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# Symmetry, full symmetry groups, and some exact solutions to a generalized Davey–Stewartson system

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The Lie symmetry algebra of a generalized Davey–Stewartson (GDS) system is obtained. The general element of this algebra depends on eight arbitrary functions of time, which has a Kac–Moody–Virasoro loop algebra structure and is isomorphic to that of the standard integrable Davey–Stewartson equations under certain conditions imposed on parameters and arbitrary functions. Then based on the symmetry group direct method proposed by Lou and Ma [J. Phys. A **38**, L129 (2005)] the full symmetry groups of the GDS system are obtained. From the full symmetry groups, both the Lie symmetry group and a group of discrete transformations can be obtained. Finally, some exact solutions involving sech-sech$^2$-sech$^2$ and tanh-tanh$^2$-tanh$^2$ type solitary wave solutions are presented by a generalized sub-equation expansion method. © *2008 American Institute of Physics.*
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## I. INTRODUCTION

In a recent study, a system of three nonlinear partial differential equations (PDEs) to model wave propagation in a bulk medium composed of an elastic material with couple stresses has been derived by Babaoglu and Erbay,[1] namely,

$$\begin{aligned} &i\psi_t + \delta\psi_{xx} + \psi_{yy} = \chi|\psi|^2\psi + \gamma(w_x + \phi_y)\psi, \\ &w_{xx} + n\phi_{xy} + m_2 w_{yy} = (|\psi|^2)_x, \\ &n w_{xy} + \lambda\phi_{xx} + m_1\phi_{yy} = (|\psi|^2)_y, \end{aligned} \tag{1.1}$$

with the condition $(\lambda-1)(m_2-m_1)=n^2$. Here $\psi(x,y,t)$ is the scaled complex amplitude of the free short transverse wave mode and $w(x,y,t)$ and $\phi(x,y,t)$ are the scaled free long longitudinal and long transverse wave modes, respectively. The parameters $\delta$, $n$, $m_1$, $m_2$, $\chi$, and $\gamma$ are real constants. Babaoglu and S. Erbay[1] showed that if $n=1-\lambda=m_1-m_2$ then (1.1) can be reduced to the classical Davey–Stewarston (DS) equations[2] by a noninvertible point transformation of dependent variables

$$\begin{aligned} &i\psi_t + \delta\psi_{xx} + \psi_{yy} = (\chi+\gamma)|\psi|^2\psi + \gamma Q\psi, \\ &Q_{xx} + m_1 Q_{yy} = (1-m_1)(|\psi|^2)_{yy}, \end{aligned} \tag{1.2}$$

where $Q=w_x+\phi_y-|\psi|^2$. Therefore, they called (1.1) the generalized DS (GDS) system. Also, in Ref. 1 some traveling type solutions of (1.1) in terms of elementary and elliptic functions are obtained. Based on some physically obvious Noetherian symmetries (time-space translations and constant change of phase), global existence and nonexistence results are established under differ-

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ent constraints on the parameters in the elliptic-elliptic-elliptic case.[3] In Ref. 4, conserved quantities, corresponding to scale invariance and pseudoconformal invariance of solutions, have been discussed. The existence of standing wave (ground state) solutions for (1.1) are given in Ref. 5. In another recent work, Eden *et al.*[6] improved the results presented in Ref. 3 on global existence and global nonexistence for the solutions of the purely elliptic GDS system (1.1), closed the gap when the coupling parameter is negative, and reduced the size of the gap when the coupling parameter is positive. In Ref. 7, some special solutions of the GDS equation (1.1) under $n=0$ are investigated by both a tanh method and a variable separation approach.

In Ref. 8, in order to study the GDS system (1.1) more conveniently from a group theoretical point of view, Güngör *et al.* differentiated the last two equations of (1.1) with respect to $x$ and $y$, respectively, and then made the substitution $w_x \to w, \phi_y \to \phi$. Also rewriting the corresponding system in a real form by separating $\psi = u + iv$ into real and imaginary parts, they changed (1.1) into a system of four real PDEs as follows:

$$\begin{aligned} u_t + \delta v_{xx} + v_{yy} &= \chi v(u^2+v^2) + \gamma v(w+\phi), \\ -v_t + \delta u_{x,x} + u_{yy} &= \chi u(u^2+v^2) + \gamma u(w+\phi), \\ w_{xx} + n\phi_{xx} + m_2 w_{yy} &= (u^2+v^2)_{xx}, \\ n w_{yy} + \lambda \phi_{xx} + m_1 \phi_{yy} &= (u^2+v^2)_{yy}, \end{aligned} \tag{1.3}$$

and also called (1.3) the GDS system. By the classical Lie group approach[9,10] and under some conditions imposed on parameters in the system, they obtained the Lie symmetry algebra of (1.3) with four arbitrary functions of $t$ and showed that it is infinite-dimensional and isomorphic to that of the standard integrable DS equations. However, during the process of our computation we find that the constraint conditions on parameters can be omitted and the general element of Lie symmetry algebra of (1.3) should divide into two cases under different parameter conditions. The first symmetry algebra includes eight arbitrary functions of $t$ with $n^2 - nm_1 - n + m_1 - \lambda m_2 \neq 0$. The second symmetry algebra involves a quadratic function of $t$, four arbitrary functions of $t$, and two arbitrary two-variable functions. We show that the first algebra of the symmetry group of (1.3) involving eight arbitrary functions has a Kac–Moody–Virasoro (KMV) loop algebra structure which is shared by the symmetry algebras of some known integrable equations in 2+1 dimensions such as the Kadomtsev–Petviashvilli (KP) equation,[11,12] generalized KP equation,[13,14] the usual integrable DS equations,[15] a general KP family,[16] the 2+1 dimensional Ablowitz-Kaup-Newell-Segur (AKNS) system,[17] the dispersive long wave (DLW) equation, and the extended self-dual Yang–Mills (SDYM) system.[18] Under the same parameters conditions in Ref. 8 and setting the four arbitrary functions among eight arbitrary functions to some special forms, the Lie symmetry algebra of (1.3) in Ref. 8 can be recovered.

However it is difficult for us to obtain the general symmetry group of (1.3) by integrating the vector fields involving eight arbitrary functions. So we turn to the symmetry group direct method recently developed by Lou and Ma,[17] which is inspired by Clarkson-Kruskal (CK's) direct method[19] and is an effective method to obtain full symmetry groups for some PDEs.[18,20,21] By the symmetry group direct method, the problems of seeking for the full symmetry groups of given PDE(s) are changed into solving a set of overdetermined nonlinear PDEs. Due to the nonlinearity of the overdetermined nonlinear PDEs, it is usually difficult for us to obtain the solutions of them. However once the full symmetry group of a given system is given, the related Lie point symmetries can be obtained simply by restricting the arbitrary functions or arbitrary constants in infinitesimal forms. At the same time, the related group of discrete transformations of the given system can also be derived from the full symmetry groups. Furthermore, using the obtained full symmetry groups and a known simple solution, one can obtain a type of group invariant solutions.

This paper is organized as follows. In Sec. II, we compute the Lie symmetry algebra of (1.3) by the classical Lie group approach and discuss their structure and the relation to the results in

Ref. 8. In Sec. III, the full symmetry groups of (1.3) are presented by the symmetry group direct method proposed by Lou and Ma.[17] The Lie point symmetries involving eight arbitrary functions in Sec. II can be derived simply by restricting the arbitrary functions in infinitesimal forms and a group of discrete transformations of (1.3) can be obtained easily. In Sec. IV, some exact solutions of (1.3) in terms of Jacobi elliptic functions are presented by a subequation expansion method.[22] Finally, a short summary and discussion are given.

## II. LIE SYMMETRY ALGEBRA OF THE GDS SYSTEM

In order to find the Lie symmetry algebra $L$ and hence the symmetry group $G$ of (1.3), we write the GDS system as a system $\Delta_i(t,x,y,u,v,w,\phi)=0$ $(i=1,2,3,4)$ and look for an algebra of vector fields of the form

$$\mathbf{V} = \tau\partial_t + \xi\partial_x + \eta\partial_y + \psi_1\partial_u + \psi_2\partial_v + \psi_3\partial_w + \psi_4\partial_\phi, \tag{2.1}$$

where the coefficients $\tau,\xi,\eta,\psi_1,\psi_2,\psi_3,\psi_4$ are functions of $t,x,y,u,v,w,\phi$. According to the general theory for symmetries of differential equations, to find these functions we prolong the vector field to fourth order derivatives and require that the second prolonged vector field annihilates $\Delta_i$ on the solution manifold of (2.1), namely,

$$\mathrm{pr}^{(2)}\mathbf{V}(\Delta(t,x,y,u,v,w,\phi))\big|_{\Delta_i=0} = 0, \quad i=1,2,3,4, \tag{2.2}$$

where $\mathrm{pr}^{(2)}\mathbf{V}$ is the second prolongation of the vector field $\mathbf{V}$. By symbolic computer system MAPLE we obtain a complicated system of determining equations, a relatively simple system of linear PDEs for $\tau$, $\xi$, $\eta$, $\psi_1$, $\psi_2$, $\psi_3$, and $\psi_4$. With the help of MAPLE, solving the determining equations we find that a general element of the symmetry algebra of the GDS system (1.3) can be written in the following two cases.

*Case 1:* When $n^2-nm_1-n+m_1-\lambda m_2\neq 0$,

$$\mathbf{V} = v_1(f)+v_2(g)+v_3(h)+v_4(k)+v_5(l)+v_6(p)+v_7(q)+v_8(s), \tag{2.3}$$

where

$$v_1(f) = f\partial_t + \frac{1}{2}f_t(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi) - \frac{x^2+\delta y^2}{8\delta}f_{tt}(v\partial_u - u\partial_v)$$
$$- \frac{(n^2-m_1m_2\delta-nm_1-\lambda m_2)x^2+(-nm_1\delta+m_1\delta+\lambda)y^2}{8\delta\gamma(n^2-nm_1-n+m_1-\lambda m_2)}f_{ttt}\partial_w$$
$$+ \frac{(-m_1m_2\delta+n-m_1)x^2+(\delta n+\delta\lambda m_2+\lambda-\delta n^2)y^2}{8\delta\gamma(n^2-nm_1-n+m_1-\lambda m_2)}f_{ttt}\partial_\phi,$$

$$v_2(g) = g\partial_x - \frac{x}{2\delta}\left[g_t(v\partial_u - u\partial_v) + \frac{1}{\gamma}g_{tt}\partial_w\right],$$

$$v_5(l) = -xyl(\partial_w - \partial_\phi), \quad v_6(p) = -xp(\partial_w - \partial_\phi),$$

$$v_4(k) = k(v\partial_u - u\partial_v) + 1/\gamma k_t\partial_w, \tag{2.4}$$

$$v_3(h) = h\partial_y - \frac{y}{2}\left[h_t(v\partial_u - u\partial_v) + \frac{1}{\gamma}h_{tt}\partial_w\right],$$

$$v_7(q) = -yq(\partial_w - \partial_\phi), \quad v_8(s) = -s(\partial_w - \partial_\phi).$$

The functions $f=f(t)$, $g=g(t)$, $h=h(t)$, $k=k(t)$, $l=l(t)$, $p=p(t)$, $q=q(t)$, and $s=s(t)$ are all arbitrary real-valued functions of class $C^\infty(I)$, $I\subseteq\mathbb{R}$, subscripts denote derivatives of $t$.

*Case 2:* When $n^2-nm_1-n+m_1-\lambda m_2=0$,

$$\widetilde{\mathbf{V}}=\widetilde{v}_1(f)+\widetilde{v}_2(g)+\widetilde{v}_3(h)+\widetilde{v}_4(k)+\widetilde{v}_5(l)+\widetilde{v}_6(p)+\widetilde{v}_7(q), \tag{2.5}$$

where

$$\widetilde{v}_1(f)=f\partial_t+\frac{1}{2}f_t\left[x\partial_x+y\partial_y-(u\partial_u+v\partial_v)+\frac{2(u^2+v^2-w)}{m_1}(\partial_w-\partial_\phi)-2(w+\phi)\partial_w\right]-\frac{(x^2+\delta y^2)}{8\delta}f_{tt}(v\partial_u-u\partial_v),$$

$$\widetilde{v}_2(g)=g\partial_x-\frac{x}{2\delta}\left[g_t(v\partial_u-u\partial_v)+\frac{1}{\gamma}g_{tt}\partial_w\right],$$

$$\widetilde{v}_3(h)=h\partial_y-\frac{y}{2}\left[h_t(v\partial_u-u\partial_v)+\frac{1}{\gamma}h_{tt}\partial_w\right],$$

$$\widetilde{v}_4(k)=k(v\partial_u-u\partial_v)+\frac{1}{\gamma}k_t\partial_w, \tag{2.6}$$

$$\widetilde{v}_5(l)=l\left(\frac{u^2+v^2-w}{m_1}-\phi\right)(\partial_w-\partial_\phi),$$

$$\widetilde{v}_6(p)=-p(\partial_w-\partial_\phi),$$

$$\widetilde{v}_7(q)=-q(\partial_w-\partial_\phi),$$

with $f=c_2t^2/2+c_1t+c_0$, and $g=g(t)$, $h=h(t)$, $k=k(t)$, $l=l(t)$ are all arbitrary real-valued functions of class $C^\infty(I)$, $I\subseteq\mathbb{R}$ and $p=p(x/2-y\lambda/(2\sqrt{\lambda(n-m_1)}),t)$ and $q=q(\sqrt{\lambda(n-m_1)}x/\lambda+y,t)$ are arbitrary functions of two independent variables, subscripts denote derivatives of $t$.

In case 1, i.e., $f(t)$ is allowed to be arbitrary, the symmetry algebra realized by the vector fields (2.3) and (2.4) is infinite dimensional and more important has the structure of a KMV algebra as we shall see below.

The nonzero commutation relations for the GDS algebra (2.3) and (2.4) are obtained as follows:

$$[v_1(f_1),v_1(f_2)]=v_1(f_1f_{2t}-f_{1t}f_2),$$

$$[v_1(f),v_2(g)]=v_2\left(fg_t-\frac{1}{2}gf_t\right)+v_6\left(-\frac{(-m_1m_2\delta+n-m_1)gf_{ttt}}{4\delta\gamma(n^2-nm_1-n+m_1-\lambda m_2)}\right),$$

$$[v_1(f),v_3(h)]=v_3\left(fh_t-\frac{1}{2}hf_t\right)+v_7\left(\frac{(-\lambda+\delta n^2-m_2\lambda\delta-\delta n)hf_{ttt}}{4\delta\gamma(n^2-nm_1-n+m_1-\lambda m_2)}\right),$$

$$[v_1(f),v_4(k)]=v_4(fk_t),\quad [v_1(f),v_5(l)]=v_5(fl_t),$$

$$[v_1(f),v_6(p)]=v_6\left(fp_t+\tfrac{3}{2}pf_t\right),$$

$$[v_1(f),v_7(p)]=v_7\left(fq_t+\tfrac{3}{2}qf_t\right), \tag{2.7}$$

$$[v_1(f),v_8(s)]=v_8(fs_t+sf_t),$$

$$[v_2(g_1),v_2(g_2)]=v_4\left(-\frac{g_1g_{2t}-g_2g_{1t}}{2\delta}\right),$$

$$[v_2(g),v_5(l)]=v_7(gl), \quad [v_2(g),v_6(p)]=v_8(gp),$$

$$[v_3(h_1),v_3(h_2)]=v_4\left(-\frac{h_1h_{2t}-h_2h_{1t}}{2}\right),$$

$$[v_3(h),v_5(l)]=v_6(hl), \quad [v_3(h),v_7(q)]=v_8(hq),$$

where $f_1=f_1(t)$, $f_2=f_2(t)$, $g_1=g_1(t)$, $g_2=g_2(t)$, $h_1=h_1(t)$, and $h_2=h_2(t)$ are all arbitrary functions of $t$, subscripts denote derivatives of $t$.

From (2.7), we see that the GDS system has a Lie symmetry algebra $L$ which allows a Levi decomposition

$$L=M \Subset N, \tag{2.8}$$

where $M=\{v_1(f)\}$ is a simple infinite-dimensional Lie algebra and

$$N=\{v_2(g),v_3(h),v_4(k),v_5(l),v_6(p),v_7(q),v_8(s)\}$$

is a nilpotent idea (nilradical). Here, $\Subset$ denotes a semidirect sum. The algebra $\{v_1(f)\}$ is isomorphic to the Lie algebra corresponding to the Lie group of diffeomorphisms of a real line.

From (2.3) and (2.4), the results (2.3) and (2.4) obtained in Ref. 8 can be recovered by restricting some parameters and some arbitrary functions as follows:

$$m_2\delta+n+1=0, \quad m_1\delta+n\delta+\lambda=0, \tag{2.9}$$

$$l=0, \quad p=-\frac{g_{tt}}{4\delta\gamma}, \quad q=-\frac{h_{tt}}{4\gamma}, \quad s=\frac{k_t}{2\gamma}. \tag{2.10}$$

We can also verify when parameters $m_1$, $m_2$, $\delta$, and $\lambda$ satisfy (2.9), then $n^2-nm_1-n+m_1-\lambda m_2 \neq 0$, i.e., $f(t)$ keeps arbitrary. At the same time, from (2.5) and (2.6), the results with $f(t)=c_2/2t^2+c_1t+c_0$ in Ref. 8 can also be reproduced by restricting some arbitrary functions as follows:

$$l=-f_t,$$

$$p=\left(\frac{x}{2}-\frac{\lambda y}{2\sqrt{\lambda(n-m_1)}}\right)\left(-\frac{g_{tt}}{4\delta\gamma}+\frac{\sqrt{\lambda(n-m_1)}}{\lambda}\frac{h_{tt}}{4\gamma}\right)+\frac{k_t}{2\gamma}, \tag{2.11}$$

$$q=\left(\frac{x}{2}+\frac{\lambda y}{2\sqrt{\lambda(n-m_1)}}\right)\left(-\frac{g_{tt}}{4\delta\gamma}-\frac{\sqrt{\lambda(n-m_1)}}{\lambda}\frac{h_{tt}}{4\gamma}\right).$$

Under the conditions (2.9) and (2.10) in **V**, the GDS system has a Lie symmetry algebra $L$ isomorphic to that of the DS symmetry algebra and also allows a Levi decomposition.[8,15] Due to the isomorphic relation between the two symmetry algebra, one can directly obtain the general symmetry group of the GDS system by the general symmetry group of the DS system known in

Ref. 15. However from the general elements (2.3) with (2.4) or (2.5) with (2.6) of the GDS Lie algebra, it is difficult for us to obtain the general symmetry group transformations for (1.3). So we shall turn to the symmetry group direct method recently developed by Lou and Ma[17] for the general symmetry groups of (1.3).

Now we summarize the main results in this section as follows.

**Theorem 2.1:** The system (1.3) is invariant under an infinite-dimensional Lie point symmetry group, the Lie algebra of which has a KMV structure isomorphic to the DS algebra if and only if the conditions (2.9) and (2.10) hold.

## III. FULL SYMMETRY GROUP TRANSFORMATIONS OF THE GDS SYSTEM

According to the symmetry group direct method[17] and considering the quadratic terms $\{u^2, v^2\}$ in the symmetry algebra (2.5) and (2.6), we uniformly set the solutions of (1.3) as follows:

$$
\begin{aligned}
u &= \alpha_1 + \beta_1 U(\xi,\eta,\tau) + r_1 V(\xi,\eta,\tau),\\
v &= \alpha_2 + \beta_2 U(\xi,\eta,\tau) + r_2 V(\xi,\eta,\tau),\\
w &= \alpha_3 + \beta_3 W(\xi,\eta,\tau) + r_3 \Phi(\xi,\eta,\tau) + \mu_1 U^2(\xi,\eta,\tau) + \mu_2 V^2(\xi,\eta,\tau),\\
\phi &= \alpha_4 + \beta_4 W(\xi,\eta,\tau) + r_4 \Phi(\xi,\eta,\tau) + \mu_3 U^2(\xi,\eta,\tau) + \mu_4 V^2(\xi,\eta,\tau),
\end{aligned}
\tag{3.1}
$$

where $\alpha_i, \beta_i, r_i, \mu_i (i=1,2,3,4)$, $\xi, \eta, \tau$ are all functions of $\{x,y,t\}$, $U(\xi,\eta,\tau)$, $V(\xi,\eta,\tau)$, $W(\xi,\eta,\tau)$, and $\Phi(\xi,\eta,\tau)$ satisfy the same equations as (1.3), i.e.,

$$
\begin{aligned}
U_\tau + \delta V_{\xi,\xi} + V_{\eta,\eta} &= \chi V(U^2+V^2) + \gamma V(W+\Phi),\\
-V_\tau + \delta U_{\xi\xi} + U_{\eta\eta} &= \chi U(U^2+V^2) + \gamma U(W+\Phi),\\
W_{\xi\xi} + n\Phi_{\xi\xi} + m_2 W_{\eta\eta} &= (U^2+V^2)_{\xi\xi},\\
nW_{\eta\eta} + \lambda\Phi_{\xi\xi} + m_1\Phi_{\eta\eta} &= (U^2+V^2)_{\eta\eta}.
\end{aligned}
\tag{3.2}
$$

Substituting (3.1) into (1.3), eliminating all terms including $U_\tau$, $V_\tau$, $W_{\xi\xi}$, and $W_{\eta\eta}$ by (3.2), we can obtain four polynomial differential equations with respect to $\{U,V,W,\Phi\}$ and their derivatives. Then collecting their coefficients of $\{U,V,W,\Phi\}$ and their derivatives, we can obtain a huge numbers of nonlinear PDEs with respect to differentiable functions: $\{\alpha_i, \beta_i, r_i, \mu_i (i=1,2,3,4), \xi, \eta, \tau\}$.

With the help of symbolic computation, MAPLE, after some thorough analysis and some quite tedious calculations, the general solutions of these huge numbers of determining equations are as follows:

*Family 1:* When $n^2 - nm_1 - n + m_1 - \lambda m_2 \neq 0$,

$$
\begin{aligned}
&\alpha_1 = \alpha_2 = r_3 = \beta_4 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,\\
&\tau = \sigma_5 \rho, \quad r_2 = \sigma_5 \beta_1, \quad r_1 = -\sigma_5 \beta_2,\\
&\xi = \sigma_1 \sqrt{\rho_t} x + \xi_0, \quad \eta = \sigma_2 \sqrt{\rho_t} y + \eta_0,\\
&\beta_1 = \sigma_3 \sqrt{\rho_t} \cos\theta, \quad \beta_2 = \sigma_4 \sqrt{\rho_t} \sin\theta,\\
&r_4 = \beta_3, \quad \beta_3 = \rho_t,
\end{aligned}
$$

$$\theta = -\frac{1}{8}\frac{\sigma_3\rho_{tt}y^2}{\sigma_4\rho_t} - \frac{1}{2}\frac{\sigma_3\eta_{0t}y}{\sigma_2\sigma_4\sqrt{\rho_t}} - \frac{1}{2}\frac{\sigma_3\xi_{0t}x}{\sigma_1\sigma_4\delta\sqrt{\rho_t}} - \frac{1}{8}\frac{\sigma_3\rho_{tt}x^2}{\sigma_4\delta\rho_t} + \theta_0,$$

$$\alpha_3 = -\alpha_4 + \frac{1}{16}\frac{(2\rho_t\rho_{ttt} - 3\rho_{tt}^2)(x^2+\delta y^2)}{\delta\gamma\rho_t^2} - \frac{1}{2}\frac{(\xi_{0t}\rho_{tt} - \rho_t\xi_{0tt})\sigma_1 x + \delta(\eta_{0t}\rho_{tt} - \rho_t\eta_{0tt})\sigma_2 y}{\rho_t^{3/2}\delta\gamma}$$
$$- \frac{1}{4}\frac{\xi_{0t}^2 + 4\sigma_3\sigma_4\delta\rho_t\theta_{0t} + \delta\eta_{0t}^2}{\rho_t\delta\gamma}, \tag{3.3}$$

$$\alpha_4 = -\frac{(2\rho_t\rho_{ttt} - 3\rho_{tt}^2)[(-m_1m_2\delta + n - m_1)x^2 + (\delta n + \delta\lambda m_2 + \lambda - \delta n^2)y^2]}{16\delta\gamma\rho_t^2(n^2 - nm_1 - n + m_1 - \lambda m_2)}$$
$$+ [F_3(t)y + F_4(t)]x + F_5(t)y + F_6(t), \tag{3.4}$$

where $\rho \equiv \rho(t)$, $\xi_0 \equiv \xi_0(t)$, $\eta_0 = \eta_0(t)$, $\theta_0 = \theta_0(t)$, $F_3(t)$, $F_4(t)$, $F_5(t)$, and $F_6(t)$ are arbitrary functions of $t$, $\rho_t \geq 0$, subscripts denote derivatives of $t$ and

$$\sigma_1^2 = 1, \quad \sigma_2^2 = 1, \quad \sigma_3^2 = 1, \quad \sigma_4^2 = 1, \quad \sigma_5^2 = 1. \tag{3.5}$$

*Family 2:* When $n^2 - nm_1 - n + m_1 - \lambda m_2 = 0$,

$$r_3 = -m_1\beta_4, \quad \beta_3 = -\beta_4 + \rho_t, \quad r_4 = m_1\beta_4 + \rho_t,$$

$$\mu_1 = \mu_2 = -\beta_4, \quad \mu_3 = \mu_4 = \beta_4, \quad \beta_4 = \nu,$$

$$\alpha_4 = F\left(\frac{x}{2} - \frac{\lambda y}{2\sqrt{\lambda(n-m_1)}}, t\right) + G\left(\frac{\sqrt{\lambda(n-m_1)}}{\lambda}x + y, t\right), \tag{3.6}$$

$$\rho = \frac{4c_2}{t+c_1} + c_0 \quad \text{or} \quad \rho = c_3 t,$$

where $F(x/2 - \lambda y/2\sqrt{\lambda(n-m_1)}, t)$ and $G(\lambda(n-m_1)/\lambda x + y, t)$ are arbitrary functions of two independent variables, $\nu = \nu(t)$ is an arbitrary function of $t$, $c_0$, $c_1$, $c_2$, and $c_3$ are arbitrary constants, and the other parameters $\{\tau, r_1, r_2, \xi, \eta, \beta_1, \beta_2, \theta, \alpha_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ are all given by (3.3) and (3.5).

From the above results, one can get the following symmetry group theorems for (1.3).

**Theorem 3.1:** If $\{U = U(x,y,t),\ V = V(x,y,t), W = W(x,y,t), \Phi = \Phi(x,y,t)\}$ is a solution of the GDS system (1.3) under $n^2 - nm_1 - n + m_1 - \lambda m_2 \neq 0$, so are $\{u, v, w, \phi\}$ with

$$u = \sqrt{\rho_t}[\sigma_3\cos(\theta)U(\xi,\eta,\tau) - \sigma_5\sigma_4\sin(\theta)V(\xi,\eta,\tau)],$$

$$v = \sqrt{\rho_t}[\sigma_4\sin(\theta)U(\xi,\eta,\tau) + \sigma_5\sigma_3\cos(\theta)V(\xi,\eta,\tau)],$$

$$w = \alpha_3 + \rho_t W(\xi,\eta,\tau) \tag{3.7}$$

$$\phi = \alpha_4 + \rho_t\Phi(\xi,\eta,\tau),$$

where $\{\rho, \xi, \eta, \tau, \alpha_3, \alpha_4, \theta, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ are determined by (3.3)–(3.5).

**Theorem 3.2:** If $\{U = U(x,y,t),\ V = V(x,y,t), W = W(x,y,t), \Phi = \Phi(x,y,t)\}$ is a solution of the GDS system (1.3) under $n^2 - nm_1 - n + m_1 - \lambda m_2 = 0$, so are $\{u, v, w, \phi\}$ with

$$u = \sqrt{\rho_t}[\sigma_3\cos(\theta)U(\xi,\eta,\tau) - \sigma_5\sigma_4\sin(\theta)V(\xi,\eta,\tau)],$$

$$v = \sqrt{\rho_t}[\sigma_4 \sin(\theta)U(\xi,\eta,\tau) + \sigma_5\sigma_3 \cos(\theta)V(\xi,\eta,\tau)],$$
$$w = \alpha_3 + (\rho_t - \nu)W(\xi,\eta,\tau) - m_1\nu\Phi(\xi,\eta,\tau) - \nu[U^2(\xi,\eta,\tau) + V^2(\xi,\eta,\tau)], \quad (3.8)$$
$$\phi = \alpha_4 + \nu W(\xi,\eta,\tau) + (\rho_t + m_1\nu)\Phi(\xi,\eta,\tau) + \nu[U^2(\xi,\eta,\tau) + V^2(\xi,\eta,\tau)],$$

where $\{\tau,\xi,\eta,\beta_1,\beta_2,\theta,\alpha_3,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}$ are determined by (3.3) and (3.5) and $\{\alpha_4,\nu,\rho\}$ are determined by (3.6).

When $n^2-nm_1-n+m_1-\lambda m_2\neq 0$, from the symmetry group Theorem 3.1, we know that for the GDS system (1.3), the symmetry group is divided into 16 classes which correspond to the following.

*Case I:* $\sigma_5=1$, i.e., $\psi\to\psi$ (or $-\psi$) and

$$\begin{aligned}
&\sigma_1=1, \quad \sigma_2=1, \quad \sigma_3=1, \quad \sigma_4=\pm 1,\\
&\sigma_1=-1, \quad \sigma_2=1, \quad \sigma_3=1, \quad \sigma_4=\pm 1,\\
&\sigma_1=1, \quad \sigma_2=-1, \quad \sigma_3=1, \quad \sigma_4=\pm 1,\\
&\sigma_1=-1, \quad \sigma_2=-1, \quad \sigma_3=1, \quad \sigma_4=\pm 1,\\
&\sigma_1=1, \quad \sigma_2=1, \quad \sigma_3=-1, \quad \sigma_4=\pm 1,\\
&\sigma_1=-1, \quad \sigma_2=1, \quad \sigma_3=-1, \quad \sigma_4=\pm 1,\\
&\sigma_1=1, \quad \sigma_2=-1, \quad \sigma_3=-1, \quad \sigma_4=\pm 1,\\
&\sigma_1=-1, \quad \sigma_2=-1, \quad \sigma_3=-1, \quad \sigma_4=\pm 1,
\end{aligned} \quad (3.9)$$

*Case II:* $\sigma_5=-1$, i.e., $\psi\to\psi^*$ (or $-\psi^*$), and $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ are determined by (3.9), respectively.

That is to say, the full symmetry group, $\mathcal{G}_{\mathrm{GDS}}$, expressed by Theorem 3.1 for the GDS system (1.3) is the product of the connected Lie point symmetry group $\mathcal{S}$ (Theorem 3.1 with $\sigma_1=\sigma_2=\sigma_3=\pm\sigma_4=\sigma_5=1$) and the discrete group $\mathcal{D}$

$$\mathcal{G}_{\mathrm{GDS}}=\mathcal{D}\otimes\mathcal{S}, \quad (3.10)$$

where

$$\mathcal{D}=\{I,\sigma^x,\sigma^y,\sigma^\psi,\sigma^{t\psi^*}\}, \quad (3.11)$$

with $I$ as the identity transformation, and

$$\sigma^x : x\to -x,$$
$$\sigma^y : y\to -y,$$
$$\sigma^\psi : \psi\to -\psi,$$
$$\sigma^{t\psi^*} : \{t,\psi\}\to\{-t,\psi^*\}.$$

It should be mentioned that the discrete groups can be read off from (3.7) by taking the continuous functions as some special constants. For instance, a group of discrete transformations generated by

$\{x,y,t,\psi,w,\phi\}\to\{-x,y,-t,\psi^*,w,\phi\}$ can be read off from $\sigma^x\sigma^t\psi^* I$ by taking $\rho=t,\xi_0=\eta_0=\theta_0=F_3(t)=F_4(t)=F_5(t)=F_6(t)=0,\sigma_1=\sigma_5=-1,\sigma_2=\sigma_3=\pm\sigma_4=1$.

From Theorem 3.1, the Lie point symmetries (2.3) and (2.4) can be reproduced by restricting

$$\sigma_1=\sigma_2=\sigma_3=\sigma_4=\sigma_5=1, \quad \rho=t+\epsilon f,$$

$$\xi_0=\epsilon g, \quad \eta_0=\epsilon h, \quad \theta_0=\epsilon k, \quad F_3(t)=\epsilon l, \tag{3.12}$$

$$F_4(t)=\epsilon p, \quad F_5(t)=\epsilon q, \quad F_6(t)=\epsilon s,$$

where $f=f(t)$, $g=g(t)$, $h=h(t)$, $k=k(t)$, $l=l(t)$, $p=p(t)$, $q=q(t)$, and $s=s(t)$ are all arbitrary functions of $t$, and $\epsilon$ is an infinitesimal parameter.

If we set

$$\xi_0=\eta_0=\theta_0=F_3(t)=F_4(t)=F_5(t)=F_6(t)=0,$$

$$\sigma_1=\sigma_2=\sigma_3=\sigma_4=\sigma_5=1, \tag{3.13}$$

$$\tau=\rho=\frac{c+dt}{a+bt}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}(2,\mathbb{R}),$$

then from (3.3), we can derive $\xi$ and $\eta$ as follows:

$$\xi=\frac{x}{a+bt}, \quad \eta=\frac{y}{a+bt},$$

$$\sqrt{\rho_t}=\frac{1}{a+bt}, \quad \theta=\frac{b(x^2+\delta y^2)}{4(a+bt)\delta}. \tag{3.14}$$

Thus from Theorem 2.1, we can obtain

$$\psi=u+iv=\frac{1}{(a+bt)}\exp\left(i\frac{b(x^2+\delta y^2)}{4(a+bt)\delta}\right)\Psi(\xi,\eta,\tau),$$

$$w=\frac{1}{(a+bt)^2}W(\xi,\eta,\tau) \tag{3.15}$$

$$\phi=\frac{1}{(a+bt)^2}\Phi(\xi,\eta,\tau),$$

here $\xi,\eta,\tau$ are determined by (3.13) and (3.14) and $\Psi(\xi,\eta,\tau)=U+iV$.

From (3.15), the results (2.12) obtained by physical symmetries $L_p$ of the GDS system in Ref. 8 can be recovered. When integrating $w(x,y,t)$ and $\phi(x,y,t)$ with respect to $x$ and $y$, respectively, the pseudoconformal transformation (8) for (1.1) found in Ref. 4 can be also recovered.

Now we turn to the case with $n^2-nm_1-n+m_1-\lambda m_2=0$. From Theorem 3.2, the Lie point symmetries (2.5) and (2.6) can be reproduced by restricting

$$F\left(\frac{x}{2}-\frac{\lambda y}{2\sqrt{\lambda(n-m_1)}},t\right)=\epsilon p\left(\frac{x}{2}-\frac{\lambda y}{2\sqrt{\lambda(n-m_1)}},t\right),$$

$$G\left(\frac{\sqrt{\lambda(n-m_1)}}{\lambda}x+y,t\right)=\epsilon q\left(\frac{\sqrt{\lambda(n-m_1)}}{\lambda}x+y,t\right),$$

$$\rho=\frac{-2(1+c_2\epsilon)}{t+c_1\epsilon}+c_0\epsilon,\quad \xi_0=\epsilon g(t),\quad \eta_0=\epsilon h(t),\tag{3.16}$$

$$\sigma_1=\sigma_2=\sigma_3=\sigma_4=\sigma_5=1,$$

$$\theta_0=\epsilon k(t),\quad \nu=\frac{\epsilon l(t)}{m_1},$$

where $\epsilon$ is an infinitesimal parameter.

It is necessary to point out that in order to obtain the general symmetry group of the Lie symmetry algebra involving the quadratic terms, even higher order terms, of dependent variables, by the symmetry group direct method, we have to assume the finite symmetry transformations to have a more complex form, even a general form. In turn, the price to pay is to integrate a very large number of nonlinear PDEs. In most cases, it is very difficult to solve these nonlinear PDEs.

## IV. SOME EXACT SOLUTIONS OF THE GDS SYSTEM

To find some types of exact solutions in high dimensions is one of the most important and difficult work. In this section, some special solutions of the GDS system (1.3) will be given in terms of Jacobi elliptic functions by a generalized subequation expansion method.[22] These solutions involve hyperbolic secant funtions and hyperbolic tangent functions as special cases. Then we can easily write down some types of group invariant solutions for (1.3) by the known exact solutions and the group transformation Theorem 3.1 and Theorem 3.2.

According to the generalized subequation expansion method,[22] we assume the solutions of (1.3) are of the following special forms:

$$\psi=[A_1(t)+B_1(t)f(\zeta)]\exp[i(g_0(t)+g_1(t)x+g_2(t)y)],$$

$$w=A_2(t)+B_2(t)f^2(\zeta),\tag{4.1}$$

$$\phi=A_3(t)+B_3(t)f^2(\zeta),$$

where $\psi=u+iv$, $\zeta$ and $f(\zeta)$ satisfy the following equations:

$$\zeta=k_1(t)x+k_2(t)y-\Omega(t),\tag{4.2}$$

$$f'(\zeta)^2=h_0+h_1f(\zeta)+h_2f^2(\zeta)+h_3f^3(\zeta)+h_4f^4(\zeta),\tag{4.3}$$

with $A_1(t)$, $B_1(t)$, $g_0(t)$, $g_1(t)$, $g_2(t)$, $A_2(t)$, $B_2(t)$, $A_3(t)$, $B_3(t)$, $k_1(t)$, $k_2(t)$, and $\Omega(t)$ are functions of $t$, $h_0$, $h_1$, $h_2$, $h_3$, and $h_4$ are real constants, and prime is derivative with respect to $t$.

Substituting (4.1) along with (4.2) and (4.3) into (1.3), removing the exponential terms, collecting coefficients of polynomials of $\{f(\zeta),f'(\zeta),t\}$ of the resulting system, then separating each coefficient to the real part and imaginary part, we obtain an ordinary differential equation

(ODE) system with respect to differentiable functions $A_1(t)$, $B_1(t)$, $g_0(t)$, $g_1(t)$, $g_2(t)$, $A_2(t)$, $B_2(t)$, $A_3(t)$, $B_3(t)$, $k_1(t)$, $k_2(t)$, and $\Omega(t)$. For simplification, we omit the ODE system in the paper.

Solving the ODE system with MAPLE, we can obtain the following results:

$$A_1(t)=h_1=h_3=0,\quad B_1(t)=c_1,\quad k_1(t)=c_4,$$

$$k_2(t)=c_5,\quad g_1(t)=c_3,\quad g_2(t)=c_2,$$

$$\Omega(t)=2(c_2c_5+\delta c_3c_4)t+c_6,$$

$$A_2(t)=-A_3(t)+\frac{-g_0'(t)-c_2^2-\delta c_3^2+h_2(\delta c_4^2+c_5^2)}{\gamma}, \tag{4.4}$$

$$B_2(t)=\frac{c_1^2((-m_1+n)c_5^2-c_4^2\lambda)c_4^2}{(n^2-m_1-m_2\lambda)c_4^2c_5^2-c_4^4\lambda-m_1c_5^4m_2},$$

$$B_3(t)=\frac{c_1^2c_5^2(-c_5^2m_2+(n-1)c_4^2)}{(n^2-m_1-m_2\lambda)c_4^2c_5^2-c_4^4\lambda-m_1c_5^4m_2},$$

$$h_4=\frac{(-\lambda(\chi+\gamma)c_4^4+((2n-m_1+1)\gamma+\chi(n^2-m_1-m_2\lambda))c_5^2c_4^2-m_2(m_1\chi+\gamma)c_5^4)c_1^2}{2(-c_4^4\lambda+(-m_1-m_2\lambda+n^2)c_5^2c_4^2-m_1c_5^4m_2)(\delta c_4^2+c_5^2)},$$

where $c_1,c_2,c_3,c_4,c_5$ are arbitrary constants, and $g_0(t)$ and $A_3(t)$ are arbitrary functions of $t$.

When $h_1=h_3=0$, (4.3) has following Jacobi elliptic function solutions:

$$f_1(\zeta)=\sqrt{\frac{-h_2m^2}{h_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\zeta;m\right),\quad h_4<0,\quad h_2>0,\quad h_0=\frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2},$$

$$f_2(\zeta)=\sqrt{\frac{-h_2}{h_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\zeta;m\right),\quad h_4<0,\quad h_2>0,\quad h_0=\frac{h_2^2(1-m^2)}{h_4(2-m^2)^2}, \tag{4.5}$$

$$f_3(\zeta)=\sqrt{\frac{-h_2m^2}{h_4(m^2+1)}}\mathrm{sn}\left(\sqrt{-\frac{h_2}{m^2+1}}\zeta;m\right),\quad h_4>0,\quad h_2<0,\quad h_0=\frac{h_2^2m^2}{h_4(m^2+1)^2},$$

where $0\le m\le 1$ is a modulus.

Thus from Eqs. (4.1), (4.2), (4.4), and (4.5), three types of exact solutions for the GDS system (1.3) are obtained as follows:

$$\psi_j=c_1f_j(\zeta)\exp[i(g_0(t)+c_3x+c_2y)],$$

$$w_j=A_2(t)+B_2(t)f_j^2(\zeta), \tag{4.6}$$

$$\phi_j=A_3(t)+B_3(t)f_j^2(\zeta),$$

where $j=1,2,3$, $f_j(\zeta)$ are determined by (4.5), $A_2(t),A_3(t),B_2(t),B_3(t)$ are determined by (4.4), and

$$\zeta=c_4x+c_5y-2(c_2c_5+\delta c_3c_4)t-c_6. \tag{4.7}$$

In the limit $m=1$, the solutions (4.6) reduce to the following solitary wave solutions in terms of hyperbolic secant function and hyperbolic tangent functions.

*Case I:* From $f_1(\zeta)$, $f_2(\zeta)$, and (4.6), the following solitary wave solutions are obtained:

$$\psi = c_1\sqrt{\frac{-h_2}{h_4}}\operatorname{sech}(\sqrt{h_2}\zeta)\exp[i(g_0(t)+c_3x+c_2y)],$$

$$w = A_2(t)+B_2(t)\frac{-h_2}{h_4}\operatorname{sech}^2(\sqrt{h_2}\zeta), \tag{4.8}$$

$$\phi = A_3(t)+B_3(t)\frac{-h_2}{h_4}\operatorname{sech}^2(\sqrt{h_2}\zeta).$$

*Case II:* From $f_3(\zeta)$ and (4.6), the following solitary wave solutions are obtained:

$$\psi = c_1\sqrt{\frac{-h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\zeta\right)\exp[i(g_0(t)+c_3x+c_2y)],$$

$$w = A_2(t)+B_2(t)\frac{-h_2}{2h_4}\tanh^2\left(\sqrt{-\frac{h_2}{2}}\zeta\right), \tag{4.9}$$

$$\phi = A_3(t)+B_3(t)\frac{-h_2}{2h_4}\tanh^2\left(\sqrt{-\frac{h_2}{2}}\zeta\right).$$

From (4.6), (4.8), and (4.9), some exact solutions of (1.1) can be derived by integrating $w_j$ and $\phi_j$ with respect to $x$ and $y$, respectively. If further setting $A_2(t)$, $A_3(t)$, and $g_0(t)$ to be some special forms and the integration functions to be zeros, the solutions of the GDS system (1) obtained in Ref. 1 can be reproduced. For example, if setting $A_2(t)=A_3(t)=0$, $g_0(t)=\Omega t$, $c_1=\pm 1$, $\int w dx \to w$, and $\int \phi dy \to \phi$, the solutions (36) in Ref. 1 can be recovered.

Now, using Theorem 3.1 and Theorem 3.2, we can derive some new types of group invariant solutions for the GDS system (1.3) and (1.1) from the solutions (4.6), (4.8), and (4.9). For simplification, we do not list them in the paper.

## V. SUMMARY AND DISCUSSION

By the classical Lie approach and symbolic computation, the Lie symmetry algebra of a GDS system is obtained. The general element of this algebra depends on eight arbitrary functions of time, which has a KMV loop algebra structure and is isomorphic to that of the standard integrable DS equations under certain conditions imposed on parameters and arbitrary functions. Then by the symmetry group direct method recently developed by Lou and Ma,[17] the full symmetry groups of the GDS system are obtained by solving the corresponding determining equations—a huge number of nonlinear PDES. From the full symmetry groups, the Lie symmetry group can be recovered and a group of discrete transformations can be obtained simultaneously. Finally, by a generalized subequation expansion method, some exact solutions in terms of Jacobi elliptic functions are presented, which include sech-sech$^2$-sech$^2$ and tanh-tanh$^2$-tanh$^2$ type solitary wave solutions.

However, when nonlinear terms of dependent variables occur in the Lie symmetry algebra of a given PDE, we have to assume the finite symmetry group transformations to have a more complex form, even a general form. In turn, the price to pay is to solve a very large number of nonlinear PDEs. How to seek for an appropriate finite symmetry group transformation for a given PDE? How to integrate a large number of nonlinear PDEs resulting from the above finite transformations? These problems should be studied further.

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