$$\begin{aligned}
&J = \operatorname{diag}\left(-\lambda, -k, \lambda\right), \\
&\Omega = \operatorname{diag}\left(-2(2k^2 - q_0^2)\lambda, -4k^3, 2(2k^2 - q_0^2)\lambda\right), \\
&\Gamma_\pm = \begin{pmatrix} -\frac{\mathbf{q}_\pm}{q_0} & \frac{(\mathbf{q}_\pm^\perp)^\dagger}{q_0} & -\frac{i\mathbf{q}_\pm}{k+\lambda} \\ \frac{iq_0}{k+\lambda} & 0 & 1 \end{pmatrix}.
\end{aligned}$$

2.2 Riemann Surface

In the scalar case, $\lambda = \sqrt{k^2 + q_0^2}$ is a branched function. The branch points are the values of $k = \pm iq_0$. We take the branch cut on the segment $[-iq_0, iq_0]$. In this case, if we set

$$k + iq_0 = r_1 e^{i\alpha_1}, \quad k - iq_0 = r_2 e^{i\alpha_2}, \quad -\frac{\pi}{2} < \alpha_j < \frac{3\pi}{2}, j = 1, 2, \tag{2.6}$$

we then obtain two single value functions

$$\lambda = \pm\sqrt{r_1 r_2} e^{\frac{i}{2}(\alpha_1 + \alpha_2)}.$$

We introduce an uniformization variable as

$$z = k + \lambda.$$

Substituting it into (2.4) gives two single-valued functions

$$k=\frac{1}{2}\left(z-\frac{q_0^2}{z}\right),\quad \lambda=\frac{1}{2}\left(z+\frac{q_0^2}{z}\right).$$

Further we can show the following relations between the Riemann surface and the k-plane.

- ▲ The region where $\mathrm{Im}\lambda>0$ come from the upper-half plane of the sheet-I and the lower-half plane of the sheet-II. The region where $\mathrm{Im}\lambda<0$ come from the upper-half plane of the sheet-II and the lower-half plane of the sheet-I.
- ▲ On the sheet-I, $z\to\infty$ as $k\to\infty$, and on the sheet-II, $z\to 0$ as $k\to\infty$.
- ▲ The real λ (real k) axes is mapped into the real z axes.
- ▲ The branch cut $i[-q_0,q_0]$ is mapped into the circle C_0 of the radius q_0 in z-plane.
- ▲ The sheet-I and sheet-II, except for the branch cut, are mapped into the exterior and the interior of C_0, respectively.

The jump contour in the complex z-plane is denoted by $\Sigma=\mathbb{R}\cup C_0$. The yellow and white regions in Fig.1 denote the analytic region $D_j (j=1,2,3,4)$, respectively

$$D_1=\{z\in\mathbb{C}:|z|^2-q_0^2>0\cap \mathrm{Im}z>0\},\quad D_2=\{z\in\mathbb{C}:|z|^2-q_0^2>0\cap \mathrm{Im}z<0\},$$
$$D_3=\{z\in\mathbb{C}:|z|^2-q_0^2<0\cap \mathrm{Im}z<0\},\quad D_4=\{z\in\mathbb{C}:|z|^2-q_0^2<0\cap \mathrm{Im}z>0\},$$

and $\bigcup_{j=1}^{4}\bar{D}_j=\mathbb{C}$.

On the z complex plane, we can rewrite the invertible matrix $\Gamma_\pm$ as

$$\Gamma_\pm=\begin{pmatrix} -\frac{\mathbf{q}_\pm}{q_0} & \frac{(\mathbf{q}_\pm^\perp)^\dagger}{q_0} & -\frac{i\mathbf{q}_\pm}{z} \\ \frac{iq_0}{z} & 0 & 1 \end{pmatrix},$$

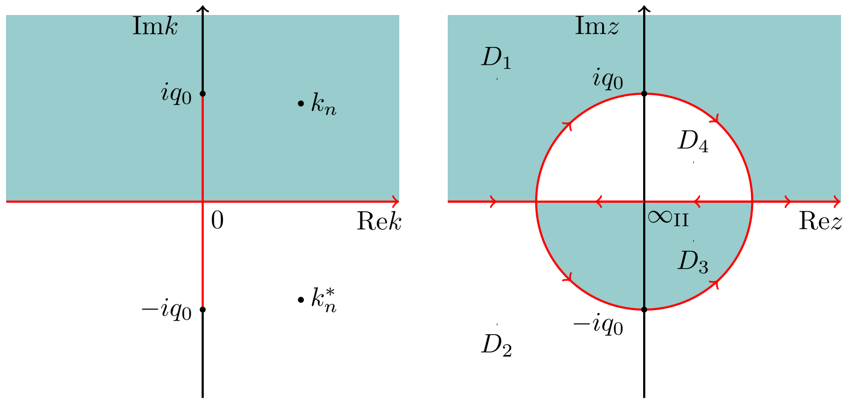


Figure 1: Left: The k-plane, showing the branch cut (red line), the branch points $\pm iq_0$, the region of Im$k > 0$ (green) and Im$k < 0$ (white); Right: The Riemann surface of z-plane, showing the analytic region D_+ (green, D_1, D_3), D_- (white, D_2, D_4).

moreover,

$$\det \Gamma_\pm = 1 + \frac{q_0^2}{z^2} \triangleq \gamma(z),$$
$$\Gamma_\pm^{-1} = \frac{1}{\gamma} D(z) \begin{pmatrix} \frac{\mathbf{q}_\pm^\dagger}{q_0} & -\frac{iq_0}{z} \\ \frac{\mathbf{q}_\pm^\perp}{q_0} & 0 \\ \frac{i\mathbf{q}_\pm^\dagger}{z} & 1 \end{pmatrix},$$
$$D(z) = \operatorname{diag}\{-1, \gamma(z), 1\}.$$

We also rewrite the Lax pair (2.1) as the polynomial form

$$\Phi_{\pm,x} = X_\pm \Phi_\pm + \Delta Q_\pm \Phi_\pm, \quad \Phi_{\pm,t} = T_\pm \Phi_\pm + \Delta \hat{Q}_\pm \Phi_\pm, \tag{2.8}$$

where

$$\Delta Q_\pm = Q - Q_\pm,$$
$$\Delta \hat{Q}_\pm = 4k^2 Q + 2ik\Lambda(Q_x - Q^2) + 2Q^3 - Q_{xx} + [Q_x, Q] - (4k^2 Q_\pm - 2ik\Lambda Q_\pm^2 + 2Q_\pm^3).$$

2.3 Analyticity

In this subsection, we can define the Jost solutions $\Phi_\pm(x, t, z)$ as the simultaneous solutions of the Lax pair (2.1) with the boundary conditions

$$\Phi_\pm(x, t, z) \sim \Gamma_\pm(z) e^{i\Theta(x,t,z)}, \qquad x \to \pm\infty, \tag{2.10}$$

where

$$\Theta(x,t,z) = Jx + \Omega t = \mathrm{diag}(\theta_1, \theta_2, -\theta_1),$$
$$\theta_1 = -\lambda[x + 2(2k^2 - q_0^2)t], \quad \theta_2 = -k(x + 4k^2 t).$$

To remove the asymptotic exponential oscillations, we introduce the following modified Jost eigenfunction

$$\mu_\pm(x,t,z) = \Phi_\pm(x,t,z)e^{-i\Theta(x,t,z)}, \tag{2.12}$$

which implies that

$$\lim_{x\to\pm\infty} \mu_\pm(x,t,z) = \Gamma_\pm(z). \tag{2.13}$$

After some simple calculation, we derive the Lax pair of $\mu_\pm$

$$(\Gamma_\pm^{-1}\mu_\pm)_x = [iJ, \Gamma_\pm^{-1}\mu_\pm] + \Gamma_\pm^{-1}\Delta Q_\pm \mu_\pm, \tag{2.14a}$$
$$(\Gamma_\pm^{-1}\mu_\pm)_t = [i\Omega, \Gamma_\pm^{-1}\mu_\pm] + \Gamma_\pm^{-1}\Delta \hat{Q}_\pm \mu_\pm, \tag{2.14b}$$

which can be written in the full derivative form

$$\mathrm{d}\big[e^{-i\Theta(x,t,z)}\Gamma_\pm^{-1}\mu_\pm e^{i\Theta(x,t,z)}\big] = e^{-i\Theta(x,t,z)}[U_1\mathrm{d}x + U_2\mathrm{d}t]e^{i\Theta(x,t,z)}, \tag{2.15}$$

where

$$U_1 = \Gamma_\pm^{-1}\Delta Q_\pm(x,t)\mu_\pm(x,t,z), \quad U_2 = \Gamma_\pm^{-1}\Delta\hat{Q}_\pm(x,t)\mu_\pm(x,t,z).$$

So we can come to the Jost integral equation from (2.15)

$$\mu_-(x,t,z) = \Gamma_-(z) + \int_{-\infty}^{x} \Gamma_-(z)e^{i(x-y)J(z)}\Gamma_-^{-1}(z)\Delta Q_-(y,t)\mu_- e^{-i(x-y)J(z)}\mathrm{d}y, \tag{2.16a}$$

$$\mu_+(x,t,z) = \Gamma_+(z) - \int_{x}^{+\infty} \Gamma_+(z)e^{i(x-y)J(z)}\Gamma_+^{-1}(z)\Delta Q_+(y,t)\mu_- e^{-i(x-y)J(z)}\mathrm{d}y. \tag{2.16b}$$

And we also derive the analyticity:

Theorem 2.1. *If $\mathbf{q}(\cdot,t) - \mathbf{q}_+ \in L^1(a,+\infty)$ or $\mathbf{q}(\cdot,t) - \mathbf{q}_- \in L^1(-\infty,a)$ for any constant $a \in \mathbb{R}$, the following columns of $\mu_+(x,t,z)$ or $\mu_-(x,t,z)$ can be analytically extended onto the corresponding regions of the complex z-plane*

Table 1: the analyticity of $\mu_{\pm,j}(j=1,2,3)$

$\mu_{+,1}$	$\mu_{+,2}$	$\mu_{+,3}$	$\mu_{-,1}$	$\mu_{-,2}$	$\mu_{-,3}$
D_4	$\mathrm{Im}z<0$	D_1	D_3	$\mathrm{Im}z>0$	D_2

And (2.12) indicates that the same analyticity and boundedness properties also hold for the columns of $\Phi_{\pm}(x,t,z)$.

Proof. We rewrite the equation (2.16a) as

$$\Gamma_-^{-1}(z)\mu_-(x,t,z)=\mathbf{I}+\int_{-\infty}^{x}e^{i(x-y)J(z)}\Gamma_-^{-1}(z)\Delta Q_-(y,t)\mu_-e^{-i(x-y)J(z)}\mathrm{d}y, \tag{2.17}$$

Next, we letting $W(x,t,z)=\Gamma_-^{-1}(z)\mu_-(x,t,z)$, and $\varpi(x,t,z)$ as the first column of matrix $W(x,t,z)$. For an arbitrary matrix $M=(m_{ij})_{3\times 3}$ with the following product notation

$$e^{i(x-y)J(z)}Me^{-i(x-y)J(z)}=\begin{pmatrix} m_{11} & e^{i(k-\lambda)(x-y)}m_{12} & e^{-2i\lambda(x-y)}m_{13} \\ e^{-i(k-\lambda)(x-y)}m_{21} & m_{22} & e^{-i(k+\lambda)(x-y)}m_{23} \\ e^{2i\lambda(x-y)}m_{31} & e^{i(k+\lambda)(x-y)}m_{32} & m_{33} \end{pmatrix}. \tag{2.18}$$

This imply the first column

$$\varpi(x,t,z)=\begin{pmatrix}1\\0\\0\end{pmatrix}+\int_{-\infty}^{x}G(x-y,z)\Delta Q_-(y,t)\Gamma_-(z)\varpi(y,t,z)\mathrm{d}y \tag{2.19}$$

where

$$G(x-y,z)=\mathrm{diag}(1,e^{-i(k-\lambda)(x-y)},e^{2i\lambda(x-y)})\Gamma_-^{-1}(z), \tag{2.20}$$

Now, one can introduce a Neumann series for $\varpi(x,t,z)$

$$\varpi(x,t,z)=\varpi^{(0)}+\varpi^{(1)}+\varpi^{(2)}+\varpi^{(3)}+\cdots, \tag{2.21}$$

where

$$\varpi^{(0)}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \varpi^{(n+1)}=\int_{-\infty}^{x}G(x-y,z)\Delta Q_-(y,t)\Gamma_-(z)\varpi^{n}(y,t,z)\mathrm{d}y, \tag{2.22}$$

and for convenience, we let $G(x-y,z)\Delta Q_-(y,t)\Gamma_-(z)=\aleph(x,y,t,z)$. The L^2 matrix norm $\parallel \aleph \parallel = \sqrt{\max\limits_{\rho_i}(\aleph^\dagger \aleph)}$, and $\max\limits_{\rho_i}(\aleph^\dagger\aleph)$ indicate the absolute maximum of eigenvalues of $\aleph^\dagger\aleph$. One has $\parallel \Gamma_- \parallel \leqslant 1+q_0/|z|$, and $\parallel \Gamma_-^{-1} \parallel \leqslant (1+\frac{q_0}{|z|})/|\gamma(z)|$ or $\parallel \Gamma_-^{-1} \parallel \leqslant 1$. This indicate that

$$\parallel \varpi^{(n+1)} \parallel \leqslant \int_{-\infty}^{x} \parallel C(x,y,t,z) \parallel \parallel \varpi^{(n)}(y,t,z) \parallel \mathrm{d}y \tag{2.23}$$

where

$$\parallel C(x,y,t,z) \parallel \leqslant \parallel \mathrm{diag}(1, e^{-i(k-\lambda)(x-y)}, e^{2i\lambda(x-y)}) \parallel \parallel \Gamma_-(z) \parallel \parallel \Delta Q(y,t) \parallel \parallel \Gamma_-^{-1}(z) \parallel \tag{2.24a}$$

$$\leqslant \varrho(z)(1+e^{(\mathrm{Im}k-\mathrm{Im}\lambda)(x-y)}+e^{-2(x-y)\mathrm{Im}\lambda}) \parallel \mathbf{q}(y,t)-\mathbf{q}_- \parallel, \tag{2.24b}$$

For $\parallel \Gamma_- \parallel \leqslant 1+q_0/|z|$ and $\parallel \Gamma_-^{-1} \parallel \leqslant (1+\frac{q_0}{|z|})/|\gamma(z)|$, $\varrho(z)=(1+q_0/|z|)^2/|\gamma(z)|$. The equation (2.18) imply that $x-y$ is always positive for μ_-. One has $1+e^{(\mathrm{Im}k-\mathrm{Im}\lambda)(x-y)}+e^{-2(x-y)\mathrm{Im}\lambda} \leqslant 3$ for $z\in\bar{D}_3$. And $\varrho(z)\to\infty$ as $z\to\pm iq_0$. So, given $\epsilon>0$, we introduce the domain $(D_3)_\epsilon = D_3\ (B_\epsilon(iq_0))\cup(B_\epsilon(-iq_0))$, where $B_\epsilon(z_0)=\{z\in\mathbb{C}||z-z_0|<\epsilon q_0\}$. It is show that $\varrho_\epsilon=\max\limits_{z\in(D_3)_\epsilon}\varrho(z)=1+2/\epsilon$. For all $z\in(D_3)_\epsilon$,

$$\parallel \varpi^{(n)}(x,t,z) \parallel \leqslant \frac{(3\varrho_\epsilon \int_{-\infty}^{x} \parallel \mathbf{q}(y,t)-\mathbf{q}_- \parallel \mathrm{d}y)^n}{n!} \tag{2.25}$$

For convenience, we let $M=3\varrho_\epsilon\int_{-\infty}^{x} \parallel \mathbf{q}(y,t)-\mathbf{q}_- \parallel \mathrm{d}y$. Then, for $n=j$, mathematical induction implies

$$\parallel \varpi^{(j+1)}(x,t,z) \parallel \leqslant \frac{3\varrho_\epsilon}{j!}\int_{-\infty}^{x} \parallel \mathbf{q}(y,t)-\mathbf{q}_- \parallel (M)^n \mathrm{d}y = \frac{1}{(j+1)!}M^{j+1}. \tag{2.26}$$

Thus, the Neumann series $\varpi(x,t,z)=\sum\limits_{n=0}^{\infty}\varpi^{(n)}$ converges absolutely and uniformly to $z\in(D_3)_\epsilon$. For $\parallel \Gamma_-^{-1} \parallel \leqslant 1$, the Neumann series is also converges absolutely and uniformly by the similar method. So μ_{-1} is analytic for $z\in D_3$. The rest results of this theorem can be proved by similar method. In particular, the second column of μ_- is analytic for $z\in \mathrm{Im}z>0$ and the second column of μ_+ is analytic for $z\in\mathrm{Im}z<0$. □

Now we introduce the scattering matrix $A(z)$. We note that $\mathrm{tr}X = -ik$ and $\mathrm{tr}T = -4\mathrm{ik}^3$. If $\Phi(x,t,z)$ is a solution of (2.1), by using Abel's theorem, we have

$$(\det\Phi)_x = \mathrm{tr}X\det\Phi, \quad (\det\Phi)_t = \mathrm{tr}T\det\Phi,$$

which can be written into the following differential equations

$$\left[e^{-i\Theta}\det\Phi\right]_x = \left[e^{-i\Theta}\det\Phi\right]_t = 0.$$

Again by using the boundary condition (2.10), we find that

$$\det\Phi_\pm(x,t,z) = \gamma(z)e^{i\theta_2(x,t,z)}. \tag{2.27}$$

$\Phi_+(x,t,z)$ and $\Phi_-(x,t,z)$ are the fundamental matrix solutions of the Lax pair. This indicates the existence of a 3×3 matrix $A(z)$, so that

$$\Phi_-(x,t,z) = \Phi_+(x,t,z)A(z), \tag{2.28}$$

where $A(z) = (a_{ij}(z))$ and $\det A(z) = 1$. And we introduce the inverse matrix $A^{-1}(z) = B(z) = (b_{ij}(z))$. In the 2×2 spectral problem, the analyticity of scattering matrix $A(z)$ can be derived from their representations as Wronskians of (2.28). In the 3×3 spectral problem, however, this approach is no longer applicable. The reason is that the second column of Jost eigenfunction is not analytic in one of analytic region D_j.

Theorem 2.2. *Under the same hypotheses as in 2.1, the diagonal scattering coefficient can be analytically extended of Σ in the following regions:*

Table 2: the analyticity of a_{jj} and b_{jj}

a_{11}	a_{22}	a_{33}	b_{11}	b_{22}	b_{33}
D_3	$\mathrm{Im}z > 0$	D_2	D_4	$\mathrm{Im}z < 0$	D_1

Proof. We note that the first part of the (2.8) is equivalent the following problem

$$\Phi_{\pm,x}(x,z) = \check{X}_\pm\Phi_\pm(x,z) + (Q(x) - Q_f(x))\Phi_\pm, \tag{2.29}$$

where

$$\check{X}(x,z) = H(x)X_+(z)+H(-x)X_-(z), \quad Q_f(x) = H(x)Q_+ + H(-x)Q_-, \tag{2.30}$$

and $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We introduce the fundamental eigenfunction $\check{\Phi}_\pm(x,z)$ as the square matrix solution of (2.33)

$$\check{\Phi}_\pm(x,z) \sim e^{xX_\pm(z)}, \qquad x \to \pm\infty. \tag{2.31}$$

By solving (2.33) in the similar way as (2.12), we introduce the following transformation

$$\check{\mu}_\pm = \check{\Phi}_\pm e^{-x\check{X}},$$

and we obtain the full derivative

$$\mathrm{d}(e^{-x\check{X}}\check{\mu}_\pm e^{x\check{X}}) = e^{-x\check{X}}(Q - Q_f)\check{\mu}_\pm e^{x\check{X}}.$$

We deriving the integral equation

$$\check{\Phi}_+(x,z) = G_f(x,0,z) - \int_x^{+\infty} G_f(x,y,z)[Q(y) - Q_f(y)]\check{\Phi}_+(y)\mathrm{d}y, \tag{2.32a}$$

$$\check{\Phi}_-(x,z) = G_f(x,0,z) + \int_{-\infty}^{x} G_f(x,y,z)[Q(y) - Q_f(y)]\check{\Phi}_-(y)\mathrm{d}y, \tag{2.32b}$$

where

$$G_f(x,y,z) = \begin{cases} e^{(x-y)X_+}, & x,y \geq 0 \\ e^{(x-y)X_-}, & x,y \leq 0 \\ e^{xX_+}e^{-yX_-}, & x,-y \geq 0 \\ e^{xX_-}e^{-yX_+}, & x,-y \leq 0 \end{cases}$$

For equation (2.32a),

$$\check{\Phi}_\pm(x,z) = G_f(x,0,z)A_\mp, \tag{2.33}$$

where

$$A_\mp = I \mp \int_{-\infty}^{+\infty} G_f(0,y,z)[Q(y) - Q_f(y)]\check{\Phi}_\pm \mathrm{d}y. \tag{2.34}$$

Considering the boundary conditions as $x \to \pm\infty$

$$\check{\Phi}_{\pm} \sim e^{xX_{\pm}} = \Gamma_{\pm} e^{i\Theta x} \Gamma_{\pm}^{-1}, \quad \Phi_{\pm} \sim \Gamma_{\pm} e^{i\Theta}. \tag{2.35}$$

Then we obtain

$$\Phi_{\pm}(x,z) \sim \check{\Phi}_{\pm}(x,z)\Gamma_{\pm}(z), \tag{2.36}$$

so (2.32) implied that

$$\Phi_{+}(x,z) = G_f(x,0,z)\Gamma_{+}(z) - \int_x^{+\infty} G_f(x,y,z)[Q(y) - Q_f(y)]\Phi_{+}(y)\mathrm{d}y, \tag{2.37a}$$

$$\Phi_{-}(x,z) = G_f(x,0,z)\Gamma_{-}(z) + \int_{-\infty}^{x} G_f(x,y,z)[Q(y) - Q_f(y)]\Phi_{-}(y)\mathrm{d}y, \tag{2.37b}$$

We compare the asymptotics as $x \to \infty$ of Φ_- from (2.33) with those of $\Phi_+ A(z)$ from (2.10) to obtain

$$A(z) = \Gamma_{+}^{-1}(z) A_{+}(z) \Gamma_{-}(z). \tag{2.38}$$

The equation (2.38) implied the following expression of scattering matrix

$$A(z) = \int_0^{+\infty} \mathrm{I}\mathrm{d}y + \int_{-\infty}^{0} \mathrm{II}\mathrm{d}y + \mathbf{I}$$

where the integrands are

$$\mathrm{I} = e^{-iyJ(z)} \Gamma_{+}^{-1}(z)[Q(y) - Q_{+}(y)]\Phi_{-}(y,z),$$
$$\mathrm{II} = \Gamma_{+}^{-1}(z)\Gamma_{-}(z) e^{-iyJ(z)} \Gamma_{-}^{-1}(z)[Q(y) - Q_{-}(z)]\Phi_{-}(y,z).$$

The diagonal elements of I and II are, respectively

$$\mathrm{I}_{11} = \sum_{j=1}^{3} \varrho_{1j} e^{iy\lambda} \Phi_{-,j1}, \quad \mathrm{I}_{22} = \sum_{j=1}^{3} \varrho_{2j} e^{iyk} \Phi_{-,j2}, \quad \mathrm{I}_{33} = \sum_{j=1}^{3} \varrho_{3j} e^{-iy\lambda} \Phi_{-,j3}$$

$$\mathrm{II}_{11} = \sum_{j=1}^{3} \left(\nu_{11}\vartheta_{1j} + \nu_{12}\vartheta_{2j} e^{iy(k-\lambda)} + \nu_{13}\vartheta_{3j} e^{-2iy\lambda} \right) e^{iy\lambda} \Phi_{-,j1}$$

$$\mathrm{II}_{22} = \sum_{j=1}^{3} \left(\nu_{21}\vartheta_{1j} + \nu_{22}\vartheta_{2j} e^{iy(k-\lambda)} + \nu_{23}\vartheta_{3j} e^{-2iy\lambda} \right) e^{iy\lambda} \Phi_{-,j2}$$

$$\mathrm{II}_{33} = \sum_{j=1}^{3} \left(\nu_{31}\vartheta_{1j} + \nu_{32}\vartheta_{2j} e^{iy(k-\lambda)} + \nu_{33}\vartheta_{3j} e^{-2iy\lambda} \right) e^{iy\lambda} \Phi_{-,j3}$$

where

$$\Gamma_+^{-1}\Gamma_- = (\nu_{ij}), \quad \Gamma_-^{-1}[Q - Q_-] = (\vartheta_{ij}), \quad \Gamma_+^{-1}[Q - Q_+] = (\varrho_{ij}).$$

Now, one can exam the individual entries of $A(z)$. In particular, recall that Φ_{-1} is always analytic for all $z \in D_3$ and bounded over $y \in \mathbb{R}$. On the other hand, $\mathrm{Im}(k - \lambda)$ and $-\mathrm{Im}\lambda$ have the same sign for $z \in D_3$. So a_{11} analytic for $z \in D_3$. The analyticity of a_{22} and a_{33} also derived as the same meethod. □

2.4 Auxiliary Eigenfunctions

The second columns of Jost solution are not analytic in a given domain. To circumvent this defect of analyticity, we introduce a modified Lax pair.

$$\widetilde{\Phi}_x = X\widetilde{\Phi}, \quad \widetilde{\Phi}_t = T\widetilde{\Phi}, \tag{2.41}$$

where $\widetilde{X} = \bar{X}(x, t, \bar{z}), \widetilde{T} = \bar{T}(x, t, \bar{z})$. And we note that

$$\begin{aligned} &\bar{Q} = -Q^T, \quad Q = -Q^\dagger, \\ &Q\Lambda = -\Lambda Q, \quad Q^T\Lambda = -\Lambda Q^T. \end{aligned}$$

Before reconstruct a solution of Lax pair (2.1) from the modified Lax pair (2.41), we need introduce the following lemma.

Lemma 2. *For $\forall \mathbf{u}, \mathbf{v} \in \mathbb{C}^3$ satisfying the following equations:*

$$\begin{aligned} &[(\Lambda\mathbf{u}) \times \mathbf{v}] + [\mathbf{u} \times (\Lambda\mathbf{v})] - [\mathbf{u} \times \mathbf{v}] - [(\Lambda\mathbf{u}) \times (\Lambda\mathbf{v})] = \mathbf{0}, \\ &\Lambda[\mathbf{u} \times \mathbf{v}] = -[(\Lambda\mathbf{u}) \times (\Lambda\mathbf{v})], \\ &Q[\mathbf{u} \times \mathbf{v}] + [(Q^T\mathbf{u}) \times \mathbf{v}] + [\mathbf{u} \times (Q^T\mathbf{v})] = \mathbf{0}, \\ &\Lambda Q^2[\mathbf{u} \times \mathbf{v}] + [(\Lambda(Q^T)^2) \times \mathbf{v}] + [\mathbf{u} \times (\Lambda(Q^T)^2)\mathbf{v}] = \mathbf{0}, \end{aligned}$$

where "$\times$" denotes the usual cross product.

Using above identities and Lemma2 one can straightforward to prove the following theorem.

Theorem 2.3. *If $\widetilde{\mathbf{v}}(x,t,z)$ and $\widetilde{\mathbf{w}}(x,t,z)$ are the two arbitrary solutions of modified Lax pair (2.41), then*

$$\mathbf{u}(x,t,z) = e^{i\theta_2(x,t,z)}[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}](x,t,z)$$

is also a solution of Lax pair (2.1).

Proof. Through some calculations, we have

$$\begin{aligned}
[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}]_x =& \widetilde{\mathbf{v}}_x \times \widetilde{\mathbf{w}} + \widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}_x \\
=&(ik\Lambda + \bar{Q})\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}} + \widetilde{\mathbf{v}} \times (ik\Lambda + \bar{Q})\widetilde{\mathbf{w}} \\
=&ik(\Lambda\widetilde{\mathbf{v}}) \times \widetilde{\mathbf{w}} + (\bar{Q}\widetilde{\mathbf{v}}) \times \widetilde{\mathbf{w}} + ik[\widetilde{\mathbf{v}} \times (\Lambda\widetilde{\mathbf{w}})] + [\widetilde{\mathbf{v}} \times (\bar{Q}\widetilde{\mathbf{w}})] \\
=&ik\{[(\Lambda\widetilde{\mathbf{v}}) \times \widetilde{\mathbf{w}}] + [\widetilde{\mathbf{v}} \times (\Lambda\widetilde{\mathbf{w}})]\} + \{[(\bar{Q}\widetilde{\mathbf{v}}) \times \widetilde{\mathbf{w}}] + [\widetilde{\mathbf{v}} \times (\bar{Q}\widetilde{\mathbf{w}})]\},
\end{aligned}$$

And we note that

$$[(\bar{Q}\widetilde{\mathbf{v}}) \times \widetilde{\mathbf{w}}] + [\widetilde{\mathbf{v}} \times (\bar{Q}\widetilde{\mathbf{w}})] = Q[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}].$$

So we derived the x-part

$$\begin{aligned}
\mathbf{u}_x(x,t,z) &= e^{i\theta_2}(i\theta_{2,x})[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] + e^{i\theta_2}[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}]_x \\
&= e^{i\theta_2}(-ik)[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] + e^{i\theta_2}(ik[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] + ik[(\Lambda\widetilde{\mathbf{v}}) \times (\Lambda\widetilde{\mathbf{w}})]) + e^{i\theta_2}Q[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] \\
&= -ik\Lambda e^{i\theta_2}[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] + e^{i\theta_2}Q[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}] \\
&= (-ik\Lambda + Q)e^{i\theta_2}[\widetilde{\mathbf{v}} \times \widetilde{\mathbf{w}}].
\end{aligned}$$

In the similar way, we can derived the t-part. □

As $x \to \infty$, we obtain the asymptotic spectral problem of (2.41)

$$\widetilde{\Phi}_{\pm,x} = \widetilde{X}_\pm \widetilde{\Phi}_\pm, \quad \widetilde{\Phi}_{\pm,t} = \widetilde{T}_\pm \widetilde{\Phi}_\pm.$$

The eigenvalues of $\widetilde{X}_\pm$ and $\widetilde{T}_\pm$ are

$$\begin{matrix} ik, & -i\lambda, & i\lambda \\ 4ik^3, & -2i\lambda(2k^2 - q_0^2), & 2i\lambda(2k^2 - q_0^2) \end{matrix}$$

We can deriving the eigenvector $\widetilde{\Gamma}_\pm(z) = \bar{\Gamma}_\pm(\bar{z})$ of matrix $\widetilde{X}_\pm$ and $\widetilde{T}_\pm$. And we note that $\det\widetilde{\Gamma}_\pm(z) = \gamma(z)$. For $z \in \Sigma$, we define the Jost solutions of (2.41) as the simultaneous solutions such that

$$\widetilde{\Phi}_\pm(x,t,z) \sim \widetilde{\Gamma}_\pm(z)e^{-i\Theta(x,t,z)}, \quad x \to \infty, \quad z \in \Sigma.$$

As in section 2.2, we introducing the modified adjoint eigenfunction

$$\widetilde{\mu}_{\pm}(x,t,z) = \widetilde{\Phi}_{\pm}(x,t,z)e^{i\Theta(x,t,z)}. \tag{2.44}$$

One can show the analyticity of $\widetilde{\mu}_{\pm}$

Table 3: the analyticity of $\widetilde{\mu}_{\pm}$

$\widetilde{\mu}_{+,1}$	$\widetilde{\mu}_{+,2}$	$\widetilde{\mu}_{+,3}$	$\widetilde{\mu}_{-,1}$	$\widetilde{\mu}_{-,2}$	$\widetilde{\mu}_{-,3}$
D_3	Im$z > 0$	D_2	D_4	Im$z < 0$	D_1

$\widetilde{\Phi}_{\pm}(x,t,z)$ are both the fundamental matrix solutions of the same problem, so we introduce the adjoint scattering matrix as

$$\widetilde{\Phi}_{-}(x,t,z) = \widetilde{\Phi}_{+}(x,t,z)\widetilde{A}(z). \tag{2.45}$$

Under the same hypotheses as in 2.1, the diagonal scattering coefficient can be analytically extended of Σ in the following regions:

Table 4: the analyticity of $\widetilde{a}_{jj}$ and $\widetilde{b}_{jj}$

$\widetilde{a}_{11}$	$\widetilde{a}_{22}$	$\widetilde{a}_{33}$	$\widetilde{b}_{11}$	$\widetilde{b}_{22}$	$\widetilde{b}_{33}$
D_4	Im$z < 0$	D_1	D_3	Im$z > 0$	D_2

In light of above results, we can define four auxiliary eigenfunctions $\chi_j, (j = 1,2,3,4)$ as the new solutions of the Lax pair (2.1)

$$\chi_1(x,t,z) = e^{i\theta_2}[\widetilde{\Phi}_{+,2} \times \widetilde{\Phi}_{-,3}](x,t,z), \tag{2.46a}$$

$$\chi_2(x,t,z) = e^{i\theta_2}[\widetilde{\Phi}_{-,2} \times \widetilde{\Phi}_{+,3}](x,t,z), \tag{2.46b}$$

$$\chi_3(x,t,z) = e^{i\theta_2}[\widetilde{\Phi}_{+,1} \times \widetilde{\Phi}_{-,2}](x,t,z), \tag{2.46c}$$

$$\chi_4(x,t,z) = e^{i\theta_2}[\widetilde{\Phi}_{-,1} \times \widetilde{\Phi}_{+,2}](x,t,z). \tag{2.46d}$$

We note that χ_j is analyticity for $z \in D_j$. And there are some relationships between the adjoint Jost solutions eigenfunction and the eigenfunctions of the Lax pair (2.1):

Lemma 3. *For $z \in \Sigma$ and for all cyclic indices j, l, and m*

$$\Phi_{\pm,j}(x,t,z) = e^{i\theta_2}[\widetilde{\Phi}_{\pm,l} \times \widetilde{\Phi}_{\pm,m}](x,t,z)/\gamma_j(z), \tag{2.47a}$$

$$\widetilde{\Phi}_{\pm,j}(x,t,z) = e^{-i\theta_2}[\Phi_{\pm,l} \times \Phi_{\pm,m}](x,t,z)/\gamma_j(z), \tag{2.47b}$$

where $\gamma_1(z) = \gamma_3(z) = 1$, $\gamma_2(z) = \gamma(z)$.

Proof. We prove (2.47a) with $j = 3$, the rest of (3)is verified in the similar method. From (2.10) and Theorem2.3, we deriving

$$\mathrm{u}_\pm(x,t,z) = e^{-i\theta_1(x,t,z)}\Gamma_{\pm,3}(z), \qquad x \to \infty.$$

The vector function $\mathrm{u}_\pm(x,t,z)$ must be linear combination of the columns of $\Phi_\pm$, then there exist three scalar functions $a_\pm(z)$, $b_\pm(z)$ and $c_\pm(z)$ so that

$$\mathrm{u}_\pm(x,t,z) = a_\pm(z)\Phi_{\pm,1}(x,t,z) + b_\pm(z)\Phi_{\pm,2}(x,t,z) + c_\pm(z)\Phi_{\pm,3}(x,t,z).$$

Comparing with the asymptotics of $\Phi_\pm(x,t,z)$ as $x \to \pm\infty$ in (2.10) yields $a_\pm(z) = b_\pm(z) = 0$ and $c_\pm(z) = 1$. □

We also note that these relations in Lemma 3 imply the Theorem2.4:

Theorem 2.4. *The scattering matrix $A(z)$ and $\widetilde{A}(z)$ admit the relation*

$$\widetilde{A}(z) = D(z)(A^{-1}(z))^T D^{-1}(z). \tag{2.48}$$

Proof. Equation (2.28) and Lemma3 yield

$$\widetilde{\Phi}_{-,1} = (a_{22}a_{33} - a_{32}a_{23})\widetilde{\Phi}_{+,1} + \gamma(a_{32}a_{13} - a_{12}a_{33})\widetilde{\Phi}_{+,2} + (a_{12}a_{23} - a_{22}a_{13})\widetilde{\Phi}_{+,3}.$$

Comparing with (2.45) yields

$$\widetilde{a}_{11} = a_{22}a_{33} - a_{32}a_{23}, \quad \widetilde{a}_{21} = \gamma(a_{32}a_{13} - a_{12}a_{33}), \quad \widetilde{a}_{31} = a_{12}a_{23} - a_{22}a_{13}.$$

In the similar way, we derived

$$\widetilde{a}_{12} = \frac{1}{\gamma}(a_{23}a_{31} - a_{33}a_{21}), \quad \widetilde{a}_{22} = a_{33}a_{11} - a_{13}a_{31}, \quad \widetilde{a}_{32} = \frac{1}{\gamma}(a_{13}a_{21} - a_{23}a_{11}),$$
$$\widetilde{a}_{13} = a_{21}a_{32} - a_{31}a_{22}, \quad \widetilde{a}_{23} = \gamma(a_{31}a_{12} - a_{11}a_{32}), \quad \widetilde{a}_{33} = a_{11}a_{22} - a_{21}a_{12}.$$

We note that

$$(A^{-1}(z))^T = \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{13}a_{32} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

We finally obtain

$$\widetilde{A}(z) = D(z)(A^{-1}(z))^T D^{-1}(z).$$

□

Next, combining Lemma 3 and the equation (2.45) in the definition (2.46) deriving Theorem2.5:

Theorem 2.5. *For $z \in \Sigma$, the Jost eigenfunction have the following relations*

$$\Phi_{-,1} = \frac{1}{a_{22}}[\chi_1 + a_{21}\Phi_{-,2}] = \frac{1}{a_{33}}[a_{31}\Phi_{-,3} + \chi_2], \tag{2.50a}$$

$$\Phi_{-,3} = \frac{1}{a_{22}}[\chi_4 + a_{23}\Phi_{-,2}] = \frac{1}{a_{11}}[a_{13}\Phi_{-,3} + \chi_3], \tag{2.50b}$$

$$\Phi_{+,1} = \frac{1}{b_{22}}[\chi_2 + b_{21}\Phi_{+,2}] = \frac{1}{b_{33}}[b_{31}\Phi_{+,3} + \chi_1], \tag{2.50c}$$

$$\Phi_{+,3} = \frac{1}{b_{22}}[\chi_3 + b_{23}\Phi_{+,2}] = \frac{1}{b_{11}}[b_{13}\Phi_{+,1} + \chi_4]. \tag{2.50d}$$

For convenience, we omit the independent variable.

Proof. Substituting (2.45) into (2.46) obtain the following equations

$$\chi_1 = \widetilde{b}_{22}e^{i\theta_2}[\widetilde{\Phi}_{-,1} \times \widetilde{\Phi}_{-,2}] + \widetilde{b}_{32}e^{i\theta_2}[\widetilde{\Phi}_{-,1} \times \widetilde{\Phi}_{-,3}], \tag{2.51a}$$

$$\chi_4 = \widetilde{b}_{12}e^{i\theta_2}[\widetilde{\Phi}_{-,1} \times \widetilde{\Phi}_{-,3}] + \widetilde{b}_{22}e^{i\theta_2}[\widetilde{\Phi}_{-,2} \times \widetilde{\Phi}_{-,3}], \tag{2.51b}$$

Substituting (2.47a) into (2.51) yields

$$\chi_1 = \widetilde{b}_{22}\Phi_{-,3} - \widetilde{b}_{32}\gamma\Phi_{-,2}, \tag{2.52a}$$

$$\chi_4 = \widetilde{b}_{12}\Phi_{-,2} + \widetilde{b}_{22}\gamma\Phi_{-,1}. \tag{2.52b}$$

Applying (2.48) to (2.52), we obtain

$$\Phi_{-,1} = \frac{1}{a_{22}}(\chi_4 + a_{21}\Phi_{-,2}), \quad \Phi_{-,3} = \frac{1}{a_{22}}(\chi_1 + a_{23}\Phi_{-,2}).$$

In the similar way we can obtain the rest of (2.5). □

To remove the exponential oscillations, we define the modified auxiliary eigenfunctions

$$m_j(x,t,z) = e^{-i\theta_1(x,t,z)}\chi_j(x,t,z), \quad j = 1,2 \tag{2.53a}$$

$$m_j(x,t,z) = e^{i\theta_1(x,t,z)}\chi_j(x,t,z), \quad j = 3,4 \tag{2.53b}$$

3 Symmetries and Asymptotics

For the zero boundary conditions with the only one the symmetry $k \mapsto \bar{k}$. However, for the nonzero boundary conditions, the symmetries are complicated because the Riemann surface introduced. It not only involved with the symmetries $k \mapsto \bar{k}$, but also involved with the symmetries $k \mapsto -\frac{q_0^2}{z}$. Namely that mapping the upper-half plane into the lower-half plane and mapping the exterior of circle C_0 of radius q_0 into the interior.

3.1 The First Symmetries

In this subsection, we firstly consider the map that $z \mapsto \bar{z}$.

Lemma 4. *If* $\Phi(x,t,z)$ *is a non-singular solution of the Lax pair (2.1), so is* $\mathrm{w}(x,t,z) = (\Phi^\dagger(x,t,\bar{z}))^{-1}$.

We also show that

Proof. If $\Phi(x,t,z)$ is a non-singular solution of the Lax pair. Then we have

$$\Phi_x^\dagger = \Phi^\dagger X^\dagger, \quad \Phi_t^\dagger = \Phi^\dagger T^\dagger. \tag{3.1}$$

If we insert X and T into (3.1), we have

$$\begin{aligned}
\mathrm{w}_x =& -[\Phi^\dagger(\bar{z})]^{-1}[\Phi^\dagger(\bar{z})]_x[\Phi^\dagger(\bar{z})]^{-1} \\
=& -[\Phi^\dagger(\bar{z})]^{-1}\Phi^\dagger(\bar{z})(ik\Lambda - Q)[\Phi^\dagger(\bar{z})]^{-1} \\
=& X\mathrm{w},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{w}_t =& -[\Phi^\dagger(\bar z)]^{-1}[\Phi^\dagger(\bar z)]_t[\Phi^\dagger(\bar z)]^{-1}\\
=& -[\Phi^\dagger(\bar z)]^{-1}\Phi^\dagger(\bar z)\left\{4ik^3\Lambda-4k^2Q-2ik(-Q_x-Q^2)\Lambda-2Q^3+Q_{xx}-[Q_x,Q]\right\}[\Phi^\dagger(\bar z)]^{-1}\\
=& T\mathrm{w}.
\end{aligned}$$

Then $\mathrm{w}(x,t,z)$ is a solution of the Lax pair. □

The Jost eigenfunctions satisfy the symmetry

Lemma 5. *The Jost eigenfunctions satisfy the following symmetry*

$$[\Phi_\pm^\dagger(x,t,\bar z)]^{-1}C(z)=\Phi_\pm(x,t,z),\quad z\in\Sigma \tag{3.4}$$

where $C(z)=\mathrm{diag}(\gamma(z),1,\gamma(z))$.

Proof. From the Lemma 4, we known that $\Phi_\pm(x,t,z)$ and $(\Phi_\pm^\dagger(x,t,\bar z))^{-1}$ are the solutions of the Lax pair. And, we note that

$$(e^{i\Theta(x,t,\bar z)})^\dagger=e^{-i\Theta(x,t,z)}.$$

$$\Phi_\pm(x,t,z)\sim\Gamma_\pm e^{i\Theta(z)},\quad x\to\pm\infty \tag{3.5}$$

and

$$\begin{aligned}
(\Phi_\pm^\dagger(x,t,\bar z))^{-1}\sim&[(\Gamma_\pm(\bar z)e^{i\Theta(\bar z)})^\dagger]^{-1}\\
&=[e^{-i\Theta(z)}\Gamma_\pm^\dagger(\bar z)]^{-1}=[\Gamma_\pm^\dagger(\bar z)]^{-1}e^{i\Theta(z)}\\
&=[C(z)\Gamma_\pm^{-1}(z)]^{-1}e^{i\Theta(z)}=\Gamma_\pm(z)C^{-1}(z)e^{i\Theta(z)}\\
&=\Gamma_\pm(z)e^{i\Theta(z)}C^{-1}(z).
\end{aligned}$$

Then, we have

$$[\Phi_\pm^\dagger(x,t,\bar z)]^{-1}C(z)\sim\Phi_\pm(x,t,z),\quad x\to\pm\infty.$$

□

And we note that between the columns of $\Phi_\pm(x,t,z)$ and $\Phi_\pm^{-1}(x,t,z)$ with special property.

Lemma 6. *The vector $\Phi_{\pm,j}(x,t,z)(j=1,2,3)$ and $\Phi_{\pm}^{-1}(x,t,z)$ with the following property*

$$[\Phi_{\pm}^{-1}(x,t,z)]^T = \frac{1}{\det\Phi_{\pm}(x,t,z)}(\Phi_{\pm,2}\times\Phi_{\pm,3},\Phi_{\pm,3}\times\Phi_{\pm,1},\Phi_{\pm,1}\times\Phi_{\pm,2})(x,t,z). \tag{3.7}$$

Using the Theorem 2.5 and Lemma 5 yields:

Lemma 7. *The Jost eigenfunctions obey the symmetry relations:*

$$\bar{\Phi}_{+,1}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{b_{22}}[\Phi_{+,2}\times\chi_3](x,t,z), \tag{3.8a}$$

$$\bar{\Phi}_{-,1}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{a_{22}}[\Phi_{-,2}\times\chi_4](x,t,z), \tag{3.8b}$$

$$\bar{\Phi}_{+,2}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{\gamma b_{11}}[\chi_4\times\Phi_{+,1}](x,t,z) = \frac{e^{-i\theta_2}}{\gamma b_{33}}[\Phi_{+,3}\times\chi_1](x,t,z), \tag{3.8c}$$

$$\bar{\Phi}_{-,2}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{\gamma a_{11}}[\chi_3\times\Phi_{-,1}](x,t,z) = \frac{e^{-i\theta_2}}{\gamma a_{33}}[\Phi_{-,3}\times\chi_2](x,t,z), \tag{3.8d}$$

$$\bar{\Phi}_{+,3}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{b_{22}}[\chi_2\times\Phi_{+,2}](x,t,z), \tag{3.8e}$$

$$\bar{\Phi}_{-,3}(x,t,\bar{z}) = \frac{e^{-i\theta_2}}{a_{22}}[\chi_1\times\Phi_{-,2}](x,t,z). \tag{3.8f}$$

Using the equation (2.28) and Lemma 5 yields:

Lemma 8. *The scattering matrix and its inverse satisfy the symmetry relation*

$$A^{\dagger}(\bar{z}) = C(z)B(z)C^{-1}(z), \qquad z\in\Sigma. \tag{3.9}$$

For all $z\in\Sigma$, the componentwise satisfy:

$$\begin{aligned}
&b_{11}(z)=\bar{a}_{11}(\bar{z}), && b_{12}(z)=\frac{\bar{a}_{21}(\bar{z})}{\gamma(z)}, && b_{13}(z)=\bar{a}_{31}(\bar{z}),\\
&b_{21}(z)=\gamma(z)\bar{a}_{12}(\bar{z}), && b_{22}(z)=\bar{a}_{22}(\bar{z}), && b_{23}(z)=\gamma(z)\bar{a}_{32}(\bar{z}),\\
&b_{31}(z)=\bar{a}_{13}(\bar{z}), && b_{32}(z)=\frac{\bar{a}_{23}(\bar{z})}{\gamma(z)}, && b_{33}(z)=\bar{a}_{33}(\bar{z}).
\end{aligned}$$

And we also derived the symmetry relations of the auxiliary eigenfunctions:

Lemma 9. *The auxiliary eigenfunctions satisfy the symmetry relations:*

$$\bar{\chi}_1(x,t,\bar{z}) = e^{-i\theta_2(z)}[\Phi_{+,2}\times\Phi_{-,3}](x,t,z), \tag{3.11a}$$

$$\bar{\chi}_2(x,t,\bar{z}) = e^{-i\theta_2(z)}[\Phi_{-,2}\times\Phi_{+,3}](x,t,z), \tag{3.11b}$$

$$\bar{\chi}_3(x,t,\bar{z}) = e^{-i\theta_2(z)}[\Phi_{+,1}\times\Phi_{-,2}](x,t,z), \tag{3.11c}$$

$$\bar{\chi}_4(x,t,\bar{z}) = e^{-i\theta_2(z)}[\Phi_{-,1}\times\Phi_{+,2}](x,t,z). \tag{3.11d}$$

The proof of Lemma 9 yields:

$$\bar{\Phi}_{\pm,j}(x,t,\bar{z}) = e^{-i\theta_2(x,t,z)}[\Phi_{\pm,l}\times\Phi_{\pm,m}](x,t,z)/\gamma_j, \tag{3.12}$$

where j, l, and m are cyclic indices and $z\in\Sigma$.

3.2 The Second Symmetries

In this subsection, we consider the map $z\mapsto -\frac{q_0^2}{z}$.

Lemma 10. *For all $z\in\Sigma$, the Jost eigenfunctions holds the symmetry relations*

$$\Phi_\pm(x,t,z) = \Phi_\pm(x,t,-\frac{q_0^2}{z})\Pi(z), \tag{3.13}$$

where

$$\Pi(z) = \begin{pmatrix} 0 & 0 & \frac{iq_0}{z} \\ 0 & 1 & 0 \\ \frac{iq_0}{z} & 0 & 0 \end{pmatrix}. \tag{3.14}$$

So we derived all of the above relations:

$$\Phi_{\pm,1}(x,t,z) = \frac{iq_0}{z}\Phi_{\pm,3}(x,t,-\frac{q_0^2}{z}), \tag{3.15a}$$

$$\Phi_{\pm,2}(x,t,z) = \Phi_{\pm,2}(x,t,-\frac{q_0^2}{z}), \tag{3.15b}$$

$$\Phi_{\pm,3}(x,t,z) = \frac{iq_0}{z}\Phi_{\pm,1}(x,t,-\frac{q_0^2}{z}). \tag{3.15c}$$

Lemma 11. *The scattering matrix holds the symmetry*

$$A(-\frac{q_0^2}{z}) = \Pi(z)A(z)\Pi^{-1}(z), \quad B(-\frac{q_0^2}{z}) = \Pi(z)B(z)\Pi^{-1}(z). \tag{3.16}$$

We have the following componentwise relations

$$a_{11}(z) = a_{33}(-\frac{q_0^2}{z}), \quad a_{12}(z) = \frac{z}{iq_0}a_{32}(-\frac{q_0^2}{z}), a_{13}(z) = a_{31}(-\frac{q_0^2}{z}),$$
$$a_{21}(z) = \frac{iq_0}{z}a_{23}(-\frac{q_0^2}{z}), a_{22}(z) = a_{22}(-\frac{q_0^2}{z}), \quad a_{23}(z) = \frac{iq_0}{z}a_{21}(-\frac{q_0^2}{z}),$$
$$a_{31}(z) = a_{13}(-\frac{q_0^2}{z}), \quad a_{32}(z) = \frac{z}{iq_0}a_{12}(-\frac{q_0^2}{z}), a_{33}(z) = a_{11}(-\frac{q_0^2}{z}).$$

Lemma 12. *The auxiliary eigenfunctions holds the symmetries relations*

$$\chi_1(z) = \frac{iq_0}{z}\chi_4(-\frac{q_0^2}{z}), \quad z \in D_1 \tag{3.18a}$$
$$\chi_2(z) = \frac{iq_0}{z}\chi_3(-\frac{q_0^2}{z}), \quad z \in D_2. \tag{3.18b}$$

3.3 The Symmetry of Reflection Coefficients

For the reasons that the reflection coefficients will be considered in the rest discussion. It is necessary that we consider the symmetry of reflection coefficients. One can derive the symmetries of reflection coefficients based on the above two symmetries.

$$\rho_1(z) = \frac{a_{21}(z)}{a_{11}(z)} = \frac{\gamma(z)\bar{b}_{12}(\bar{z})}{\bar{b}_{11}(\bar{z})}, \quad \rho_2(z) = \frac{a_{31}(z)}{a_{11}(z)} = \frac{\bar{b}_{13}(\bar{z})}{\bar{b}_{11}(\bar{z})}, \tag{3.19a}$$
$$\rho_3(z) = \frac{a_{32}(z)}{a_{22}(z)} = \frac{\bar{b}_{23}(\bar{z})}{\gamma(z)\bar{b}_{22}(\bar{z})}, \quad \rho_1(-\frac{q_0^2}{z}) = \frac{z}{iq_0}\frac{a_{23}(z)}{a_{33}(z)} = \frac{z}{iq_0}\frac{\gamma(z)\bar{b}_{32}(\bar{z})}{\bar{b}_{33}(\bar{z})}, \tag{3.19b}$$
$$\rho_2(-\frac{q_0^2}{z}) = \frac{a_{13}(z)}{a_{33}(z)} = \frac{\bar{b}_{31}(\bar{z})}{\bar{b}_{33}(\bar{z})}, \quad \rho_3(-\frac{q_0^2}{z}) = \frac{iq_0}{z}\frac{a_{12}(z)}{a_{22}(z)} = \frac{iq_0}{z}\frac{\bar{b}_{21}(\bar{z})}{\gamma(z)\bar{b}_{22}(\bar{z})}, \tag{3.19c}$$

3.4 Asymptotic Behavior

In order to normalize the Riemann-Hilbert problem, it is necessary to consider the asymptotic behavior of the eigenfunctions and scattering coefficients as $k \to \infty$, which corresponds to the behavior both $z \to \infty$ and $z \to 0$. So we consider the following expansion for $\Gamma_\pm^{-1}\mu_\pm$

$$\Gamma_\pm^{-1}\mu_\pm = E_0 + \frac{E_1}{z} + \frac{E_2}{z^2} + \cdots \quad z \to \infty, \tag{3.20a}$$
$$\Gamma_\pm^{-1}\mu_\pm = F_0 + F_1 z + F_2 z^2 + \cdots \quad z \to 0, \tag{3.20b}$$

where E_j and F_j are 3×3 matrices and independent of z. Through some explicitly evaluate, we obtain the asymptotic behavior. For the convenience of expression, the block diagonal and block off-diagonal matrixs about a arbitrary 3×3 matrix S are defined as following:

$$S_{bd}=\begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad S_{bo}=\begin{pmatrix} 0 & 0 & s_{13} \\ 0 & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix}, \quad [S_{bd}]_o=\begin{pmatrix} 0 & s_{12} & 0 \\ s_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And we note that for arbitrary two matrixes S_1 and S_2 with the following qualities

$$[UV]_{bd}=U_{bd}V_{bd}+U_{bo}V_{bo}, \qquad [UV]_{bo}=U_{bd}V_{bo}+U_{bo}V_{bd},$$
$$[U_{bd}V_{bd}]_d=U_dV_d+[U_{bd}]_o[V_{bd}]_o, \quad [U_{bd}V_{bd}]_o=U_d[V_{bd}]_o+[U_{bd}]_oV_d.$$

So we show the following theorem:

Theorem 3.1. *For $z\to\infty$ and $z\to 0$, the asymptotic of $\mu_\pm$ as following*

$$\mu_\pm(x,t,z)=\begin{pmatrix} -\frac{\mathbf{q}_\pm}{q_0} & \frac{\mathbf{q}_\pm^\perp}{q_0} & -\frac{i\mathbf{q}}{z} \\ \frac{i\mathbf{q}_\pm^\dagger\mathbf{q}}{zq_0} & \frac{i(\mathbf{q}_\pm^\perp)^\dagger\mathbf{q}}{zq_0} & 1 \end{pmatrix}+\frac{1}{z}[\Gamma_\pm]_{bd}[E_1]_{bd}+o(\frac{1}{z^2}), \quad z\to\infty. \tag{3.22a}$$

$$\mu_\pm(x,t,z)=\begin{pmatrix} -\frac{\mathbf{q}}{q_0} & \frac{\mathbf{q}_\pm^\perp}{q_0} & -\frac{i\mathbf{q}_\pm}{z} \\ \frac{iq_0}{z} & 0 & \frac{\mathbf{q}_\pm^\dagger\mathbf{q}}{q_0^2} \end{pmatrix}+[\Gamma_\pm]_{bd}[F_0]_{bo}+o(1), \quad z\to 0. \tag{3.22b}$$

Theorem 3.2. *For $z\to\infty$, the asymptotic of m_j as following*

$$m_1(x,t,z)=\begin{pmatrix} -\frac{\mathbf{q}_+}{q_0} \\ \frac{i\mathbf{q}_+^\dagger\mathbf{q}}{zq_0} \end{pmatrix}+o(\frac{1}{z^2}), \quad m_2(x,t,z)=\begin{pmatrix} -\frac{\mathbf{q}_-}{q_0} \\ \frac{i\mathbf{q}_-^\dagger\mathbf{q}}{zq_0} \end{pmatrix}+o(\frac{1}{z^2}), \tag{3.23a}$$

$$m_3(x,t,z)=\begin{pmatrix} -\frac{i\mathbf{q}}{zq_0^2}[\mathbf{q}_-^\dagger\mathbf{q}_+] \\ \frac{1}{q_0^2}[\mathbf{q}_-^\dagger\mathbf{q}_+] \end{pmatrix}+o(\frac{1}{z^2}), \quad m_4(x,t,z)=\begin{pmatrix} -\frac{i\mathbf{q}}{zq_0^2}[\mathbf{q}_-^\dagger\mathbf{q}_+] \\ \frac{1}{q_0^2}[\mathbf{q}_-^\dagger\mathbf{q}_+] \end{pmatrix}+o(\frac{1}{z^2}) \tag{3.23b}$$

Theorem 3.3. *For $z \to 0$, the asymptotic of m_j as following*

$$m_1(x,t,z) = \begin{pmatrix} \mathbf{0} \\ \frac{i}{zq_0}[\mathbf{q}_-^\dagger \mathbf{q}_+] \end{pmatrix} + o(1), \quad m_2(x,t,z) = \begin{pmatrix} \mathbf{0} \\ \frac{i}{zq_0}[\mathbf{q}_-^\dagger \mathbf{q}_+] \end{pmatrix} + o(1), \tag{3.24a}$$

$$m_3(x,t,z) = \begin{pmatrix} -\frac{i\mathbf{q}_-}{z} \\ 0 \end{pmatrix} + o(1), \quad m_4(x,t,z) = \begin{pmatrix} -\frac{i\mathbf{q}_\pm}{z} \\ 0 \end{pmatrix} + o(1) \tag{3.24b}$$

Proof. Substituting (3.20) into the system (2.14) and compare the coefficients of z^j, we deriving the following results

$$\begin{aligned}
&E_0^{(12)} = E_0^{(13)} = E_0^{(21)} = E_0^{(31)} = E_0^{(23)} = E_0^{(32)} = 0,\\
&E_0^{(11)} = E_0^{(22)} = E_0^{(33)} = 1,\\
&E_1^{(13)} = \frac{i}{q_0}[\bar{q}q_\pm + q\bar{q}_\pm - q_0^2], \quad E_1^{(23)} = \frac{i}{q_0}[\bar{q}q_\pm - q\bar{q}_\pm],\\
&E_1^{(31)} = \frac{i}{q_0}[\bar{q}q_\pm + q\bar{q}_\pm - q_0^2], \quad E_1^{(32)} = \frac{i}{q_0}[q\bar{q}_\pm - \bar{q}q_\pm].\\
&F_0^{(12)} = F_0^{(13)} = F_0^{(31)} = F_0^{(32)} = F_1^{(12)} = F_1^{(32)} = 0,\\
&F_0^{(11)} = F_0^{(22)} = F_0^{(33)} = 1,\\
&F_0^{(21)} = \frac{\bar{q}q_\pm - \bar{q}_\pm q}{q_0^2}, \quad F_0^{'(23)} = \frac{q^2\bar{q}_\pm^2 - q_\pm^2\bar{q}}{q_0^3},\\
&F_1^{(13)} = F_1^{(31)} = -\frac{i}{q_0^3}[\bar{q}q_\pm + \bar{q}_\pm q - q_0^2].
\end{aligned}$$

□

Theorem 3.4. *For $z \to \infty$, the asymptotic of a_{jj} as following*

$$a_{11} = \frac{\mathbf{q}_+^\dagger \mathbf{q}_-}{q_0^2} + o(\frac{1}{z}), \quad a_{22} = \frac{\mathbf{q}_+^\dagger \mathbf{q}_-}{q_0^2} + o(\frac{1}{z}), \quad a_{33} = 1 + o(\frac{1}{z}), \tag{3.26a}$$

$$b_{11} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(\frac{1}{z}), \quad b_{22} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(\frac{1}{z}), \quad b_{33} = 1 + o(\frac{1}{z}), \tag{3.26b}$$

Theorem 3.5. *For $z \to 0$, the asymptotic of a_{jj} as following*

$$a_{11} = 1 + o(z), \quad a_{22} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(z), \quad a_{33} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(z), \tag{3.27a}$$

$$b_{11} = 1 + o(z), \quad b_{22} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(z), \quad b_{33} = \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_0^2} + o(z), \tag{3.27b}$$

4 Riemann-Hilbert Problem

4.1 Distribution of Discrete Spectral

For the 3×3 matrix spectral problem, the characters of discrete spectral is more complicate than 2×2 matrix spectral problem. Firstly, we introduce the following 3×3 matrix:

$$\begin{aligned}
\Xi_1(z) &= (\chi_1(z), \Phi_{-,2}(z), \Phi_{+,3}(z)), \qquad && z \in D_1 \\
\Xi_2(z) &= (\chi_2(z), \Phi_{+,2}(z), \Phi_{-,3}(z)), \qquad && z \in D_2 \\
\Xi_3(z) &= (\Phi_{-,1}(z), \Phi_{+,2}(z), \chi_3(z)), \qquad && z \in D_3 \\
\Xi_4(z) &= (\Phi_{+,1}(z), \Phi_{-,2}(z), \chi_4(z)), \qquad && z \in D_4
\end{aligned}$$

Then the determinant of $\Xi_j(z)$ can be derived via some explicit calculation:

$$\begin{aligned}
&\mathrm{Wr}\Xi_1(z) = a_{22}(z)b_{33}(z)\gamma e^{i\theta_2}, \quad \mathrm{Wr}\Xi_2(z) = a_{33}(z)b_{22}(z)\gamma e^{i\theta_2}, \\
&\mathrm{Wr}\Xi_3(z) = a_{11}(z)b_{22}(z)\gamma e^{i\theta_2}, \quad \mathrm{Wr}\Xi_4(z) = a_{22}(z)b_{11}(z)\gamma e^{i\theta_2}.
\end{aligned}$$

The columns of $\Xi_1(z)$ is linearly dependent at the zeros of $a_{22}(z)$ and $b_{33}(z)$. In the similar way, the other equations can be obtain the similarly properties. For the existence of symmetries of scattering coefficient, we known that the zeros is dependent of each other.

Lemma 13. *Let* $\mathrm{Im} z_0 > 0$*, then*

$$a_{22}(z_0) = 0 \Longleftrightarrow b_{22}(\bar{z}_0) = 0 \Longleftrightarrow b_{22}(-\frac{q_0^2}{\bar{z}_0}) = 0 \Longleftrightarrow a_{22}(-\frac{q_0^2}{z_0}) = 0.$$

Lemma 14. *Let* $\mathrm{Im} z_0 > 0$ *and* $|z_0| \geq q_0$ *then*

$$b_{33}(z_0) = 0 \Longleftrightarrow a_{33}(\bar{z}_0) = 0 \Longleftrightarrow a_{11}(-\frac{q_0^2}{\bar{z}_0}) = 0 \Longleftrightarrow b_{11}(-\frac{q_0^2}{z_0}) = 0.$$

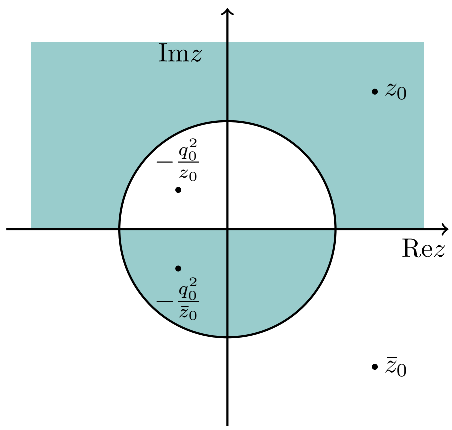


Figure 2: The distribution of discrete spectral.

From Lemma13 and Lemma14 we derive the conclusion that the discrete eigenvalues appear in the four kind points: z_0, $\bar{z}_0$, $-\frac{q_0^2}{z_0}$, $-\frac{q_0^2}{\bar{z}_0}$. The distribution of discrete spectral can be shown in Figure 2. It is similar with the 2×2 matrix spectral problem. So, it is enough to just study the zeros of $a_{22}(z)$ and $b_{33}(z)$. The zeros are divided into the following three types:

- ▲ the first kind eigenvalue: $a_{22}(z_0) = 0$ and $b_{33}(z_0) \neq 0$.
- ▲ the second kind eigenvalue: $a_{22}(z_0) \neq 0$ and $b_{33}(z_0) = 0$.
- ▲ the third kind eigenvalue: $a_{22}(z_0) = 0$ and $b_{33}(z_0) = 0$.

Some results can be obtained:

Lemma 15. *For* $\mathrm{Im} z_0 > 0$ *and* $|z_0| > q_0$*, the following results are equivalent*

(1) $\chi_1(z_0) = 0$*;*

(2) $\chi_4(-\frac{q_0^2}{z_0}) = 0$*;*

(3) $\exists b_0$*, so that* $\Phi_{+,2}(\bar{z}_0) = b_0\Phi_{-,3}(\bar{z}_0)$*;*

(4) $\exists \tilde{b}_0$*, so that* $\Phi_{+,2}(-\frac{q_0^2}{\bar{z}_0}) = \tilde{b}_0\Phi_{-,1}(-\frac{q_0^2}{\bar{z}_0})$*;*

Proof. (1) $\Longleftrightarrow$ (2): From (3.18a), it can be proofed.

(1) $\Longleftrightarrow$ (3): From (2.46) and (3.8), it can be derived.

(1) $\Longleftrightarrow$ (4): From (3.15), it can be obtained. □

In the similar way, we have

Lemma 16. *For* $\mathrm{Im} z_0 > 0$ *and* $|z_0| > q_0$*, the following results are equivalent*

(1) $\chi_2(\bar{z}_0) = 0$*;*

(2) $\chi_3(-\frac{q_0^2}{\bar{z}_0}) = 0$*;*

(3) $\exists \hat{b}_0$*, so that* $\Phi_{-,2}(z_0) = \hat{b}_0 \Phi_{+,3}(z_0)$*;*

(4) $\exists \check{b}_0$*, so that* $\Phi_{-,2}(-\frac{q_0^2}{z_0}) = \check{b}_0 \Phi_{+,1}(-\frac{q_0^2}{z_0})$*;*

Lemma 17. *For* $\mathrm{Im} z_0$ *and* $|z_0| > q_0$*,* $a_{22}(z_0) b_{33}(z_0) = 0$*,*

• *If* z_0 *is the first kind eigenvalue, then*

$$\begin{aligned} &\Phi_{-,2}(z_0) = h_0 \chi_1(z_0), && \chi_2(\bar{z}_0) = \hat{h}_0 \Phi_{+,2}(\bar{z}_0), \\ &\chi_3(-\frac{q_0^2}{\bar{z}_0}) = \check{h}_0 \Phi_{+,2}(-\frac{q_0^2}{\bar{z}_0}), && \Phi_{-,2}(-\frac{q_0^2}{z_0}) = \tilde{h}_0 \chi_4(-\frac{q_0^2}{z_0}). \end{aligned}$$

• *If* z_0 *is the second kind eigenvalue, then*

$$\begin{aligned} &\chi_1(z_0) = f_0 \Phi_{+,3}(z_0), && \Phi_{-,3}(\bar{z}_0) = \hat{f}_0 \chi_2(\bar{z}_0), \\ &\Phi_{-,1}(-\frac{q_0^2}{\bar{z}_0}) = \check{f}_0 \chi_3(-\frac{q_0^2}{\bar{z}_0}), && \chi_4(-\frac{q_0^2}{z_0}) = \tilde{f}_0 \Phi_{+,1}(-\frac{q_0^2}{z_0}). \end{aligned}$$

• *If* z_0 *is the third kind eigenvalue, then* $\chi_1(z_0) = \chi_2(\bar{z}_0) = 0$ *and*

$$\begin{aligned} &\Phi_{-,2}(z_0) = g_0 \Phi_{+,3}(z_0), \quad \Phi_{-,3}(\bar{z}_0) = \hat{g}_0 \Phi_{+,2}(\bar{z}_0), \\ &\Phi_{-,1}(-\frac{q_0^2}{\bar{z}_0}) = \check{g}_0 \Phi_{+,2}(-\frac{q_0^2}{\bar{z}_0}) \quad \Phi_{-,2}(-\frac{q_0^2}{z_0}) = \tilde{g}_0 \Phi_{+,1}(-\frac{q_0^2}{z_0}). \end{aligned}$$

Proof. If $a_{22}(z_0) = 0$ and $b_{33}(z_0) \neq 0$, we can derived $\Phi_{-,2}(z_0)$ and $\chi_1(z_0)$ are linear dependent. So there exist a constant value h_0, such that $\Phi_{-,2}(z_0) = h_0 \chi_1(z_0)$. The rest equations of Lemma17 can be proofed in the similar way. □

For deriving the residue conditions is convenient, Lemma17 can be rewrite in terms of the modified eigenfunctions.

• If $\{z_n | n = 1, 2, \cdots N_1\}$ are the first kind eigenvalues

$$\begin{aligned} &\mu_{-,2}(z_n) = h_n e^{i(\theta_1 - \theta_2)(z_n)} m_1(z_n), && m_2(\bar{z}_n) = \hat{h}_n e^{-i(\theta_1 - \theta_2)(\bar{z}_n)} \mu_{+,2}(\bar{z}_n), \\ &m_3(-\frac{q_0^2}{\bar{z}_n}) = \check{h}_n e^{i(\theta_1 + \theta_2)(-\frac{q_0^2}{\bar{z}_n})} \mu_{+,2}(-\frac{q_0^2}{\bar{z}_n}), && \mu_{-,2}(-\frac{q_0^2}{z_n}) = \tilde{h}_n e^{-i(\theta_1 + \theta_2)(-\frac{q_0^2}{z_n})} m_4(-\frac{q_0^2}{z_n}). \end{aligned}$$

- If $\{\zeta_n | n = 1, 2, \cdots N_2\}$ is the second kind eigenvalues, then

$$m_1(\zeta_n) = f_n e^{-2i\theta_1(\zeta_n)}\mu_{+,3}(\zeta_n), \qquad \mu_{-,3}(\bar{\zeta}_n) = \hat{f}_n e^{2i\theta_1(\bar{\zeta}_n)} m_2(\bar{\zeta}_n),$$
$$\mu_{-,1}(-\frac{q_0^2}{\bar{\zeta}_n}) = \check{f}_n e^{-2i\theta_1(-\frac{q_0^2}{\zeta_n})} m_3(-\frac{q_0^2}{\bar{\zeta}_n}), \quad m_4(-\frac{q_0^2}{\zeta_n}) = \tilde{f}_n e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}\mu_{+,1}(-\frac{q_0^2}{\zeta_n}).$$

- If $\{\omega_n | n = 1, 2, \cdots N_3\}$ is the third kind eigenvalues

$$\mu_{-,2}(\omega_n) = g_n e^{-i(\theta_1+\theta_2)(\omega_n)}\mu_{+,3}(\omega_n), \quad \mu_{-,3}(\bar{\omega}_n) = \hat{g}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)}\mu_{+,2}(\bar{\omega}_n),$$
$$\mu_{-,1}(-\frac{q_0^2}{\bar{\omega}_n}) = \check{g}_n e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})}\mu_{+,2}(-\frac{q_0^2}{\bar{\omega}_n}), \quad \mu_{-,2}(-\frac{q_0^2}{\omega_n}) = \tilde{g}_n e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}\mu_{+,1}(-\frac{q_0^2}{\omega_n}).$$

4.2 Riemann-Hilbert Problem

In this section, we transform solving the asymptotic matrix spectral problem into solving an appropriate RH problem. So we need construct a jump condition that is similar with 2×2. The difference is that the jump matrixes are constructed four times. For convenience, we make some expressions that $D_+ = D_1 \cup D_3$ and $D_- = D_2 \cup D_4$.

Theorem 4.1. *The meromorphic function $M(x,t,z) = M_j(x,t,z)$, $z \in D_j (j = 1, 2, 3, 4)$*

$$M^+(x,t,z) = \begin{cases} M_1(x,t,z) = \left(\frac{m_1}{b_{33}}, \frac{\mu_{-,2}}{a_{22}}, \mu_{+,3}\right), & z \in D_1 \\ M_3(x,t,z) = \left(\frac{\mu_{-,1}}{a_{11}}, \mu_{+,2}, \frac{m_3}{b_{22}}\right), & z \in D_3 \end{cases}$$

$$M^-(x,t,z) = \begin{cases} M_2(x,t,z) = \left(\frac{m_2}{b_{22}}, \mu_{+,2}, \frac{\mu_{-,3}}{a_{33}}\right), & z \in D_2 \\ M_4(x,t,z) = \left(\mu_{+,1}, \frac{\mu_{-,2}}{a_{22}}, \frac{m_4}{b_{11}}\right), & z \in D_4 \end{cases}$$

and $M_j(x,t,z)$ hold the jump conditions

$$M^+(x,t,z) = M^-(x,t,z)[I - e^{i\Theta(x,t,z)}L(z)e^{-i\Theta(x,t,z)}], \quad z \in \Sigma$$

where $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, here $\Sigma_j = \bar{D}_j \cap \bar{D}_{j+1} (j = 1, 2, 3, 4)$. The matrix

$L(z)$ are given as

$$L(z)=\begin{pmatrix} -\rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}}) & \rho_3(z)\rho_2(-\frac{q_0^2}{z})-\frac{z}{iq_0}\rho_3(-\frac{q_0^2}{z}) & \rho_2(-\frac{q_0^2}{z}) \\ l_1 & l_2 & l_3 \\ \bar{\rho}_2(-\frac{q_0^2}{\bar{z}}) & -\rho_3(z) & 0 \end{pmatrix}, \quad z\in\Sigma_1, \tag{4.9a}$$

$$L(z)=\begin{pmatrix} \rho_2(z)\rho_2(-\frac{q_0^2}{z}) & 0 & \rho_2(-\frac{q_0^2}{z}) \\ l_4 & 0 & l_5 \\ \rho_2(z) & 0 & 0 \end{pmatrix}, \quad z\in\Sigma_2, \tag{4.9b}$$

$$L(z)=\begin{pmatrix} l_6 & l_7 & l_8 \\ -\rho_1(z) & 0 & \bar{\rho}_3(\bar{z})\gamma(z) \\ -\rho_2(z)+\rho_1(z)\rho_3(z) & \rho_3(z) & -\gamma z\bar{\rho}_3(\bar{z})\rho_3(z) \end{pmatrix}, \quad z\in\Sigma_3, \tag{4.9c}$$

$$L(z)=\begin{pmatrix} \bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\bar{\rho}_2(\bar{z}) & 0 & -\bar{\rho}_2(\bar{z}) \\ 0 & 0 & 0 \\ \bar{\rho}_2(-\frac{q_0^2}{\bar{z}}) & 0 & 0 \end{pmatrix}, \quad z\in\Sigma_4, \tag{4.9d}$$

where

$$l_1 = -\left(\gamma(z)\frac{z}{-iq_0}(\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}})) + \bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\rho_1(-\frac{q_0^2}{z})\frac{iq_0}{z}\right),$$

$$l_2 = \gamma(z)\frac{z}{-iq_0}\left(\rho_3(z)\rho_2(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) - \frac{z}{iq_0}\rho_3(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}})\right) + \frac{iq_0}{z}\rho_3(z)\rho_1(-\frac{q_0^2}{z}),$$

$$l_3 = \gamma(z)\frac{z}{-iq_0}\rho_2(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \frac{iq_0}{z}\rho_1(-\frac{q_0^2}{z}),$$

$$l_4 = \gamma(z)\frac{z}{-iq_0}(\rho_2(z)\rho_2(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) - \bar{\rho}_3(-\frac{q_0^2}{\bar{z}})) + \frac{iq_0}{z}\rho_2(z)\rho_1(-\frac{q_0^2}{z}) - \rho_1(z),$$

$$l_5 = \gamma(z)\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \gamma(z)\frac{z}{-iq_0}\rho_2(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \frac{iq_0}{z}\rho_1(-\frac{q_0^2}{z}),$$

$$l_6 = \frac{z}{iq_0}\rho_1(z)\rho_3(-\frac{q_0^2}{z}) - \rho_2(z)\bar{\rho}_2(\bar{z}) + \rho_1(z)\rho_3(z)\bar{\rho}_2(\bar{z}),$$

$$l_7 = \frac{z}{iq_0}\rho_3(-\frac{q_0^2}{z}) + \rho_3(z)\bar{\rho}_2(\bar{z}),$$

$$l_8 = -\gamma(z)\frac{z}{iq_0}\bar{\rho}_3(\bar{z})\rho_3(-\frac{q_0^2}{z}) - \gamma(z)\rho_3(z)\bar{\rho}_3(\bar{z})\bar{\rho}_2(\bar{z}) - \bar{\rho}_2(\bar{z}).$$

Proof. From (2.28) and Theorem 2.5, we can obtain

$$\frac{\Phi_{-,2}}{a_{22}} = \left(\frac{a_{12}}{a_{22}} - \frac{a_{32}}{a_{22}}\frac{a_{13}}{a_{33}}\right)\frac{\chi_2}{b_{22}} + \left(1 + \frac{a_{12}}{a_{22}}\frac{b_{21}}{b_{22}} - \frac{a_{32}}{a_{22}}\frac{a_{13}}{a_{33}}\frac{b_{21}}{b_{22}} - \frac{a_{32}}{a_{22}}\frac{a_{23}}{a_{33}}\right)\Phi_{+,2} + \frac{a_{32}}{a_{22}}\frac{\Phi_{-,3}}{a_{33}},$$

$$\Phi_{+,3} = -\frac{a_{13}}{a_{33}}\frac{\chi_2}{b_{22}} + \left(-\frac{a_{13}}{a_{33}}\frac{b_{21}}{b_{22}} - \frac{a_{23}}{a_{33}}\right)\Phi_{+,2} + \frac{\Phi_{-,3}}{a_{33}},$$

$$\frac{\chi_1}{b_{33}} = \left(1 + \frac{b_{31}}{b_{33}}\frac{a_{13}}{a_{33}}\right)\frac{\chi_2}{b_{22}} + \left(\frac{b_{21}}{b_{22}} + \frac{b_{31}}{b_{33}}\frac{a_{13}}{a_{33}}\frac{b_{21}}{b_{22}} + \frac{b_{31}}{b_{33}}\frac{a_{23}}{a_{33}}\right)\Phi_{+,2} - \frac{b_{31}}{b_{33}}\frac{\Phi_{-,3}}{a_{33}}.$$

Then we can obtain the matrix $L(z)$ as $z \in \Sigma_1$. The others can be derived in the similar way. □

From the asymptotic behavior of $\mu_\pm$, m_j and scattering coefficient in Section 5, the asymptotic behavior of $M(x,t,z)$ can be obtained:

$$M(x,t,z) = \begin{pmatrix} -\frac{\mathbf{q}_+}{q_0} & \frac{\mathbf{q}_+^\perp}{q_0} & 0 \\ 0 & 0 & 1 \end{pmatrix} + o(\frac{1}{z}), \quad z \to \infty, \tag{4.12}$$

$$M(x,t,z) = \begin{pmatrix} 0 & 0 & -\frac{i\mathbf{q}_+}{z} \\ \frac{iq_0}{z} & 0 & 0 \end{pmatrix} + o(1), \quad z \to 0, \tag{4.13}$$

For convenience, we define the notation $M^{\pm} = (m_1^{\pm}, m_2^{\pm}, m_3^{\pm})$, and $M^{\pm}_{-1,\nu}(x,t)$ represent the residue of $M^{\pm}$ at $z = \nu$.

Theorem 4.2. *The meromorphic functions $M^{\pm}$ satisfy the residue conditions*

$$M^{+}_{-1,z_n} = \left(0, H_n m_1^{+}(z_n) e^{i(\theta_1-\theta_2)(z_n)}, 0\right), \tag{4.14a}$$

$$M^{-}_{-1,\bar{z}_n} = \left(\hat{H}_n m_2^{-}(\bar{z}_n) e^{-i(\theta_1-\theta_2)(\bar{z}_n)}, 0, 0\right) \tag{4.14b}$$

$$M^{+}_{-1,-\frac{q_0^2}{\bar{z}_n}} = \left(0, 0, \check{H}_n m_2^{-}(\bar{z}_n) e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})}\right), \tag{4.14c}$$

$$M^{-}_{-1,-\frac{q_0^2}{z_n}} = \left(0, \frac{z_n}{iq_0}\tilde{H}_n m_1^{+}(z_n) e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_n})}, 0\right) \tag{4.14d}$$

$$M^{+}_{-1,\zeta_n} = \left(F_n m_3^{+}(\zeta_n) e^{-2i\theta_1(\zeta_n)}, 0, 0\right), \tag{4.14e}$$

$$M^{-}_{-1,\bar{\zeta}_n} = \left(0, 0, \hat{F}_n m_1^{-}(\bar{\zeta}_n) e^{2i\theta_1(\bar{\zeta}_n)}\right) \tag{4.14f}$$

$$M^{+}_{-1,-\frac{q_0^2}{\zeta_n}} = \left(\frac{\bar{\zeta}_n}{iq_0}\check{F}_n m_1^{-}(\bar{\zeta}_n) e^{-2i\theta_1(-\frac{q_0^2}{\zeta_n})}, 0, 0\right) \tag{4.14g}$$

$$M^{-}_{-1,-\frac{q_0^2}{\zeta_n}} = \left(0, 0, \frac{\zeta_n}{iq_0}\tilde{F}_n m_3^{+}(\zeta_n) e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}\right) \tag{4.14h}$$

$$M^{+}_{-1,\omega_n} = \left(0, G_n m_3^{+}(\omega_n) e^{-i(\theta_1+\theta_2)(\omega_n)}, 0\right), \tag{4.14i}$$

$$M^{-}_{-1,\bar{\omega}_n} = \left(0, 0, \hat{G}_n m_2^{-}(\bar{\omega}_n) e^{i(\theta_1+\theta_2)(\bar{\omega}_n)}\right) \tag{4.14j}$$

$$M^{+}_{-1,-\frac{q_0^2}{\bar{\omega}_n}} = \left(\check{G}_n m_2^{-}(\bar{\omega}_n) e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})}, 0, 0\right) \tag{4.14k}$$

$$M^{-}_{-1,-\frac{q_0^2}{\omega_n}} = \left(0, \frac{\omega_n}{iq_0}\tilde{G}_n m_3^{+}(\omega_n) e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}, 0\right) \tag{4.14l}$$

where

$$H_n = \frac{h_n b_{33}(z_n)}{a'_{22}(z_n)}, \quad \hat{H}_n = \frac{\hat{h}_n}{b'_{22}(\bar{z}_n)}, \quad \check{H}_n = \frac{\check{h}_n}{b'_{22}(-\frac{q_0^2}{\bar{z}_n})}, \quad \tilde{H}_n = \frac{\tilde{h}_n b_{33}(z_n)}{a'_{22}(-\frac{q_0^2}{z_n})} \tag{4.15a}$$

$$F_n = \frac{f_n}{b'_{33}(\zeta_n)}, \quad \hat{F}_n = \frac{\hat{f}_n b_{22}(\bar{\zeta}_n)}{a'_{33}(\bar{\zeta}_n)}, \quad \check{F}_n = \frac{\check{f}_n b_{22}(\bar{\zeta}_n)}{a'_{11}(-\frac{q_0^2}{\zeta_n})}, \quad \tilde{F}_n = \frac{\tilde{f}_n}{b'_{11}(-\frac{q_0^2}{\zeta_n})} \tag{4.15b}$$

$$G_n = \frac{g_n}{a'_{22}(\omega_n)}, \quad \hat{G}_n = \frac{\hat{g}_n}{a'_{33}(\bar{\omega}_n)}, \quad \check{G}_n = \frac{\check{g}_n}{a'_{11}(-\frac{q_0^2}{\bar{\omega}_n})}, \quad \tilde{G}_n = \frac{\tilde{g}_n}{a'_{22}(-\frac{q_0^2}{\omega_n})} \tag{4.15c}$$

Lemma 18. *The constants in Theorem17 obey the following relations*

$$\tilde{h}_n = \frac{iq_0}{z_n} h_n, \quad \hat{h}_n = \frac{iq_0}{\bar{z}_n} \check{h}_n = -\gamma(\bar{z}_n) a_{33}(\bar{z}_n) \bar{h}_n \tag{4.16a}$$

$$\tilde{f}_n = f_n, \quad \hat{f}_n = \check{f}_n = -\frac{\bar{f}_n}{\bar{a}_{22}(\zeta_n)} \tag{4.16b}$$

$$\tilde{g}_n = \frac{iq_0}{\omega_n} g_n, \quad \hat{g}_n = \frac{iq_0}{\bar{\omega}_n} \check{g}_n = -\frac{a'_{33}(\bar{\omega}_n)}{b'_{22}(\bar{\omega}_n)} \gamma(\bar{\omega}_n) \bar{g}_n. \tag{4.16c}$$

Lemma 19. *The constants in Theorem4.2 obey the following relations*

$$\hat{H}_n = -\gamma(\bar{z}_n) \bar{H}_n, \quad \check{H}_n = \frac{iq_0}{\bar{z}_n} \gamma(\bar{z}_n) \bar{H}_n, \quad \tilde{H}_n = -\frac{z_n}{iq_0} H_n, \tag{4.17a}$$

$$\hat{F}_n = -\bar{F}_n, \quad \check{F}_n = -\frac{q_0^2}{\bar{\zeta}_n^2} \bar{F}_n, \quad \tilde{F}_n = \frac{q_0^2}{\zeta_n^2} F_n, \tag{4.17b}$$

$$\hat{G}_n = -\gamma(\bar{\omega}_n) \bar{G}_n, \quad \check{G}_n = -\frac{\bar{\omega}_n^3}{iq_0^3} \gamma(\bar{\omega}_n) \bar{G}_n, \quad \tilde{G}_n = -\frac{\omega_n}{iq_0} G_n \tag{4.17c}$$

Proof. From equation (3.7) and the first symmetry of Jost eigenfunctions

$$(\Phi^{\dagger}_{\pm}(x,t,\bar{z}))^{-1} C(z) = \Phi_{\pm}(x,t,z), \tag{4.18}$$

we obtain

$$\Phi_{-,3}(\bar{z}) = \frac{\gamma(\bar{z})}{\det \bar{\Phi}_{-}(z)} [\bar{\Phi}_{-,1}(z) \times \bar{\Phi}_{-,2}(z)], \tag{4.19a}$$

$$\Phi_{+,2}(\bar{z}) = \frac{1}{\det \bar{\Phi}_{+}(z)} [\bar{\Phi}_{+,3}(z) \times \bar{\Phi}_{+,1}(z)]. \tag{4.19b}$$

Substituting them into the first two equations of (3) in Lemma 17 and combine the relation (2.28), we deriving the result

$$-g_0 \bar{\gamma}(\bar{z}_0)[a_{11}(z_0)(\Phi_{+,3}(z_0) \times \Phi_{+,1}(z_0)) + a_{21}(z_0)(\Phi_{+,3}(z_0) \times \Phi_{+,2}(z_0))] = \bar{\bar{g}}_0 \Phi_{+,3}(z_0) \times \Phi_{+,1}(z_0). \tag{4.20}$$

One can dot product the vector $\Phi_{+,2}(z_0)$ on the both sides of equation (4.20). Then $\hat{g}_0 = -\bar{g}_0 \gamma(\bar{z}_0) \bar{a}_{11}(z_0)$. □

5 Trace Formula and theta Condition

For $B = A^{-1}$ and $\det A = \det B = 1$, one can obtain

$$a_{22} b_{22} = 1/[1 + \frac{a_{12} b_{21} + a_{32} b_{23}}{a_{22} b_{22}}]. \tag{5.1}$$

Furthermore, taking logarithmic of both sides of (5.1) and combined with the reflection coefficients (3.19), we can get

$$\log a_{22}(z) - \log \frac{1}{b_{22}(z)} = J_0(z). \tag{5.2}$$

where

$$J_0(z) = \log[1 + \gamma(z)\rho_3(-\frac{q_0^2}{z})\bar{\rho}_3(-\frac{q_0^2}{\bar{z}}) + \gamma(z)\rho_3(z)\bar{\rho}_3(\bar{z})].$$

Since a_{22} and b_{22} are analytic in the upper-half plane and lower-half plane, equation (5.2) is a jump condition for a scalar, additive Riemann-Hilbert problem. To circumvent the pole singularities coming from the zeros of a_{22} and b_{22}, we define two analytic function

$$\beta^+(z) = a_{22}(z)e^{i\Delta\theta}\prod_{n=1}^{N_1}\frac{z-\bar{z}_n}{z-z_n}\frac{z-(-\frac{q_0^2}{\bar{z}_n})}{z-(-\frac{q_0^2}{z_n})}\prod_{n=1}^{N_3}\frac{z-\bar{\omega}_n}{z-\omega_n}\frac{z-(-\frac{q_0^2}{\bar{\omega}_n})}{z-(-\frac{q_0^2}{\omega_n})}, \quad z\in\mathbb{C}^+ \tag{5.3a}$$

$$\beta^-(z) = \frac{1}{b_{22}}(z)e^{i\Delta\theta}\prod_{n=1}^{N_1}\frac{z-\bar{z}_n}{z-z_n}\frac{z-(-\frac{q_0^2}{\bar{z}_n})}{z-(-\frac{q_0^2}{z_n})}\prod_{n=1}^{N_3}\frac{z-\bar{\omega}_n}{z-\omega_n}\frac{z-(-\frac{q_0^2}{\bar{\omega}_n})}{z-(-\frac{q_0^2}{\omega_n})}, \quad z\in\mathbb{C}^- \tag{5.3b}$$

where $\Delta\theta = \theta_+ - \theta_-$, and $\beta^{\pm}(z)$ no zeros in $\mathbb{C}^{\pm}$ and approaches 1 as $z\to\infty$ in the analytic region in z-plane. Using the Plemej's formula and the projectors $P^{\pm}$ we obtain $\log\beta(z) = P(\log[\beta^+\beta^-])$ for $z\in\mathbb{C}\setminus\Sigma$, where P is the Cauchy projectors

$$P_{\pm}(f)(z) = \frac{1}{2\pi i}\int_{\Sigma}\frac{f(\xi)}{\xi-(z\pm i0)}\xi. \tag{5.4}$$

Combine (5.2) and taking the exponentials,

$$a_{22}(z) = \exp\left(-i\Delta\theta - \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{J_0(\xi)}{\xi-z}\mathrm{d}\xi\right)\prod_{n=1}^{N_1}\frac{z-z_n}{z-\bar{z}_n}\frac{z-(-\frac{q_0^2}{z_n})}{z-(-\frac{q_0^2}{\bar{z}_n})}\prod_{n=1}^{N_3}\frac{z-\omega_n}{z-\bar{\omega}_n}\frac{z-(-\frac{q_0^2}{\omega_n})}{z-(-\frac{q_0^2}{\bar{\omega}_n})} \tag{5.5}$$

In the similar way as above yields

$$\log b_{33}(z) - \log\frac{1}{a_{33}(z)} = -\log\left[1+\frac{1}{\gamma(z)}\rho_1(-\frac{q_0^2}{z})\bar{\rho}_1(-\frac{q_0^2}{\bar{z}}) + \rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\right], \tag{5.6a}$$

$$\log a_{11}(z) - \log\frac{1}{b_{11}(z)} = -\log\left[1+\gamma(z)\rho_1(z)\bar{\rho}_1(\bar{z}) + \rho_2(z)\bar{\rho}_2(\bar{z})\right], \tag{5.6b}$$

However, $b_{33}(z)$, $a_{33}(z)$, $a_{11}(z)$ and $b_{11}(z)$ are only analytic in D_1, D_2, D_3, D_4. This is the reason that this situation is complicated than above.To formula a appropriate Riemann-Hilbert problem, we have to introduce a sectionally function which is analytic on the whole z-plane. So we obtain

$$\log b_{33}(z) - \log \frac{1}{b_{11}(z)} = \log a_{22}(z) - \log\left[1 - \bar{\rho}_2(\bar{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\right], \tag{5.7a}$$

$$\log a_{11}(z) - \log \frac{1}{a_{33}(z)} = \log b_{22}(z) - \log\left[1 - \rho_2(z)\rho_2(-\frac{q_0^2}{z})\right], \tag{5.7b}$$

Next, we define the following sectionally analytic function

$$\hat{\beta}^+(z) = \begin{cases} \beta_1(z), & z \in D_1 \\ \beta_3(z), & z \in D_3 \end{cases} \quad \hat{\beta}^-(z) = \begin{cases} \beta_2(z), & z \in D_2 \\ \beta_4(z), & z \in D_4 \end{cases}$$

where

$$\beta_1(z) = \frac{b_{33}(z)}{s_1(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \bar{z}_n} \frac{z - (-\frac{q_0^2}{\bar{z}_n})}{z - (-\frac{q_0^2}{z_n})} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n} \frac{z - (-\frac{q_0^2}{\zeta_n})}{z - (-\frac{q_0^2}{\bar{\zeta}_n})}, \tag{5.8a}$$

$$\beta_2(z) = \frac{\bar{s}_1(\bar{z})}{a_{33}(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \bar{z}_n} \frac{z - (-\frac{q_0^2}{\bar{z}_n})}{z - (-\frac{q_0^2}{z_n})} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n} \frac{z - (-\frac{q_0^2}{\zeta_n})}{z - (-\frac{q_0^2}{\bar{\zeta}_n})}, \tag{5.8b}$$

$$\beta_3(z) = \frac{a_{11}(z)}{\bar{s}_2(\bar{z})} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \bar{z}_n} \frac{z - (-\frac{q_0^2}{\bar{z}_n})}{z - (-\frac{q_0^2}{z_n})} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n} \frac{z - (-\frac{q_0^2}{\zeta_n})}{z - (-\frac{q_0^2}{\bar{\zeta}_n})}, \tag{5.8c}$$

$$\beta_4(z) = \frac{s_2(z)}{b_{11}(z)} \prod_{n=1}^{N_1} \frac{z - z_n}{z - \bar{z}_n} \frac{z - (-\frac{q_0^2}{\bar{z}_n})}{z - (-\frac{q_0^2}{z_n})} \prod_{n=1}^{N_2} \frac{z - \bar{\zeta}_n}{z - \zeta_n} \frac{z - (-\frac{q_0^2}{\zeta_n})}{z - (-\frac{q_0^2}{\bar{\zeta}_n})}, \tag{5.8d}$$

where

$$s_1(z) = \prod_{n=1}^{N_1} \frac{z - z_n}{z - \bar{z}_n} \prod_{n=1}^{N_3} \frac{z - \omega_n}{z - \bar{\omega}_n}, \quad s_2(z) = \prod_{n=1}^{N_1} \frac{z - (-\frac{q_0^2}{z_n})}{z - (-\frac{q_0^2}{\bar{z}_n})} \prod_{n=1}^{N_3} \frac{z - (-\frac{q_0^2}{\omega_n})}{z - (-\frac{q_0^2}{\bar{\omega}_n})}, \tag{5.9}$$

Then (5.6) can be rewritten as

$$\log \beta_1(z) - \log \beta_2(z) = J_1(z), \quad \log \beta_3(z) - \log \beta_4(z) = J_3(z), \tag{5.10}$$

where

$$J_1(z) = -\log\left[1 + \frac{1}{\gamma(z)}\rho_1(-\frac{q_0^2}{z})\bar{\rho}_1(-\frac{q_0^2}{\bar{z}}) + \rho_2(-\frac{q_0^2}{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\right], \tag{5.11a}$$

$$J_3(z) = -\log\left[1 + \gamma(z)\rho_1(z)\bar{\rho}_1(\bar{z}) + \rho_2(z)\bar{\rho}_2(\bar{z})\right], \tag{5.11b}$$

Combine (5.5) and $\beta_j(z)$, (5.7) can be rewritten as

$$\log\beta_3(z)-\log\beta_2(z)=J_2(z),\quad \log\beta_1(z)-\log\beta_4(z)=J_4(z), \tag{5.12}$$

where

$$J_2(z)=\frac{1}{2\pi i}\log\int_{\mathbb{R}}\frac{\bar{J}_0(\xi)}{\xi-z}\mathrm{d}\xi-\log\left[1-\bar{\rho}_2(\bar{z})\bar{\rho}_2(-\frac{q_0^2}{\bar{z}})\right], \tag{5.13a}$$

$$J_4(z)=-\frac{1}{2\pi i}\log\int_{\mathbb{R}}\frac{J_0(\xi)}{\xi-z}\mathrm{d}\xi-\log\left[1-\rho_2(z)\rho_2(-\frac{q_0^2}{z})\right], \tag{5.13b}$$

Now, (5.10) and (5.12) form the jump conditions for the Riemann-Hilbert problem with the analytic functions $\hat{\beta}^{\pm}(z)$. It can be precisely rewritten as

$$\log\hat{\beta}^{+}(z)-\log\hat{\beta}^{-}(z)=J_i(z),\quad J_i(z)=\Sigma_i(i=1,2,3,4). \tag{5.14}$$

Applying the projectors $P^{\pm}$ and Plemelj's formula yields

$$\log\hat{\beta}(z)=\frac{1}{2\pi i}\int_{\Sigma}\frac{J(\xi)}{\xi-z}\mathrm{d}\xi,\quad z\in\mathbb{C}\backslash\Sigma \tag{5.15}$$

Taking the exponential of both sides with $z\in D_1$yields

$$\frac{b_{33}(z)}{s_1(z)}=\exp\left(\frac{1}{2\pi i}\int_{\Sigma}\frac{J(\xi)}{\xi-z}\mathrm{d}\xi\right)\prod_{n=1}^{N_1}\frac{z-\bar{z}_n}{z-z_n}\frac{z-(-\frac{q_0^2}{z_n})}{z-(-\frac{q_0^2}{\bar{z}_n})}\prod_{n=1}^{N_2}\frac{z-\zeta_n}{z-\bar{\zeta}_n}\frac{z-(-\frac{q_0^2}{\bar{\zeta}_n})}{z-(-\frac{q_0^2}{\zeta_n})}, \tag{5.16}$$

Next, we obtain

$$b_{33}(z)=\exp\left(\frac{1}{2\pi i}\int_{\Sigma}\frac{J(\xi)}{\xi-z}\mathrm{d}\xi\right)\prod_{n=1}^{N_1}\frac{z-(-\frac{q_0^2}{z_n})}{z-(-\frac{q_0^2}{\bar{z}_n})}\prod_{n=1}^{N_2}\frac{z-\zeta_n}{z-\bar{\zeta}_n}\frac{z-(-\frac{q_0^2}{\bar{\zeta}_n})}{z-(-\frac{q_0^2}{\zeta_n})}\prod_{n=1}^{N_3}\frac{z-\omega_n}{z-\bar{\omega}_n}, \tag{5.17}$$

The asymptotic phase difference $\Delta\theta=\theta_{+}-\theta_{-}$ is given by

$$\Delta\theta=\frac{1}{2\pi}\int_{\Sigma}\frac{J(\xi)}{\xi}\xi+2\sum_{n=1}^{N_1}\arg z_n-4\sum_{n=1}^{N_2}\arg\zeta_n-2\sum_{n=1}^{N_3}\arg\omega_n \tag{5.18}$$

6 Solution of Riemann-Hilbert Problem

In this section, we reconstruct the solution of RH for deriving the solutions of SS equation. Firstly, it is necessary to introduce the following theorem:

Theorem 6.1. *Let ν_n is the set of all discrete spectral, the solution of the RH problem in Theorem 4.1 is given by*

$$M(x,t,z)=\Gamma_+(z)+\sum_{n=1}^{N}\left(\frac{M^+_{-1,\nu_n}}{z-\nu_n}+\frac{M^-_{-1,\bar\nu_n}}{z-\bar\nu_n}+\frac{M^+_{-1,-\frac{q_0^2}{\bar\nu_n}}}{z-(-\frac{q_0^2}{\bar\nu_n})}+\frac{M^-_{-1,-\frac{q_0^2}{\nu_n}}}{z-(-\frac{q_0^2}{\nu_n})}\right)-\frac{1}{2\pi i}\int_\Sigma\frac{M^-(\xi)\hat L(\xi)}{\xi-z}\mathrm{d}\xi, \tag{6.1}$$

where $N=N_1+N_2+N_3$, $\hat L=e^{i\Theta}Le^{-i\Theta}$. The modified eigenfunctions in the residue conditions (4.14) can be given as

$$\begin{aligned}m_2^-(z)=&\begin{pmatrix}\frac{\mathbf{q}_+^\perp}{q_0}\\0\end{pmatrix}+\sum_{n=1}^{N_1}\left(\frac{H_ne^{i(\theta_1-\theta_2)(z_n)}}{z-z_n}+\frac{z_n}{iq_0}\frac{\tilde H_ne^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_n})}}{z-(-\frac{q_0^2}{z_n})}\right)m_1^+(z_n)\\&+\sum_{n=1}^{N_3}\left(\frac{G_ne^{-i(\theta_1+\theta_2)(\omega_n)}}{z-\omega_n}+\frac{\omega_n}{iq_0}\frac{\tilde G_ne^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}}{z-(-\frac{q_0^2}{\omega_n})}\right)m_3^+(\omega_n)-\frac{1}{2\pi i}\int_\Sigma\frac{(M^-\hat L(\xi))_2}{\xi-z}\mathrm{d}\xi\end{aligned}, \tag{6.2a}$$

$$\begin{aligned}m_1^+(z)=&\begin{pmatrix}-\frac{\mathbf{q}_+}{q_0}\\\frac{iq_0}{z}\end{pmatrix}+\sum_{n=1}^{N_1}\frac{\hat H_ne^{-i(\theta_1-\theta_2)(\bar z_n)}}{z-\bar z_n}m_2^-(\bar z_n)+\sum_{n=1}^{N_2}\frac{F_ne^{-2i\theta_1(\zeta_n)}}{z-\zeta_n}m_3^+(\zeta_n)\\&+\sum_{n=1}^{N_2}\frac{\bar\zeta_n}{iq_0}\frac{\check F_ne^{-2i\theta_1(-\frac{q_0^2}{\bar\zeta_n})}}{z-(-\frac{q_0^2}{\bar\zeta_n})}m_1^-(\bar\zeta_n)+\sum_{n=1}^{N_3}\frac{\check G_ne^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar\omega_n})}}{z-(-\frac{q_0^2}{\bar\omega_n})}m_2^-(\bar\omega_n)-\frac{1}{2\pi i}\int_\Sigma\frac{(M^-\hat L(\xi))_1}{\xi-z}\mathrm{d}\xi\end{aligned} \tag{6.2b}$$

$$\begin{aligned}m_1^-(z)=&\begin{pmatrix}-\frac{\mathbf{q}_+}{q_0}\\\frac{iq_0}{z}\end{pmatrix}+\sum_{n=1}^{N_1}\frac{\hat H_ne^{-i(\theta_1-\theta_2)(\bar z_n)}}{z-\bar z_n}m_2^-(\bar z_n)+\sum_{n=1}^{N_2}\frac{F_ne^{-2i\theta_1(\zeta_n)}}{z-\zeta_n}m_3^+(\zeta_n)\\&+\sum_{n=1}^{N_2}\frac{\bar\zeta_n}{iq_0}\frac{\check F_ne^{-2i\theta_1(-\frac{q_0^2}{\bar\zeta_n})}}{z-(-\frac{q_0^2}{\bar\zeta_n})}m_1^-(\bar\zeta_n)+\sum_{n=1}^{N_3}\frac{\check G_ne^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar\omega_n})}}{z-(-\frac{q_0^2}{\bar\omega_n})}m_2^-(\bar\omega_n)-\frac{1}{2\pi i}\int_\Sigma\frac{(M^-\hat L(\xi))_1}{\xi-z}\mathrm{d}\xi\end{aligned} \tag{6.2c}$$

$$\begin{aligned}m_3^+(z)=&\begin{pmatrix}-\frac{i\mathbf{q}_+}{z}\\1\end{pmatrix}+\sum_{n=1}^{N_1}\frac{\check H_ne^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar z_n})}}{z-(-\frac{q_0^2}{\bar z_n})}m_2^-(\bar z_n)+\sum_{n=1}^{N_2}\frac{\hat F_ne^{2i\theta_1(\bar\zeta_n)}}{z-\bar\zeta_n}m_1^-(\bar\zeta_n)\\&+\sum_{n=1}^{N_2}\frac{\zeta_n}{iq_0}\frac{\tilde F_ne^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{z-(-\frac{q_0^2}{\zeta_n})}m_3^+(\zeta_n)+\sum_{n=1}^{N_3}\frac{\hat G_ne^{i(\theta_1+\theta_2)(\bar\omega_n)}}{z-\bar\omega_n}m_2^-(\bar\omega_n)-\frac{1}{2\pi i}\int_\Sigma\frac{(M^-\hat L(\xi))_3}{\xi-z}\mathrm{d}\xi\end{aligned} \tag{6.2d}$$

So then, according to the asymptotics of $\mu_\pm$ as $z\to\infty$, one can obtain the reconstruction of potential function.

Theorem 6.2. *The solution of Riemann-Hilbert problem in Theorem 6.1 is reconstructed as*

$$\begin{aligned} q(x,t) = q_+ &+ i\sum_{n=1}^{N_1}\check{H}_n e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})} m_{12}^-(\bar{z}_n) + i\sum_{n=1}^{N_2}\frac{\zeta_n}{iq_0}\tilde{F}_n e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})} m_{13}^+(\zeta_n) \\ &+ i\sum_{n=1}^{N_2}\hat{F}_n e^{2i\theta_1(\bar{\zeta}_n)} m_{11}^-(\bar{\zeta}_n) + i\sum_{n=1}^{N_3}\hat{G}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)} m_{12}^-(\bar{\omega}_n) \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} m_{12}^-(\bar{z}_{i'}) =& \frac{q_+}{q_0} + \sum_{n=1}^{N_1}\left(\frac{H_n e^{i(\theta_1-\theta_2)(z_n)}}{\bar{z}_{i'}-z_n} + \frac{z_n}{iq_0}\frac{\tilde{H}_n e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_n})}}{\bar{z}_{i'}-(-\frac{q_0^2}{z_n})}\right) m_{11}^+(z_n) \\ &+ \sum_{n=1}^{N_3}\left(\frac{G_n e^{-i(\theta_1+\theta_2)(\omega_n)}}{\bar{z}_{i'}-\omega_n} + \frac{\omega_n}{iq_0}\frac{\tilde{G}_n e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}}{\bar{z}_{i'}-(-\frac{q_0^2}{\omega_n})}\right) m_{13}^+(\omega_n) \end{aligned}, \tag{6.4a}$$

$$\begin{aligned} m_{12}^-(\bar{\omega}_{l'}) =& \frac{q_+}{q_0} + \sum_{n=1}^{N_1}\left(\frac{H_n e^{i(\theta_1-\theta_2)(z_n)}}{\bar{\omega}_{l'}-z_n} + \frac{z_n}{iq_0}\frac{\tilde{H}_n e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_n})}}{\bar{\omega}_{l'}-(-\frac{q_0^2}{z_n})}\right) m_{11}^+(z_n) \\ &+ \sum_{n=1}^{N_3}\left(\frac{G_n e^{-i(\theta_1+\theta_2)(\omega_n)}}{\bar{\omega}_{l'}-\omega_n} + \frac{\omega_n}{iq_0}\frac{\tilde{G}_n e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}}{\bar{\omega}_{l'}-(-\frac{q_0^2}{\omega_n})}\right) m_{13}^+(\omega_n) \end{aligned}, \tag{6.4b}$$

$$\begin{aligned} m_{11}^+(z_{i'}) =& -\frac{q_+}{q_0} + \sum_{n=1}^{N_1}\frac{\hat{H}_n e^{-i(\theta_1-\theta_2)(\bar{z}_n)}}{z_{i'}-\bar{z}_n} m_{12}^-(\bar{z}_n) + \sum_{n=1}^{N_2}\frac{F_n e^{-2i\theta_1(\zeta_n)}}{z_{i'}-\zeta_n} m_{13}^+(\zeta_n) \\ &+ \sum_{n=1}^{N_2}\frac{\bar{\zeta}_n}{iq_0}\frac{\check{F}_n e^{-2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{z_{i'}-(-\frac{q_0^2}{\zeta_n})} m_{11}^-(\bar{\zeta}_n) + \sum_{n=1}^{N_3}\frac{\check{G}_n e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})}}{z_{i'}-(-\frac{q_0^2}{\bar{\omega}_n})} m_{12}^-(\bar{\omega}_n) \end{aligned} \tag{6.4c}$$

$$\begin{aligned} m_{11}^-(\bar{\zeta}_{j'}) =& -\frac{q_+}{q_0} + \sum_{n=1}^{N_1}\frac{\hat{H}_n e^{-i(\theta_1-\theta_2)(\bar{z}_n)}}{\bar{\zeta}_{j'}-\bar{z}_n} m_{12}^-(\bar{z}_n) + \sum_{n=1}^{N_2}\frac{F_n e^{-2i\theta_1(\zeta_n)}}{\bar{\zeta}_{j'}-\zeta_n} m_{13}^+(\zeta_n) \\ &+ \sum_{n=1}^{N_2}\frac{\bar{\zeta}_n}{iq_0}\frac{\check{F}_n e^{-2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{\bar{\zeta}_{j'}-(-\frac{q_0^2}{\zeta_n})} m_{11}^-(\bar{\zeta}_n) + \sum_{n=1}^{N_3}\frac{\check{G}_n e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})}}{\bar{\zeta}_{j'}-(-\frac{q_0^2}{\bar{\omega}_n})} m_{12}^-(\bar{\omega}_n) \end{aligned} \tag{6.4d}$$

$$m_{13}^{+}(\zeta_{j'}) = -\frac{iq_+}{\zeta_{j'}} + \sum_{n=1}^{N_1} \frac{\check{H}_n e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{\zeta_{j'} - (-\frac{q_0^2}{\bar{z}_n})} m_{12}^{-}(\bar{z}_n) + \sum_{n=1}^{N_2} \frac{\hat{F}_n e^{2i\theta_1(\bar{\zeta}_n)}}{\zeta_{j'} - \bar{\zeta}_n} m_{11}^{-}(\bar{\zeta}_n)$$
$$+ \sum_{n=1}^{N_2} \frac{\zeta_n}{iq_0} \frac{\tilde{F}_n e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{\zeta_{j'} - (-\frac{q_0^2}{\zeta_n})} m_{13}^{+}(\zeta_n) + \sum_{n=1}^{N_3} \frac{\hat{G}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)}}{\zeta_{j'} - \bar{\omega}_n} m_{12}^{-}(\bar{\omega}_n) \tag{6.4e}$$

$$m_{13}^{+}(\omega_{l'}) = -\frac{iq_+}{\omega_{l'}} + \sum_{n=1}^{N_1} \frac{\check{H}_n e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{\omega_{l'} - (-\frac{q_0^2}{\bar{z}_n})} m_{12}^{-}(\bar{z}_n) + \sum_{n=1}^{N_2} \frac{\hat{F}_n e^{2i\theta_1(\bar{\zeta}_n)}}{\omega_{l'} - \bar{\zeta}_n} m_{11}^{-}(\bar{\zeta}_n)$$
$$+ \sum_{n=1}^{N_2} \frac{\zeta_n}{iq_0} \frac{\tilde{F}_n e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{\omega_{l'} - (-\frac{q_0^2}{\zeta_n})} m_{13}^{+}(\zeta_n) + \sum_{n=1}^{N_3} \frac{\hat{G}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)}}{\omega_{l'} - \bar{\omega}_n} m_{12}^{-}(\bar{\omega}_n) \tag{6.4f}$$

and $i' = 1, 2, \cdots, N_1, \quad j' = 1, 2, \cdots, N_2, \quad l' = 1, 2, \cdots, N_3.$

For convenience, we define the following notations:

$$\Delta_n^{(1)}(x,t,z) = \frac{H_n e^{i(\theta_1-\theta_2)(z_n)}}{z - z_n} + \frac{z_n}{iq_0} \frac{\tilde{H}_n e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_n})}}{z - (-\frac{q_0^2}{z_n})} \tag{6.5a}$$

$$\Delta_n^{(2)}(x,t,z) = \frac{G_n e^{-i(\theta_1+\theta_2)(\omega_n)}}{z - \omega_n} + \frac{\omega_n}{iq_0} \frac{\tilde{G}_n e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_n})}}{z - (-\frac{q_0^2}{\omega_n})} \tag{6.5b}$$

$$\Delta_n^{(3)}(x,t,z) = \frac{\hat{H}_n e^{-i(\theta_1-\theta_2)(\bar{z}_n)}}{z - \bar{z}_n}, \quad \Delta_n^{(4)}(x,t,z) = \frac{F_n e^{-2i\theta_1(\zeta_n)}}{z - \zeta_n} \tag{6.5c}$$

$$\Delta_n^{(5)}(x,t,z) = \frac{\bar{\zeta}_n}{iq_0} \frac{\check{F}_n e^{-2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{z - (-\frac{q_0^2}{\zeta_n})}, \quad \Delta_n^{(6)}(x,t,z) = \frac{\check{G}_n e^{-i(\theta_1-\theta_2)(-\frac{q_0^2}{\bar{\omega}_n})}}{z - (-\frac{q_0^2}{\bar{\omega}_n})} \tag{6.5d}$$

$$\Delta_n^{(7)}(x,t,z) = \frac{\check{H}_n e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_n})}}{z - (-\frac{q_0^2}{\bar{z}_n})}, \quad \Delta_n^{(8)}(x,t,z) = \frac{\hat{F}_n e^{2i\theta_1(\bar{\zeta}_n)}}{z - \bar{\zeta}_n} \tag{6.5e}$$

$$\Delta_n^{(9)}(x,t,z) = \frac{\zeta_n}{iq_0} \frac{\tilde{F}_n e^{2i\theta_1(-\frac{q_0^2}{\zeta_n})}}{z - (-\frac{q_0^2}{\zeta_n})}, \quad \Delta_n^{(10)}(x,t,z) = \frac{\hat{G}_n e^{i(\theta_1+\theta_2)(\bar{\omega}_n)}}{z - \bar{\omega}_n} \tag{6.5f}$$

Substituting (6.4c) and (6.4f) into (6.4a) and (6.4b), we obtain

$$
\begin{aligned}
m_{12}^{-}(z) = \frac{q_+}{q_0} - \sum_{n=1}^{N_1} \frac{q_+}{q_0} \Delta_n^{(1)}(z) - \sum_{n=1}^{N_3} \frac{iq_+}{\omega_n} \Delta_n^{(2)}(z) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(1)}(z) \Delta_{n'}^{(4)}(z_n) m_{13}^{+}(\zeta_{n'}) \\
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_2} \Delta_n^{(1)}(z) \Delta_{n'}^{(5)}(z_n) m_{11}^{-}(\bar{\zeta}_{n'}) + \sum_{n=1}^{N_1} \sum_{n'=1}^{N_1} \Delta_n^{(1)}(z) \Delta_{n'}^{(3)}(z_n) m_{12}^{-}(\bar{z}_{n'}) \\
+ \sum_{n=1}^{N_1} \sum_{n'=1}^{N_3} \Delta_n^{(1)}(z) \Delta_{n'}^{(6)}(z_n) m_{12}^{-}(\bar{\omega}_{n'}) + \sum_{n=1}^{N_3} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_{n'}^{(9)}(\omega_n) m_{13}^{+}(\zeta_{n'}) \\
+ \sum_{n=1}^{N_3} \sum_{n'=1}^{N_2} \Delta_n^{(2)}(z) \Delta_{n'}^{(8)}(\omega_n) m_{11}^{-}(\bar{\zeta}_{n'}) + \sum_{n=1}^{N_3} \sum_{n'=1}^{N_1} \Delta_n^{(2)}(z) \Delta_{n'}^{(7)}(\omega_n) m_{12}^{-}(\bar{z}_{n'}) \\
+ \sum_{n=1}^{N_3} \sum_{n'=1}^{N_3} \Delta_n^{(2)}(z) \Delta_{n'}^{(10)}(\omega_n) m_{12}^{-}(\bar{\omega}_{n'}).
\end{aligned}
\tag{6.6}
$$

where $z = \bar{z}_{i'}$ and $z = \bar{\omega}_{l'}$. Together with (6.4d), (6.4f) and (6.6)comprise a linear equations.To solve this equations by Gramer's rule, we define the vector $\mathbf{x} = (x_1, \cdots, x_{N_1+2N_2+N_3})$,

$$
x_k = \begin{cases} m_{12}^{-}(\bar{z}_k), & k = 1, \cdots, N_1, \\ m_{13}^{+}(\zeta_{k-N_1}), & k = N_1+1, \cdots, N_1+N_2, \\ m_{11}^{-}(\bar{\zeta}_{k-N_1-N_2}), & k = N_1+N_2+1, \cdots, N_1+2N_2, \\ m_{12}^{-}(\bar{\omega}_{k-N_1-2N_2}), & k = N_1+2N_2+1, \cdots, N_1+2N_2+N_3, \end{cases}
$$

So we obtain a systems equations

$$
(\mathbf{I} - \mathbf{F})\mathbf{x}_k = \mathbf{W}\mathbf{x}_k = \mathbf{b}_k, \tag{6.7}
$$

where

$$
b_k = \begin{cases}
\dfrac{q_+}{q_0} - \displaystyle\sum_{n=1}^{N_1} \dfrac{q_+}{q_0} \Delta_n^{(1)}(\bar{z}_k) \\
\qquad - \displaystyle\sum_{n=1}^{N_3} \dfrac{iq_+}{\omega_n} \Delta_n^{(2)}(\bar{z}_k), & k = 1, \cdots, N_1, \\
-\dfrac{iq_+}{\zeta_{k-N_1}}, & k = N_1 + 1, \cdots, N_1 + N_2, \\
-\dfrac{q_+}{q_0}, & k = N_1 + N_2 + 1, \cdots, N_1 + 2N_2, \\
\dfrac{q_+}{q_0} - \displaystyle\sum_{n=1}^{N_1} \dfrac{q_+}{q_0} \Delta_n^{(1)}(\bar{\omega}_{k-N_1-2N_2}) \\
\qquad - \displaystyle\sum_{n=1}^{N_3} \dfrac{iq_+}{\omega_n} \Delta_n^{(2)}(\bar{\omega}_{k-N_1-2N_2}), & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3,
\end{cases}
$$

$$
y_k = \begin{cases}
-i\check{H}_k e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_k})}, & k = 1, \cdots, N_1, \\
-\dfrac{\zeta_{k-N_1}}{q_0} \tilde{F}_{k-N_1} e^{2i\theta_1(-\frac{q_0^2}{\zeta_{k-N_1}})}, & k = N_1 + 1, \cdots, N_1 + N_2, \\
-i\hat{F}_{k-N_1-N_2} e^{2i\theta_1(\bar{\zeta}_{k-N_1-N_2})}, & k = N_1 + N_2 + 1, \cdots, N_1 + 2N_2, \\
-i\hat{G}_{k-N_1-2N_2} e^{i(\theta_1+\theta_2)(\bar{\omega}_{k-N_1-2N_2})}, & k = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3.
\end{cases} \tag{6.8}
$$

The matrix **F** are defined as follows:

For $i, j = 1, \cdots, N_1$

$$
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i)\Delta_j^{(3)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i)\Delta_j^{(7)}(\omega_n) \tag{6.9}
$$

For $i = 1, \cdots, N_1$ and $j = N_1 + 1, \cdots, N_1 + N_2$

$$
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i)\Delta_{j-N_1}^{(4)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i)\Delta_{j-N_1}^{(9)}(\omega_n) \tag{6.10}
$$

For $i = 1, \cdots, N_1$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i)\Delta_{j-N_1-N_2}^{(5)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i)\Delta_{j-N_1-N_2}^{(8)}(\omega_n) \tag{6.11}
$$

For $i = 1, \cdots, N_1$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$
F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{z}_i)\Delta_{j-N_1-2N_2}^{(6)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{z}_i)\Delta_{j-N_1-2N_2}^{(10)}(\omega_n) \tag{6.12}
$$

For $i = N_1 + 1, \cdots, N_2$ and $j = 1, \cdots, N_1$

$$F_{ij} = \Delta_j^{(7)}(\zeta_{i-N_1}) \tag{6.13}$$

For $i, j = N_1 + 1, \cdots, N_2$

$$F_{ij} = \Delta_{j-N_1}^{(9)}(\zeta_{i-N_1}) \tag{6.14}$$

For $i = N_1 + 1, \cdots, N_2$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \Delta_{j-N_1-N_2}^{(8)}(\zeta_{i-N_1}) \tag{6.15}$$

For $i = N_1 + 1, \cdots, N_2$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \Delta_{j-N_1-2N_2}^{(10)}(\zeta_{i-N_1}) \tag{6.16}$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = 1, \cdots, N_1$

$$F_{ij} = \Delta_j^{(3)}(\bar{\zeta}_{i-N_1-N_2}) \tag{6.17}$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = N_1 + 1, \cdots, N_1 + N_2$

$$F_{ij} = \Delta_{j-N_1}^{(4)}(\bar{\zeta}_{i-N_1-N_2}) \tag{6.18}$$

For $i, j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \Delta_{j-N_1-N_2}^{(5)}(\bar{\zeta}_{i-N_1-N_2}) \tag{6.19}$$

For $i = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$ and $j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \Delta_{j-N_1-2N_2}^{(6)}(\bar{\zeta}_{i-N_1-N_2}) \tag{6.20}$$

For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = 1, \cdots, N_1$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2})\Delta_j^{(3)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2})\Delta_j^{(7)}(\omega_n) \tag{6.21}$$

For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = N_1 + 1, \cdots, N_1 + N_2$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1}^{(4)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1}^{(9)}(\omega_n) \tag{6.22}$$

For $i = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$ and $j = N_1 + N_2 + 1, \cdots, N_1 + 2N_2$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1-N_2}^{(5)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1-N_2}^{(8)}(\omega_n) \tag{6.23}$$

For $i, j = N_1 + 2N_2 + 1, \cdots, N_1 + 2N_2 + N_3$

$$F_{ij} = \sum_{n=1}^{N_1} \Delta_n^{(1)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1-2N_2}^{(6)}(z_n) + \sum_{n=1}^{N_3} \Delta_n^{(2)}(\bar{\omega}_{i-N_1-2N_2})\Delta_{j-N_1-2N_2}^{(10)}(\omega_n) \tag{6.24}$$

By the Gramer's rule, one can derive that the components of the solutions of system (6.7)

$$x_k = \frac{\det \mathbf{W}_k^{aug}}{\det \mathbf{W}}, \quad k = 1, 2, \cdots, N_1 + 2N_2 + N_3, \tag{6.25}$$

where $\mathbf{W}_k^{aug} = (\mathbf{W}_1, \cdots, \mathbf{W}_{k-1}, \mathbf{b}, \mathbf{W}_{k+1}, \cdots, \mathbf{W}_{N_1+2N_2+N_3})$ Substituting x_k into the reconstruction formula (6.26) yields the potential function

$$q(x,t) = \frac{\det \mathbf{W}^{aug}}{\det \mathbf{W}} \tag{6.26}$$

and

$$\mathbf{W}^{aug} = \begin{pmatrix} q_+ & \mathbf{y}^T \\ \mathbf{b} & \mathbf{W} \end{pmatrix}, \tag{6.27}$$

where

$$\mathbf{b} = (b_1, \cdots, b_{N_1+2N_2+N_3}), \quad \mathbf{y} = (y_1, \cdots, y_{N_1+2N_2+N_3})^T \tag{6.28}$$

7 The Soliton Solutions of Sasa-Satsuma equation

In this section, we introduced three kinds of solutions and plot their graphic models. Certainly, one can obtain more solutions from the equation (6.26).

7.1 Case I

In this subsection, we assume the eigenvalues ν_n is the first kind eigenvalues. This implies that $N_1 = 1$ and $N_2 = N_3 = 0$. And we also assume that $\mathbf{q}_+ =$

$(1,1)^T, \quad H_1 = e^{\alpha+i\eta}, \quad z = \rho e^{i\delta}, (0 < \delta < \pi)$. So we derived the soliton solution

$$q(x,t) = \frac{\det\begin{pmatrix} q_+ & y_1 \\ b_1 & 1-F_{11} \end{pmatrix}}{1-F_{11}},$$

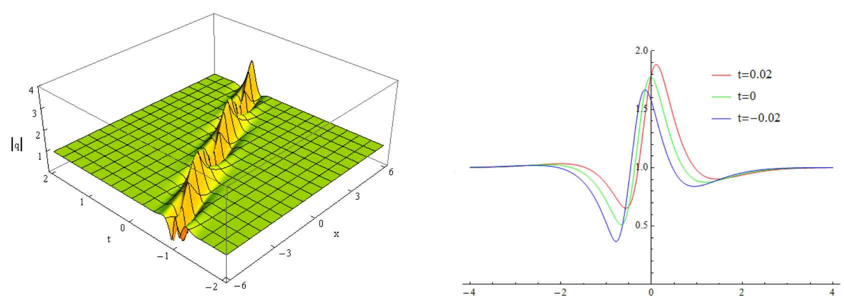
where

$$F_{11} = \Delta_1^{(1)}(\bar{z}_1)\Delta_1^{(3)}(z_1), \quad b_1 = \frac{q_+}{q_0} - \frac{q_+}{q_0}\Delta_1^{(1)}(\bar{z}_1), \quad y_1 = -i\check{H}_1 e^{i(\theta_1+\theta_2)(-\frac{q_0^2}{\bar{z}_1})},$$

$$\Delta_1^{(1)}(\bar{z}_1) = \frac{H_1 e^{i(\theta_1-\theta_2)(z_1)}}{\bar{z}_1 - z_1} + \frac{z_1}{iq_0}\frac{\tilde{H}_1 e^{-i(\theta_1+\theta_2)(-\frac{q_0^2}{z_1})}}{\bar{z}_1 - (-\frac{q_0^2}{z_1})},$$

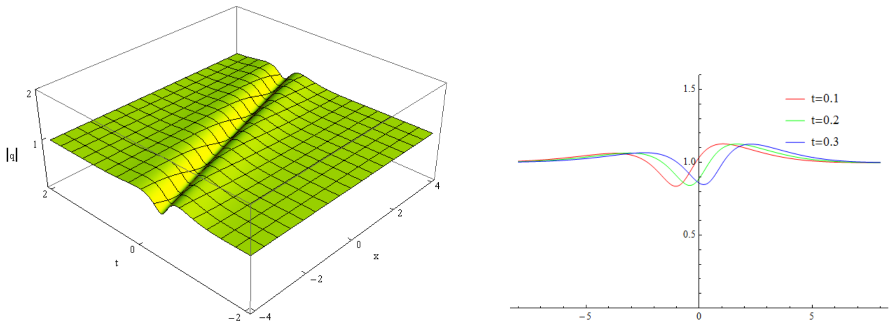
$$\Delta_1^{(3)}(z_1) = \frac{\hat{H}_1 e^{-i(\theta_1-\theta_2)(\bar{z}_1)}}{z_1 - \bar{z}_1}.$$

Figure 3-4 showing the perspective view and propagation view of the one soliton solution as different parameters. Figure 3 showing the situation of $\rho < q_0$. Figure 4 showing $\rho > q_0$.



(a) (b)

Fig. 3. $q_0 = \sqrt{2}$, $\rho = 1$, $\delta = \frac{\pi}{3}$, $\alpha = 0$, $\eta = -\frac{\pi}{2}$. (a) the perspective view of one soliton. (b) the propagation view of soliton solution in different time.



(a) (b)

Fig. 4. $q_0 = \sqrt{2}$, $\rho = 3$, $\delta = \frac{\pi}{3}$, $\alpha = 0$, $\eta = -\frac{\pi}{2}$. (a) the perspective view of one soliton. (b) the propagation view of soliton solution in different time.

7.2 Case II

In this subsection, we assume the eigenvalues ν_n is the second kind eigenvalues. This implies that $N_2 = 1$ and $N_1 = N_3 = 0$. And we also assume that $\mathbf{q}_+ = (1,1)^T$, $\quad H_1 = e^{\alpha + i\eta}$, $\quad \zeta = i\rho e^{i\delta}$. So we derived the soliton solution

$$q(x,t) = \frac{\det\begin{pmatrix} q_+ & y_1 & y_2 \\ b_1 & 1 - F_{11} & -F_{12} \\ b_2 & -F_{21} & 1 - F_{22} \end{pmatrix}}{\det\begin{pmatrix} 1 - F_{11} & -F_{12} \\ -F_{21} & 1 - F_{22} \end{pmatrix}},$$

where

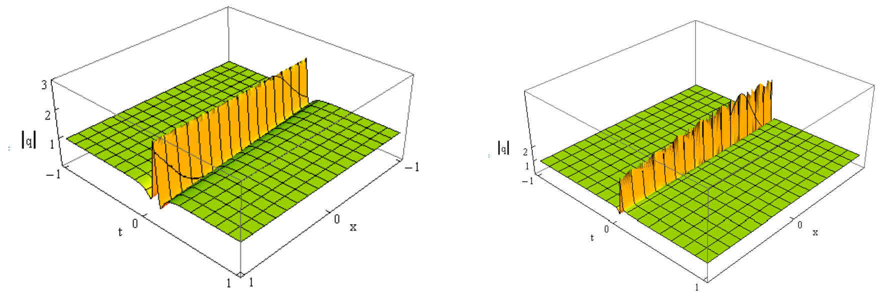
$$F_{11} = \Delta_1^{(9)}(\zeta_1), \quad F_{12} = \Delta_1^{(8)}(\zeta_1), \quad F_{21} = \Delta_1^{(4)}(\bar{\zeta}_1), \quad F_{22} = \Delta_1^{(5)}(\bar{\zeta}_1),$$

$$b_1 = -\frac{iq_+}{\zeta_1}, \quad b_2 = -\frac{q_+}{q_0}, \quad y_1 = -\frac{\zeta_1}{q_0}\tilde{F}_1 e^{2i\theta_1(-\frac{q_0^2}{\zeta_1})}, \quad y_2 = -i\hat{F}_1 e^{2i\theta_1(\bar{\zeta}_1)},$$

$$\Delta_1^{(9)}(\zeta_1) = \frac{\zeta_1}{iq_0}\frac{\tilde{F}_1 e^{2i\theta_1(-\frac{q_0^2}{\zeta_1})}}{\zeta_1 - (-\frac{q_0^2}{\zeta_1})}, \quad \Delta_1^{(8)}(\zeta_1) = \frac{\hat{F}_1 e^{2i\theta_1(\bar{\zeta}_1)}}{\zeta_1 - \bar{\zeta}_1},$$

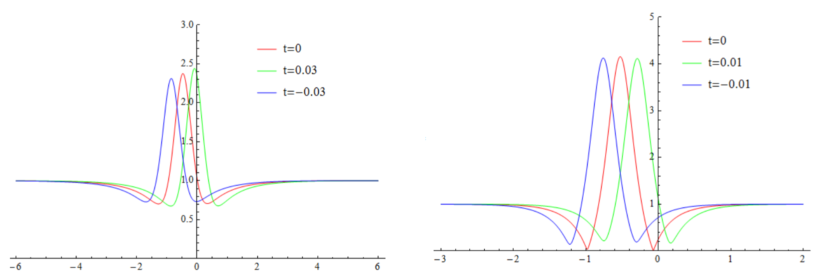
$$\Delta_1^{(4)}(\bar{\zeta}_1) = \frac{F_1 e^{-2i\theta_1(\zeta_1)}}{\bar{\zeta} - \zeta_1}, \quad \Delta_1^{(5)}(\bar{\zeta}_1) = \frac{\bar{\zeta}_1}{iq_0}\frac{\check{F}_1 e^{-2i\theta_1(-\frac{q_0^2}{\zeta_1})}}{\bar{\zeta}_1 - (-\frac{q_0^2}{\bar{\zeta}_1})}.$$

Figure 5 showing the perspective view one soliton solution as $\rho > 0$ and $\rho < 0$. Figure 6 showing the propagation view of one soliton solution as different time.



(a) (b)

Fig. 5. (a)the perspective view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{24}$.(b)the perspective view of one soliton solution $\rho = 4$, $\delta = \frac{\pi}{24}$.



(a) (b)

Fig. 6. (a)the propagation view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{24}$.(b)the propagation view of soliton solution as $\rho = 4$, $\delta = \frac{\pi}{24}$.

7.3 Case III

In this subsection, we assume the eigenvalues ν_n is the third kind eigenvalues. This implies that $N_3 = 1$ and $N_1 = N_2 = 0$. And we also assume that $\mathbf{q}_+ = (1,1)^T, \quad H_1 = e^{\alpha + i\eta}, \quad \omega_1 = \rho e^{i\delta}$. So we derived the soliton solution

$$q(x,t) = \frac{\det \begin{pmatrix} q_+ & y_1 \\ b_1 & 1 - F_{11} \end{pmatrix}}{1 - F_{11}},$$

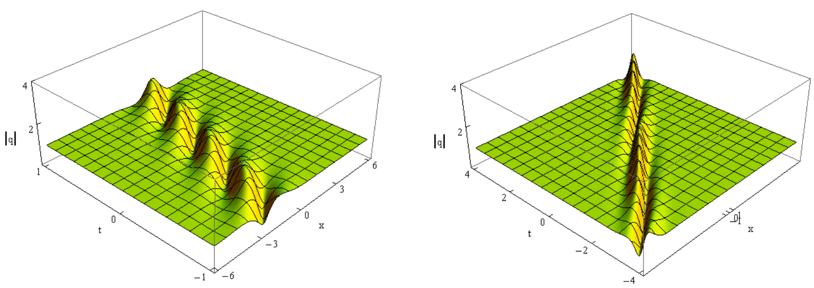
where

$$F_{11} = \Delta_1^{(2)}(\bar{\omega}_1)\Delta_1^{(10)}(\omega_1), \quad b_1 = -\frac{iq_+}{\omega_1}\Delta_1^{(2)}(\bar{\omega}_1), \quad y_1 = -i\hat{G}_1 e^{i(\theta_1+\theta_2)(\bar{\omega}_1)},$$

$$\Delta_1^{(2)}(\bar{\omega}_1) = \frac{G_1 e^{-i(\theta_1+\theta_2)(\omega_n)}}{\bar{\omega}_1 - \omega_1} + \frac{\omega_1}{iq_0}\frac{\tilde{G}_1 e^{i(\theta_1-\theta_2)(-\frac{q_0^2}{\omega_1})}}{\bar{\omega}_1 - (-\frac{q_0^2}{\omega_1})},$$

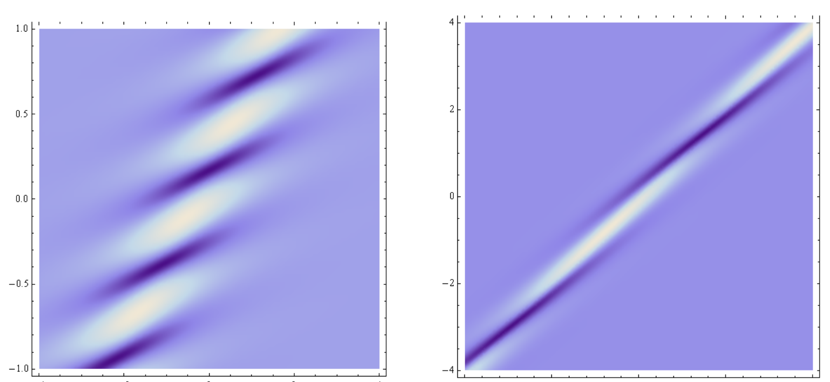
$$\Delta_1^{10}(\omega_1) = \frac{\hat{G}_1 e^{i(\theta_1+\theta_2)(\bar{\omega}_1)}}{\omega_1 - \bar{\omega}_1}.$$

Figure 7 and Figure 8 showing the perspective view and propagation view of one soliton solution.



(a) (b)

Fig. 7. (a)the perspective view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{4}$.(b)the perspective view of soliton solution as $\rho = 1$, $\delta = \frac{\pi}{4}$.



(a) (b)

Fig. 8. (a)the density view of one soliton solution as $\rho = 2$, $\delta = \frac{\pi}{4}$.(b)the density view of soliton solution $\rho = 1$, $\delta = \frac{\pi}{4}$.

Acknowledgments

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