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The long-time asymptotics of the derivative nonlinear Schrödinger equation with step-like initial value

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ABSTRACT

Consideration in this present paper is the long-time asymptotics of solutions to the derivative nonlinear Schrödinger (DNLS) equation with the step-like initial value

$$q(x,0)=q_0(x)=\begin{cases}A_1e^{i\phi}e^{2iBx}, & x<0,\\ A_2e^{-2iBx}, & x>0,\end{cases}$$

by Deift–Zhou method. The step-like initial problem is described by a matrix Riemann–Hilbert problem. A crucial ingredient used in this paper is to introduce the g-function mechanism for solving the problem of the entries of the jump matrix growing exponentially as $t\to\infty$. It is shown that the leading order term of the long-time asymptotics solution of the DNLS equation is expressed by the Theta function Θ about the Riemann-surface of genus 3 and the subleading order term expressed by parabolic cylinder and Airy functions.

1. Introduction and main result

This paper is devoted to the long-time asymptotics behavior of solutions to the derivative nonlinear Schrödinger (DNLS) equation with a step-like initial value

$$iq_t+q_{xx}-iq^2\overline{q}_x+\frac{1}{2}|q|^4q=0, \tag{1.1}$$

$$q(x,0)=\begin{cases}A_1e^{i\phi_1}e^{-2iB_1x}, & x<0,\\ A_2e^{i\phi_2}e^{-2iB_2x}, & x>0.\end{cases} \tag{1.2}$$

Eq. (1.1) alternatively termed by DNLS-III equation and sometimes referred as the Gerdjikov–Ivanov equation, to model weakly nonlinear dispersive water waves, Alfvén waves propagating along with the constant magnetic field in cold plasmas and ultrafast waves in optical fibers [1–3]. Here and after, the overbar denotes the complex conjugation and the subscript denotes the differential with respect to the corresponding variables. The DNLS-I,-II equations

$$iq_t+q_{xx}+i(|q|^2q)_x=0, \tag{1.3}$$

$$iq_t+q_{xx}+i|q|^2q_x=0, \tag{1.4}$$

or termed by Kaup–Newell equation and Chen–Lee–Liu equation are also the canonical models of the DNLS equation. There exist a chain of gauge transformations to relate DNLS-I,-II,-III equation with each other [4]. Theoretically, a solution $\check{q}(x,t)$ of DNLS-I equation (1.3),

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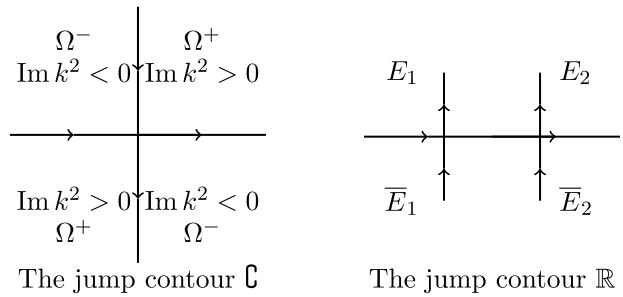

Fig. 1. The jump contour $\mathbb{C} = \mathbb{R} \cup i\mathbb{R}$ and $\mathbb{R}$.

the invertible gauge transformation $q(x,t) = \check{q}(x,t)\exp\left(\mp i\int_{-\infty}^{x}|\check{q}|^2(y,t)\mathrm{d}y\right)$ maps the solutions of the DNLS equation (1.1) with $t \to \frac{t}{2}$. However, it is very hard to be done explicitly due to the involved indefinite integration. Therefore, one needs to work with these equations separately.

Note that the problematic term $(|q|^2q)_x$ in (1.3) is replaced by the quintic term $|q|^2q$ without derivative and a derivative term $q^2\overline{q}_x$ with a better convolution structure in (1.1). Therefore, the present paper is concerned with the long-time asymptotics under the step-like (asymmetric) initial value problem for $x \lessgtr 0$ of the DNLS equation (1.1). The now well-known method of nonlinear steepest descent for studying the long-time asymptotics of solutions of integrable nonlinear equations with initial value was introduced in the early 1990s in a seminal paper by Deift and Zhou [5], building on earlier works of Manakov [6] and Its [7]. For a detailed historical review of this method please see [8] and further extended by Deift, Venakides and Zhou [9,10]. This method of increasing perfection is based on the development of the nonlinear steepest descent method for Riemann–Hilbert (RH) problem associated with integrable nonlinear equations. The intermeshing of the RH formalism and the Deift–Zhou approach to the step-like initial value problems has gradually been the subject of more works [11–14]. This idea was adapted by Venakides to problems in the shock problem with initial data for the integrable equation [9]. Buckingham, Boutet de Monvel, Biondini, Minakov and Grava considered the long-time asymptotics for the step-like initial value [15–24].

Before stating our assumptions and result more precisely, we recall known results concerning the DNLS equation (1.1). The multiple soliton solutions are addressed for the DNLS equation (1.1) under the initial value with zero/nonzero boundary conditions as $x \to \pm\infty$ by analyzing a matrix Riemann–Hilbert (RH) problem [25]. The Dirichlet initial–boundary value problem for the DNLS equation (1.1) is exhibited to be locally well-posed in $H^s(\mathbb{R}^+)$ for $s \in (1/2, 5/2)$ and $s = 3/2$ on the half-line [26]. For the nondecaying boundary value, the existence of a solution is classified for the DNLS equation (1.1) with asymptotical time-periodic boundary values and two particular families of parameters lying on the quarter plane $\{(x,t) \in \mathbb{R}^2 | x \geq 0, t \geq 0\}$ [27]. Due to its integrability, many explicit solutions in the closed form (including solitons and algebraic solitons, breathers and rogue wave solutions, algebro-geometric solutions), Hamiltonian structures, integrable decompositions and similarity reductions have been presented for the DNLS equation (1.1) [28–36]. The global existence for the DNLS equation (1.1) was proved by inverse scattering method in [37].

In the context of inverse scattering, the long-time asymptotics was studied for the DNLS equation (1.1) with step-like initial values $q(x,0) = \begin{cases} Ae^{i\phi}e^{-2iBx}, & x \leq 0, \\ 0, & x > 0, \end{cases}$ time-periodic initial value on the quarter plane and the nonzero boundary condition by the nonlinear steepest descent method [38–40]. Liu studied the long-time behavior of solutions to the DNLS equation (1.1) for soliton-free initial data [41]. These works gave the leading order asymptotics where error is $\mathcal{O}(t^{-1/2})$. However, the subleading order asymptotics not derived.

In the present paper, we consider the long-time asymptotics in a shock case of DNLS equation (1.1) with the more general step-like initial value conditions (1.2). Moreover, we derive the leading order and the subleading order asymptotics where the error is $\mathcal{O}(t^{-1/2}\ln(t))$.

In the initial value (1.2), $\{A_j, B_j, \phi_j\}_1^2 \in \mathbb{R}$ and $A_j > 0$. Eq. (1.1) with initial condition (1.2) admits the plane wave solution $q_j^{\pm\infty}(x,t) = A_j e^{i\phi_j}e^{-2iB_jx+2i\omega_jt}$, where $\omega_j = \frac{1}{4}(A_j^4 + 4A_j^2B_j - 8B_j^2)$. For $B_1 > B_2$ (the rarefaction case), the asymptotics does not depend on the values of D_j (D_j defined in Section 2.2), for details see [38]. For $B_1 < B_2$ (the shock case), the asymptotic is influenced by $D_j/(B_2 - B_1)$. For simplicity, we consider the symmetric shock $D_1 = D_2 = D > 0$ and $B_2 = -B_1 = B > 0$. The infinite branch of $\mathrm{Im}\, g = 0$ passes through the points E_1 and $\overline{E}_1$ before the two real zeros μ_1 and μ_2 of $\mathrm{Im}\, g = 0$ directly lead to the genus 3. The distribution of the asymptotical region was shown in Fig. 3, where ξ_{E_1} denotes $\xi = \xi_{E_1} = 2(B + \sqrt{D^2 + B^2})$ as the infinite branch pass through the points E_1 and $\overline{E}_1$, ξ_{mer} denotes $\xi = \xi_{mer}$ the two real zeros $\mu = \mu_1 = \mu_2$ of $\mathrm{Im}\, g = 0$. And the two zeros merge, where $\xi_{mer} = 4(-B + \sqrt{2}D)$. So the infinite branch pass through the two points E_1 and $\overline{E}_1$ before the zeros merge if and only if $\xi_{E_1} > \xi_{mer}$, i.e. $D/B < \frac{4+6\sqrt{2}}{7}$.

The present paper is devoted to studying the long-time asymptotic of the genus 3 for the DNLS equation (1.1) by means of the matrix RH problem (RH problems in this paper are 2×2 matrix-valued). A critical step in the nonlinear steepest descent method consists in deforming the contour associated to the RH problem in a way adapted to the structure of the phase function that defines the oscillatory dependence on parameters. When the entries of the jump matrix are not analytic, they must be approximated by rational functions so that the deformation can be carried out. Therefore, we bring in the g-function mechanism which is introduced when the entries of the jump matrix grow exponentially or oscillate as $t \to \infty$ [42]. The core idea of the g-function mechanism is to transform the phase function θ of the basic RH problem to a g-function so that the jump matrix of the RH problem is constant or decayed to an identity matrix by some matrix deformations.

For this purpose, we fix some notations for this paper. We define $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and lower plane of the complex plane $\mathbb{C}$, as shown in right panel of Fig. 1. All RH problems in this paper are considered in the L^2-RH problem [43–48].

Our main result is addressed as follows:

Theorem 1.1. *The long-time asymptotics of the solution to the DNLS equation (1.1) with the initial value condition (1.2) is given by the following formula*

$$q(x,t) = q_0 + \frac{q_1}{\sqrt{t}} + \mathcal{O}(t^{-1}\ln t), \quad t \to \infty, \tag{1.5}$$

where the leading order term shown as

$$q_0 = e^{2i(tg^{(0)}+h(\infty))}\mathrm{Im}(E_1+E_2+\alpha+\beta)\frac{\Theta(\varphi(\infty^+)+d)\Theta(\varphi(\infty^+)-v(t)-d)}{\Theta(\varphi(\infty^+)+v(t)+d)\Theta(\varphi(\infty^+)-d)}, \tag{1.6}$$

and the subleading order term shown as

$$q_1 = -2ie^{2i(tg^{(0)}+h(\infty))}\frac{(Y_\mu(x,t,\mu)m_1^{pc}Y_\mu^{-1}(x,t,\mu))_{12}}{\psi_\mu(\mu)}. \tag{1.7}$$

The constants $g^{(0)}$ and $h(\infty)$ are given by (2.49) and (3.20). The Riemann Theta function Θ and Abel map φ are defined by (4.4) and (4.7), respectively. The matrices Y_μ and m_1^{pc} are given by (4.35) and (4.39). The constant ψ_μ defined by (4.32).

2. Preliminaries

In this section, we mainly introduce some preparations for studying the long-time asymptotics of the DNLS equation (1.1) with the initial value (1.2), such as Jost solutions, scattering datas, basic RH problem, g-function.

2.1. Jost solution

It is well known that the DNLS equation (1.1) can be represented as the compatibility condition of two linear spectral problem (Lax pair). The Lax pair makes it possible to reduce the long-time asymptotics of the solutions with the initial value problem for Eq. (1.1) to the matrix RH problem, which involves the Jost solutions of the Lax pair. The DNLS equation (1.1) admits the Lax pair [49]

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi, \tag{2.1}$$

where Ψ is a 2×2 matrix-valued function of (x,t,k), $k\in\mathbb{C}$ is the spectral parameter and

$$U = -ik^2\sigma_3 + kQ + \frac{i}{2}|q|^2\sigma_3,$$
$$V = -2ik^4\sigma_3 + 2k^3Q + ik^2|q|^2\sigma_3 - ikQ_x\sigma_3 + \frac{i}{4}|q|^4\sigma_3 + \frac{1}{2}(q\bar{q}_x - \bar{q}q_x)\sigma_3,$$

with

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Setting $\Psi_j^{\pm\infty}$ (for convenience we omit x,t,k) are the solutions of the Lax pair (2.1) as $x\to\pm\infty$. And $\Psi_j^{\pm\infty}$ satisfy the following systems

$$\Psi_{jx}^{\pm\infty} = U_j^{\pm\infty}\Psi_j^{\pm\infty}, \quad \Psi_{jt}^{\pm\infty} = V_j^{\pm\infty}\Psi_j^{\pm\infty},$$

where $U_j^{\pm\infty}$ and $V_j^{\pm\infty}$ are defined by U and V with the q instead by $q_j^{\pm\infty}$. Furtherly, one can derive $\Psi_j^{\pm\infty}$ with the form

$$\Psi_j^{\pm\infty} = e^{(-iB_jx+i\omega_jt)\sigma_3}Y_j(k)e^{(-ixX_j(k)-it\Omega_j(k))\sigma_3},$$

where

$$X_j(k) = \sqrt{(k^2 - B_j - \frac{A_j^2}{2}) + k^2A_j^2}, \quad \Omega_j(k) = 2(k^2+B_j)X_j(k),$$
$$Y_j(k) = \frac{1}{2}e^{\frac{i\phi_j}{2}\sigma_3}\begin{pmatrix} y_j(k)+y_j^{-1}(k) & y_j(k)-y_j^{-1}(k) \\ y_j(k)-y_j^{-1}(k) & y_j(k)+y_j^{-1}(k) \end{pmatrix}e^{-\frac{i\phi_j}{2}\sigma_3},$$
$$y_j(k) = \left(\frac{k^2 - B_j - \frac{A_j^2}{2} - ikA_j}{k^2 - B_j - \frac{A_j^2}{2} + ikA_j}\right)^{\frac{1}{4}}.$$

The branch cuts for X_j and y_j are taken along the segment $\gamma_j\cup\overline{\gamma}_j = \{k\in\mathbb{C}|\,\mathrm{Re}^2k - \mathrm{Im}^2k = B_j, \mathrm{Re}^2k \le B_j + \frac{A_j^2}{2}\}$, where $\gamma_j = \{k\in\mathbb{C}|\,\mathrm{Re}^2k - \mathrm{Im}^2k = B_j, \mathrm{Re}^2k \le B_j + \frac{A_j^2}{2}, \mathrm{Im}\,k^2 > 0\}$. X_j and y_j with the asymptotics

$$X_j(k) = k^2 - B + \mathcal{O}(k^{-2}), \quad y_j(k) = 1 + \mathcal{O}(k^{-1}), \quad k\to\infty.$$

For $k^2 \in \gamma_j \cup \overline{\gamma}_j$, $y_{j+}(k) = iy_{j-}(k)$ and $Y_j(k)$ satisfy the jump condition

$$Y_{j+}(k) = Y_{j-}(k)\begin{pmatrix} 0 & ie^{i\phi_j} \\ ie^{-i\phi_j} & 0 \end{pmatrix}.$$

Introduce the transformation $\Psi_j(x,t,k) = \mu_j(x,t,k)e^{-ix(X_j+B_j)\sigma_3 - it(\Omega_j-\omega_j)\sigma_3}$. The Lax pair (2.1) rewritten as a new version about Jost solution Ψ_j

$$\Psi_{jx} = (U - U_j^{\pm\infty})\Psi_j + U_j^{\pm\infty}\Psi_j, \quad \Psi_{jt} = (V - V_j^{\pm\infty})\Psi_j + V_j^{\pm\infty}\Psi_j, \tag{2.4}$$

and one can note that the version of Lax pair (2.1) about μ_j

$$\mu_{jx} = i(X_j + B_j)\mu_j\sigma_3 + U\mu_j, \quad \mu_{jt} = i(\Omega_j - \omega_j)\mu_j\sigma_3 + V\mu_j. \tag{2.5}$$

Multiply both sides by $(\Psi_j^{\pm\infty})^{-1}$ for the equation of (2.4) and derive the full derivative form

$$\mathrm{d}\left[(\Psi_j^{\pm\infty})^{-1}\Psi_j\right] = (\Psi_j^{\pm\infty})^{-1}(U - U_j^{\pm\infty})\Psi_j\mathrm{d}x + (\Psi_j^{\pm\infty})^{-1}(V - V_j^{\pm\infty})\Psi_j\mathrm{d}t.$$

Furthermore, the solutions $\Psi_j(x,t,k)$ and $\mu_j(x,t,k)$ can be represented as the Volterra integral equations

$$\begin{aligned} \Psi_j(x,t,k) &= \Psi_j^{\pm\infty}(x,t,k) + \int_{\pm\infty}^{x} \Lambda(x,y,t,k)\Lambda^{\natural}(y,t,k)\Psi_j(y,t,k)\mathrm{d}y, \\ \mu_j(x,t,k) &= e^{i(\omega_j t - B_j x)\widehat{\sigma}_3}Y_j(k) \\ &\quad + \int_{\pm\infty}^{x} \Lambda(x,y,t,k)\Lambda^{\natural}(y,t,k)\mu_j(y,t,k)e^{-i(X_j+B_j)(y-x)\sigma_3}\mathrm{d}y, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \Lambda(x,y,t,k) &= \Psi_j^{\pm\infty}(x,t,k)(\Psi_j^{\pm\infty})^{-1}(y,t,k), \\ \Lambda^{\natural}(y,t,k) &= k(Q - Q_j^{\pm\infty})(y,t) + \frac{i}{2}|q|^2(y,t)\sigma_3 - \frac{i}{2}A_j^2\sigma_3. \end{aligned} \tag{2.7}$$

The existence, analyticity and differentiation of Ψ_j and μ_j can be proven directly, here we just list their properties, for details, see [38].

Lemma 2.1. *The Jost solutions $\Psi_j(x,t,k)$ of (2.1) admit the following properties:*

► *$\Psi_1^{(1)}(x,t,k)$ and $\Psi_2^{(2)}$ are analytic in $\{k \in \mathbb{C} \mid \mathrm{Im}k^2 > 0 \setminus \{\gamma_1 \cup \gamma_2\}\}$ $\Psi_1^{(2)}(x,t,k)$ and $\Psi_2^{(1)}$ are analytic in $\{k \in \mathbb{C} \mid \mathrm{Im}k^2 < 0 \setminus \{\overline{\gamma}_1 \cup \overline{\gamma}_2\}\}$, where $\gamma_j \cup \overline{\gamma}_j = \{k \in \mathbb{C} \mid \mathrm{Re}^2k - \mathrm{Im}^2k = B, \mathrm{Re}^2k \le B_j + A_j^2/4\}$. The distribution of asymptotic region is shown in the left panel of Fig. 1.*

► *$\Psi_j(k)$ satisfies the symmetries*

$$\Psi_j(k) = \sigma_1\sigma_3\overline{\Psi_j(\overline{k})}\sigma_3\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

► *$\left((\Psi_j^{\pm\infty})^{-1}\Psi_j\right)^{(1)}(x,t,k)$ and $\left((\Psi_j^{\pm\infty})^{-1}\Psi_j\right)^{(2)}(x,t,k)$ admit the asymptotics*

$$\begin{aligned} \left((\Psi_j^{\pm\infty})^{-1}\Psi_j\right)^{(1)}(x,t,k) &= e^{(1)} + \mathcal{O}(k^{-1}), \quad k \to \infty, \\ \left((\Psi_j^{\pm\infty})^{-1}\Psi_j\right)^{(2)}(x,t,k) &= e^{(2)} + \mathcal{O}(k^{-1}), \quad k \to \infty, \end{aligned}$$

where the superscripts $\left((\Psi_j^{\pm\infty})^{-1}\Psi_j\right)^{(j)}$ denote the jth column of the matrix $(\Psi_j^{\pm\infty})^{-1}\Psi_j$, $e^{(j)}$ denotes the jth column of a identity matrix $I_{2\times 2}$.

2.2. Scattering data

For the reason that $\Psi_j(x,t,k)$, $(j = 1,2)$ are solutions of the Lax pair (2.1), there exists a scattering matrix $S(\lambda)$ obey the scattering relation

$$\Psi_2(x,t,k) = \Psi_1(x,t,k)S(k), \quad k \in \mathbb{R}, \quad k \ne B_j. \tag{2.9}$$

From the symmetries (2.8), one can derive the following form of $\Psi_j(k)$,

$$\Psi_j(k) = \begin{pmatrix} \Psi_j^{(11)}(k) & \Psi_j^{(12)}(k) \\ \Psi_j^{(21)}(k) & \Psi_j^{(22)}(k) \end{pmatrix} = \begin{pmatrix} \overline{\Psi_j^{(22)}(\overline{k})} & \Psi_j^{(12)}(k) \\ -\overline{\Psi_j^{(12)}(\overline{k})} & \Psi_j^{(22)}(k) \end{pmatrix}, \quad (j = 1,2). \tag{2.10}$$

Eq. (2.9) implies that

$$S(k) = \Psi_1^{-1}(0,0,k)\Psi_2(0,0,k). \tag{2.11}$$

Inserting (2.10) into (2.11), one can derive the scattering matrix $S(k)$ with the following structure

$$S(k) = \begin{pmatrix} \overline{a(\overline{k})} & b(k) \\ -\overline{b(\overline{k})} & a(k) \end{pmatrix},$$

and $\det S(k) = 1$. From the analyticities of $\Psi_j(x,t,k)$, one can derive the analyticities of $a(k)$ and $\overline{a}(k)$ are in $\mathbb{C}^+\backslash\{\gamma_j \cup \overline{\gamma}_j\}_{j=1}^2$ and $\mathbb{C}^-\backslash\{\gamma_j \cup \overline{\gamma}_j\}_{j=1}^2$, respectively. The scatter coefficient $r(k) = \frac{\overline{b(\overline{k})}}{a(k)}$. And $a(k)$ with the asymptotics $a(k) = 1 + \mathcal{O}(k^{-1})$ for $k \to \infty$. Generally, the map $q \to \{a, b, r\}$ is the direct scattering map.

From the analyticities and asymptotics of $\Psi_j(x,t,k)$, a piecewise matrix function $m(x,t,k)$ is given by

$$m(x,t,k) = \begin{cases} \left(\dfrac{\Psi_1^{(1)}e^{it\theta}}{a}, \Psi_2^{(2)}e^{-it\theta}\right), & \Omega^+ \setminus \gamma_j, \\ \left(\Psi_2^{(1)}e^{it\theta}, \dfrac{\Psi_1^{(2)}e^{-it\theta}}{\overline{a}}\right), & \Omega^- \setminus \overline{\gamma}_j, \end{cases} \tag{2.12}$$

where $\theta = 2k^4 + \xi k^2$, $\xi = x/t$ and the analyticity regions are defined by $\Omega^\pm = \{k \in \mathbb{C} \mid \pm \mathrm{Im} k^2 > 0\}$. This matrix function $m(x,t,k)$ admits the jump condition

$$m_+(x,t,k) = m_-(x,t,k)J(x,t,k), \quad k^2 \in \mathbb{C}. \tag{2.13}$$

The jump contour $\mathbb{C}$ can be viewed as the boundary of the regions $\Omega^\pm$. The jump matrix $J(x,t,k)$ is given by

$$J(x,t,k) = \begin{pmatrix} 1 + r\overline{r} & \overline{r} \\ r & 1 \end{pmatrix}. \tag{2.14}$$

Due to the multi-value of spectrum parameter k^2, we reduce by symmetry from scattering data on the oriented contour $\mathbb{C}$ to scattering data on the oriented contour $\mathbb{R}$. Both contours with orientation are shown in Fig. 1. Introduce the transformation [50,51]

$$\tilde{m}(x,t,\lambda) = \nabla k^{-\frac{\widehat{\sigma}_3}{2}} m(x,t,k), \tag{2.15}$$

where $\nabla = \begin{pmatrix} 1 & 0 \\ -\frac{i}{2}\overline{q} & 1 \end{pmatrix}$ and $\widehat{\sigma}_3$ is a 2×2 matrix with $\widehat{\sigma}_3 A = \sigma_3 A \sigma_3^{-1}$. Using transformation (2.15), we can now reduce the spectral problem with $k \in \mathbb{C}$ to $\lambda \in \mathbb{R}$. And $\tilde{m}$ satisfies the asymptotic $\tilde{m} \to I$ as $\lambda \to \infty$. The modified scattering coefficient $\varrho(\lambda) = \frac{r(k)}{k}$. X_j and Ω_j are rewritten as

$$X_j(\lambda) = \sqrt{(\lambda - B_j)^2 + \frac{A_j^4}{4} + A_j^2 B_j}, \quad \Omega_j(\lambda) = 2(\lambda + B_j)X_j(\lambda).$$

The branch points $E_j = B_j + iD_j$, where $D_j^2 = \frac{A_j^4}{4} + A_j^2 B_j$. The branch cuts $\gamma_j = [B_j, E_j]$ and $\overline{\gamma}_j = [\overline{E}_j, B_j]$. For $\lambda \to \infty$, $X_j(\lambda)$ and $\Omega_j(\lambda)$ with the asymptotics

$$X_j(\lambda) = \lambda - B + \mathcal{O}(\lambda^{-1}), \quad \Omega_j(\lambda) = 2\lambda^2 + \omega_j + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty.$$

Functions $\tilde{m}_\pm(x,t,\lambda)$ are analyticity in the regions $\mathbb{C}^\pm\backslash\{\gamma_j \cup \overline{\gamma}_j\}$ (γ_j and $\overline{\gamma}_j$ see Fig. 2) and satisfy the jump condition

$$\tilde{m}_+(x,t,\lambda) = \tilde{m}_-(x,t,\lambda)e^{-it(\xi\lambda + 2\lambda^2)\widehat{\sigma}_3}\tilde{J}(x,t,\lambda), \tag{2.16}$$

where the definition of $\widehat{\sigma}_3$ is shown in (2.15). The jump matrix $\tilde{J}(x,t,\lambda)$ is given by

$$\tilde{J}(x,t,\lambda) = \begin{pmatrix} 1 + \lambda\varrho\overline{\varrho} & \overline{\varrho} \\ \lambda\varrho & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{2.17}$$

Now, we successfully map the spectral problem of k-plane to λ-plane. For the reason that there exist branch cuts $\gamma_j \cup \overline{\gamma}_j$, the function $\tilde{m}(x,t,\lambda)$ does not continue as $\lambda \in \gamma_j \cup \overline{\gamma}_j$. We consider the jump condition about the branch cuts $\gamma_j \cup \overline{\gamma}_j$ as shown in Lemma 2.2:

Lemma 2.2. *For $\lambda \in \gamma_j$ or $\lambda \in \overline{\gamma}_j$, the jump matrices are given by*

$$\tilde{J} = \begin{cases} \begin{pmatrix} 1 & 0 \\ \lambda f(\lambda) & 1 \end{pmatrix}, & \lambda \in \gamma_1, \\ \begin{pmatrix} \frac{\tilde{a}_-}{\overline{\tilde{a}}_+} & ie^{i\phi_2} \\ 0 & \frac{\tilde{a}_+}{\overline{\tilde{a}}_-} \end{pmatrix}, & \lambda \in \gamma_2, \end{cases} \qquad \tilde{J} = \begin{cases} \begin{pmatrix} 1 & -\overline{f(\overline{\lambda})} \\ 0 & 1 \end{pmatrix}, & \lambda \in \overline{\gamma}_1, \\ \begin{pmatrix} \frac{\overline{\tilde{a}}_+}{\overline{\tilde{a}}_-} & 0 \\ ie^{-i\phi_2} & \frac{\overline{\tilde{a}}_-}{\overline{\tilde{a}}_+} \end{pmatrix}, & \lambda \in \overline{\gamma}_2, \end{cases}$$

where $f(\lambda) = \varrho_+(\lambda) - \varrho_-(\lambda)$.

Proof. For $\lambda \in \gamma_j \cup \overline{\gamma}_j$, introduce the

$$\aleph_j(x,t,\lambda) = I + \int_{\pm\infty}^{x} \Lambda(x,y,t,\lambda)\Lambda^\natural(y,t,\lambda)\aleph_j(y,t,\lambda)\Psi_j^{\pm\infty}(y,t,\lambda)(\Psi_j^{\pm\infty})^{-1}(x,t,\lambda)\mathrm{d}y. \tag{2.18}$$

For every fixed (y,t), the function $\Psi_j^{\pm\infty}(x,t,\lambda)(\Psi_j^{\pm\infty})^{-1}(y,t,\lambda)$ is a solution of the x-part with q replaced by $q_j^{\pm\infty}$. Since this solution equals the identity matrix at $x = y$ and the matrix M in the Lax pair (2.1) is a polynomial in λ, we conclude that

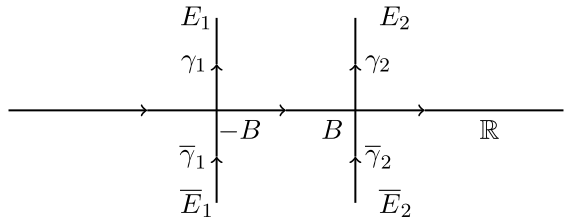

Fig. 2. The jump contour $\Sigma = \mathbb{R} \cup \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2$.

$\Psi_j^{\pm\infty}(x,t,\lambda)(\Psi_j^{\pm\infty})^{-1}(y,t,\lambda)$ is an entire function of λ, well defined for $\lambda \in \gamma_1 \cup \gamma_2$. Thus, $\Psi_{j\pm}$ and $\aleph_j \Psi_{j\pm}^{\pm\infty}$ solve the same integral equation for $\lambda \in \Sigma_{j'}, j' \neq j$. Hence, $\Psi_{1\pm}(x,t,\lambda)$ and $\Psi_{2\pm}(x,t,\lambda)$ can be written as follows for $\lambda \in \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2$:

$$\Psi_{1\pm} = \aleph_1 \Psi_{1\pm}^{-\infty}, \quad \Psi_2 = \aleph_2 \Psi_2^{+\infty}, \quad \lambda \in \gamma_1 \cup \overline{\gamma}_1, \tag{2.19a}$$

$$\Psi_{2\pm} = \aleph_2 \Psi_{2\pm}^{+\infty}, \quad \Psi_1 = \aleph_1 \Psi_1^{-\infty}, \quad \lambda \in \gamma_2 \cup \overline{\gamma}_2. \tag{2.19b}$$

The scattering matrix $\tilde{S}_\pm(\lambda)$ on branch cuts $\gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2$ and $\det \tilde{S}_\pm(\lambda) = 1$. There exist the relations

$$\Psi_{2\pm}(x,t,\lambda) = \Psi_1(x,t,\lambda)\tilde{S}_\pm(\lambda), \quad \lambda \in \gamma_2 \cup \overline{\gamma}_2, \tag{2.20a}$$

$$\Psi_2(x,t,\lambda) = \Psi_{1\pm}(x,t,\lambda)\tilde{S}_\pm(\lambda), \quad \lambda \in \gamma_1 \cup \overline{\gamma}_1. \tag{2.20b}$$

For $\lambda \in \gamma_2 \cup \overline{\gamma}_2$, one can derive $\tilde{S}_\pm(\lambda) = \Psi_1^{-1}(x,t,\lambda)\aleph_2(x,t,\lambda)\Psi_{2\pm}^{+\infty}(x,t,\lambda)$. Letting $x = t = 0$, one have $\tilde{S}_\pm(\lambda) = P_2(\lambda)Y_{2\pm}(\lambda)$, where $P_2(\lambda) = \Psi_1^{-1}(0,0,\lambda)\aleph_2(0,0,\lambda)$. Hence

$$\tilde{S}_+(\lambda) = \tilde{S}_-(\lambda)\begin{pmatrix} 0 & ie^{i\phi_2} \\ ie^{-i\phi_2} & 0 \end{pmatrix}, \quad \lambda \in \gamma_2 \cup \overline{\gamma}_2. \tag{2.21}$$

This implies that

$$\tilde{S}_{12+} = ie^{i\phi_2}\tilde{S}_{11-}, \quad \tilde{S}_{22+} = ie^{i\phi_2}\tilde{S}_{21-}. \tag{2.22}$$

The jump relation across γ_2 follows

$$\begin{pmatrix} \frac{\Psi_1^{(1)}}{\tilde{a}_+} & \Psi_{2+}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\Psi_1^{(1)}}{\tilde{a}_-} & \Psi_{2-}^{(2)} \end{pmatrix} \begin{pmatrix} \frac{\tilde{a}_-}{\tilde{a}_+} & c_2 \\ 0 & \frac{\tilde{a}_+}{\tilde{a}_-} \end{pmatrix}, \tag{2.23}$$

where c_2 is a function with respect to λ. Furtherly,

$$\frac{\Psi_{2+}^{(2)}}{\tilde{a}_+} - \frac{\Psi_{2-}^{(2)}}{\tilde{a}_-} = \frac{c_2}{\tilde{a}_+\tilde{a}_-}\Psi_1^{(1)}. \tag{2.24}$$

From (2.20),

$$\Psi_{2\pm}^{(2)} = \tilde{S}_{12\pm}\Psi_1^{(1)} + \tilde{S}_{22\pm}\Psi_1^{(2)}. \tag{2.25}$$

One can derive $\tilde{S}_{22\pm} = \det\begin{pmatrix} \Psi_1^{(1)} & \Psi_{2\pm}^{(2)} \end{pmatrix} = \tilde{a}_\pm$ and

$$\frac{\Psi_{2+}^{(2)}}{\tilde{a}_+} - \frac{\Psi_{2-}^{(2)}}{\tilde{a}_-} = \left(\frac{\tilde{S}_{12+}}{\tilde{S}_{22+}} - \frac{\tilde{S}_{12-}}{\tilde{S}_{22-}}\right)\Psi_1^{(1)}. \tag{2.26}$$

Hence

$$\frac{\tilde{S}_{12+}}{\tilde{S}_{22+}} - \frac{\tilde{S}_{12-}}{\tilde{S}_{22-}} = ie^{i\phi_2}\frac{\tilde{S}_{11-}\tilde{S}_{22-} - \tilde{S}_{12-}\tilde{S}_{21-}}{\tilde{S}_{22+}\tilde{S}_{22-}} = \frac{ie^{i\phi_2}}{\tilde{a}_+\tilde{a}_-}. \tag{2.27}$$

Thus $c_2 = ie^{i\phi_2}$.

For $k \in \gamma_1 \cup \overline{\gamma}_1$, one can derive $\tilde{S}_\pm(\lambda) = (\Phi_{1\pm}^{-\infty})^{-1}(x,t,\lambda)\aleph_1(x,t,\lambda)\Phi_2(x,t,\lambda)$. Letting $x = t = 0$, one have $\tilde{S}_\pm(\lambda) = Y_{1\pm}^{-1}(\lambda)P_1(k)$, where $P_1(\lambda) = \aleph_1^{-1}(0,0,\lambda)\Phi_2(0,0,\lambda)$. Hence

$$\tilde{S}_-(\lambda)\tilde{S}_+^{-1}(\lambda) = Y_{1-}^{-1}Y_{1+} = \begin{pmatrix} 0 & ie^{i\phi_1} \\ ie^{-i\phi_1} & 0 \end{pmatrix}. \tag{2.28}$$

This implies that

$$\tilde{S}_-(\lambda) = \begin{pmatrix} 0 & ie^{i\phi_1} \\ ie^{-i\phi_1} & 0 \end{pmatrix}\tilde{S}_+(k). \tag{2.29}$$

That is

$$\tilde{S}_{21-} = ie^{-i\phi_1}\tilde{S}_{11+}, \quad \tilde{S}_{22-} = ie^{-i\phi_1}\tilde{S}_{12+}, \quad \lambda \in \gamma_1 \cup \overline{\gamma}_1. \tag{2.30}$$

According the jump relation across γ_1 as

$$\begin{pmatrix} \frac{\Psi_{1+}^{(1)}}{\tilde{a}_+} & \Psi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\Psi_{1-}^{(1)}}{\tilde{a}_-} & \Psi_2^{(2)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \tag{2.31}$$

where c_1 is a function with respect to λ.

$$\frac{\Psi_{1+}^{(1)}}{\tilde{a}_+} - \frac{\Psi_{1-}^{(1)}}{\tilde{a}_-} = c_1 \Psi_2^{(2)}. \tag{2.32}$$

From (2.20),

$$\Psi_{1\pm}^{(1)} = \tilde{S}_{22\pm}\Psi_2^{(1)} - \tilde{S}_{21\pm}\Psi_2^{(2)}. \tag{2.33}$$

One can derive $\tilde{S}_{22\pm} = \det\begin{pmatrix} \Psi_1^{(1)} & \Psi_{2\pm}^{(2)} \end{pmatrix} = \tilde{a}_\pm$. Thus

$$\frac{\Psi_{1+}^{(1)}}{\tilde{a}_+} - \frac{\Psi_{1-}^{(1)}}{\tilde{a}_-} = \left(\frac{\tilde{S}_{21-}}{\tilde{S}_{22-}} - \frac{\tilde{S}_{21+}}{\tilde{S}_{22+}}\right)\Psi_2^{(2)}. \tag{2.34}$$

As above, one can derive

$$\frac{\tilde{S}_{21-}}{\tilde{S}_{22-}} - \frac{\tilde{S}_{21+}}{\tilde{S}_{22+}} = \frac{ie^{i\phi_1}}{\tilde{a}_+\tilde{a}_-}. \tag{2.35}$$

Thus $c_1 = \lambda f(\lambda)$. □

Furthermore, we note that the scattering datas obey the following relationships from the proof of Lemma 2.2

$$\begin{cases} \tilde{a}_+ = -ie^{-i\phi_1}\tilde{b}_-, \\ \tilde{b}_+ = -ie^{i\phi_1}\tilde{a}_-, \end{cases} \quad \lambda \in \gamma_1 \cup \overline{\gamma}_1, \qquad \begin{cases} \tilde{a}_+ = -ie^{i\phi_2}\overline{\tilde{b}}_-, \\ \tilde{b}_+ = ie^{i\phi_2}\overline{\tilde{a}}_-, \end{cases} \quad \lambda \in \gamma_2 \cup \overline{\gamma}_2.$$

From Lemma 2.2, we summarize the function $\tilde{m}(x,t,\lambda)$ satisfies the RH problem:

RH problem 2.1. *$\tilde{m}(x,t,\lambda)$ satisfies the RH problem*

- *$\tilde{m}(x,t,\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma$, where jump contour Σ see Fig. 2.*
- *$\tilde{m}(x,t,\lambda)$ satisfies the jump condition*

$$\tilde{m}_+(x,t,\lambda) = \tilde{m}_-(x,t,\lambda)e^{-it(\xi\lambda+2\lambda^2)\hat{\sigma}_3}\tilde{J}(x,t,\lambda), \tag{2.36}$$

where the jump matrix $\tilde{J}(x,t,\lambda)$ is given by

$$\tilde{J}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 1+\lambda\varrho\overline{\varrho} & \overline{\varrho} \\ \lambda\varrho & 1 \end{pmatrix}, & \lambda \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ \lambda f(\lambda) & 1 \end{pmatrix}, & \lambda \in \gamma_1, \\ \begin{pmatrix} \frac{\tilde{a}_-}{\overline{\tilde{a}}_+} & i \\ 0 & \frac{\tilde{a}_+}{\overline{\tilde{a}}_-} \end{pmatrix}, & \lambda \in \gamma_2, \\ \begin{pmatrix} 1 & -\overline{f(\overline{\lambda})} \\ 0 & 1 \end{pmatrix}, & \lambda \in \overline{\gamma}_1, \\ \begin{pmatrix} \frac{\overline{\tilde{a}}_+}{\overline{\tilde{a}}_-} & 0 \\ i & \frac{\overline{\tilde{a}}_-}{\overline{\tilde{a}}_+} \end{pmatrix}, & \lambda \in \overline{\gamma}_2. \end{cases}$$

- *$\tilde{m}(x,t,\lambda)$ satisfies the asymptotic behavior*

$$\tilde{m}(x,t,\lambda) \to I, \quad \lambda \to \infty. \tag{2.37}$$

2.3. The basic Riemann-Hilbert problem

It is necessary to regularize function $\tilde{m}(x,t,\lambda)$ to get the RH problem of subsequent deformations in the L^2. Define function $\hat{m}(x,t,\lambda)$ by

$$\hat{m}(x,t,\lambda) = \tilde{m}(x,t,\lambda)\nu^{-\sigma_3}(\lambda),$$

where $\nu = \left(\frac{\lambda-E_1}{\lambda-\overline{E}_1}\right)^{1/4}$ admits $\nu_+ = i\nu_-$. This transformation implies that $\hat{a}\overline{\hat{a}} = a\overline{a}$, $\hat{b}\overline{\hat{b}} = b\overline{b}$, $\hat{\varrho} = \varrho\nu^{-2}$, $\hat{\varrho}\overline{\hat{\varrho}} = \varrho\overline{\varrho} = |\varrho|^2$. $\hat{a}$, $\hat{b}$ and $\frac{\overline{\hat{\varrho}}}{1+\lambda\hat{\varrho}\overline{\hat{\varrho}}} = \hat{a}\hat{b}$ are bounded near E_1, $\hat{\varrho}$ is bounded near E_2.

In order to study the long-time asymptotics of the step-like initial value problem for the DNLS equation (1.1) via the Deift–Zhou method, we refresh the RH problem based on the previous analyses as follows:

RH problem 2.2. *$\hat{m}(x,t,\lambda)$ satisfies the RH problem*

- *$\hat{m}(x,t,\lambda)$ is analytical in $\mathbb{C}\backslash\Sigma$, where jump contour Σ see Fig. 2.*
- *$\hat{m}(x,t,\lambda)$ satisfies the jump condition*

$$\hat{m}_+(x,t,\lambda)=\hat{m}_-(x,t,\lambda)\hat{J}(\lambda), \tag{2.38}$$

where the jump matrix

$$\hat{J}(x,t,\lambda)=e^{-it\theta(\lambda)\sigma_3}\hat{J}^{(0)}(\lambda)e^{it\theta(\lambda)\sigma_3}, \quad \lambda\in\Sigma, \tag{2.39}$$

and

$$\hat{J}^{(0)}(\lambda)=\begin{cases}\begin{pmatrix}1+\lambda\hat{\varrho}\hat{\bar{\varrho}} & \hat{\bar{\varrho}}\\ \lambda\hat{\varrho} & 1\end{pmatrix}, & \lambda\in\mathbb{R},\\ \begin{pmatrix}-i & 0\\ \lambda\hat{f} & i\end{pmatrix}, & \lambda\in\gamma_1,\\ \begin{pmatrix}-i & -\hat{\bar{f}}\\ 0 & i\end{pmatrix}, & \lambda\in\overline{\gamma}_1,\\ \begin{pmatrix}\frac{\hat{a}_-}{\hat{\bar{a}}_+} & i\nu^2\\ 0 & \frac{\hat{a}_+}{\hat{\bar{a}}_-}\end{pmatrix}, & \lambda\in\gamma_2,\\ \begin{pmatrix}\frac{\hat{\bar{a}}_+}{\hat{a}_-} & 0\\ i\nu^{-2} & \frac{\hat{\bar{a}}_-}{\hat{a}_+}\end{pmatrix}, & \lambda\in\overline{\gamma}_2.\end{cases}$$

- *$\hat{m}(x,t,\lambda)$ satisfies the asymptotic behavior*

$$\hat{m}(x,t,\lambda)\to I, \quad \lambda\to\infty. \tag{2.40}$$

The solution $q(x,t)$ of the DNLS equation (1.1) with the initial value is reconstructed by

$$q(x,t)=2i\lim_{\lambda\to\infty}\lambda\hat{m}_{12}(x,t,\lambda). \tag{2.41}$$

The jump matrices $\hat{J}(x,t,\lambda)$ and $\hat{m}(x,t,\lambda)$ admit the following symmetries

$$\hat{m}(x,t,\lambda)=\sigma_1\sigma_3\overline{\hat{m}(x,t,\bar{\lambda})}\sigma_3\sigma_1, \qquad \lambda\in\mathbb{C}\backslash\Sigma,$$

$$\hat{J}(x,t,\lambda)=\begin{cases}\sigma_1\sigma_3\overline{\hat{J}(x,t,\bar{\lambda})}\sigma_3\sigma_1, & \lambda\in\gamma_1\cup\overline{\gamma}_1\cup\gamma_2\cup\overline{\gamma}_2,\\ \sigma_1\sigma_3\overline{\hat{J}(x,t,\bar{\lambda})}^{-1}\sigma_3\sigma_1, & \lambda\in\mathbb{R}.\end{cases}$$

2.4. g-Function

Consider the initial value $D_1=D_2=D>0$ and $B_2=-B_1=B>0$. Let $g=\Omega_2(\lambda)+\xi X_2(\lambda)$ be the g-function, the two real zeros of $\mathrm{Im}(g)$ given by

$$\mu_1=\frac{B}{2}-\frac{\xi}{8}-\frac{1}{8}\sqrt{(\xi+4B)^2-32D^2}, \tag{2.43}$$

$$\mu_2=\frac{B}{2}-\frac{\xi}{8}+\frac{1}{8}\sqrt{(\xi+4B)^2-32D^2}, \tag{2.44}$$

and

$$\frac{\mathrm{d}g}{\mathrm{d}\lambda}=\frac{4(\lambda-\mu_1)(\lambda-\mu_2)}{\sqrt{(\lambda-E_2)(\lambda-\bar{E}_2)}}. \tag{2.45}$$

As ξ decreases, the infinite branch of the curve $\mathrm{Im}\,g=0$ moves to the right. There are three possibilities:
Case 1: The finite branch hits E_1 and $\bar{E}_1$ before the two real zeros μ_1 and μ_2 merge.
Case 2: The two real zeros μ_1 and μ_2 merge before the finite branch hits E_1 and $\bar{E}_1$.
Case 3: The finite branch hits E_1 and $\bar{E}_1$ at the same time as the two real zeros μ_1 and μ_2 merge. We define some notations:

- The infinite branch hits E_1 and $\bar{E}_1$ for $\xi=\xi_{E_1}=2(B+\sqrt{D^2+B^2})$.
- The two real zeros μ_1 and μ_2 merge for $\xi=\xi_{mer}=4(-B+\sqrt{2}D)$.
- The infinite branch hits E_1 and $\bar{E}_1$ before the zeros merge if $\xi_{E_1}>\xi_{mer}$, i.e.

$$\textbf{Case 1}: \quad D/B<\frac{4+6\sqrt{2}}{7},$$

$$\textbf{Case 2}: \quad D/B>\frac{4+6\sqrt{2}}{7},$$

$$\textbf{Case 3}: \quad D/B=\frac{4+6\sqrt{2}}{7}.$$

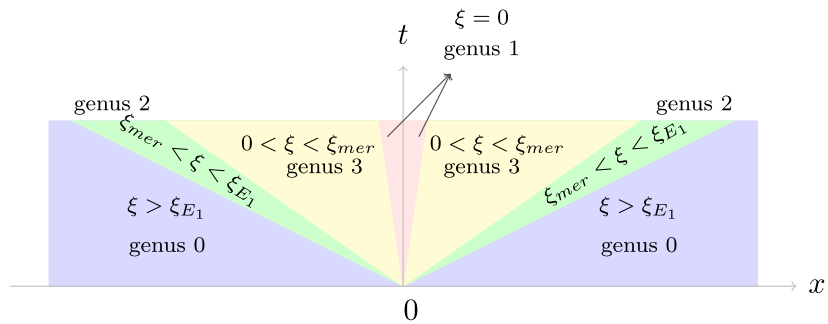

Fig. 3. The space–time region of x and t for $\frac{4+6\sqrt{2}}{7} > \frac{D}{B} > 1$, where the blue region denotes the genus 0 region; the green region denotes the genus 2 region; the yellow region denotes the genus 3 region; the red region denotes the genus 1 region.

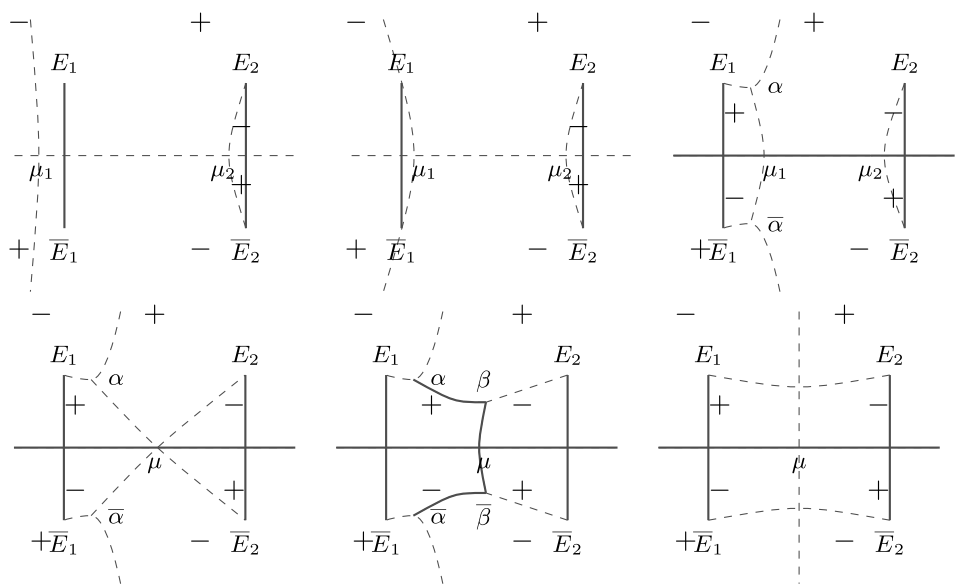

Fig. 4. The signature tables of $\mathrm{Im}\, g$: $\xi > \xi_{E_1}$, $\xi = \xi_{E_1}$, $\xi_{mer} < \xi < \xi_{E_1}$, $\xi = \xi_{mer}$, $0 < \xi < \xi_{mer}$, $\xi = 0$.

Each of these cases signifies the ending of the plane wave sector, because the g-function g_2 stops to provide a signature table appropriate for subsequent deformations. Thus a more complicated g-function is required. Next, we pay attention to the situation small ξ. Let parameters be $D_1 = D_2 = D > 0$, $B_2 = -B_1 = B > 0$, $\phi_1 = \phi$ and $\phi_2 = 0$, for $1 < \frac{D}{B} < \frac{4+6\sqrt{2}}{7}$. For the distribution of region see Fig. 3. For the signature tables see Fig. 4.

In this paper, we focus on region $0 < \xi < \xi_{mer}$. For $\varepsilon < |\xi| < \xi_0$, ξ_0 is a positive constant, $\varepsilon \in (0, \xi_0)$, an appropriate g-function has a derivative of the form

$$\frac{\mathrm{d}g}{\mathrm{d}\lambda} = \frac{\Upsilon}{\hbar}, \tag{2.46}$$

where

$$\Upsilon = 4(\lambda - \mu)(\lambda - \alpha)(\lambda - \overline{\alpha})(\lambda - \beta)(\lambda - \overline{\beta}),$$
$$\hbar = \left[(\lambda - E_1)(\lambda - \overline{E}_1)(\lambda - E_2)(\lambda - \overline{E}_2)(\lambda - \alpha)(\lambda - \overline{\alpha})(\lambda - \beta)(\lambda - \overline{\beta})\right]^{1/2}.$$

And this g-function generating genus 3 asymptotics. The genus 3 structure because the derivative of g-function has five zeros: one real zeros μ, two complex conjugate pairs of zeros $\{\alpha, \overline{\alpha}\}$ and $\{\beta, \overline{\beta}\}$ lie in $\mathbb{C}^+ \backslash (\Sigma_1 \cup \Sigma_2)$. And dg defined by (2.46) where $\mu \in \mathbb{R}$, $\alpha = \mathrm{Re}\alpha + i\mathrm{Im}\alpha$, $\beta = \mathrm{Re}\beta + i\mathrm{Im}\beta$ which are determined by

$$\int_{a_1} \hat{\mathrm{d}g} = \int_{a_2} \hat{\mathrm{d}g} = \int_{a_3} \hat{\mathrm{d}g} = 0, \tag{2.47a}$$

$$\lim_{\lambda\to\infty} \left(\frac{\mathrm{d}g}{\mathrm{d}\lambda} - 4\lambda\right) = \xi, \quad \lim_{\lambda\to\infty} \lambda \left(\frac{\mathrm{d}g}{\mathrm{d}\lambda} - 4\lambda - \xi\right) = 0. \tag{2.47b}$$

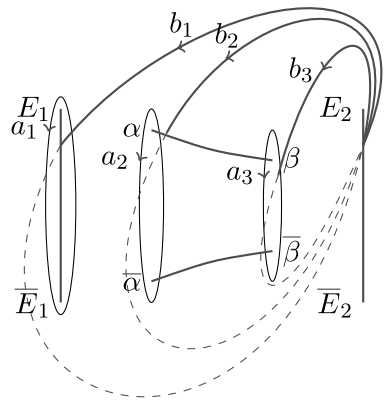

Fig. 5. The contour a_j and b_j of Riemann surface with genus 3.

The contour a_j see Fig. 5. And $\hat{\mathrm{d}g}$ denotes the differential on the Riemann surface given by $\mathrm{d}g$ on the upper sheet and by $-\mathrm{d}g$ on the lower sheet ($\hat{\mathrm{d}g}(\lambda^{\pm}) = \pm\mathrm{d}g(\lambda)$ for $\lambda \in \mathbb{C}\backslash\{\gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})}\}$). Then $\hat{\mathrm{d}g}$ is a meromorphic differential on the Riemann surface which is holomorphic everywhere except for two poles at $\infty^{\pm}$. The solvability of system (2.47) of equations characterizes a genus 3 sector. Since $\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\lambda) = \overline{\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\overline{\lambda})}$, we have $\int_D^B \mathrm{d}g = \overline{\int_{\overline{D}}^{\overline{B}} \mathrm{d}g}$ where the second integral is the complex conjugate of the first one. This implies that $\int_{a_j} \hat{\mathrm{d}g} \in i\mathbb{R}$. So the first third in (2.47) are three real conditions. The precise definition of g-function on the Riemann surface M (see Fig. 5) will be given in the following:

$$g(\lambda) = \int_{\overline{E}_2}^{\lambda} \mathrm{d}g, \quad \lambda \in \mathbb{C}\backslash\Sigma^{mod}, \tag{2.48}$$

where $\Sigma^{mod} = \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}$. The systems of (2.47) ensure that

$$\begin{aligned} &\mathrm{d}g(\lambda) = 4\lambda + \xi + \mathcal{O}(\lambda^{-2}), \quad \lambda \to \infty, \\ &g(\lambda) = \theta(\lambda) + g^{(0)} + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty. \end{aligned} \tag{2.49}$$

Lemma 2.3. *The g-function defined by (2.48) with the following properties:*

- *$g(\lambda) - \theta(\lambda)$ is analytic and bounded for $\lambda \in \hat{\mathbb{C}}\backslash\Sigma^{mod}$ with continuous boundary values on Σ^{mod}, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*
- *$g(\lambda)$ admits the symmetry $g(\lambda) = \overline{g(\overline{\lambda})}$.*
- *$g(\lambda)$ admits the RH problem*

$$\begin{aligned} &g_+(\lambda) + g_-(\lambda) = \begin{cases} 2\Delta_1, & \lambda \in \gamma_1 \cup \overline{\gamma}_1, \\ 2\Delta_2, & \lambda \in \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\beta,\alpha)}, \\ 0, & \lambda \in \gamma_2 \cup \overline{\gamma}_2, \end{cases} \\ &g_+(\lambda) - g_-(\lambda) = 2\Delta_3, \qquad \lambda \in \gamma_{(\overline{\beta},\beta)}, \end{aligned} \tag{2.50}$$

where

$$\Delta_1 = g(E_1)g(\overline{E}_1), \quad \Delta_2 = g(\alpha)g(\overline{\alpha}), \quad \Delta_3 = \frac{g_+(\beta) - g_-(\beta)}{2} = \frac{g_+(\overline{\beta}) - g_-(\overline{\beta})}{2}. \tag{2.51}$$

Proof. For $\lambda \in \mathbb{C}\backslash\Sigma^{mod}$, function $g(\lambda)$ is analytic function with bounded and continuous boundary values on Σ^{mod}. Eq. (2.49) implies that $g(\lambda) - \theta(\lambda)$ is analytic also at infinity. The first one follows.

Using that

$$\left(\frac{\mathrm{d}g}{\mathrm{d}\lambda}\right)_+ = \left(\frac{\mathrm{d}g}{\mathrm{d}\lambda}\right)_- \begin{cases} -1, & \lambda \in \Sigma^{mod} \setminus \gamma_{(\overline{\beta},\beta)}, \\ 1, & \lambda \in \gamma_{(\overline{\beta},\beta)}, \end{cases} \tag{2.52}$$

one can find that $g_+ + g_-$ and $g_+ - g_-$ are constants on each curve in (2.50). And these constant values are fixed by evaluation at the endpoints. Since $\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\lambda) = \overline{\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\overline{\lambda})}$ and $g(E_2) = g(\overline{E}_2) = 0$, (2.50) derived. The second one follows.

In Eq. (2.47) and $\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\lambda) = \overline{\frac{\mathrm{d}g}{\mathrm{d}\lambda}(\overline{\lambda})}$ ensure that $\mathrm{Im}\, g = 0$ on $\mathbb{R}$ and that $\mathrm{Im}\, g(E_1) = \mathrm{Im}\, g(\alpha) = \mathrm{Im}\, g_+(\beta) = \mathrm{Im}\, g_-(\beta) = 0$. Hence the constants Δ_j in (2.51) are real. The third one follows. □

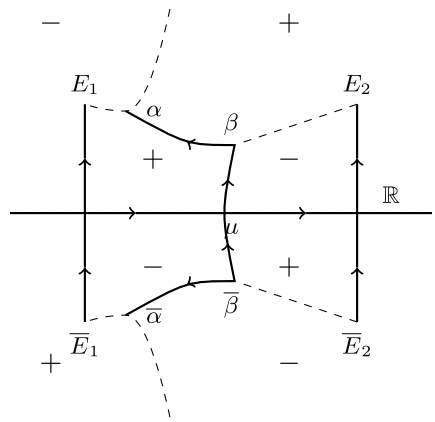

Fig. 6. The branch cuts and the level set $\mathrm{Img} = 0$ (the dash line and $\mathbb{R} \cup \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)} \cup \gamma_{(\beta,\alpha)}$). The region of $\mathrm{Img} > 0$ is "+" and $\mathrm{Img} < 0$ is "−". The jump contour $\Sigma^{(1)} = \mathbb{R} \cup \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)} \cup \gamma_{(\beta,\alpha)}$.

3. Deformation of the jump contour

In this section, our main purpose is to re-normalize Problem 2.2 such that it is well-behaved as $t \to \infty$ along any characteristic by establishing transformation $\hat{m} \to \hat{m}^{(j)}$. For deriving the long-time asymptotics of Eq. (1.1), the jump matrices need to be transformed as a constant matrix or decay to identity matrix. Now we will perform five transformations for the RH problem.

Introduce the matrix function $\hat{m}^{(1)}(x, t, \lambda)$

$$\hat{m}^{(1)}(x, t, \lambda) = e^{-itg^{(0)}\sigma_3}\hat{m}(x, t, \lambda)e^{-it(g(\lambda)-\theta(\lambda))\sigma_3}, \tag{3.1}$$

where $g^{(0)} = (g - \theta)(\xi, \infty)$. Function $\hat{m}^{(1)}(x, t, \lambda)$ admits the following RH problem:

RH problem 3.1. *Function $\hat{m}^{(1)}(x, t, \lambda)$ satisfies the following jump condition:*

- *$\hat{m}^{(1)}(x, t, \lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(1)}$, where $\Sigma^{(1)}$ see Fig. 6.*
- *$\hat{m}^{(1)}(x, t, \lambda)$ satisfies the jump condition*

$$\hat{m}^{(1)}_+ = \hat{m}^{(1)}_-\hat{j}^{(1)}, \tag{3.2}$$

where the jump matrix

$$\hat{j}^{(1)} = \begin{cases} \begin{pmatrix} 1 & \hat{\overline{\varrho}}e^{-2itg} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda\hat{\varrho}e^{2itg} & 1 \end{pmatrix}, & \lambda \in (\mu, +\infty), \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda\hat{\varrho}e^{2itg(k)}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}} & 1 \end{pmatrix}\begin{pmatrix} 1+\lambda\hat{\varrho}\hat{\overline{\varrho}} & 0 \\ 0 & \frac{1}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}} \end{pmatrix}\begin{pmatrix} 1 & \frac{\hat{\overline{\varrho}}e^{-2itg(k)}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}} \\ 0 & 1 \end{pmatrix}, & \lambda \in (-\infty, \mu), \\ \begin{pmatrix} -ie^{it(g_+-g_-)} & 0 \\ \lambda\hat{f}e^{it(g_++g_-)} & ie^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_1, \\ \begin{pmatrix} \frac{\hat{a}_-}{\hat{a}_+}e^{it(g_+-g_-)} & i\nu^2e^{-it(g_+-g_-)} \\ 0 & \frac{\hat{a}_+}{\hat{a}_-}e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_2, \\ \begin{pmatrix} -ie^{it(g_+-g_-)} & -\hat{\overline{f}}e^{-it(g_++g_-)} \\ 0 & ie^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \overline{\gamma}_1, \\ \begin{pmatrix} \frac{\hat{\overline{a}}_+}{\hat{\overline{a}}_-}e^{it(g_+-g_-)} & 0 \\ i\nu^{-2}e^{it(g_++g_-)} & \frac{\hat{\overline{a}}_-}{\hat{\overline{a}}_+}e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \overline{\gamma}_2, \\ \begin{pmatrix} e^{it(g_+-g_-)} & 0 \\ 0 & e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}. \end{cases}$$

- *$\hat{m}^{(1)}(x, t, \lambda)$ satisfies the asymptotic behavior*

$$\hat{m}^{(1)}(x, t, \lambda) \to I, \quad \lambda \to \infty. \tag{3.3}$$

There is a bad factorization in jump matrix $\hat{J}^{(1)}$ for $\lambda \in (-\infty, \mu)$. To eliminate the intermediate matrix, we define function $\hat{m}^{(2)}(x, t, \lambda)$ by

$$\hat{m}^{(2)}(x, t, \lambda) = \hat{m}^{(1)}(x, t, \lambda)\delta^{-\sigma_3}(\lambda), \tag{3.4}$$

where

$$\delta(\lambda) = \exp\left[\frac{-i}{2\pi}\int_{-\infty}^{\mu}\frac{\ln(1+s|\hat{\varrho}(s)|^2)}{s-\lambda}ds\right], \quad \lambda \in \mathbb{C}\backslash(-\infty, \mu]. \tag{3.5}$$

The function $\delta(\lambda)$ with the scalar RH problem:

Lemma 3.1. *The function $\delta(\lambda)$ admits the following properties:*

- ▸ *$\delta(\lambda)$ and $\delta^{-1}(\lambda)$ are bounded and analytic for $\lambda \in \mathbb{C}\backslash(-\infty, \mu]$ with continuous boundary values on $(-\infty, \mu)$.*
- ▸ *δ admits the symmetry $\delta(\lambda) = \overline{\delta(\bar{\lambda})}^{-1}$.*
- ▸ *δ admits the jump condition*

$$\begin{aligned} &\delta_+ = \delta_-(1+\lambda|\hat{\varrho}|^2), \quad (-\infty, \mu),\\ &\delta_+ = \delta_-, \qquad\qquad\quad (\mu, +\infty). \end{aligned}$$

- ▸ *$\delta(\lambda)$ admits the asymptotic behavior*

$$\delta(\lambda) = 1 + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty. \tag{3.6}$$

Lemma 3.1 implies that δ^{σ_3} satisfies a L^2-RH problem. Hence $\hat{m}$ satisfies Problem 2.2 iff $\hat{m}^{(2)}$ satisfies the RH problem:

RH problem 3.2. *$\hat{m}^{(2)}(x, t, \lambda)$ satisfies the following properties:*

- *$\hat{m}^{(2)}(x, t, \lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(2)}$, where $\Sigma^{(2)}$ see Fig. 6.*
- *$\hat{m}^{(2)}(x, t, \lambda)$ satisfies the jump condition*

$$\hat{m}_+^{(2)} = \hat{m}_-^{(2)}\hat{J}^{(2)}, \tag{3.7}$$

where the jump matrix $\hat{J}^{(2)}$ is given by

$$\hat{J}^{(2)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{\lambda\hat{\varrho}}{1+\lambda\hat{\varrho}\hat{\bar{\varrho}}}\delta_-^{-2}e^{2itg} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\hat{\bar{\varrho}}}{1+\lambda\hat{\varrho}\hat{\bar{\varrho}}}\delta_+^2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \lambda \in (-\infty, \mu),\\ \begin{pmatrix} 1 & \hat{\bar{\varrho}}\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda\hat{\varrho}\delta^{-2}e^{2itg} & 1 \end{pmatrix}, & \lambda \in (\mu, +\infty),\\ \begin{pmatrix} -ie^{it(g_+-g_-)} & 0 \\ \lambda\hat{f}\delta^{-2}e^{it(g_++g_-)} & ie^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_1,\\ \begin{pmatrix} \frac{\hat{a}_-}{\hat{\bar{a}}_+}e^{it(g_+-g_-)} & i\nu^2\delta^2 e^{-it(g_++g_-)} \\ 0 & \frac{\hat{a}_+}{\hat{\bar{a}}_-}e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_2,\\ \begin{pmatrix} -ie^{it(g_+-g_-)} & -\hat{\bar{f}}\delta^2 e^{-it(g_++g_-)} \\ 0 & ie^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \overline{\gamma}_1,\\ \begin{pmatrix} \frac{\hat{\bar{a}}_+}{\hat{\bar{a}}_-}e^{it(g_+-g_-)} & 0 \\ i\nu^{-2}\delta^{-2}e^{it(g_++g_-)} & \frac{\hat{\bar{a}}_-}{\hat{\bar{a}}_+}e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \overline{\gamma}_2,\\ \begin{pmatrix} e^{it(g_+-g_-)} & 0 \\ 0 & e^{-it(g_+-g_-)} \end{pmatrix}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}. \end{cases}$$

- *$\hat{m}^{(2)}(x, t, \lambda)$ satisfies the asymptotic behavior*

$$\hat{m}^{(2)}(x, t, \lambda) \to I, \quad \lambda \to \infty. \tag{3.8}$$

The purpose of the third deformation of the RH problem is to extend the jump matrix off the real axis. Then, the complex plane $\mathbb{C}$ is separated into six sectors which are respectively denoted by $U_j (j = 1, 2, \ldots, 6)$. The distributions of U_j are shown in Fig. 7. With this deformation, we define a new function $\hat{m}^{(3)}$ that deforms the oscillation term along the real axis onto new contours. Along the new

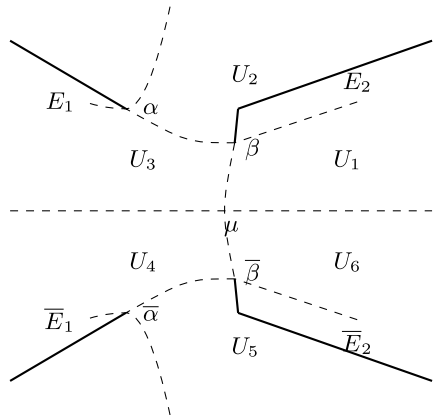

Fig. 7. The distribution of $U_j, j = 1, 2, \ldots, 6$.

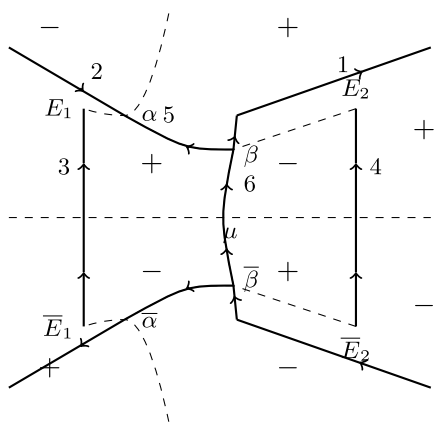

Fig. 8. The jump contour $\Sigma^{(3)}$.

contours, the deformed oscillation term is decaying. The function $\hat{m}^{(3)}$ defined by

$$\hat{m}^{(3)} = \hat{m}^{(2)} \begin{cases} \begin{pmatrix} 1 & 0 \\ -\lambda\hat{\varrho}\delta^{-2}e^{2itg} & 1 \end{pmatrix}, & \lambda \in U_1, \\ \begin{pmatrix} 1 & -\frac{\hat{\overline{\varrho}}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}}\delta^{2}e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \lambda \in U_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda\hat{\varrho}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}}\delta^{-2}e^{2itg} & 1 \end{pmatrix}, & \lambda \in U_4, \\ \begin{pmatrix} 1 & \hat{\overline{\varrho}}\delta^{2}e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \lambda \in U_6, \\ I, & \lambda \in U_2 \cup U_5. \end{cases}$$

By using the identity

$$i = \lambda\hat{f}\left(\frac{\hat{\overline{\varrho}}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}}\right)_-, \quad -i = \lambda\hat{f}\left(\frac{\hat{\overline{\varrho}}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}}\right)_+,$$
$$\frac{\hat{a}_-}{\hat{a}_+} + i\lambda\hat{\rho}_+\nu^2 = 0, \quad \frac{\hat{a}_+}{\hat{a}_-} + i\lambda\hat{\rho}_-\nu^2 = 0,$$

the function $\hat{m}^{(3)}$ admits the RH problem:

RH problem 3.3. *$\hat{m}^{(3)}(x, t, \lambda)$ satisfies the following properties:*

- *$\hat{m}^{(3)}(x, t, \lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(3)}$, where $\Sigma^{(3)}$ see Fig. 8.*
- *$\hat{m}^{(3)}(x, t, \lambda)$ satisfies the jump condition*

$$\hat{m}^{(3)}_+ = \hat{m}^{(3)}_-\hat{J}^{(3)}, \tag{3.9}$$

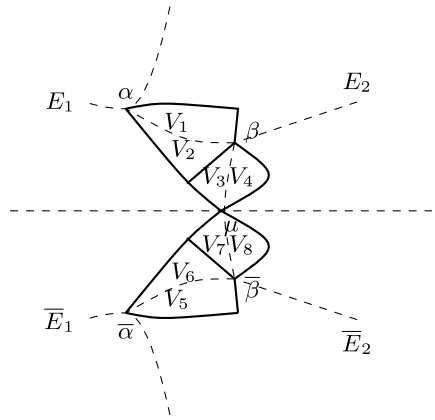

Fig. 9. The distribution of V_j, $j = 1, 2, \ldots, 8$.

where the jump matrix $\hat{J}^{(3)} = \hat{J}_j^{(3)}$ *in the upper half-plane rewritten as*

$$\hat{J}_1^{(3)} = \begin{pmatrix} 1 & 0 \\ -\lambda\hat{\varrho}\delta^{-2}e^{2itg} & 1 \end{pmatrix}, \quad \hat{J}_2^{(3)} = \begin{pmatrix} 1 & -\frac{\hat{\overline{\varrho}}}{1+\lambda\hat{\varrho}\hat{\overline{\varrho}}}\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix},$$

$$\hat{J}_3^{(3)} = \begin{pmatrix} 0 & (\lambda\hat{f})^{-1}\delta^2 e^{-it(g_+-g_-)} \\ \lambda\hat{f}\delta^{-2}e^{it(g_++g_-)} & 0 \end{pmatrix},$$

$$\hat{J}_4^{(3)} = \begin{pmatrix} 0 & i\nu^2\delta^2 e^{-it(g_++g_-)} \\ i\nu^{-2}\delta^{-2}e^{it(g_++g_-)} & 0 \end{pmatrix},$$

$$\hat{J}_5^{(3)} = e^{-itg_-\sigma_3}\begin{pmatrix} 1 & -i(\lambda\hat{f})^{-1}\delta^2 \\ 0 & 1 \end{pmatrix} e^{itg_+\sigma_3},$$

$$\hat{J}_6^{(3)} = e^{-itg_-\sigma_3}\begin{pmatrix} 1 & -i(\lambda\hat{f})^{-1}\delta^2 \\ \lambda\hat{\varrho}\delta^{-2} & \hat{a}\hat{\overline{a}} \end{pmatrix} e^{itg_+\sigma_3}.$$

where the subscript of $\hat{J}_j^{(3)}$ *denotes the jump contour in Fig. 8.*

- $\hat{m}^{(3)}(x, t, \lambda)$ *satisfies the asymptotic behavior*

$$\hat{m}^{(3)}(x, t, \lambda) \to I, \quad \lambda \to \infty. \tag{3.11}$$

Our purpose for performing the fourth deformation of the jump contour is to transform the jump matrix across $\gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}$ to a diagonal or off-diagonal matrix. Then, the branch cuts $\gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}$ is separated into eight jump contours in the upper half-plane which are respectively denoted by $(5, 6, \ldots, 12)$ and there exist eight jump contours in the lower half-plane, see Fig. 10. The analytic regions enclosed by these jump contours named by $V_j (j = 1, 2, \ldots, 8)$, see Fig. 9. First, we deformate $\hat{J}_5^{(3)}$ and $\hat{J}_6^{(3)}$ as

$$\hat{J}_5^{(3)} = e^{-itg_-\sigma_3}\begin{pmatrix} 1 & 0 \\ i\lambda\hat{f}\delta^{-2} & 1 \end{pmatrix}\begin{pmatrix} 0 & -i(\lambda\hat{f})^{-1}\delta^2 \\ -i\lambda\hat{f}\delta^{-2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ i\lambda\hat{f}\delta^{-2} & 1 \end{pmatrix} e^{-itg_+\sigma_3},$$

$$\hat{J}_6^{(3)} = e^{-itg_-\sigma_3}\begin{pmatrix} 1 & \frac{-i(\lambda\hat{f})^{-1}\delta^2}{\hat{a}\hat{\overline{a}}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\hat{a}\hat{\overline{a}}} & 0 \\ 0 & \hat{a}\hat{\overline{a}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\lambda\hat{\varrho}\delta^{-2}}{\hat{a}\hat{\overline{a}}} & 1 \end{pmatrix} e^{-itg_+\sigma_3}.$$

We define the function $\hat{m}^{(4)}$ in the upper half-plane by

$$\hat{m}^{(4)} = \hat{m}^{(3)}\begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{e^{2itg}}{\hat{a}\hat{b}\delta^2} & 1 \end{pmatrix}, & \lambda \in V_1, \\ \begin{pmatrix} 1 & 0 \\ \frac{e^{2itg}}{\hat{a}\hat{b}\delta^2} & 1 \end{pmatrix}, & \lambda \in V_2, \\ \begin{pmatrix} 1 & 0 \\ -\frac{\lambda\hat{\varrho}}{\hat{a}\hat{\overline{a}}}\delta^{-2}e^{-2itg} & 1 \end{pmatrix}, & \lambda \in V_3, \\ \begin{pmatrix} 1 & -\hat{\overline{\varrho}}\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \lambda \in V_4, \\ I, & elsewhere, \end{cases}$$

and $\hat{m}^{(4)}(x, t, \lambda)$ admits the RH problem:

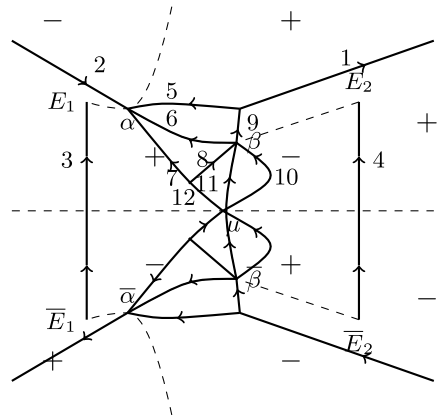

Fig. 10. The jump contours $\Sigma^{(4)}$ and $\Sigma^{(5)}$.

RH problem 3.4. *$\hat{m}^{(4)}(x,t,\lambda)$ satisfies the following properties:*

- *$\hat{m}^{(4)}(x,t,\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(4)}$, where $\Sigma^{(4)}$ see Fig. 10.*
- *$\hat{m}^{(4)}(x,t,\lambda)$ satisfies the jump condition*

$$\hat{m}_+^{(4)} = \hat{m}_-^{(4)}\hat{J}^{(4)}, \tag{3.13}$$

where the jump matrix $\hat{J}^{(4)} = \hat{J}_j^{(4)}$ in the upper half-plane rewritten as

$$\hat{J}_1^{(4)} = \hat{J}_1^{(3)}, \quad \hat{J}_2^{(4)} = \hat{J}_2^{(3)}, \quad \hat{J}_3^{(4)} = \hat{J}_3^{(3)}, \quad \hat{J}_4^{(4)} = \hat{J}_4^{(3)},$$

$$\hat{J}_5^{(4)} = \hat{J}_7^{(4)} = \begin{pmatrix} 1 & 0 \\ -\frac{e^{2itg}}{\hat{a}\hat{b}\delta^2} & 1 \end{pmatrix}, \quad \hat{J}_6^{(4)} = \begin{pmatrix} 0 & -\hat{a}\hat{b}\delta^2 e^{-it(g_+ + g_-)} \\ \frac{1}{\hat{a}\hat{b}\delta^2 e^{-it(g_+ + g_-)}} & 0 \end{pmatrix},$$

$$\hat{J}_8^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\hat{a}^2\hat{\bar{a}}\hat{b}}\delta^{-2}e^{2itg} & 1 \end{pmatrix}, \quad \hat{J}_9^{(4)} = \begin{pmatrix} 1 & 0 \\ -\frac{\hat{\bar{a}}}{\hat{\bar{b}}}\delta^{-2}e^{2itg} & 1 \end{pmatrix},$$

$$\hat{J}_{10}^{(4)} = \begin{pmatrix} 1 & -\frac{\hat{b}}{\hat{\bar{a}}}\delta^2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, \quad \hat{J}_{11}^{(4)} = \begin{pmatrix} \frac{e^{it(g_+ - g_-)}}{\hat{a}\hat{\bar{a}}} & 0 \\ 0 & \hat{a}\hat{\bar{a}}e^{-it(g_+ - g_-)} \end{pmatrix},$$

$$\hat{J}_{12}^{(4)} = \begin{pmatrix} 1 & 0 \\ \frac{\hat{\bar{b}}}{\hat{a}^2\hat{\bar{a}}}\delta^{-2}e^{2itg} & 1 \end{pmatrix},$$

where the subscript of $\hat{J}_j^{(4)}$ denotes the jth jump contour in Fig. 10.

- *$\hat{m}^{(4)}(x,t,\lambda)$ satisfies the asymptotic behavior*

$$\hat{m}^{(4)}(x,t,\lambda) \to I, \quad \lambda \to \infty. \tag{3.15}$$

For making the jump matrix across the branch cuts $\gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\beta,\alpha)}$ and across $\gamma_{(\overline{\beta},\beta)}$ constant in λ. We introduce the matrix function $\hat{m}^{(5)}$ by

$$\hat{m}^{(5)}(x,t,\lambda) = e^{-ih(\infty)\sigma_3}\hat{m}^{(4)}(x,t,\lambda)e^{ih(\lambda)\sigma_3}. \tag{3.16}$$

The function $h(\lambda)$ in $\hat{m}^{(5)}$ is defined as

$$h(\lambda) = \frac{\hbar(\lambda)}{2i\pi}\int_{\Sigma^{mod}} \frac{H(s)}{s-\lambda}\mathrm{d}s, \tag{3.17}$$

where

$$H(k) = \begin{cases} \frac{2\tau_1 + h_1}{\hbar_+}, & \lambda \in \gamma_1 \cup \overline{\gamma}_1, \\ \frac{2\tau_2 + h_2}{\hbar_+}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})}, \\ \frac{2\tau_3 + h_3}{\hbar}, & \lambda \in \gamma_{(\overline{\beta},\beta)}, \\ \frac{h_4}{\hbar_+}, & \lambda \in \gamma_2 \cup \overline{\gamma}_2, \end{cases} \tag{3.18}$$

with

$$\begin{aligned}
h_1 &= -i\ln(\hat{a}_+\hat{a}_-\delta^2 e^{i\phi}), && \lambda\in\gamma_1,\\
h_2 &= -i\ln(i\hat{a}_+\hat{a}_-\delta^2), && \lambda\in\gamma_{(\beta,\alpha)},\\
h_3 &= -i\ln(\hat{a}\hat{\bar{a}}), && \lambda\in\gamma_{(\mu,\beta)},\\
h_4 &= -i\ln(\mu^2\delta^2), && \lambda\in\gamma_2,
\end{aligned}$$

and τ_j are real constants, $h_j=\overline{h}_j$. $h(\lambda)$ satisfies Lemma 3.2.

Lemma 3.2. *$h(\lambda)$ admits the following properties:*

- *$h(\lambda)$ admits the symmetry*

$$h(\lambda)=\overline{h(\overline{\lambda})},\quad \lambda\in\hat{\mathbb{C}}\backslash\Sigma^{mod}. \tag{3.19}$$

- *$h(\lambda)$ admits the asymptotic*

$$\begin{aligned}
&h(\lambda)=h(\infty)+\mathcal{O}(\lambda^{-1}),\quad \lambda\to\infty,\\
&h(\infty)=-\frac{1}{2i\pi}\int_{\Sigma^{mod}} s^3H(s)\mathrm{d}s.
\end{aligned} \tag{3.20}$$

- *$e^{ih\sigma_3}$ is bounded and analytic for $k\in\hat{\mathbb{C}}\backslash\Sigma^{mod}$.*
- *$h(\lambda)$ satisfies the jump conditions*

$$h_+ + h_- = \begin{cases} 2\tau_1 - i\ln(\hat{a}_+\hat{a}_-\delta^2 e^{i\phi}), & \lambda\in\gamma_1,\\ 2\tau_1 + i\ln(\hat{\bar{a}}_+\hat{\bar{a}}_-\delta^{-2}e^{-i\phi}), & \lambda\in\overline{\gamma}_1,\\ 2\tau_2 - i\ln(i\hat{a}\hat{b}\delta^2), & \lambda\in\gamma_{(\beta,\alpha)},\\ 2\tau_2 + i\ln(-i\hat{\bar{a}}\hat{\bar{b}}\delta^{-2}), & \lambda\in\gamma_{(\overline{\alpha},\overline{\beta})},\\ -i\ln(\mu^2\delta^2), & \lambda\in\gamma_2\cup\overline{\gamma}_2,\end{cases}$$

$$h_+ - h_- = \begin{cases} 2\tau_3 - i\ln(\hat{a}\hat{\bar{a}}), & \lambda\in\gamma_{(\mu,\beta)},\\ 2\tau_3 + i\ln(\hat{a}\hat{\bar{a}}), & \lambda\in\gamma_{(\overline{\beta},\mu)}.\end{cases}$$

RH problem 3.5. *Function $\hat{m}^{(5)}(x,t,\lambda)$ satisfies the following properties:*

- *$\hat{m}^{(5)}(x,t,\lambda)$ is analytic in $\mathbb{C}\backslash\Sigma^{(5)}$, where $\Sigma^{(5)}$ see Fig. 10.*
- *$\hat{m}^{(5)}(x,t,\lambda)$ satisfies the jump condition*

$$\hat{m}^{(5)}_+ = \hat{m}^{(5)}_-\hat{J}^{(5)}, \tag{3.21}$$

where the jump matrix $\hat{J}^{(5)}=\hat{J}^{(5)}_j$ is given by

$$\hat{J}^{(5)}_1=\begin{pmatrix}1 & 0\\ \frac{\hat{\bar{b}}}{\hat{\bar{a}}}\delta^{-2}e^{2itg}e^{2ih} & 1\end{pmatrix},\quad \hat{J}^{(5)}_2=\begin{pmatrix}1 & \hat{a}\hat{b}\delta^2e^{-2itg}e^{-2ih}\\ 0 & 1\end{pmatrix},$$

$$\hat{J}^{(5)}_3=\begin{pmatrix}0 & ie^{-2i(t\Delta_1+\tau_1)}\\ ie^{2i(t\Delta_1+\tau_1)} & 0\end{pmatrix},\quad \hat{J}^{(5)}_4=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},$$

$$\hat{J}^{(5)}_5=\hat{J}^{(5)}_7=\begin{pmatrix}1 & 0\\ -\frac{e^{2itg}}{\hat{a}\hat{b}\delta^2}e^{2ih} & 1\end{pmatrix}\quad \hat{J}^{(5)}_6=\begin{pmatrix}0 & ie^{-2i(t\Delta_2+\tau_2)}\\ ie^{2i(t\Delta_2+\tau_2)} & 0\end{pmatrix},$$

$$\hat{J}^{(5)}_8=\begin{pmatrix}1 & 0\\ \frac{1}{\hat{a}^2\hat{\bar{a}}\hat{b}}\delta^{-2}e^{2itg}e^{2ih} & 1\end{pmatrix},\quad \hat{J}^{(5)}_9=\begin{pmatrix}1 & 0\\ -\frac{\hat{\bar{a}}}{\hat{\bar{b}}}\delta^{-2}e^{2itg}e^{2ih} & 1\end{pmatrix},$$

$$\hat{J}^{(5)}_{10}=\begin{pmatrix}1 & -\frac{\hat{b}}{\hat{\bar{a}}}\delta^2e^{-2itg}e^{-2ih}\\ 0 & 1\end{pmatrix},\quad \hat{J}^{(5)}_{11}=\begin{pmatrix}e^{2i(t\Delta_3+\tau_3)} & 0\\ 0 & e^{-2i(t\Delta_3+\tau_3)}\end{pmatrix},$$

$$\hat{J}^{(5)}_{12}=\begin{pmatrix}1 & 0\\ \frac{\hat{\bar{b}}}{\hat{a}^2\hat{\bar{a}}}\delta^{-2}e^{2itg}e^{2ih} & 1\end{pmatrix},$$

where the subscript of $\hat{J}^{(5)}_j$ denotes the jth jump contour in Fig. 10.

- *$\hat{m}^{(5)}(x,t,\lambda)$ satisfies the asymptotic behavior*

$$\hat{m}^{(5)}(x,t,\lambda)\to I,\quad \lambda\to\infty. \tag{3.23}$$

The lower half-plane can be derived by the symmetries.

4. The long-time asymptotic

Let $D_\varepsilon(\alpha)$, $D_\varepsilon(\overline{\alpha})$, $D_\varepsilon(\beta)$, $D_\varepsilon(\overline{\beta})$ and $D_\varepsilon(\mu)$ be the small disks of α, $\overline{\alpha}$, β, $\overline{\beta}$ and μ. Define $\mathcal{D} = D_\varepsilon(\alpha) \cup D_\varepsilon(\overline{\alpha}) \cup D_\varepsilon(\beta) \cup D_\varepsilon(\overline{\beta}) \cup D_\varepsilon(\mu)$. The approximate solution is

$$m^{app} = \begin{cases} m^{\alpha}, & \lambda \in D_\varepsilon(\alpha), \\ m^{\beta}, & \lambda \in D_\varepsilon(\beta), \\ m^{\overline{\alpha}}, & \lambda \in D_\varepsilon(\overline{\alpha}), \\ m^{\overline{\beta}}, & \lambda \in D_\varepsilon(\overline{\beta}), \\ m^{\mu}, & \lambda \in D_\varepsilon(\mu), \\ m^{mod}, & elsewhere, \end{cases} \tag{4.1}$$

and the jump contour $\Sigma^{app} = \Sigma^{mod} \cup \partial\mathcal{D} \cup \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{Z} \cup \overline{\mathcal{Z}} \cup \mathcal{X}$. $\mathcal{Y}$, $\overline{\mathcal{Y}}$, $\mathcal{Z}$, $\overline{\mathcal{Z}}$ and $\mathcal{X}$ are defined in the next subsections.

4.1. The model problem

From the fifth deformation of the jump contour, the jump matrix approaches the identity matrix as $t \to \infty$ on $\Sigma^{(5)} \setminus (\Sigma_1 \cup \Sigma_2 \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\beta},\beta)})$. Inspired by this idea, for $t \to \infty$, the solution of function $\hat{m}^{(5)}$ approaches the solution of function m^{mod}. Function m^{mod} admits the RH problem

$$m_+^{mod}(x,t,\lambda) = m_-^{mod}(x,t,\lambda) v^{mod}(x,t,\lambda), \quad \lambda \in \Sigma^{mod}, \tag{4.2}$$

where we define $\Sigma^{mod} = \gamma_1 \cup \overline{\gamma}_1 \cup \gamma_2 \cup \overline{\gamma}_2 \cup \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})} \cup \gamma_{(\overline{\beta},\beta)}$ and the jump matrix

$$v^{mod} = \begin{cases} \begin{pmatrix} 0 & ie^{-2i(t\Delta_1+\tau_1)} \\ ie^{2i(t\Delta_1+\tau_1)} & 0 \end{pmatrix}, & \lambda \in \gamma_1 \cup \overline{\gamma}_1, \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \lambda \in \gamma_2 \cup \overline{\gamma}_2, \\ \begin{pmatrix} 0 & ie^{-2i(t\Delta_2+\tau_2)} \\ ie^{2i(t\Delta_2+\tau_2)} & 0 \end{pmatrix}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})}, \\ \begin{pmatrix} e^{2i(t\Delta_3+\tau_3)} & 0 \\ 0 & e^{-2i(t\Delta_3+\tau_3)} \end{pmatrix}, & \lambda \in \gamma_{(\overline{\beta},\beta)}, \end{cases}$$

Define a vector valued function $\mathbf{M}(\lambda, u, e)$ by

$$\mathbf{M}(\lambda, u, e) = \left(\frac{\Theta(\varphi(\lambda^+) + u + e)}{\Theta(\varphi(\lambda^+) + e)}, \frac{\Theta(\varphi(\lambda^+) - u - e)}{\Theta(\varphi(\lambda^+) - e)} \right), \quad \lambda \in \mathbb{C} \setminus \Sigma^{mod}, \quad u, e \in \mathbb{C}^3, \tag{4.3}$$

where Θ is the Riemann theta function

$$\Theta(z) = \sum_{N \in \mathbb{Z}^3} e^{2i\pi(\frac{1}{2}N^T\tau N + N^T z)}, \quad z \in \mathbb{C}^3. \tag{4.4}$$

Riemann theta function Θ has the properties for all $z \in \mathbb{C}^3$,

$$\Theta(z + e^{(j)}) = \Theta(z), \quad \Theta(z + \tau^{(j)}) = e^{2i\pi(-z_j - \frac{\tau_{ij}}{2})}\Theta(z), \quad \Theta(z) = \Theta(-z), \quad j = 1, 2, 3, \tag{4.5}$$

where $\tau = (\tau_{jl})_{3\times 3}$ is a 3×3 period matrix. τ defined by $\tau_{jl} = \int_{b_j} \zeta_l$ (see [52] for its detailed informations), where $\int_{a_i} \zeta_j = \delta_{ij}$. ζ_j is given by $\zeta_j = \sum_{l=1}^{3} A_{jl}\hat{\zeta}_l$, where $\hat{\zeta}_l = \hbar^{-1}\lambda^{l-1}$ and $(A^{-1})_{jl} = \int_{a_j} \hat{\zeta}_l$. Since $\hbar$ admits $\hbar(\lambda^+) = -\hbar(\lambda^-)$, we have the symmetry for $\zeta(\lambda^+) = -\zeta(\lambda^-)$. And

$$\int_{a_j^+} \zeta = \frac{1}{2}\int_{a_j} \zeta = \frac{1}{2}e^{(j)}, \quad \int_{b_j^+} \zeta = \frac{1}{2}\int_{b_j} \zeta = \frac{1}{2}\tau^{(j)}, \tag{4.6}$$

a_j^+ and b_j^+ are the restrictions of a_j and b_j to the upper sheet, $e^{(j)}$ is the jth column of the identity matrix $I_{3\times 3}$, $\tau^{(j)}$ is also. φ in (4.3) is Abel map $\varphi : M \to \mathbb{C}^3$ with base point $\overline{E}_2$,

$$\varphi(P) = \int_{\overline{E}_2}^{P} \zeta, \quad P \in M. \tag{4.7}$$

And φ satisfies the jump conditions

$$\varphi_+(\lambda^+) + \varphi_-(\lambda^+) = -\varphi_+(\lambda^-) - \varphi_-(\lambda^-) = \begin{cases} \tau^{(1)}, & \lambda \in \gamma_1 \cup \overline{\gamma}_1, \\ \tau^{(2)}, & \lambda \in \gamma_{(\beta,\alpha)}, \\ \tau^{(2)} + e^{(1)} + e^{(2)}, & \lambda \in \gamma_{(\overline{\alpha},\overline{\beta})}, \\ 0, & \lambda \in \gamma_2 \cup \overline{\gamma}_2, \end{cases}$$

$$\varphi_+(\lambda^+) - \varphi_-(\lambda^+) = \tau^{(2)} - \tau^{(3)} + e^{(1)} + e^{(2)}, \quad \lambda \in \gamma_{(\overline{\beta},\beta)},$$

where $\varphi_+(\lambda^\pm)$ and $\varphi_-(\lambda^\pm)$ are the boundary value of $\varphi(l^\pm)$ for $l \in \mathbb{C}$ approaches λ from the right and left of the contour, respectively.

The vector valued function $\mathbf{M}(\lambda, u, e)$ satisfies the jump condition

$$\mathbf{M}_+(\lambda, u, e) = \mathbf{M}_-(\lambda, u, e)\begin{cases}\begin{pmatrix} 0 & e^{2i\pi u_1} \\ e^{-2i\pi u_1} & 0 \end{pmatrix}, & \lambda \in \gamma_1 \cup \overline{\gamma}_1,\\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \lambda \in \gamma_2 \cup \overline{\gamma}_2,\\ \begin{pmatrix} 0 & e^{2i\pi u_2} \\ e^{-2i\pi u_2} & 0 \end{pmatrix}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})},\\ \begin{pmatrix} e^{-2i\pi(u_2-u_3)} & 0 \\ 0 & e^{2i\pi(u_2-u_3)} \end{pmatrix}, & \lambda \in \gamma_{(\overline{\beta},\beta)}.\end{cases}$$

Define a vector valued function $\mathbf{N}(\lambda, u, e)$ by

$$\mathbf{N}(\lambda, u, e) = \frac{1}{2}\begin{pmatrix} (\nu_1 + \nu_1^{-1})\mathbf{M}_1(\lambda, u, e) & (\nu_1 - \nu_1^{-1})\mathbf{M}_2(\lambda, u, e) \\ (\nu_1 - \nu_1^{-1})\mathbf{M}_1(\lambda, u, -e) & (\nu_1 + \nu_1^{-1})\mathbf{M}_2(\lambda, u, -e) \end{pmatrix}, \quad \lambda \in \mathbb{C}\backslash\Sigma^{mod}, \quad u, e \in \mathbb{C}^3,$$

where

$$\nu_1(\lambda) = \left(\frac{(\lambda - E_1)(\lambda - E_2)(\lambda - \alpha)(\lambda - \beta)}{(\lambda - \overline{E}_1)(\lambda - \overline{E}_2)(\lambda - \overline{\alpha})(\lambda - \overline{\beta})}\right)^{1/4}, \quad \lambda \in \mathbb{C}\backslash\Sigma^{mod}. \tag{4.8}$$

For $k \to \infty$, $\nu_1(\lambda) = 1 + O(\lambda^{-1})$. Let $\hat{\nu}_1$ denote the function $M \to \hat{\mathbb{C}}$ which is given by ν_1^2 on the upper sheet and by $-\nu_1^2$ on the lower sheet of M. $\hat{\nu}_1(\lambda^{\pm}) = \pm\nu_1^2(\lambda)$ for $\lambda \in \Sigma^{mod}$. Then $\hat{\nu}_1$ is a meromorphic function on M. Noting that $\hat{\nu}_1$ has four simple zeros at E_1, E_2, α and β, we see that $\hat{\nu}_1$ has degree four. Hence, function $\hat{\nu}_1 - 1$ has four zeros on M counting multiplicity. These zeros are $\infty^+, P_1, P_2, P_3 \in M$. Let $\mathfrak{D}$ be the divisor $\mathfrak{D} = P_1P_2P_3$ on M. $\mathbf{N}(\lambda, u, e)$ satisfies the jump condition

$$\mathbf{N}_+(\lambda, u, e) = \mathbf{N}_-(\lambda, u, e)\begin{cases}\begin{pmatrix} 0 & ie^{2i\pi u_1} \\ ie^{-2i\pi u_1} & 0 \end{pmatrix}, & \lambda \in \gamma_1 \cup \overline{\gamma}_1,\\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \lambda \in \gamma_2 \cup \overline{\gamma}_2,\\ \begin{pmatrix} 0 & ie^{2i\pi u_2} \\ ie^{-2i\pi u_2} & 0 \end{pmatrix}, & \lambda \in \gamma_{(\beta,\alpha)} \cup \gamma_{(\overline{\alpha},\overline{\beta})},\\ \begin{pmatrix} -e^{-2i\pi(u_2-u_3)} & 0 \\ 0 & -e^{2i\pi(u_2-u_3)} \end{pmatrix}, & \lambda \in \gamma_{(\overline{\beta},\beta)}.\end{cases}$$

Define the complex vector $d(\xi) \in \mathbb{C}^3$ by $d = \varphi(\mathfrak{D}) + \mathcal{K}$, where $\varphi(\mathfrak{D}) = \sum_1^3 \varphi(P_j)$ and $\mathcal{K} = \frac{1}{2}(e^{(1)} + e^{(3)} + \tau^{(1)} + \tau^{(2)} + \tau^{(3)})$. Define the vector function $v(\xi, t) = v(t) = -\frac{1}{\pi}(t\,\triangle_1 + \tau_1, t\,\triangle_2 + \tau_2, t(\triangle_2 - \triangle_3) + \tau_2 - \tau_3 + \frac{\pi}{2})$.

The solution of the model RH problem is shown as follows:

Lemma 4.1. *For each choice of the constants $\{\triangle_j, \tau_j\}_{j=1}^3$ and for $t \geq 0$, the function $m^{mod}(x, t, \lambda)$ defined by*

$$m^{mod}(x, t, \lambda) = \mathbf{N}(\infty, v(t), d)^{-1}\mathbf{N}(\lambda, v(t), d), \quad \lambda \in \hat{\mathbb{C}}\backslash\Sigma^{mod}, \tag{4.9}$$

is the unique solution of the RH problem (4.2). And this solution satisfies

$$\lim_{\lambda\to\infty} \lambda m_{12}^{mod}(x, t, \lambda) = -\frac{i}{2}\mathrm{Im}(E_1 + E_2 + \alpha + \beta)\frac{\Theta(\varphi(\infty^+) + d)\Theta(\varphi(\infty^+) - v(t) - d)}{\Theta(\varphi(\infty^+) + v(t) + d)\Theta(\varphi(\infty^+) - d)}.$$

Proof. Define a multivalued meromorphic function $\mathbf{P}_j(P)$, $j = 1, 2$ by

$$\mathbf{P}_1(P) = (\hat{\nu}_1(P) - 1)\frac{\Theta(\varphi(P) - v(t) - d)}{\Theta(\varphi(P) - d)}, \quad \mathbf{P}_2(P) = (\hat{\nu}_1(P) - 1)\frac{\Theta(\varphi(P) + v(t) - d)}{\Theta(\varphi(P) - d)}.$$

Using the symmetry $\varphi(\lambda^+) = -\varphi(\lambda^-)$, then

$$\mathbf{N}(\lambda, v(t), d) = \frac{1}{2\nu(\lambda)}\begin{pmatrix} -\mathbf{P}_1(\lambda^-) & \mathbf{P}_1(\lambda^+) \\ \mathbf{P}_2(\lambda^+) & -\mathbf{P}_2(\lambda^-) \end{pmatrix}, \quad \lambda \in \hat{\mathbb{C}}\backslash\Sigma^{mod}. \tag{4.10}$$

The function $\Theta(\varphi(\lambda^{\pm}) \pm v(t) - d)$ is bounded on $\hat{\mathbb{C}}\backslash\Sigma^{mod}$. And $\Theta(\varphi(P) - d)$ has zero divisor which is a factor of $\hat{\nu}_1 - 1$. $\mathbf{N}(\lambda, v(t), d)$ is an analytic function which is bounded away from the eight branch points. Thus one can derive

$$|\mathbf{N}(\lambda, v(t), d)| \leq C|\lambda - \lambda_0|^{-1/4}, \quad \lambda \in \hat{\mathbb{C}}\backslash\Sigma^{mod}, \tag{4.11}$$

where λ_0 is one of the eight branch points. For $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} \mathbf{N}(\lambda, v(t), d) = \mathbf{N}(\infty, v(t), d) = \begin{pmatrix} \mathbf{M}_1(\infty, v(t), d) & 0 \\ 0 & \mathbf{M}_2(\infty, v(t), -d) \end{pmatrix},$$

where

$$\mathbf{M}_1(\infty, v(t), d) = \frac{\Theta(\varphi(\infty^+) + v(t) + d)}{\Theta(\varphi(\infty^+) + d)}, \quad \mathbf{M}_2(\infty, v(t), d) = \frac{\Theta(\varphi(\infty^+) - v(t) + d)}{\Theta(\varphi(\infty^+) + d)}.$$

The values $\Theta(\varphi(\infty^+)+v(t)+d)$ and $\Theta(\varphi(\infty^+)+d)$ are finite and nonzero [52,53]. These imply that $\mathbf{N}(\infty, v(t), d)$ is invertible. Eq. (4.9) is successfully proved:

$$\begin{aligned}
&\lim_{\lambda\to\infty} \lambda m_{12}^{mod}(x,t,\lambda)\\
&= \frac{1}{\mathbf{M}_1(\infty, v(t), d)} \lim_{\lambda\to\infty} \lambda \mathbf{N}_{12}(\lambda, v(t), d)\\
&= \frac{\Theta(\varphi(\infty^+)+d)}{\Theta(\varphi(\infty^+)+v(t)+d)} \lim_{\lambda\to\infty} \frac{\lambda(\nu_1(\lambda)-\nu_1^{-1}(\lambda))}{2} \frac{\Theta(\varphi(\infty^+)-v(t)-d)}{\Theta(\varphi(\infty^+)-d)} \quad \square \quad \square\\
&= -\frac{i}{2}\mathrm{Im}(E_1+E_2+\alpha+\beta)\frac{\Theta(\varphi(\infty^+)+d)\Theta(\varphi(\infty^+)-v(t)-d)}{\Theta(\varphi(\infty^+)+v(t)+d)\Theta(\varphi(\infty^+)-d)}.
\end{aligned}$$

4.2. The local model near α, β and μ

The jump matrix $\hat{m}^{(5)}$ of the fifth deformation with the property that $\hat{v}^{(5)}-I$ decays to zero for $\lambda \in \Sigma^{(5)} \setminus \Sigma^{mod}$ as $t \to \infty$. But this decay is not uniform decay as λ approaches Σ^{mod}. So, for the parts of $\Sigma^{(5)}$ that lie near Σ^{mod}, it is necessary to introduce the local solutions which are better approximations of $\hat{m}^{(5)}$ than m^{mod}. These local approximations help us derive the approximate error estimates.

local model near α: We define a function $m^{(\alpha 0)}(x,t,k)$ for k near α by

$$m^{(\alpha 0)}(x,t,\lambda) = \hat{m}^{(5)}(x,t,\lambda)e^{-i(\frac{i}{2}\ln(-\delta(\lambda))^2\hat{a}(\lambda)\hat{b}(\lambda)+tg(\alpha)+h(\lambda))\sigma_3}, \quad \lambda \in D_\varepsilon(\alpha)\backslash\Sigma^{(5)}.$$

Lemma 3.1 and Lemma 3.2 imply that the exponential of $m^{(\alpha 0)}$ is bounded and analytic for $\lambda \in D_\varepsilon(\alpha)\backslash\Sigma^{(5)}$. The function $m^{(\alpha 0)}$ satisfies the following jump condition

$$m_+^{(\alpha 0)}(x,t,\lambda) = m_-^{(\alpha 0)}(x,t,\lambda)v^{(\alpha 0)}(x,t,\lambda), \quad \lambda \in \Sigma^{(5)} \cap D_\varepsilon(\alpha), \tag{4.12}$$

where

$$v^{(\alpha 0)} = \begin{cases} \begin{pmatrix} 1 & -e^{-2itg_\alpha} \\ 0 & 1 \end{pmatrix}, & \lambda \in \mathcal{A}_1, \\ \begin{pmatrix} 1 & 0 \\ e^{2itg_\alpha} & 1 \end{pmatrix}, & \lambda \in \mathcal{A}_2 \cup \mathcal{A}_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \lambda \in \mathcal{A}_3, \end{cases}$$

with $g_\alpha(\lambda) = g(\lambda) - g(\alpha)$ for $\lambda \in D_\varepsilon(\alpha)\backslash\gamma_{(\beta,\alpha)}$. And $\mathcal{A}_j = \overline{\mathcal{S}}_{j-1} \cap \overline{\mathcal{S}}_j$, $\overline{\mathcal{S}}_0 = \overline{\mathcal{S}}_4$. Let $\mathcal{A} = \cup \mathcal{A}_j$, $\overline{\mathcal{A}}$ is the conjugate of $\mathcal{A}$. For the jump contour of RH problem (4.12) and $\mathcal{S}_j$ please see Fig. 11.

For relating this RH problem to the Airy function of Eq. (A.5), we introduce $\zeta(\lambda) = \left(\frac{3itg_\alpha(\lambda)}{2}\right)^{2/3}$. We define a parametrix $m^\alpha(x,t,\lambda)$ for $\hat{m}^{(5)}$ near α by

$$m^\alpha(x,t,\lambda) = Y_\alpha(x,t,\lambda)m^{Ai}(\zeta(\lambda))e^{-i(\frac{i}{2}\ln(-\delta(\lambda))^2\hat{a}(\lambda)\hat{b}(\lambda)+tg(\alpha)+h(\lambda))\sigma_3}, \quad \lambda \in D_\varepsilon(\alpha)\backslash\Sigma^{(5)},$$

where

$$Y_\alpha(x,t,\lambda) = m^{mod}e^{-i(\frac{i}{2}\ln(-\delta(\lambda))^2\hat{a}(\lambda)\hat{b}(\lambda)+tg(\alpha)+h(\lambda))\sigma_3}(m_{as,N}^{Ai}\zeta(\lambda))^{-1}, \quad N \geq 0. \tag{4.13}$$

Function $m_{as,N}^{Ai}\zeta(\lambda)$ is analytic near α for the jump contour $\mathcal{Y}_3$ satisfies (A.6). And the first second terms in (4.13), i.e. m^{mod} $e^{-i(\frac{i}{2}\ln(-\delta(\lambda))^2\hat{a}(\lambda)\hat{b}(\lambda)+tg(\alpha)+h(\lambda))\sigma_3}$ also satisfies (A.6) on $\mathcal{Y}_3$. According Theorem A.1, in Eq. (A.5), we have

$$m^\alpha(\lambda)(m^{mod}(\lambda))^{-1} = I + O(t^{-N-1}), \quad t \to \infty, \quad \lambda \in \partial D_\varepsilon(\alpha), \quad N \geq 0. \tag{4.14}$$

local model near β: Define $m^{(\beta 0)}(x,t,\lambda)$ for λ near β as

$$m^{(\beta 0)}(x,t,\lambda) = \hat{m}^{(5)}(x,t,\lambda)e^{-i(\frac{i}{2}\ln(\delta(\lambda))^2\hat{a}(\lambda)\hat{b}(\lambda)+h(\lambda))\sigma_3}, \quad \lambda \in D_\varepsilon(\beta)\backslash\Sigma^{(5)}.$$

Function $m^{(\beta 0)}(x,t,\lambda)$ satisfies the following jump condition

$$m_+^{(\beta 0)}(x,t,\lambda) = m_-^{(\beta 0)}(x,t,\lambda)v^{(\beta 0)}(x,t,\lambda), \quad \lambda \in \Sigma^{(5)} \cap D_\varepsilon(\beta), \tag{4.15}$$

where

$$v^{(\beta 0)}=\begin{cases}\begin{pmatrix}1 & -\frac{1}{\hat{a}\hat{\bar{a}}}e^{-2itg}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{Z}_1,\\ \begin{pmatrix}1 & 0\\ \hat{a}\hat{\bar{a}}e^{2itg} & 1\end{pmatrix}, & \lambda\in\mathcal{Z}_2,\\ \begin{pmatrix}0 & e^{-it(g_+-g_-)}\\ -e^{it(g_+-g_-)} & 0\end{pmatrix}, & \lambda\in\mathcal{Z}_3,\\ \begin{pmatrix}1 & 0\\ \frac{1}{\hat{a}\hat{\bar{a}}}e^{2itg} & 1\end{pmatrix}, & \lambda\in\mathcal{Z}_4,\\ \begin{pmatrix}\frac{e^{it(g_+-g_-)}}{a\bar{a}} & 0\\ 0 & \hat{a}\hat{\bar{a}}e^{-it(g_+-g_-)}\end{pmatrix}, & \lambda\in\mathcal{Z}_5.\end{cases}$$

Let $\mathcal{Z}=\cup\mathcal{Z}_j$, $\overline{\mathcal{Z}}$ be the conjugate of $\mathcal{Z}$. For the jump contour $\mathcal{Z}_j$ and $\mathcal{Z}_j=\overline{\mathcal{T}}_{j-1}\cap\overline{\mathcal{T}}_j$ please see Fig. 12. Define $g_\beta(\lambda)$ as

$$g_\beta(\lambda)=\int_\beta^\lambda \mathrm{d}g=\begin{cases}g(\lambda)-g_-(\beta), & \lambda\in\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_5,\\ g(\lambda)-g_+(\beta), & \lambda\in\mathcal{T}_3\cup\mathcal{T}_4.\end{cases}\tag{4.16}$$

Introduce a transformation

$$m^{(\beta 1)}(x,t,\lambda)=m^{(\beta 0)}(x,t,\lambda)A(\lambda),\quad \lambda\in D_\varepsilon(\beta)\backslash\Sigma^{(5)},\tag{4.17}$$

where

$$A(\lambda)=\begin{cases}(\hat{a}\hat{\bar{a}})^{-\frac{\sigma_3}{2}}e^{-itg_-(\beta)\sigma_3}, & \lambda\in\mathcal{T}_1\cup\mathcal{T}_2\cup\mathcal{T}_5,\\ (\hat{a}\hat{\bar{a}})^{\frac{\sigma_3}{2}}e^{-itg_+(\beta)\sigma_3}, & \lambda\in\mathcal{T}_3\cup\mathcal{T}_4.\end{cases}\tag{4.18}$$

We can derive that $m^{(\beta 1)}(x,t,\lambda)$ satisfies a jump condition

$$m_+^{(\beta 1)}(x,t,\lambda)=m_-^{(\beta 1)}(x,t,\lambda)v^{(\beta 1)}(x,t,\lambda),\tag{4.19}$$

where the jump matrix

$$v^{(\beta 1)}=\begin{cases}\begin{pmatrix}1 & -e^{-2itg_\beta}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{Z}_1,\\ \begin{pmatrix}1 & 0\\ e^{2itg_\beta} & 1\end{pmatrix}, & \lambda\in\mathcal{Z}_2\cup\mathcal{Z}_4,\\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & \lambda\in\mathcal{Z}_3,\\ I, & \lambda\in\mathcal{Z}_5.\end{cases}$$

Similar with local near α, we need to transform this model to Airy function of Eq. (A.5). Let $\zeta(\lambda)=(\frac{3it}{2}g_\beta(\lambda))^{2/3}$. Define $m^\beta(x,t,\lambda)$ by

$$m^\beta(x,t,\lambda)=Y_\beta(x,t,\lambda)m^{Ai}(\zeta(\lambda))A^{-1}(\lambda)e^{i(\frac{i}{2}\ln(\hat{a}\hat{b}\delta^2)+tg(\beta)+h(\lambda))\sigma_3},\quad \lambda\in D_\varepsilon(\beta)\backslash\Sigma^{(5)},$$

where

$$Y_\beta(x,t,\lambda)=m^{mod}e^{-i(\frac{i}{2}\ln(\hat{a}\hat{b}\delta^2)+tg(\beta)+h(\lambda))\sigma_3}A(\lambda)(m^{Ai}_{as,N}(\zeta(\lambda)))^{-1}.$$

According to the asymptotic function (A.6) of Theorem A.1 in Eq. (A.5), one can derive

$$m^\beta(\lambda)(m^{mod}(\lambda))^{-1}=I+O(t^{-N-1}),\quad t\to\infty,\quad \lambda\in\partial D_\varepsilon(\beta),\quad N\geq 0.\tag{4.20}$$

local model near μ: Define $m^{(\mu 0)}(x,t,\lambda)$ for λ near μ by

$$m^{(\mu 0)}(x,t,\lambda)=\hat{m}^{(5)}(x,t,\lambda)e^{-ih\sigma_3}B(\lambda),\quad \lambda\in D_\varepsilon(\mu)\backslash\Sigma^{(5)},\tag{4.21}$$

where

$$B(\lambda)=\begin{cases}e^{-itg_-(\mu)\sigma_3}, & \lambda\in\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_6,\\ e^{-itg_+(\mu)\sigma_3}, & \lambda\in\mathcal{R}_3\cup\mathcal{R}_4\cup\mathcal{R}_5,\end{cases}\tag{4.22}$$

and function $g_\mu(\lambda)$ for λ near μ, i.e. $\lambda\in D_\varepsilon(\mu)$, defined by

$$g_\mu(\lambda)=\int_\mu^\lambda \mathrm{d}g=\begin{cases}g(\lambda)-g_-(\mu), & \lambda\in\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_6,\\ g(\lambda)-g_+(\mu), & \lambda\in\mathcal{R}_3\cup\mathcal{R}_4\cup\mathcal{R}_5.\end{cases}\tag{4.23}$$

Function $m^{(\mu 0)}(x,t,\lambda)$ satisfies jump condition

$$m_+^{(\mu 0)}(x,t,\lambda)=m_-^{(\mu 0)}(x,t,\lambda)v^{(\mu 0)}(x,t,\lambda),\tag{4.24}$$

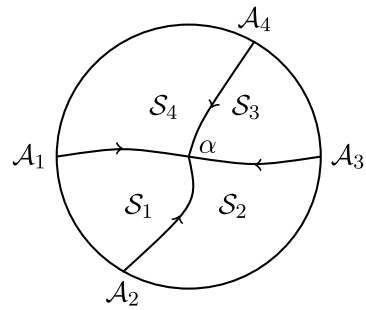

Fig. 11. The jump contour $\mathcal{A}_j$ and $\mathcal{S}_j$.

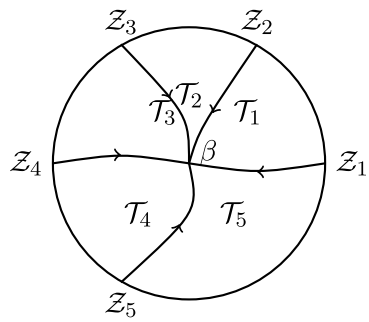

Fig. 12. The jump contour $\mathcal{Z}_j$ and $\mathcal{T}_j$.

where the jump matrix

$$v^{(\mu 0)}=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{\hat{\bar{b}}}{\hat{\bar{a}}}\delta^{-2}e^{2it g_\mu} & 1\end{pmatrix}, & \lambda\in\mathcal{X}_1,\\ \begin{pmatrix}1 & -\frac{\hat{b}}{\hat{\bar{a}}}\delta^{2}e^{-2it g_\mu}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{X}_2,\\ \begin{pmatrix}\frac{1}{a\bar{a}} & 0\\ 0 & a\bar{a}\end{pmatrix}, & \lambda\in\mathcal{X}_3,\\ \begin{pmatrix}1 & 0\\ \frac{\hat{\bar{b}}}{\hat{a}^2\hat{\bar{a}}}\delta^{-2}e^{2it g_\mu} & 1\end{pmatrix}, & \lambda\in\mathcal{X}_4,\\ \begin{pmatrix}1 & -\frac{\hat{b}}{\hat{a}\hat{\bar{a}}^2}\delta^{2}e^{-2it g_\mu}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{X}_5,\\ \begin{pmatrix}\hat{a}\hat{\bar{a}} & 0\\ 0 & \frac{1}{\hat{a}\hat{\bar{a}}}\end{pmatrix}, & \lambda\in\mathcal{X}_6.\end{cases}$$

For the jump contour please see Fig. 13. For eliminating the jump across $\gamma_{(\bar{\beta},\beta)}$, we introduce a function

$$\tilde{\delta}(\lambda)=\exp\left[\frac{i}{2\pi}\int_{\gamma_{(\mu,\beta)}}\frac{\ln\hat{a}(s)\hat{\bar{a}}(s)}{s-\lambda}\mathrm{d}s\right]\exp\left[\frac{-i}{2\pi}\int_{\gamma_{(\bar{\beta},\mu)}}\frac{\ln\hat{a}(s)\hat{\bar{a}}(s)}{s-\lambda}\mathrm{d}s\right],\quad \lambda\in\mathbb{C}\backslash\gamma_{(\bar{\beta},\beta)}.$$

Lemma 4.2. *The function $\tilde{\delta}(\lambda)$ admits the following properties:*

- $\tilde{\delta}(\lambda)$ *and* $\tilde{\delta}^{-1}(\lambda)$ *are bounded and analytic functions for* $\lambda\in D_\varepsilon(\mu)\backslash\gamma_{(\bar{\beta},\beta)}$.
- $\tilde{\delta}(\lambda)$ *admits the symmetry*

$$\tilde{\delta}=(\bar{\tilde{\delta}})^{-1},\quad \lambda\in\mathbb{C}\backslash\gamma_{(\bar{\beta},\beta)}. \tag{4.25}$$

- $\tilde{\delta}(\lambda)$ *admits the following jump condition*

$$\tilde{\delta}_+=\tilde{\delta}_-\begin{cases}\frac{1}{\hat{a}\hat{\bar{a}}}, & \lambda\in\gamma_{(\mu,\beta)},\\ \hat{a}\hat{\bar{a}}, & \lambda\in\gamma_{(\bar{\beta},\mu)}.\end{cases} \tag{4.26}$$

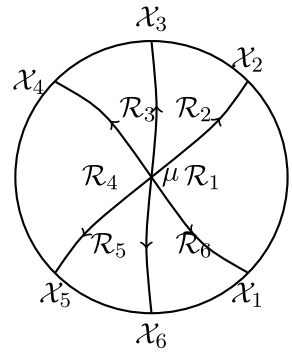

Fig. 13. The jump contour $\mathcal{R}_j$ and $\mathcal{X}_j$, $j = 1, 2, 3, 4, 5, 6$.

► $\tilde{\delta}(\lambda)$ *can be rewritten as*

$$\tilde{\delta}(\lambda) = exp\left[i\nu_2[\ln_\beta(\lambda-\mu) + \ln_{\overline{\beta}}(\lambda-\mu)] + \tilde{\chi}(\lambda)\right], \tag{4.27}$$

where $\nu_2 = \frac{\ln(1+|q|^2)}{2\pi}$ *and*

$$\begin{aligned}\tilde{\chi} =& \frac{1}{2i\pi}L_\beta(\beta,\lambda)\ln(1+\varrho(\beta)\overline{\varrho}(\beta)) + \frac{1}{2i\pi}L_{\overline{\beta}}(\overline{\beta},\lambda)\ln(1+\varrho(\overline{\beta})\overline{\varrho}(\overline{\beta}))\\ &+ \frac{1}{2i\pi}\int_{\gamma_{\mu,\beta}} L_\beta(s,\lambda)\ln(1+\varrho(s)\overline{\varrho}(s)) - \frac{1}{2i\pi}\int_{\gamma_{\overline{\beta},\mu}} L_\beta(s,\lambda)\ln(1+\varrho(s)\overline{\varrho}(s)).\end{aligned}$$

Some symbols denote respectively

$$\begin{aligned}&\ln_\beta(\lambda-\mu) = \ln(\lambda-\mu), \quad \lambda \in D_\varepsilon\backslash\gamma_{(\mu,\beta)},\\ &\ln_{\overline{\beta}}(\lambda-\mu) = \ln(\lambda-\mu), \quad \lambda \in D_\varepsilon\backslash\gamma_{(\overline{\beta},\mu)},\\ &\mathrm{L}_\beta(s,\lambda) = \ln(\lambda-s), \quad s \in \gamma_{(\mu,\beta)}, \quad \lambda \in D_\varepsilon\backslash\gamma_{(\mu,\beta)},\\ &\mathrm{L}_{\overline{\beta}}(s,\lambda) = \ln(\lambda-s), \quad s \in \gamma_{(\overline{\beta},\mu)}, \quad \lambda \in D_\varepsilon\backslash\gamma_{(\overline{\beta},\mu)}.\end{aligned}$$

Remark: Function δ in Lemma 3.1 with a new version

$$\delta(\lambda) = exp\left[-i\nu_2\ln(\lambda-\mu) + \chi(\lambda)\right], \quad \lambda \in \mathbb{C}\backslash(-\infty,\mu], \tag{4.28}$$

where $\nu_2 = \frac{\ln(1+|q|^2)}{2}$ and

$$\begin{aligned}\chi(\lambda) =& \frac{1}{2i\pi}\ln(\lambda+1)\ln\left(\frac{1+|\varrho_+(-1)|^2}{1+|\varrho_-(-1)|^2}\right) -\\ &\frac{1}{2i\pi}\left(\int_{-\infty}^{-1} + \int_{-1}^{\mu}\right)\ln(\lambda-s)\ln(1+|\varrho(s)|^2), \quad \lambda \in \mathbb{C}\backslash(-\infty,\mu].\end{aligned}$$

$\varrho_+(-1)$ and $\varrho_-(-1)$ denote the values of $\varrho(\lambda)$ on the left and right sides of $\gamma_1 \cup \overline{\gamma}_1$. Moreover, functions $\tilde{\delta}$ and δ satisfy the following relations:

Lemma 4.3. *Let* $\delta_2 = \delta\tilde{\delta}$, *thus*

$$\delta_2(\lambda) = p(z(\lambda))\delta_0(\lambda)\delta_1(\lambda), \quad \lambda \in D_\varepsilon(\mu)\backslash((-\infty,\mu] \cup \gamma_{(\overline{\beta},\beta)}), \tag{4.29}$$

where

$$\begin{aligned}&p(z) = exp\left[-i\nu_2[\ln_{-\frac{\pi}{2}} z - \ln z - \ln_0 z]\right], \quad z \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R}_-),\\ &\delta_0(t) = e^{\frac{\pi\nu_2}{2}} t^{-\frac{i\nu_2}{2}} exp^{-i\nu_2\ln\psi_\mu(\mu)} e^{\chi(\mu)+\tilde{\chi}(\mu)}, \quad t > 0,\\ &\delta_1(\lambda) = e^{-i\nu_2\ln\frac{\psi_\mu(\lambda)}{\psi_\mu(\mu)}} e^{\chi(\lambda)-\chi(\mu)+\tilde{\chi}(\lambda)-\tilde{\chi}(\mu)}, \quad \lambda \in D_\varepsilon(\mu).\end{aligned}$$

Define $m^{(\mu 1)}(x,t,\lambda)$ by

$$m^{(\mu 1)}(x,t,\lambda) = m^{(\mu 0)}(x,t,\lambda)\tilde{\delta}(\lambda)^{-\sigma_3}, \quad \lambda \in D_\varepsilon(\mu). \tag{4.30}$$

$m^{(\mu 1)}(x,t,\lambda)$ satisfies a jump condition

$$m_+^{(\mu 1)}(x,t,\lambda) = m_-^{(\mu 1)}(x,t,\lambda)v^{(\mu 1)}(x,t,\lambda), \tag{4.31}$$

the jump matrix is given by

$$v^{(\mu 1)}=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{\hat{\hat{b}}}{\hat{\hat{a}}}\delta_2^{-2}e^{2itg_\mu} & 1\end{pmatrix}, & \lambda\in\mathcal{X}_1,\\ \begin{pmatrix}1 & -\frac{\hat{b}}{\hat{\hat{a}}}\delta_2^{2}e^{-2itg_\mu}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{X}_2,\\ \begin{pmatrix}1 & 0\\ \frac{\hat{\hat{b}}}{\hat{a}^2\hat{\hat{a}}}\delta_2^{-2}e^{2itg_\mu} & 1\end{pmatrix}, & \lambda\in\mathcal{X}_4,\\ \begin{pmatrix}1 & -\frac{\hat{b}}{\hat{a}\hat{\hat{a}}^2}\delta_2^{2}e^{-2itg_\mu}\\ 0 & 1\end{pmatrix}, & \lambda\in\mathcal{X}_5,\\ I, & \lambda\in\mathcal{X}_3\cup\mathcal{X}_6.\end{cases}$$

Let $\mathcal{X}=\mathcal{X}_1\cup\mathcal{X}_2\cup\mathcal{X}_4\cup\mathcal{X}_5$. For relating $m^{(\mu 1)}$ to the solution of the parabolic cylinder functions in Appendix B, we introduce z for $2iz^2=2itg_\mu$. For the reason that $g_\mu(\lambda)$ has a double zero at $\lambda=\mu$, we let

$$z=it^{1/2}(\lambda-\mu)\psi_\mu(\lambda), \tag{4.32}$$

where $\psi_\mu(\lambda)$ is analytic function for $\lambda\in D_\varepsilon(\mu)$.

Define $m^{(\mu 2)}(x,t,z(\lambda))$ by

$$m^{(\mu 2)}(x,t,z(\lambda))=m^{(\mu 1)}(x,t,\lambda)\delta_0(t)^{\sigma_3},\quad \lambda\in D_\varepsilon(\mu)\backslash\Sigma^{(5)}. \tag{4.33}$$

Function $m^{(\mu 2)}(x,t,z(\lambda))$ satisfies the RH problem $m_+^{(\mu 2)}(x,t,\lambda)=m_-^{(\mu 2)}(x,t,\lambda)v^{(\mu 2)}(x,t,\lambda)$, where

$$v^{(\mu 2)}=\begin{cases}\begin{pmatrix}1 & 0\\ \hat{r}\delta_1^{-2}\rho^{-2}(z)e^{2iz^2} & 1\end{pmatrix}, & \arg z=\frac{\pi}{4},\\ \begin{pmatrix}1 & \hat{\hat{r}}\delta_1^{2}\rho^{2}(z)e^{-2iz^2},\\ 0 & 1\end{pmatrix}, & \arg z=\frac{3\pi}{4},\\ \begin{pmatrix}1 & 0\\ -\frac{\hat{r}(1+\hat{r}\hat{\hat{r}})}{(1+|q|^2)^2}\delta_1^{-2}\rho^{-2}(z)e^{2iz^2} & 1\end{pmatrix}, & \arg z=\frac{5\pi}{4},\\ \begin{pmatrix}1 & -\frac{\hat{\hat{r}}(1+\hat{r}\hat{\hat{r}})}{(1+|q|^2)^2}\delta_1^{2}\rho^{2}(z)e^{-2iz^2}\\ 0 & 1\end{pmatrix}, & \arg z=\frac{7\pi}{4}.\end{cases}$$

We define a matrix function $m^\mu(x,t,\lambda)$ for $\hat{m}^{(5)}$ near μ by

$$m^\mu(x,t,\lambda)=Y_\mu(x,t,\lambda)m^{pc}(q,z(\lambda))\delta_0(t)^{-\sigma_3}\tilde{\delta}^{\sigma_3}(\lambda)B^{-1}(\lambda)e^{ih(\lambda)\sigma_3}, \tag{4.34}$$

where $m^{pc}(q,z(\lambda))$ is the solution of the RH problem (B.1) in Appendix B. Function $Y_\mu(x,t,\lambda)$ is analytic for $\lambda\in D_\varepsilon(\mu)$ and defined by

$$Y_\mu(x,t,\lambda)=m^{mod}(x,t,\lambda)e^{-ih(\lambda)\sigma_3}B(\lambda)\tilde{\delta}^{-\sigma_3}\delta_0^{\sigma_3}(t). \tag{4.35}$$

Function $m^\mu(x,t,\lambda)$ admits the following lemma:

Lemma 4.4. *Function m^μ satisfies the jump condition*

$$m_+^\mu(x,t,\lambda)=m_-^\mu(x,t,\lambda)v^\mu(x,t,\lambda), \tag{4.36}$$

where the jump matrix

$$v^\mu=\hat{J}^{(5)},\quad \lambda\in\gamma_{(\bar{\beta},\beta)}\cap D_\varepsilon(\mu). \tag{4.37}$$

For $t\to\infty$,

$$\begin{aligned}&\|\hat{v}^{(5)}-v^\mu\|_{L^1(\chi)}=\mathcal{O}(t^{-1}\ln(t)),\\ &\|\hat{v}^{(5)}-v^\mu\|_{L^2(\chi)}=\mathcal{O}(t^{-3/4}\ln(t)),\\ &\|\hat{v}^{(5)}-v^\mu\|_{L^\infty(\chi)}=\mathcal{O}(t^{-1/2}\ln(t)),\\ &\|m^{mod}(m^\mu)^{-1}-I\|_{L^\infty(\partial D_\varepsilon(\mu))}=\mathcal{O}(t^{-1/2}),\end{aligned}$$

$$\frac{1}{2i\pi}\int_{\partial D_\varepsilon(\mu)}(m^{mod}(m^\mu)^{-1}-I)\mathrm{d}\lambda=\frac{Y_\mu(x,t,\mu)m_1^{pc}Y_\mu^{-1}(x,t,\mu)}{\sqrt{t}\psi_\mu(\mu)}+\mathcal{O}(t^{-1}), \tag{4.38}$$

where

$$m_1^{pc}=\begin{pmatrix}0 & -e^{-\pi\nu_2}\beta^{pc}(q)\\ e^{-\pi\nu_2}\overline{\beta^{pc}(q)} & 0\end{pmatrix}. \tag{4.39}$$

4.3. The asymptotic solution

Function m^{err} defined by

$$m^{err} = \hat{m}^{(5)}(m^{app})^{-1}, \tag{4.40}$$

satisfies RH problem

$$m_+^{err}(x,t,\lambda) = m_-^{err}(x,t,\lambda)v^{err}(x,t,\lambda), \quad k \in \Sigma^{err}, \tag{4.41}$$

where $\Sigma^{err} = (\Sigma^{(5)} \backslash (\Sigma^{mod} \cup \overline{\mathcal{D}})) \cup \partial D \cup \mathcal{X}$, $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_4 \cup \mathcal{X}_5$ and

$$v^{err} = \begin{cases} m^{mod}\hat{J}^{(5)}(m^{mod})^{-1}, & \lambda \in \Sigma^{err} \backslash \overline{\mathcal{D}}, \\ m^{mod}(m^{\alpha})^{-1}, & \lambda \in \partial D_{\varepsilon}(D_{\alpha}), \\ m^{mod}(m^{\beta})^{-1}, & \lambda \in \partial D_{\varepsilon}(D_{\beta}), \\ m^{mod}(m^{\overline{\alpha}})^{-1}, & \lambda \in \partial D_{\varepsilon}(D_{\overline{\alpha}}), \\ m^{mod}(m^{\overline{\beta}})^{-1}, & \lambda \in \partial D_{\varepsilon}(D_{\overline{\beta}}), \\ m^{mod}(m^{\mu})^{-1}, & \lambda \in \partial D_{\varepsilon}(D_{\mu}), \\ m_-^{\mu}\hat{J}^{(5)}(m_+^{\mu})^{-1}, & \lambda \in \mathcal{X}. \end{cases}$$

Let $\hat{\omega} = v^{err} - I$, and $\hat{\omega}$ admits the following lemma:

Lemma 4.5 ([47]). *For $t \to \infty$, $\hat{\omega}$ satisfies*

$$\|\hat{\omega}\|_{(L^1 \cap L^2 \cap L^{\infty})(\Sigma^{err} \backslash \overline{\mathcal{D}})} = \mathcal{O}(e^{-ct}), \quad c > 0. \tag{4.42}$$

Eqs. (4.14) and (4.20) imply that

$$\|\hat{\omega}\|_{L^{\infty}(\partial D_{\varepsilon}(\alpha) \cup \partial D_{\varepsilon}(\beta) \cup \partial D_{\varepsilon}(\overline{\alpha}) \cup \partial D_{\varepsilon}(\overline{\beta}))} = \mathcal{O}(t^{-N}), \quad t \to \infty, \quad N \geq 1. \tag{4.43}$$

For $t \to \infty$, Lemma 4.4 yields that

$$\begin{aligned} &\|\hat{\omega}\|_{L^1(\mathcal{X})} = \mathcal{O}(t^{-1}\ln t), \quad \|\hat{\omega}\|_{L^2(\mathcal{X})} = \mathcal{O}(t^{-3/4}\ln t), \\ &\|\hat{\omega}\|_{L^{\infty}(\mathcal{X})} = \mathcal{O}(t^{-1/2}\ln t), \quad \|\hat{\omega}\|_{L^{\infty}(\partial D_{\varepsilon}(\mu))} = \mathcal{O}(t^{-1/2}). \end{aligned} \tag{4.44}$$

Thus for $t \to \infty$, we have

$$\|\hat{\omega}\|_{(L^1 \cap L^2)(\Sigma^{err})} = \mathcal{O}(t^{-1/2}), \quad \|\hat{\omega}\|_{L^{\infty}(\Sigma^{err})} = \mathcal{O}(t^{-1/2}\ln t). \tag{4.45}$$

Let $\hat{\mathcal{C}}$ denote the Cauchy operator associated with Σ^{err}, for function $f(\lambda)$

$$(\hat{\mathcal{C}}f)(\lambda) = \frac{1}{2i\pi}\int_{\Sigma^{err}} \frac{f(s)}{s-\lambda}\mathrm{d}s, \quad \lambda \in \mathbb{C} \backslash \Sigma^{err}.$$

Define $\hat{\mathcal{C}}_{\hat{\omega}}$: $L^2(\Sigma^{err}) \to L^2(\Sigma^{err})$ by $\hat{\mathcal{C}}_{\hat{\omega}}f = \hat{\mathcal{C}}_-(f\hat{\omega})$ [47,48]. We have the lemma:

Lemma 4.6. *In the Banach space $\mathcal{B}(L^2(\Sigma^{err}))$, we have*

$$\|\hat{\mathcal{C}}_{\hat{\omega}}\|_{\mathcal{B}(L^2(\Sigma^{err}))} \leq C\|\hat{\omega}\|_{L^{\infty}(\Sigma^{err})} = \mathcal{O}(t^{-1/2}\ln t), \quad t \to \infty, \tag{4.46}$$

and $I - \hat{\mathcal{C}}_{\hat{\omega}} \in \mathcal{B}(L^2(\Sigma^{err}))$ is invertible for large enough t.

Define a 2×2-matrix function $\hat{\mu}(x,t,\lambda)$ for a large t by

$$\hat{\mu} = I + (I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1}\hat{\mathcal{C}}_{\hat{\omega}}I \in I + L^2(\Sigma^{err}), \tag{4.47}$$

we consider the Neumann series representation of $(I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1}$ as $(I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1} = \sum_{j=0}^{\infty}\hat{\mathcal{C}}_{\hat{\omega}}^j$. we have

$$\|(I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1}\|_{\mathcal{B}(L^2(\Sigma^{err}))} \leq \sum_{j=0}^{\infty}\|\hat{\mathcal{C}}_{\hat{\omega}}\|^j_{\mathcal{B}(L^2(\Sigma^{err}))} = \left(1 - \|\hat{\mathcal{C}}_{\hat{\omega}}\|_{\mathcal{B}(L^2(\Sigma^{err}))}\right)^{-1}, \tag{4.48}$$

as $\|\hat{\mathcal{C}}_{\hat{\omega}}\|_{\mathcal{B}(L^2(\Sigma^{err}))} < I$. Using Lemma 4.6, we have

$$\begin{aligned} \|\hat{\mu} - I\|_{L^2(\Sigma^{err})} &= \|(I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1}\hat{\mathcal{C}}_{\hat{\omega}}I\|_{L^2(\Sigma^{err})} \\ &\leq \|(I - \hat{\mathcal{C}}_{\hat{\omega}})^{-1}\|_{\mathcal{B}(L^2(\Sigma^{err}))}\|\hat{\mathcal{C}}_{\hat{\omega}}I\|_{L^2(\Sigma^{err})} \\ &\leq \frac{C\|\hat{\omega}\|_{(L^2(\Sigma^{err}))}}{I - I - \|\hat{\mathcal{C}}_{\hat{\omega}}\|_{\mathcal{B}(L^2(\Sigma^{err}))}} \\ &\leq C\|\hat{\omega}\|_{(L^2(\Sigma^{err}))}. \end{aligned} \tag{4.49}$$

Using Eq. (4.45), we have the estimate shown as follows:

Lemma 4.7. *Function $\hat{\mu}$ satisfies*

$$\|\hat{\mu} - I\|_{L^2(\Sigma^{err})} = \mathcal{O}(t^{-1/2}), \quad t \to \infty. \tag{4.50}$$

The definition 4.7 implies that $\hat{\mu}-I=\hat{\mathcal{C}}_{\hat{\omega}}\hat{\mu}$, thus $m^{err}=I+\hat{\mathcal{C}}(\hat{\mu}\hat{\omega})$ satisfies a L^2-RH problem. And this L^2-RH problem with a unique solution is shown as follows:

Lemma 4.8. *There exist the unique solution of the RH problem (4.41) given by*

$$m^{err}(x,t,\lambda)=I+\frac{1}{2i\pi}\int_{\Sigma^{err}}\frac{\hat{\mu}(s)\hat{\omega}(s)}{s-\lambda}\mathrm{d}s. \tag{4.51}$$

From Lemma 4.8, we have

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda(m^{err}(x,t,\lambda)-I)&=\lim_{\lambda\to\infty}\frac{1}{2i\pi}\int_{\Sigma^{err}}\frac{\lambda\hat{\mu}(s)\hat{\omega}(s)}{s-\lambda}\mathrm{d}s\\&=-\frac{1}{2i\pi}\int_{\Sigma^{err}}\hat{\mu}(s)\hat{\omega}(s)\mathrm{d}s.\end{aligned} \tag{4.52}$$

This implies that

$$\begin{aligned}&\lim_{\lambda\to\infty}\lambda(m^{err}(x,t,\lambda)-I)\\&=-\frac{1}{2i\pi}\int_{\partial D_\varepsilon(\mu)}\hat{\omega}(x,t,\lambda)\mathrm{d}\lambda-\frac{1}{2i\pi}\int_{\partial D_\varepsilon(\mu)}(\hat{\mu}(x,t,\lambda)-I)\hat{\omega}(x,t,\lambda)\mathrm{d}\lambda\\&=-\frac{1}{2i\pi}\int_{\partial D_\varepsilon(\mu)}(m^{mod}(m^{\mu})^{-1}-I)\mathrm{d}\lambda+\mathcal{O}(\|\hat{\mu}-I\|_{L^2(\partial D_\varepsilon(\mu))}\|\hat{\omega}\|_{L^2(\partial D_\varepsilon(\mu))})\\&=-\frac{Y_\mu(x,t,\mu)m_1^{pc}Y_\mu^{-1}(x,t,\mu)}{\sqrt{t}\psi_\mu(\mu)}+\mathcal{O}(t^{-1}),\quad t\to\infty.\end{aligned}$$

Eq. (4.44) and Lemma 4.7 imply that the contribution of $\mathcal{X}$ to the right hand side of Eq. (4.52) is

$$\mathcal{O}(\|\hat{\omega}\|_{L^1(\mathcal{X})})+\mathcal{O}(\|\hat{\mu}-I\|_{L^2(\mathcal{X})}\|\hat{\omega}\|_{L^2(\mathcal{X})})=\mathcal{O}(t^{-1}\ln t),\quad t\to\infty. \tag{4.53}$$

Thus, we have the following limit

$$\lim_{\lambda\to\infty}\lambda(m^{err}(x,t,\lambda)-I)=-\frac{Y_\mu(x,t,\mu)m_1^{pc}Y_\mu^{-1}(x,t,\mu)}{\sqrt{t}\psi_\mu(\mu)}+\mathcal{O}(t^{-1}\ln t),\quad t\to\infty. \tag{4.54}$$

Combining with the five transformations in Section 3, we have

$$\begin{aligned}&\lim_{\lambda\to\infty}\lambda(\hat{m}(x,t,\lambda)-I)\\&=e^{itg^{(0)}\sigma_3}e^{ih(\infty)\sigma_3}\lim_{\lambda\to\infty}\lambda(m^{mod}-I+(m^{err}-I)m^{mod})e^{-ih(\infty)\sigma_3}e^{-itg^{(0)}\sigma_3}\\&=e^{i(tg^{(0)}+h(\infty))\sigma_3}\left(\lim_{\lambda\to\infty}\lambda(m^{mod}-I)+\lim_{\lambda\to\infty}\lambda(m^{err}-I)\right)e^{-i(tg^{(0)}+h(\infty))\sigma_3}.\end{aligned}$$

Thus, we have the solution of the DNLS equation (1.1) given by

$$\begin{aligned}q(x,t)&=2i\lim_{\lambda\to\infty}(\lambda\hat{m}(x,t,\lambda))_{12}\\&=2ie^{2i(tg^{(0)}+h(\infty))}\left(\lim_{\lambda\to\infty}\lambda m_{12}^{mod}(x,t,\lambda)+\lim_{\lambda\to\infty}\lambda m_{12}^{err}(x,t,\lambda)\right).\end{aligned} \tag{4.55}$$

Insert Lemma 4.1 and Eq. (4.54) into Eq. (4.55), Theorem 1.1, can be derived.

CRediT authorship contribution statement

Lili Wen: Conceptualization, Software, Data creation, Writing – original draft, Visualization, Investigation, Validation, Writing – review & editing. **Yong Chen:** Methodology, Supervision. **Jian Xu:** Methodology, Supervision.

Declaration of competing interest

The authors of this paper do not have any possible conflicts of interest.

Data availability

The data that supports the findings of this study are available within the article.

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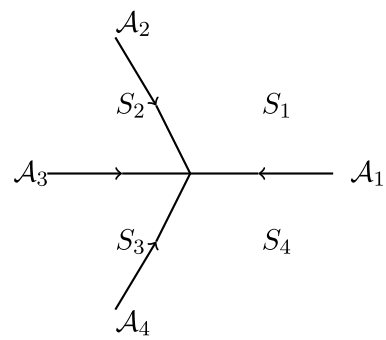

Fig. A.14. The contour of the Airy RH problem.

Appendix A. Airy function

For $\zeta \in \mathbb{C}\backslash Y$, we define the function $m^{Ai}(\zeta)$ as

$$m^{Ai}(\zeta) = \Psi(\zeta)\cdot\begin{cases} e^{\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3}, & \zeta \in S_1 \cup S_4, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3}, & \zeta \in S_2, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} e^{\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3}, & \zeta \in S_3, \end{cases} \tag{A.1}$$

where

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} Ai(\zeta) & Ai(\omega^2\zeta) \\ Ai'(\zeta) & \omega^2 Ai'(\omega^2\zeta) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & \zeta \in \mathbb{C}^+, \\ \begin{pmatrix} Ai(\zeta) & -\omega^2 Ai(\omega\zeta) \\ Ai'(\zeta) & -Ai'(\omega\zeta) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & \zeta \in \mathbb{C}^-, \end{cases} \tag{A.2}$$

and $\mathcal{A} = \cup\mathcal{A}_j \subset \mathbb{C}$, $j = 1, 2, 3, 4$, and

$$\mathcal{A}_1 = \{y|0 \le y \le \infty\}, \quad \mathcal{A}_2 = \{ye^{\frac{2i\pi}{3}}|0 \le y \le \infty\},$$
$$\mathcal{A}_3 = \{-y|0 \le y \le \infty\}, \quad \mathcal{A}_4 = \{ye^{\frac{-2i\pi}{3}}|0 \le y \le \infty\},$$

see Fig. A.14.

Define the asymptotic approximation $m^{Ai}_{as,N}(\zeta)$ as

$$m^{Ai}_{as,N}(\zeta) = \frac{e^{\frac{i\pi}{12}}}{2\sqrt{\pi}}\zeta^{-\frac{\sigma_3}{4}}\sum_{k=0}^{N}\left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k}\begin{pmatrix} (-1)^k u_k & u_k \\ -(-1)^k v_k & v_k \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}, \quad \zeta \in \mathbb{C}\backslash\mathcal{A}, \tag{A.3}$$

where N is the positive integer number, u_k and v_k are the real constants

$$u_0 = v_0 = 1, \quad u_k = \frac{(2k+1)(2k+3)\cdots(6k-1)}{(216)^k k!}, \quad v_k = -\frac{6k+1}{6k-1}u_k, \quad k = 1, 2, 3. \tag{A.4}$$

Theorem A.1. *The function m^{Ai} satisfies the following properties:*

► *The function m^{Ai} is analytic for $\zeta \in \mathbb{C}\backslash\mathcal{A}$ and admits the RH problem*

$$m^{Ai}_{+}(\zeta) = m^{Ai}_{-}(\zeta)v^{Ai}(\zeta), \quad \zeta \in \mathcal{A}\backslash\{0\}, \tag{A.5}$$

where

$$v^{Ai}(\zeta) = \begin{cases} \begin{pmatrix} 1 & -e^{-\frac{4}{3}\zeta^{\frac{3}{2}}} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathcal{A}_1, \\ \begin{pmatrix} 1 & 0 \\ e^{\frac{4}{3}\zeta^{\frac{3}{2}}} & 1 \end{pmatrix}, & \zeta \in \mathcal{A}_2 \cup \mathcal{A}_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \mathcal{A}_3. \end{cases}$$

► *The function $m^{Ai}_{as,N}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash(-\infty, 0]$ and admits the jump condition*

$$m^{Ai}_{as,N+}(\zeta) = m^{Ai}_{as,N-}(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta < 0. \tag{A.6}$$

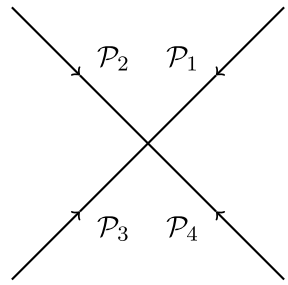

Fig. B.15. The contour of the Parabolic Cylinder RH problem.

► *The function $m^{Ai}_{as,N}(\zeta)$ approximates m^{Airy} as $\zeta \to \infty$*

$$(m^{Ai}_{as,N}(\zeta))^{-1} m^{Ai}(\zeta) = I + \mathcal{O}(\zeta^{-\frac{3(N+1)}{2}}), \quad \zeta \to \infty, \tag{A.7}$$

where the error term is uniform with respect to $\arg\zeta \in [1, 2\pi]$.

Appendix B. Parabolic cylinder function

Define the RH problem

$$m^{pc}_{+}(q,z) = m^{pc}_{-}(q,z) v^{pc}(q,z), \quad z \in \mathcal{P}, \tag{B.1}$$

where $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ and

$$v^{pc}(q,z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ q\rho(q,z)^{-2}e^{2iz^2} & 1 \end{pmatrix}, & z \in \mathcal{P}_1, \\ \begin{pmatrix} 1 & \bar{q}\rho(q,z)^2 e^{-2iz^2} \\ 0 & 1 \end{pmatrix}, & z \in \mathcal{P}_2, \\ \begin{pmatrix} 1 & 0 \\ -\frac{q}{1+|q|^2}\rho(q,z)^{-2}e^{2iz^2} & 1 \end{pmatrix}, & z \in \mathcal{P}_3, \\ \begin{pmatrix} 1 & -\frac{\bar{q}}{1+|q|^2}\rho(q,z)^2 e^{-2iz^2} \\ 0 & 1 \end{pmatrix}, & z \in \mathcal{P}_4, \end{cases}$$

with $\nu : \mathbb{C} \to [0, \infty)$ and $\nu(q) = \frac{1}{2\pi}\ln(1+|q|^2)$. Let $\rho(q,z) = e^{i\nu(q)\ln_{-\pi/2} z}$, i.e. $\rho(q,z) = z^{i\nu(q)}$. The rays $\mathcal{P} = \cup \mathcal{P}_j \subset \mathbb{C}$, $j = 1, 2, 3, 4$,

$$\mathcal{P}_1 = \{se^{\frac{i\pi}{4}} | 0 \le s \le \infty\}, \quad \mathcal{P}_2 = \{se^{\frac{3i\pi}{4}} | 0 \le s \le \infty\},$$
$$\mathcal{P}_3 = \{se^{\frac{-3i\pi}{4}} | 0 \le s \le \infty\}, \quad \mathcal{P}_4 = \{se^{\frac{-3i\pi}{4}} | 0 \le s \le \infty\},$$

see Fig. B.15.

Theorem B.1. *The RH problem with a unique solution $m^{pc}(q,z)$*

$$m^{pc}(q,z) = I + \frac{i}{z}\begin{pmatrix} 0 & -e^{-\pi\nu}\beta^{\mathcal{P}}(q) \\ e^{\pi\nu}\overline{\beta^{\mathcal{P}}(q)} & 0 \end{pmatrix} + \mathcal{O}(\frac{1}{z^2}), \quad z \to \infty, \quad q \in \mathbb{C}, \tag{B.2}$$

where the error term is uniform with respect to $\arg z \in [0, 2\pi]$ *and q is compact subset of $\mathbb{C}$, and $\beta^{\mathcal{P}}(q)$ is given by*

$$\beta^{\mathcal{P}}(q) = \frac{\sqrt{\nu(q)}}{2} e^{i(-\frac{3\pi}{4} - 2\nu(q)\ln 2 - \arg q + \arg\Gamma(i\nu(q)))}, \quad q \in \mathbb{C}. \tag{B.3}$$

And for compact subset $K \subset \mathbb{C}$,

$$\sup_{q\in K}\sup_{z\in\mathbb{C}\setminus\mathcal{P}} |m^{pc}(q,z)| < \infty. \tag{B.4}$$

References

[1] W. Mio, T. Ogino, K. Minami, S. Takeda, Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas, J. Phys. Soc. Japan 41 (1976) 265–271.
[2] E. Mjolhus, On the modulational instability of hydromagnetic waves parallel to the magnetic field, J. Plasma Phys. 16 (1976) 321–334.
[3] Y. Kodama, Optical solitons in a monomode fiber, J. Stat. Phys. 39 (1985) 597–614.
[4] M. Wadati, K. Sogo, Gauge transformation in soliton theory, J. Phys. Soc. Japan 52 (1983) 394–398.
[5] P.A. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems: Asymptotics for the mKdV equation, Ann. of Math. 137 (2) (1993) 295–368.
[6] S.V. Manakov, Nonlinear Fraunhofer diffraction, Zh. Eksp. Teor. Fiz. 65 (1973) 1392–1398, (Russian);Sov.Phys.JETP. 38(1974)693-696.

[7] A.R. Its, Asymptotic behavior of the solution to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations, Dokl. Akad. Nauk SSSR 261 (1981) 14–18, (Russian); Soviet. Math. Dokl. 24(1982)452-456(English).
[8] P.A. Deift, A.R. Its, X. Zhou, Long-time asymptotics for integrable nonlinear wave equations, in: Important Developments in Soliton Theory, in: Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993, pp. 181–204.
[9] P.A. Deift, S. Venakides, X. Zhou, The collisionless shock region for the long-time behavior of solutions of the KdV equation, Comm. Pure Appl. Math. 47 (1994) 199–206.
[10] P.A. Deift, S. Venakides, X. Zhou, New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems, Int. Math. Res. Not. IMRN 6 (1997) 286–299.
[11] A.V. Gurevich, L.P. Pitaevskii, Nonstationary structure of a collisionless shock-wave, Zhurnal Eksperimentalnoi I Teoreticheskoi Fiziki 65 (2) (1973) 590–604.
[12] E.Y. Khruslov, Asymptotic behavior of the solution of the Cauchy problem for the Korteweg–de Vries equation with step like initial data, Sb. Math. 99 (1976) 261–281.
[13] V.P. Kotlyarov, E.Y. Khruslov, Solitons of the nonlinear Schrödinger equation generated by the continuous spectrum, Teor. Math. phys. 68 (1986) 751–761.
[14] S. Venakides, Long time asymptotics of the Korteweg–de Vries equation, Trans. Amer. Math. Soc. 293 (1986) 411–419.
[15] A. Boutet de Monvel, V.P. Kotlyarov, D. Shepelsky, Focusing NLS equation: long-time dynamics of step-like initial data, Int. Math. Res. Not. 7 (2011) 1613–1653.
[16] R. Buckingham, S. Venakides, Long-time asymptotics of the nonlinear Schrödinger equation shock problem, Commun. Pur. Appl. Math. 60 (9) (2007) 1349–1414.
[17] A. Boutet de Monvel, A.R. Its, V.P. Kotlyarov, Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line, Comm. Math. Phys. 290 (2) (2009) 479–522.
[18] A. Boutet de Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: the genus 3 sector, Comm. Math. Phys. 390 (3) (2022) 1081–1148.
[19] A. Boutet de Monvel, J. Lenells, D. Shepelsky, The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics, Comm. Math. Phys. 383 (2) (2021) 893–952.
[20] G. Biondini, D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions at infinity and asymptotic stage of modulational instability, Commun. Pur. Appl. Math. 70 (12) (2017) 2300–2365.
[21] G. Biondini, S.T. Li, D. Mantzavinos, Long-time asymptotics for the focusing nonlinear Schrödinger equation with nonzero boundary conditions in the presence of a discrete spectrum, Comm. Math. Phys. 382 (3) (2021) 1495–1577.
[22] G. Biondini, G. Kovačič, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys. 55 (2014) 031506.
[23] A. Minakov, Long-time behavior of the solution to the mKdV equation with step-like initial data, J. Phys. A 44 (2011) 085206.
[24] T. Grava, A. Minakov, On the long-time asymptotic behavior of the modified Korteweg–de Vries equation with step-like ininial data, SIAM J. Math. Anal. 52 (2020) 5892–5993.
[25] Z.C. Zhang, E.G. Fan, Inverse scattering transform and multiple high-order pole solutions for the Gerdjikov-Ivanov equation under the zero/nonzero background, Z. Angew. Math. Phys. 72 (2021) 153.
[26] M.B. Erdoğan, T.B. Gürel, N. Tzirakis, The derivative nonlinear Schrödinger equation on the half line, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 35 (2018) 1947–1973.
[27] S. Fromm, Admissible boundary values for the Gerdjikov-Ivanov equation with asymptotically time-periodic boundary data, Symmetry Integr. Geom. 16 (2020) 079.
[28] E.G. Fan, A family of completely integrable multi-Hamiltonian systems explicitly related to some celebrated equations, J. Math. Phys. 42 (2001) 4327–4344.
[29] H. Yilmaz, Exact solutions of the Gerdjikov-Ivanov equation using Darboux transformations, J. Nonlinear Math. Phys. 22 (2015) 32–46.
[30] S.S. Zhang, T. Xu, M. Li, X.F. Zhang, High-order algebraic solitons of Gerdjikov-Ivanov equation: Asymptotic analysis and emergence of rogue waves, Physica D 432 (2022) 133128.
[31] S.W. Xu, J.S. He, The rogue wave and breather solution of the Gerdjikov-Ivanov equation, J. Math. Phys. 53 (2012) 063507.
[32] Y. Hou, E.G. Fan, P. Zhao, Algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy, J. Math. Phys. 54 (2013) 073505.
[33] P. Zhao, E.G. Fan, Finite gap integration of the derivative nonlinear Schrödinger equation: a Riemann-Hilbert method, Physica D 402 (2020) 132213.
[34] S. Kakei, T. Kikuchi, Affine Lie group approach to a derivative nonlinear Schrödinger equation and its similarity reduction, Int. Math. Res. Not. 78 (2004) 4181–4209.
[35] S. Kakei, T. Kikuchi, Solutions of a derivative nonlinear Schrödinger hierarchy and its similarity reduction, Glasg. Math. J. 47A (2005) 99–107.
[36] J.B. Chen, Y.P. Zhen, The complex Hamiltonian system in the Gerdjikov-Ivanov equation and its applications, Anal. Math. Phys. 12 (2022) 100.
[37] J.Q. Liu, P.A. Perry, C. Sulem, Global existence for the derivative nonlinear Schrödinger equation by the method of inverse scattering, Commun. Part. Diff. Eq. 41 (11) (2016) 1692–1760.
[38] J. Xu, E.G. Fan, Y. Chen, Long-time asymptotic for the derivative nonlinear Schrödinger equation with step-like initila value, Math. Phys. Anal. Geom. 16 (2013) 253–288.
[39] B.L. Guo, N. Liu, The Gerdjikov-Ivanov-type derivative nonlinear Schrödinger equation: Long-time dynamics of nonzero boundary conditions, Math. Methods Appl. Sci. 42 (2019) 4839–4861.
[40] S.F. Tian, T.T. Zhang, Long-time Asymptotic Behavior for the Gerdjikov-Ivanov type of derivative Nonlinear Schrödinger equation with time-periodic boundary condition, P. Am. Math. Soc. 146 (4) (2018) 1713–1729.
[41] J.Q. Liu, P.A. Perry, C. Sulem, Long-time behavior of solutions to the derivative nonlinear Schrödinger equation for soliton-free initial data, Ann. I. H. Poincaré AN 35 (2018) 217–265.
[42] P. Deift, S. Venakides, X. Zhou, The collisionless shock region for the long-time behavior of solutions of the KdV equation, Commun. Pur. Appl. Math. 47 (2) (1994) 199–206.
[43] P.A. Deift, Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach, in: Courant Lecture Notes in Mathematics, vol. 3, New York University, Courant Institute of Mathematical Sciences, American Mathematical Society, New York, Providence, RI, 1999.
[44] P.A. Deift, X. Zhou, A priori L^p-estimates for solutions of Riemann-Hilbert problems, Int. Math. Res. Not. IMRN 40 (2002) 2121–2154.
[45] A.S. Fokas, A.R. Its, A.A. Kapaev, V.Y. Novokshenov, Painlevé transcendents: the Riemann-Hilbert approach, in: Mathematical Surveys and Monographs, vol. 128, American Mathematical Society, Providence, RI, 2006.
[46] X. Zhou, The Riemann-Hilbert problem and inverse scattering, SIAM J. Math. Anal. 20 (4) (1989) 966–986.
[47] J. Lenells, The nonlinear steepest descent method for Riemann-Hilbert problems of low regularity, Indiana Math. J. 66 (4) (2017) 1287–1332.
[48] J. Lenells, Matrix Riemann-Hilbert problems with jumps across Carleson contours, Monatsh. Math. 186 (1) (2018) 111–152.
[49] A.V. Kitaev, A.H. Vartanian, Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: solitonless sector, Inverse Problems 13 (1997) 1311–1339.
[50] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (1978) 789–801.
[51] L.K. Arruda, J. Lenells, Long-time asymptotics for the derivative nonlinear Schrödinger equation on the half-line, Nonlinearity 30 (2017) 4141–4172.
[52] H.M. Farkas, I. Kra, Riemann surfaces, in: Graduate Texts in Mathematics, Vol. 71, 2nd ed., Springer-Verlag, New York, 1992.
[53] J.D. Fay, Theta functions on Riemann surfaces, in: Lecture Notes in Mathematics, vol. 352, Springer-Verlag, Berlin, New York, 1973.