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The complete convergence theorem holds for contact processes in a random environment on $\mathbb{Z}^d \times \mathbb{Z}^+$

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Abstract

In this article, we consider the basic contact process in a static random environment on the half space $\mathbb{Z}^d \times \mathbb{Z}^+$ where the recovery rates are constants and the infection rates are independent and identically distributed random variables. We show that, for almost every environment, the complete convergence theorem holds. This is a generalization of the known result for the classical contact process in the half space case.

MSC: 60K35

Keywords: Contact process; Random environment; Half space; Graphical representation; Block condition; Dynamic renormalization; Complete convergence theorem

1. Introduction

The aim of this paper is to obtain the complete convergence theorem for the contact process in a random environment on the half space $(\mathbb{H}, \mathbb{E})$. The vertex set is $\mathbb{H} = \mathbb{Z}^d \times \mathbb{Z}^+$ $(d \geq 1)$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ denotes the set of integers and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers. And the edge set is $\mathbb{E} = \{(x, y) : x, y \in \mathbb{H}, \|x - y\| = 1\}$, where $\| \cdot \|$ denotes the Euclidean norm. Here, we treat the graph as unoriented; that is, (x, y) and (y, x) denote the same edge for all $x, y \in \mathbb{H}$ satisfying $\|x - y\| = 1$. The environment is

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given by $\lambda = (\lambda_e)_{e\in\mathbb{E}}$, a collection of nonnegative random variables which are indexed by the edges in $\mathbb{E}$. The random variable λ_e gives the infection rate on edge e. We let the law of $(\lambda_e)_{e\in\mathbb{E}}$ be independent and identically distributed with law μ, which puts mass 1 on $[0, +\infty)$. To describe the environment more formally, we consider the following probability space. We take $\Omega_1 = [0, +\infty)^{\mathbb{E}}$ as the sample space, whose elements are represented by $\omega = (\omega(e) : e \in \mathbb{E})$. The value $\omega(e)$ corresponds to the infection rate on edge e;, that is, $\lambda_e(\omega) = \omega(e)$ for every $e \in \mathbb{E}$. We take $\mathscr{F}_1$ to be the σ-field of subsets of Ω_1 generated by the finite-dimensional cylinders. Finally, we take product measure on $(\Omega_1, \mathscr{F}_1)$; this is the measure $\mathbf{P}^\mu = \prod_{e\in\mathbb{E}} \mu_e$, where μ_e is a measure on $[0, +\infty)$ satisfying $\mu_e(\omega(e) \in \cdot) = \mu(\cdot)$ for every $e \in \mathbb{E}$. The probability space $(\Omega_1, \mathscr{F}_1, \mathbf{P}^\mu)$ describes the environment.

Next, we fix the environment $\lambda = (\lambda_e)_{e\in\mathbb{E}}$ and consider the basic contact process under this environment. The state space of the contact process $\xi = \xi(\lambda)$ is $\{A : A \subseteq \mathbb{H}\}$, and the transition rates are as follows:

$$\begin{cases} \xi_t \to \xi_t \setminus \{x\} & \text{for } x \in \xi_t \text{ at rate } 1, \\ \xi_t \to \xi_t \cup \{x\} & \text{for } x \notin \xi_t \text{ at rate } \displaystyle\sum_{y:\|y-x\|=1} \lambda_{(y,x)} \mathbf{1}_{\{y\in\xi_t\}}. \end{cases}$$

Readers can refer to the standard references Liggett [9] and Durrett [6] for how these rates rigorously determine a Markov process $\xi(\lambda)$ on $(\Omega_2, \mathscr{F}_2, \mathbf{P}_\lambda)$ and for much on the contact process as well as other interacting particle systems. Denote by $\xi^A(\lambda)$ the process with initial state A. If λ is random, then the transition rates are random variables, and therefore $\mathbf{P}_\lambda$ becomes a random measure. We say that ξ^A survives if $\xi_t^A \neq \emptyset$ for all $t \geq 0$, while ξ^A dies out if there exists $t > 0$ such that $\xi_t^A = \emptyset$.

The model in several special environments have been studied before. For example, Bezuidenhout and Grimmett [1] studied the case when $\mu(\{c\}) = 1$ for some $c > 0$. (In fact, this is an almost nonrandom environment.) Bramson et al. [2] studied the case when $\mu(\{a, b\}) = 1$ for some $0 < a < b$. Chen and Yao [4] studied the case when $\mu(\{0, c\}) = 1$ for some $c > 0$. All the above models belong to static environments; that is, the environment does not change as time goes. There are some models concerning contact processes in dynamic environments; see, for example, Broman [3], Remenik [11], and Steif and Warfheimer [12].

Regarding complete convergence, Bezuidenhout and Grimmett [1] showed that the complete convergence theorem holds for the basic contact process on $\mathbb{Z}^d$. Chen and Yao [4] showed that the complete convergence theorem holds for the contact process on open clusters of half space $\mathbb{Z}^d \times \mathbb{Z}^+$. In this paper, we will show that, for the general model described above, the complete convergence theorem still holds for almost every environment. It generalizes the results of Bezuidenhout and Grimmett [1] and Chen and Yao [4] in the half space case. Denote by ν_λ the upper invariant measure, that is, the weak limit of the distribution of $\xi_t^{\mathbb{H}}(\lambda)$ as $t \to \infty$, and denote by $\delta_\emptyset$ the probability measure which puts mass one on the empty set. Note that, since λ is random, ν_λ is a random measure. We then have the following complete convergence theorem, which is the main result of this paper.

Theorem 1.1. *Suppose μ puts mass 1 on $[0, \infty)$. Then there exists $\Omega_0 \subseteq \Omega_1$ with $\mathbf{P}^\mu(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$ and $A \subseteq \mathbb{H}$,*

$$\xi_t^A(\lambda) \Rightarrow \nu_\lambda \cdot \mathbf{P}_\lambda(\xi^A(\lambda)\text{survives}) + \delta_\emptyset \cdot \mathbf{P}_\lambda(\xi^A(\lambda)\text{dies out})$$

as t tends to infinity, where '$\Rightarrow$' stands for $\mathbf{P}_\lambda$-weak convergence.

The main purpose of this paper is to prove Theorem 1.1, which will be specified in the following sections. The rest of this paper is organized as follows. In Section 2, we give some preliminaries including some basic notation, together with an introduction to the important 'graphical representation'. In Section 3, we prove the 'block conditions' which are essential to the proof of Theorem 1.1. We prove it under three different cases. In Section 4, we use these blocks to construct the route and use the renormalization method to make further preparations. Finally, in Section 5, we prove Theorem 1.1 by checking the two equivalent conditions in Theorem 1.12 of [10].

The main idea of the whole procedure is enlightened by Bezuidenhout and Grimmett [1]. But there are some big differences. In order to make good use of some symmetric properties, we need to consider the annealed law first (Sections 3 and 4), then go back to the quenched law to get the desired result (Section 5). The fact is, under the annealed law, the process is not Markovian, but events depending on disjoint subgraphs are relatively independent. In consequence, we can only get 'space blocks' rather than 'space–time blocks' as in Bezuidenhout and Grimmett [1]. Furthermore, we can only use these 'space blocks' to obtain the result in the half space case. We believe that the result will hold for the whole space case, but we cannot construct the independent 'restart process' as in Bezuidenhout and Grimmett [1] by adopting the method of this paper.

2. Preliminaries

We only prove the case $d = 1$; that is, $\mathbb{H} = \mathbb{Z} \times \mathbb{Z}^+$. Our technique still works for the case $d \geq 2$ after trivial modifications. In this section, we introduce some basic notation for the following analysis.

When $d = 1$, for simplicity we use a complex number $a + b\mathrm{i}$ to denote the vertex $(a, b) \in \mathbb{H} = \mathbb{Z} \times \mathbb{Z}^+$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Furthermore, we use the notation $\lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor$ to denote the rectangle

$$[\min\{a, c\}, \max\{a, c\}] \times [\min\{b, d\}, \max\{b, d\}];$$

that is, $a+b\mathrm{i}$ and $c+d\mathrm{i}$ are diagonal sites of this rectangle. The notation $\lceil\cdot\rfloor$ can be used in a more flexible way. If $a = c$ (respectively, $b = d$), then $\lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor$ denotes a vertical (respectively, horizontal) line. We can also let a, b, c, or d be infinity. For example, $\lceil -3, 3 + \infty\mathrm{i}\rfloor$ denotes the infinite 'rectangle' $[-3, 3] \times [0, +\infty)$.

Now, we introduce a special notation $\langle\cdot,\cdot\rangle$. For $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{Z}^+$, define

$$\langle a + b\mathrm{i}, c + d\mathrm{i}\rangle := \begin{cases} \{(u, v) \in \mathbb{E} : u, v \in \lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor, \{\Re(u), \Re(v)\} \not\subseteq \{a, c\}\}, \\ \qquad \text{if } |a - c| \geq 2|b - d|, \\ \{(u, v) \in \mathbb{E} : u, v \in \lceil a + b\mathrm{i}, c + d\mathrm{i}\rfloor, \{\Im(u), \Im(v)\} \not\subseteq \{b, d\}\}, \\ \qquad \text{if } 2|a - c| \leq |b - d|. \end{cases}$$

Then $\langle a + b\mathrm{i}, c + d\mathrm{i}\rangle$ is an edge set. See Fig. 1.

For a real number a, let $[a]$ be the largest integer which is no larger than a. Then, for $x \in \mathbb{H}$ and $M \in \mathbb{Z}^+$, set

$$B_x(M) := \lceil x - M - M\mathrm{i}, x + M + M\mathrm{i}\rfloor \cap \mathbb{H}$$

to be the 'ball' centered at x and with radius M (but restricted on $\mathbb{H}$).

Denote by $\mathbf{P}$ a probability measure which satisfies

$$\mathbf{P}(\xi^A \in \cdot) = \int \mathbf{P}_\lambda(\xi^A(\lambda) \in \cdot)\mathbf{P}^\mu(d\omega).$$

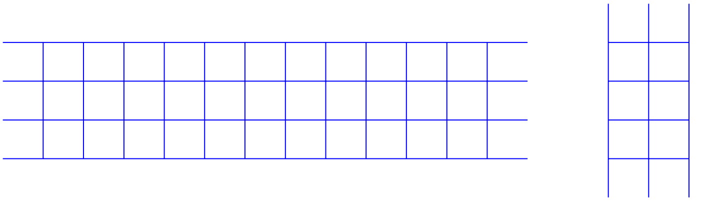

Fig. 1. $\langle a + b\mathrm{i}, c + d\mathrm{i}\rangle$.

We call $\mathbf{P}$ the annealed (average) law and $\mathbf{P}_\lambda$ the quenched law. Note that the contact process is Markovian under the quenched law, while it is not Markovian under the annealed law.

We shall make abundant use of the graphical representation of the contact process which was first proposed in Harris [8]. We follow the notation of Bezuidenhout and Grimmett [1]. Fix λ, and think of the process as being imbedded in space–time. Along each 'time-line' $x \times [0, \infty)$ are positioned 'deaths' at the points of a Poisson process with intensity 1, and between each ordered pair $x_1 \times [0, \infty)$, $x_2 \times [0, \infty)$ of adjacent time-lines are positioned edges directed from the first to the second having centers forming a Poisson processes of intensity $\lambda_{(x_1,x_2)}$ on the set $\frac{1}{2}(x_1 + x_2) \times [0, \infty)$. These Poisson processes are taken to be independent of each other. The random graph obtained from $\mathbb{H} \times [0, \infty)$ by deleting all points at which a death occurs and adding in all directed edges can be used as a percolation superstructure on which a realization of the contact process is built. We shall make free use of the language of percolation. For example, for $A, B \subseteq \mathbb{H} \times [0, \infty)$, we say that A is joined to B if there exists $a \in A$ and $b \in B$ such that there exists a path from a to b traversing time-lines in the direction of increasing time (but crossing no death) and directed edges between such lines; for $C \subseteq \mathbb{H} \times [0, \infty)$, we say that A is joined to B within C if such a path exists using segments of time-lines lying entirely in C. We next extend the notion 'within' in this paper. For $A, B \subseteq \mathbb{H} \times [0, \infty)$ and $C \subseteq \mathbb{H}$, we say that A is joined to B within C if such a path exists using segments of time-lines lying entirely in $C \times [0, \infty)$; for $D \subseteq \mathbb{E}$, we say that A is joined to B within D if such a path exists using directed edges having centers lying entirely in $D' \times [0, \infty)$, where $D' = \{\frac{x_1+x_2}{2} : (x_1, x_2) \in D\}$.

For $x \in \mathbb{H}$, $r \in \mathbb{Z}^+$ and $t \in [0, \infty)$, we call $(x \times t)_r$ a horizontal (respectively, vertical) seed with $2r + 1$ sites if all sites in $\lceil x - r, x + r \rfloor$ (respectively, $\lceil x - r\mathrm{i}, x + r\mathrm{i} \rfloor$) are infected at time t. We say that a horizontal seed $(x \times s)_r$ is joined to a vertical seed $(y \times t)_r$ if $\lceil x - r, x + r \rfloor \times s$ is joined to $z \times t$ for all $z \in \lceil y - r\mathrm{i}, y + r\mathrm{i} \rfloor$. The word 'seed' comes from Grimmett [7].

3. Block conditions

To prove Theorem 1.1, we need to get the 'block conditions' for the survival of the process. The construction is enlightened by Bezuidenhout and Grimmett [1], and was used successfully in the proof of the complete convergence theorem for contact processes on open clusters of $\mathbb{Z}^d \times \mathbb{Z}^+$; see Chen and Yao [4]. We first introduce some notation we will need.

For $h, w \in \mathbb{N}$, define the random set

$$\begin{aligned}&\Phi^R(h, w)\\ &\quad := \{x \in \lceil w, w + h\mathrm{i} \rfloor : \lceil -r, r \rfloor \times 0 \text{ is joined to } x \times [0, \infty) \text{ within } \lceil -w, w + h\mathrm{i} \rfloor\}.\end{aligned}$$

Hence, $\Phi^R(h, w)$ is a subset of the right side of the box $\lceil -w, w+h\mathrm{i} \rfloor$. Similarly, define $\Phi^L(h, w)$ as a subset of the left side. Define the random set $\Phi^{UR}(h, w)$, which is a subset of the right part of the up side, as follows:

$$\Phi^{UR}(h,w) := \{x \in \lceil h\text{i}, w+h\text{i}\rfloor : \lceil -r, r\rfloor \times 0 \text{ is joined to } x \times [0,\infty) \text{ within } \lceil -w, w+h\text{i}\rfloor \times [0,\infty)\}.$$

Similarly, define $\Phi^{UL}(h,w)$ as the subset of the left part. Furthermore, denote

$$\Phi(h,w) := \Phi^L(h,w) \cup \Phi^R(h,w) \cup \Phi^{UL}(h,w) \cup \Phi^{UR}(h,w). \tag{3.1}$$

Then, we have

$$\begin{aligned} |\Phi(h,w)| &\le |\Phi^L(h,w)| + |\Phi^{UL}(h,w)| + |\Phi^{UR}(h,w)| + |\Phi^R(h,w)| \\ &\le |\Phi(h,w)| + 3. \end{aligned} \tag{3.2}$$

Next, we present the 'block conditions' in the following proposition.

Proposition 3.1. *Suppose that* $\mathbf{P}(\xi^0 \textit{survives}) > 0$. *Then, for any* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *sufficiently small, one of the following two assertions must be true.*

(1) *There exist constants* h, w *with* $w = 4h$, *such that*

$$\mathbf{P}(|\Phi^R(h,w)| > N) > 1-\varepsilon, \qquad \mathbf{P}(|\Phi^R(h,2w)| > N) > 1-\varepsilon. \tag{3.3}$$

(2) *There exist constants* h, w *with* $8h \ge w$, *such that*

$$\mathbf{P}(|\Phi^{UR}(h,w)| > N) > 1-\varepsilon, \qquad \mathbf{P}(|\Phi^R(2h,w)| > N) > 1-\varepsilon. \tag{3.4}$$

Here, $|\cdot|$ *denotes the cardinality of a set.*

The content of Proposition 3.1 is quite similar to Lemma 3.2 in Chen and Yao [4], but things are much more difficult here. In the Bernoulli bond percolation model, it is easy to get the property that the existence of crossing from bottom to top of a box is small if the ratio of the height to the width of the box is large enough. However, in the model presented in this paper, this property is not obvious. So we need to develop some new ideas to make the construction. In detail, we consider the following three cases, which will be proved in Sections 3.1–3.3, respectively. Here and henceforth, for any $A, B \subseteq \mathbb{H}$, we say that ξ^A survives within B if, for any $t > 0$, there exists $x \in B$ such that $A \times 0$ is joined to $x \times t$ within B, while we say that ξ^A dies out within B otherwise.

Case 1. $\mu(\{0\}) > 0$.

Case 2. $\mu(\{0\}) = 0$ and ξ^0 cannot survive within any 'slab' $\lceil -k, k+\infty\text{i}\rfloor$ with positive probability.

Case 3. ξ^0 survives within some 'slab' with positive probability.

The following lemma is important to the analysis throughout this paper. The idea of its proof comes from the Remark on page 347 of [12].

Lemma 3.1. *If* $\mathbf{P}(\xi^0 \textit{survives}) > 0$, *then*

$$\lim_{r\to\infty} \mathbf{P}(\xi^{\lceil -r,r\rfloor} \textit{ survives}) = 1. \tag{3.5}$$

Proof. Let $Y_x := \mathbf{1}_{\{\xi^x \text{survives}\}}$ for any $x \in (-\infty, +\infty)$. Then, by our assumption, we have

$$\mathbf{P}(Y_x = 1) = \mathbf{P}(\xi^0 \text{ survives})$$

for any $x \in (-\infty, +\infty)$. Furthermore, it follows from the graphical representation that $\{Y_x\}_{x\in(-\infty,+\infty)}$ is ergodic. So

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ survives}) = \mathbf{P}(\exists x \in \lceil -r, r\rfloor \text{ s.t. } \xi^x \text{ survives})$$
$$\to \mathbf{P}(\exists x \in (-\infty, +\infty) \text{ s.t. } Y_x = 1) = 1$$

as r tends to infinity, as desired. □

3.1. *Proof of Case 1*

In this subsection, we shall prove that the block conditions hold if $\mu(\{0\}) > 0$. By Lemma 3.1, for any $\varepsilon > 0$ sufficiently small we can take some $r \in \mathbb{N}$ such that

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ survives}) > 1 - \frac{\varepsilon^6}{4}. \tag{3.6}$$

Set $w_n = 2^n$ and $h_n = 2^{w_n^2}$ for each $n > 100r$. Since $\mu(\{0\}) > 0$, we have that, for sufficiently large n, with large probability there exists $1 < h < h_n - 1$ such that $\lambda_{(x,x+\mathrm{i})} = 0$ for all $x \in \lceil -w_n + h\mathrm{i}, w_n + h\mathrm{i}\rfloor$. Obviously, if $\lambda_{(x,x+\mathrm{i})} = 0$ for all $x \in \lceil -w_n + h\mathrm{i}, w_n + h\mathrm{i}\rfloor$, then

$$\Phi^{UL}(h_n, w_n) = \Phi^{UR}(h_n, w_n) = \emptyset.$$

So we can conclude that there exists n_0 such that, for $n > n_0$,

$$\mathbf{P}(|\Phi^{UL}(h_n, w_n)| + |\Phi^{UR}(h_n, w_n)| = 0) > 1 - \frac{\varepsilon^6}{2}. \tag{3.7}$$

Let $\mathscr{F}_n$ denote the σ-field generated by the graphical representation within $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ ($n = 1, 2, \ldots$). Note that, for any $n \in \mathbb{N}$, if $\lambda_e = 0$ for all $e \in \{(x, y) : x \in \Phi(h_n, w_n), y \notin \lceil -w_n, w_n + h_n\mathrm{i}\rfloor\}$, then $\xi^{\lceil -r,r\rfloor}$ must die out, since no sites outside $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ can be infected. This implies that

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ dies out} \mid \mathscr{F}_n) \geq [\mu(\{0\})]^{|\Phi(h_n,w_n)|+2}$$

for any $n \in \mathbb{N}$. By the martingale convergence theorem,

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ dies out} \mid \mathscr{F}_n) \to \mathbf{1}_{\{\xi^{\lceil -r,r\rfloor} \text{ dies out}\}} \quad \text{a.s.}$$

as n tends to infinity. Since $0 < \mu(\{0\}) < 1$, it follows that

$$\lim_{n\to\infty} |\Phi(h_n, w_n)| = \infty \quad \text{almost surely on } \{\xi^{\lceil -r,r\rfloor} \text{ survives}\}.$$

Therefore,

$$\mathbf{P}(\exists m, \forall n > m, |\Phi(h_n, w_n)| > 2N \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1.$$

Hence there exists $n_1 > n_0$ such that, for $n > n_1$,

$$\mathbf{P}(|\Phi(h_n, w_n)| > 2N \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) > 1 - \frac{\varepsilon^6}{4}. \tag{3.8}$$

By (3.6) and (3.8), if $n > n_1$, then

$$\mathbf{P}(|\Phi(h_n, w_n)| > 2N) > 1 - \frac{\varepsilon^6}{2}. \tag{3.9}$$

Furthermore, from (3.2) we can see that $|\Phi(h_n, w_n)| > 2N$ and $|\Phi^{UL}(h_n, w_n)| + |\Phi^{UR}(h_n, w_n)| = 0$ together imply that $|\Phi^L(h_n, w_n)| + |\Phi^R(h_n, w_n)| > 2N$. Therefore, by (3.7) and (3.9), we get that, if $n > n_1$, then

$$\begin{aligned}
&\mathbf{P}(|\Phi^L(h_n, w_n)| + |\Phi^R(h_n, w_n)| > 2N)\\
&\quad\geq \mathbf{P}(|\Phi(h_n, w_n)| > 2N, |\Phi^{UL}(h_n, w_n)| + |\Phi^{UR}(h_n, w_n)| = 0)\\
&\quad\geq \mathbf{P}(|\Phi(h_n, w_n)| > 2N) + \mathbf{P}(|\Phi^{UL}(h_n, w_n)| + |\Phi^{UR}(h_n, w_n)| = 0) - 1\\
&\quad> 1 - \varepsilon^6.
\end{aligned}$$

Using the Fortuin–Kasteleyn–Ginibre (FKG) inequality (see Theorem 2.4 of Grimmett [7]) and the symmetric property, we can get

$$\begin{aligned}
\varepsilon^6 &> \mathbf{P}(|\Phi^L(h_n, w_n)| + |\Phi^R(h_n, w_n)| \leq 2N)\\
&\geq \mathbf{P}(|\Phi^L(h_n, w_n)| \leq N, |\Phi^R(h_n, w_n)| \leq N)\\
&\geq [\mathbf{P}(|\Phi^R(h_n, w_n)| \leq N)]^2.
\end{aligned}$$

Consequently, when n is large,

$$\mathbf{P}(|\Phi^R(h_n, w_n)| > N) > 1 - \varepsilon^3 \quad (> 1 - \varepsilon^2 > 1 - \varepsilon). \tag{3.10}$$

Similarly, we have

$$\mathbf{P}(|\Phi^R(h_n, 2w_n)| > N) > 1 - \varepsilon^3 \quad (> 1 - \varepsilon^2 > 1 - \varepsilon) \tag{3.11}$$

when n is large.

Comparing (3.10) and (3.11) with (3.3), we see that the ratio of h_n to w_n is much larger than we want. Hence we need to reduce the height. Let $k'_n = w_n^2 - n + 2$ and $h'_n = h_n/2^{k'_n}$ for $n = 1, 2, \ldots$. Then $4h'_n = w_n$. If

$$\mathbf{P}(|\Phi^R(h'_n, w_n)| > N) > 1 - \varepsilon^2, \qquad \mathbf{P}(|\Phi^R(h'_n, 2w_n)| > N) > 1 - \varepsilon^2$$

for some n, then (1) is true. Otherwise, at least one of the two following statements must be true.

(3) There exists a subsequence (n_i) such that $\mathbf{P}(|\Phi^R(h'_{n_i}, w_{n_i})| > N) \leq 1 - \varepsilon^2$.

(4) There exists a subsequence (n_i) such that $\mathbf{P}(|\Phi^R(h'_{n_i}, 2w_{n_i})| > N) \leq 1 - \varepsilon^2$.

For $i = 1, 2, \ldots$, take $w'_{n_i} = w_{n_i}$ if (3) is true, and take $w'_{n_i} = 2w_{n_i}$ if (4) is true. Then, for any i, we have

$$w'_{n_i} \leq 8h'_{n_i} \quad \text{and} \quad \mathbf{P}(|\Phi^R(h'_{n_i}, w'_{n_i})| > N) \leq 1 - \varepsilon^2.$$

Meanwhile, from (3.10) and (3.11), we get

$$\mathbf{P}(|\Phi^R(h_{n_i}, w'_{n_i})| > N) > 1 - \varepsilon^2$$

for any i. So, for any i, there exists $0 \leq k \leq k'_{n_i}$ such that

$$\mathbf{P}\left(\left|\Phi^R\left(\frac{h_{n_i}}{2^{k+1}}, w'_{n_i}\right)\right| > N\right) \leq 1 - \varepsilon^2, \qquad \mathbf{P}\left(\left|\Phi^R\left(\frac{h_{n_i}}{2^{k}}, w'_{n_i}\right)\right| > N\right) > 1 - \varepsilon^2.$$

Set $h_i^* = h_{n_i}/2^{k+1}$ and $w_i^* = w'_{n_i}$. It follows that

$$w_i^* \le 8h_i^*, \qquad \mathbf{P}(|\Phi^R(2h_i^*, w_i^*)| > N) > 1-\varepsilon^2$$
$$\text{and} \quad \mathbf{P}(|\Phi^R(h_i^*, w_i^*)| > N) \le 1-\varepsilon^2 \tag{3.12}$$

for any i.

We next show that there exists i_0 such that

$$\mathbf{P}(|\Phi^{UL}(h_{i_0}^*, w_{i_0}^*)| + |\Phi^{UR}(h_{i_0}^*, w_{i_0}^*)| > 2N) > 1-\varepsilon^2. \tag{3.13}$$

In fact, if no such i_0 exists, then $\mathbf{P}(|\Phi^{UL}(h_i^*, w_i^*)| + |\Phi^{UR}(h_i^*, w_i^*)| > 2N) \le 1-\varepsilon^2$ for all i. Using (3.2), (3.12), and the FKG inequality, we can get that, for any i,

$$\begin{aligned}\mathbf{P}(|\Phi(h_i^*, w_i^*)| \le 4N-3) &\ge \mathbf{P}(|\Phi^L(h_i^*, w_i^*)| \le N, |\Phi^R(h_i^*, w_i^*)| \\ &\le N, |\Phi^{UL}(h_i^*, w_i^*)| + |\Phi^{UR}(h_i^*, w_i^*)| \le 2N) \\ &\ge \mathbf{P}(|\Phi^L(h_i^*, w_i^*)| \le N) \cdot \mathbf{P}(|\Phi^R(h_i^*, w_i^*)| \\ &\le N) \cdot \mathbf{P}(|\Phi^{UL}(h_i^*, w_i^*)| + |\Phi^{UR}(h_i^*, w_i^*)| \le 2N) \\ &\ge \varepsilon^6.\end{aligned}$$

However, h_i^* tends to infinity as $i \to \infty$. This implies that there exists a strictly increasing subsequence $(h_{i_j}^*)$ such that

$$\mathbf{P}(|\Phi(h_{i_j}^*, w_{i_j}^*)| \le 4N-3) \ge \varepsilon^6. \tag{3.14}$$

On the other hand, by an argument similar to that of (3.8), we have that, when j is sufficiently large,

$$\mathbf{P}(|\Phi(h_{i_j}^*, w_{i_j}^*)| > 4N-3 \mid \xi^{\lceil -r,r \rfloor} \text{ survives}) > 1 - \frac{3\varepsilon^6}{4}. \tag{3.15}$$

(3.6) and (3.15) together imply that, when j is sufficiently large,

$$\mathbf{P}(|\Phi(h_{i_j}^*, w_{i_j}^*)| > 4N-3) > 1-\varepsilon^6. \tag{3.16}$$

(3.16) contradicts (3.14). As a result, (3.13) is true for some i_0.

Let $h^* = h_{i_0}^*$, $w^* = w_{i_0}^*$. Then (3.13) together with the FKG inequality and the symmetric property lead to

$$\mathbf{P}(|\Phi^{UL}(h^*, w^*)| > N) = \mathbf{P}(|\Phi^{UR}(h^*, w^*)| > N) > 1-\varepsilon.$$

So (2) is true, and the proof of Case 1 is completed. □

3.2. *Proof of Case 2*

In this subsection we shall prove that the block conditions hold if $\mu(\{0\}) = 0$ and if ξ^0 cannot survive within any 'slab' $\lceil -k, k+\infty\mathrm{i} \rfloor$ with positive probability. Fix $N \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small. By Lemma 3.1, we can take some $r \in \mathbb{N}$ such that

$$\mathbf{P}(\xi^{\lceil -r,r \rfloor} \text{ survives}) > 1 - \frac{\varepsilon^2}{16}. \tag{3.17}$$

Set

$$E := \mathbf{P}(0 \times 0 \text{ is joined to } z \times 1 \text{ within } \{0\} \cup \lceil 1, 4N + N\mathrm{i} \rfloor \\ \text{for all } z \in \lceil 4N, 4N + N\mathrm{i} \rfloor) \tag{3.18}$$

and

$$\alpha := \mathbf{P}(E);$$

then $\alpha > 0$. Let U be large enough to ensure that, in $[U/20N]$ or more independent trials of an experiment with success probability α, the probability of obtaining at least one success exceeds $1 - \frac{\varepsilon}{4}$. Let a be the minimal value which satisfies $\mu([a, \infty)) > 1 - \frac{\varepsilon}{200N^2}$. Then, for any set $A \subset \mathbb{E}$ with $\#A \le 20N^2$,

$$\mathbf{P}^{\mu}(\lambda_e \ge a, e \in A) = (\mu([a, \infty)))^{\#A} > \left(1 - \frac{\varepsilon}{200N^2}\right)^{20N^2} > 1 - \frac{\varepsilon}{8}. \tag{3.19}$$

The value of a is strictly larger than 0, since $\mu((0, \infty)) = 1$. Set

$$\beta := \mathbf{P}(0 \times 0 \text{ is joined to } z \times 1 \text{ within } \{0\} \cup \lceil 1, 4N + N\mathrm{i} \rfloor \\ \text{for all } z \in \lceil 4N, 4N + N\mathrm{i} \rfloor \mid \lambda_e = a \text{ for all } e \in \mathbb{E}).$$

Then $\beta > 0$, since $a > 0$. Let V be large enough to ensure that, in $[V/2U]$ or more independent trials of an experiment with success probability β, the probability of obtaining at least one success exceeds $1 - \frac{\varepsilon}{8}$. For $h, w \in \mathbb{N}$ with $h, w > 100r$, define

$$\Theta^R(h, w) := \{t : \lceil -r, r \rfloor \times 0 \text{ is joined to } \lceil w + h\mathrm{i}, w \rfloor \times t \text{ within } \lceil -w, w + h\mathrm{i} \rfloor\}.$$

And denote by $\mathbf{m}(\cdot)$ the Lebesgue measure on $[0, \infty)$. Then $\mathbf{m}(\Theta^R(h, w))$ is the length of infected time of the right side of the box $\lceil -w, w + h\mathrm{i} \rfloor$. Define Θ^L, Θ^{UL}, and Θ^{UR} similarly. Note that, for any $D \in \{L, R, UL, UR\}$ and $h, w \in \mathbb{N}$,

$$\{\Phi^D(h, w) = \emptyset\} = \{\Theta^D(h, w) = \emptyset\}. \tag{3.20}$$

First, we will prove the following lemma.

Lemma 3.2. *One of the following two assertions must be true.*

(1′) *There exist constants h, w with $w = 4h > 100r$, such that*

$$\mathbf{P}(|\Phi^R(h, w)| + \mathbf{m}(\Theta^R(h, w)) > U + V) > 1 - \frac{\varepsilon}{2},$$
$$\mathbf{P}(|\Phi^R(h, 2w)| + \mathbf{m}(\Theta^R(h, 2w)) > U + V) > 1 - \frac{\varepsilon}{2}.$$

(2′) *There exist constants h, w with $8h \ge w$, such that*

$$\mathbf{P}(|\Phi^{UR}(h, w)| + \mathbf{m}(\Theta^{UR}(h, w)) > U + V) > 1 - \frac{\varepsilon}{2},$$
$$\mathbf{P}(|\Phi^R(2h, w)| + \mathbf{m}(\Theta^R(2h, w)) > U + V) > 1 - \frac{\varepsilon}{2}.$$

Proof. Set $w_n = 2^n$ for each $n > 100r$. Since $\mathbf{P}(\xi^0$ dies out within $\lceil -k, k + \infty\mathrm{i} \rfloor) = 1$ for all $k \in \mathbb{N}$, we have, for every $n > 100r$,

$$\mathbf{P}(\xi^{\lceil -r, r \rfloor} \text{ dies out within } \lceil -w_n, w_n + \infty\mathrm{i} \rfloor) = 1.$$

This implies that we can find some $h_n \in \{2^{w_n}, 2^{w_n+1}, 2^{w_n+2}, \ldots\}$, such that

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ dies out within } \lceil -w_n, w_n + h_n\mathrm{i} - \mathrm{i}\rfloor) > 1 - \frac{\varepsilon^2}{8}. \tag{3.21}$$

Without loss of generality, we suppose (h_n) to be a strictly increasing sequence. Then all sites being joined with $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ are contained in $\lceil -w_{n+1}, w_{n+1} + h_{n+1}\mathrm{i}\rfloor$. By (3.21), we have

$$\mathbf{P}(|\Phi^{UL}(h_n, w_n)| + |\Phi^{UR}(h_n, w_n)| = 0 \mid |\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n)) > 2U + 2V) > 1 - \frac{\varepsilon^2}{8} \tag{3.22}$$

for all $n > 100r$. For $h, w \in \mathbb{N}$ with $h, w > 100r$, denote

$$\Theta(h, w) := \Theta^R(h, w) \cup \Theta^L(h, w) \cup \Theta^{UR}(h, w) \cup \Theta^{UL}(h, w).$$

As before, let $\mathscr{F}_n$ be the σ-field generated by the graphical representation within $\lceil -w_n + h_n\mathrm{i}, w_n + h_n\mathrm{i}\rfloor$ $(n = 1, 2, \ldots)$. Note that, for any $n \in \mathbb{N}$, if there is no flow passing through the edges

$$\Xi(h_n, w_n) := \{(x, y) \in \mathbb{E} : x \in \Phi(h_n, w_n), y \notin \lceil -w_n, w_n + h_n\mathrm{i}\rfloor\}$$

for every $t \in \Theta(h_n, w_n)$, then $\xi^{\lceil -r,r\rfloor}$ must die out, since no sites outside $\lceil -w_n, w_n + h_n\mathrm{i}\rfloor$ can be infected. Here, $\Phi(\cdot,\cdot)$ is defined as in (3.1). Note that $|\Xi(h_n, w_n)| = |\Phi(h_n, w_n)| + 2$ for any $n \in \mathbb{N}$. And, for any $n \in \mathbb{N}$, $A \subseteq \lceil -w_n, -w_n + h_n\mathrm{i}\rfloor \cup \lceil -w_n + h_n\mathrm{i}, w_n + h_n\mathrm{i}\rfloor \cup \lceil w_n + h_n\mathrm{i}, w_n\rfloor$, and $B \subseteq [0, \infty)$, we have $\Phi(h_n, w_n), \Theta(h_n, w_n) \in \mathscr{F}_n$, and

$$\begin{aligned}&\mathbf{P}(\text{there is no flow passing through the edges in } \Xi(h_n, w_n),\ \Phi(h_n, w_n) = A,\\ &\quad \times \Theta(h_n, w_n) = B \mid \mathscr{F}_n) \geq \mathbf{1}_{\{\Phi(h_n,w_n)=A,\,\Theta(h_n,w_n)=B\}} \cdot [\mathbf{E}(\exp\{-\mathbf{m}(B)\cdot\xi\})]^{|A|+2},\end{aligned}$$

where ξ is a random variable with law μ. So

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ dies out}|\mathscr{F}_n) \geq [\mathscr{L}(\mathbf{m}(\Theta(h_n, w_n)))]^{|\Phi(h_n,w_n)|+2}$$

for any $n \in \mathbb{N}$, where $\mathscr{L}(t) := \mathbf{E}e^{-t\xi}$ is the Laplace transform of the random variable ξ. By the martingale convergence theorem,

$$\mathbf{P}(\xi^{\lceil -r,r\rfloor} \text{ dies out}|\mathscr{F}_n) \to \mathbf{1}_{\{\xi^{\lceil -r,r\rfloor} \text{ dies out}\}} \quad \text{a.s.}$$

as n tends to infinity. So

$$\lim_{n\to\infty} [\mathscr{L}(\mathbf{m}(\Theta(h_n, w_n)))]^{|\Phi(h_n,w_n)|+2} = 0 \text{ almost surely on } \{\xi^{\lceil -r,r\rfloor} \text{ survives}\}.$$

But $\lim_{n\to\infty}[\mathscr{L}(\mathbf{m}(\Theta(h_n, w_n)))]^{|\Phi(h_n,w_n)|+2} = 0$ implies that $\lim_{n\to\infty}[|\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n))] = \infty$. So

$$\lim_{n\to\infty} [|\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n))] = \infty \text{ almost surely on } \{\xi^{\lceil -r,r\rfloor} \text{ survives}\}.$$

Therefore,

$$\mathbf{P}(\exists m, \forall n > m, |\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n)) > 2U + 2V \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1.$$

Hence there exists $n_0 > 100r$ such that, for $n > n_0$,

$$\mathbf{P}(|\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n)) > 2U + 2V \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) > 1 - \frac{\varepsilon^2}{16}. \tag{3.23}$$

By (3.17) and (3.23), we get, for $n > n_0$,

$$\mathbf{P}(|\Phi(h_n, w_n)| + \mathbf{m}(\Theta(h_n, w_n)) > 2U + 2V) > 1 - \frac{\varepsilon^2}{8}. \tag{3.24}$$

By (3.20), (3.22) and (3.24), we have, for large n,

$$\mathbf{P}(|\Phi^R(h_n, w_n)| + |\Phi^L(h_n, w_n)| + \mathbf{m}(\Theta^R(h_n, w_n)) + \mathbf{m}(\Theta^L(h_n, w_n)) > 2U + 2V) > 1 - \frac{\varepsilon^2}{4}.$$

Using the FKG inequality and the symmetric property again, we have

$$\mathbf{P}(|\Phi^R(h_n, w_n)| + \mathbf{m}(\Theta^R(h_n, w_n)) > U + V) > 1 - \frac{\varepsilon}{2} \tag{3.25}$$

for any sufficient large n. By (3.25), we can conclude that one of the following two assertions must be true.

(1′) There exist constants r, h, w with $w = 4h$, such that

$$\mathbf{P}(|\Phi^R(h, w)| + \mathbf{m}(\Theta^R(h, w)) > U + V) > 1 - \frac{\varepsilon}{2},$$
$$\mathbf{P}(|\Phi^R(h, 2w)| + \mathbf{m}(\Theta^R(h, 2w)) > U + V) > 1 - \frac{\varepsilon}{2}.$$

(2′) There exist constants h, w with $8h \geq w$, such that

$$\mathbf{P}(|\Phi^{UR}(h, w)| + \mathbf{m}(\Theta^{UR}(h, w)) > U + V) > 1 - \frac{\varepsilon}{2},$$
$$\mathbf{P}(|\Phi^R(2h, w)| + \mathbf{m}(\Theta^R(2h, w)) > U + V) > 1 - \frac{\varepsilon}{2}.$$

The argument is a little modification from the proof of Case 1 to reduce the height, and is omitted here. We have finished the proof of the lemma. □

Comparing Lemma 3.2 with Case 2, we only need to prove the following.

(a) If h and w satisfy (1′), then

$$\mathbf{P}(|\Phi^R(h + N, w + 4N)| > N \mid |\Phi^R(h, w)| + \mathbf{m}(\Theta^R(h, w)) > U + V) > 1 - \frac{\varepsilon}{2} \tag{3.26}$$

and

$$\mathbf{P}(|\Phi^R(h + N, 2w + 8N)| > N \mid |\Phi^R(h, 2w)| + \mathbf{m}(\Theta^R(h, 2w)) > U + V) > 1 - \frac{\varepsilon}{2}. \tag{3.27}$$

(b) If h and w satisfy (2′), then

$$\mathbf{P}(|\Phi^{UR}(h + N, w + 8N)| > N \mid |\Phi^{UR}(h, w)| + \mathbf{m}(\Theta^{UR}(h, w)) > U + V) > 1 - \frac{\varepsilon}{2} \tag{3.28}$$

and

$$\mathbf{P}(|\Phi^{UR}(h + N, 2w + 16N)| > N \mid |\Phi^{UR}(h, 2w)| + \mathbf{m}(\Theta^{UR}(h, 2w)) > U + V) > 1 - \frac{\varepsilon}{2}. \tag{3.29}$$

We only prove (3.26), since the proofs of (3.27)–(3.29) are similar. Note that, if

$$\mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h,w)| > U) \geq \left(1-\frac{\varepsilon}{4}\right) \cdot \mathbf{P}(|\Phi^R(h,w)| > U) \quad (3.30)$$

and

$$\mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h,w)| < U, \mathbf{m}(\Theta^R(h,w)) > V) \\ \geq \left(1-\frac{\varepsilon}{4}\right) \cdot \mathbf{P}(|\Phi^R(h,w)| < U, \mathbf{m}(\Theta^R(h,w)) > V), \quad (3.31)$$

then (3.26) holds. Therefore, to prove (3.26), it suffices to prove (3.30) and (3.31).

Proof of (3.30). Let h and w satisfy $(1')$. Let t_1 be the first time that a site in $\lceil w, w+(h-2N)\mathrm{i}\rfloor$ is infected. That is,

$$t_1 := \inf\{t : \lceil -r, r\rfloor \times 0 \text{ is joined to } \lceil w, w+(h-2N)\mathrm{i}\rfloor \times t \\ \text{within } \lceil -w, w+h\mathrm{i}\rfloor \times [0,\infty)\}.$$

If $t_1 < \infty$, then, with probability 1, there exists a unique infected site $x_1 \in \lceil w, w+(h-2N)\mathrm{i}\rfloor$ such that $\lceil -r, r\rfloor \times 0$ is joined to $x_1 \times t_1$ within $\lceil -w, w+h\mathrm{i}\rfloor \times [0,\infty)$. Generally, let t_k be the first time that a site in $\lceil w, w+(h-2N)\mathrm{i}\rfloor \setminus (\cup_{i=1}^{k-1} \lceil x_i - 2N\mathrm{i}, x_i + N\mathrm{i}\rfloor)$ is infected, and let x_k be the corresponding infected site if $t_k < \infty$. Denote by E_k the event that $x_k \times t_k$ is joined to every site of $\lceil x_k + 4N, x_k + 4N + N\mathrm{i}\rfloor \times (t_k + 1)$ within $\{x_k\} \cup \lceil x_k + 1, x_k + 4N + N\mathrm{i}\rfloor$. If E_k occurs, then $|\Phi^R(h+N, w+4N)| > N$. By transitivity and rotation invariance of the space, we know that $(\mathbf{1}_{E_k} | t_k < \infty)_{k=1}^{\infty}$ has the same distribution as $\mathbf{1}_E$, where E is defined in (3.18). Let

$$Y_k = \begin{cases} \mathbf{1}_{E_k}, & \text{if } t_k < \infty, \\ \text{an independent random variable with the same distribution as } \mathbf{1}_E, & \text{if } t_k = \infty. \end{cases}$$

Then $\mathbf{P}(Y_k = 1) = 1 - \mathbf{P}(Y_k = 0) = \alpha$.

Note that $Y_1, Y_2, \ldots$ are independent with respect to $\mathbf{P}$, since they are measurable with respect to the σ-fields generated by the graphical representations within mutually disjoint edge sets. Also, there exists $t_1 < \cdots < t_{[U/20N]} < \infty$ almost surely if $|\Phi^R(h,w)| > U$. Moreover, $\{|\Phi^R(h,w)| > U\}$ and $\{\sum_{k=1}^{[U/20N]} Y_k \geq 1\}$ are increasing events. Therefore, by the FKG inequality,

$$\begin{aligned} &\mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h,w)| > U) \\ &\quad\geq \mathbf{P}(\text{some } E_k \text{ occurs }, |\Phi^R(h,w)| > U) \\ &\quad= \mathbf{P}\left(|\Phi^R(h,w)| > U, \sum_{k=1}^{[U/20N]} Y_k \geq 1\right) \\ &\quad\geq \mathbf{P}(|\Phi^R(h,w)| > U) \cdot \mathbf{P}\left(\sum_{k=1}^{[U/20N]} Y_k \geq 1\right) \\ &\quad\geq \left(1-\frac{\varepsilon}{4}\right) \cdot \mathbf{P}(|\Phi^R(h,w)| > U). \end{aligned}$$

Then (3.30) holds, as desired. □

Proof of (3.31). For any $x \in \lceil w, w+h\mathrm{i}\rfloor$, set

$$T(x) := \mathbf{m}(\{t \geq 0 : \lceil -r, r\rfloor \times 0 \text{ is joined to } x \times t \text{ within } \lceil -w, w+h\mathrm{i}\rfloor\})$$

to be the Lebesgue measure of the total infection time of x. So, if $|\Phi^R(h,w)| < U$ and $\mathbf{m}(\Theta^R(h,w)) > V$, then there exists $x \in \lceil w, w+h\mathrm{i}\rfloor$ such that $T(x) > \frac{V}{U}$. Define random events

$$A_0 = \left\{T(w) > \frac{V}{U}\right\},$$
$$A_1 = \left\{T(w) \le \frac{V}{U}, T(w+\mathrm{i}) > \frac{V}{U}\right\},$$
$$A_2 = \left\{T(w) \le \frac{V}{U}, T(w+\mathrm{i}) \le \frac{V}{U}, T(w+2\mathrm{i}) > \frac{V}{U}\right\},$$
$$\dots$$
$$A_h = \left\{T(w) \le \frac{V}{U}, \dots, T(w+(h-1)\mathrm{i}) \le \frac{V}{U}, T(w+h\mathrm{i}) > \frac{V}{U}\right\}.$$

For any $0 \le k \le h$, suppose that A_k occurs. We set $s_0 = 0$ and

$$s_i = \inf\{t \in (s_{i-1}+1, \infty) : \mathbf{m}(\{s_{i-1}+1 < s < t : x_k \text{ is infected at time } s\}) = 1\}$$

for $i = 1, 2, \dots$ inductively. (Here, $\inf \emptyset$ is defined to be $+\infty$.) Define

$$D_i := (s_{i-1}+1, s_i) \cap \{t \ge 0 : x_k \text{ is infected at time } t\}$$

for $i = 1, 2, \dots$. Then $\mathbf{m}(D_i) = 1$ if $s_i < +\infty$. And $d(D_{i-1}, D_i) \ge 1$ for $i = 1, 2, \dots$, where

$$d(A, B) := \inf\{|a-b| : a \in A, b \in B\}$$

for any $A, B \subset \mathbf{R}$. Furthermore, for $i = 1, 2, \dots$, define

$$\tau_i := \inf\{t \in (s_{i-1}+1, s_i) : x_k \text{ is infected at time } t\}.$$

Note that $s_i < \infty$ implies that $\tau_i < \infty$ for $i = 1, 2, \dots$.

Denote by F_i the event that $x_k \times \tau_i$ is joined to every site of $\lceil x_k+4N, x_k+4N+N\mathrm{i}\rfloor \times (\tau_i+1)$ within $\{x_k\} \cup \lceil x_k+1, x_k+4N+N\mathrm{i}\rfloor$. If F_i occurs, then $|\Phi^R(h+N, w+4N)| > N$. By transitivity and rotation invariance of the space, we know that $(\mathbf{1}_{F_i} | s_i < \infty)_{i=1}^{\infty}$ has the same distribution as $\mathbf{1}_F$, where F is the event that 0×0 is joined to every site of $\lceil 4N, 4N+N\mathrm{i}\rfloor \times 1$ within $\{0\} \cup \lceil 1, 4N+N\mathrm{i}\rfloor$. Let

$$Z_i = \begin{cases} \mathbf{1}_{F_i}, & \text{if } s_i < \infty, \\ \text{an independent random variable with the same distribution as } \mathbf{1}_F, & \text{if } s_i = \infty. \end{cases}$$

By the strong Markov property under the quenched law, we know that $Z_1, Z_2, \dots$ are independent with respect to $\mathbf{P}_\lambda$ for any fixed environment λ. And for any environment λ such that $\lambda_e \ge a$ for all $e \in \{x_k\} \cup \lceil x_k+1, x_k+4N+N\mathrm{i}\rfloor$, we have

$$\mathbf{P}_\lambda(Z_i = 1) = 1 - \mathbf{P}_\lambda(Z_i = 0) \ge \beta$$

by the monotonicity of the contact process. So, by our choice of V and U,

$$\mathbf{P}_\lambda\left(\sum_{i=1}^{[V/2U]} Z_i \ge 1\right) \ge 1 - \frac{\varepsilon}{8}. \tag{3.32}$$

Turning to the annealed law, we get from (3.19) and (3.32) that

$$\mathbf{P}\left(\sum_{i=1}^{[V/2U]} Z_i \ge 1\right) \ge \mathbf{P}^\mu(\lambda_e \ge a \text{ for all } e \in \{x_k\} \cup \lceil x_k+1, x_k+4N+N\mathrm{i}\rfloor) \cdot \left(1 - \frac{\varepsilon}{8}\right)$$

$$\geq 1 - \frac{\varepsilon}{4}.$$

Furthermore, note that there exists $s_1 < \cdots < s_{[V/2U]} < \infty$ almost surely if $|\Phi^R(h, w)| < U$ and $\mathbf{m}(\Theta^R(h, w)) > V$. Therefore,

$$\begin{aligned}
&\mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k) \\
&\quad \geq \mathbf{P}(\text{some } F_i \text{ occurs}, |\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k) \\
&\quad = \mathbf{P}\left(|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k, \sum_{i=1}^{[V/2U]} Z_i \geq 1\right) \\
&\quad = \mathbf{P}(|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k) \cdot \mathbf{P}\left(\sum_{i=1}^{[V/2U]} Z_i \geq 1\right) \\
&\quad \geq \left(1 - \frac{\varepsilon}{4}\right) \cdot \mathbf{P}(|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k).
\end{aligned}$$

Here, the third equality holds because the event $\{|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k\}$ is measurable with respect to the σ-field generated by the graphical representation within $G_1 = \lceil -w, w + h\mathrm{i} \rfloor$, while the event $\{\sum_{i=1}^{[V/2U]} Z_i \geq 1\}$ is measurable with respect to the σ-field generated by the graphical representation within $G_2 = \{x_k\} \cup \lceil x_k + 1, x_k + 4N + N\mathrm{i} \rfloor$. The two events are independent, since G_1 and G_2 are disjoint edge sets which share no common edges. Next, note that

$$\{|\Phi^R(h, w)| < U\} \cap \{\mathbf{m}(\Theta^R(h, w)) > V\} \subseteq \bigcup_{k=0}^{h} A_k,$$

and $A_k, k = 0, 1, \ldots, h$, are mutually exclusive events. Therefore,

$$\begin{aligned}
&\mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V) \\
&\quad = \sum_{k=0}^{h} \mathbf{P}(|\Phi^R(h+N, w+4N)| > N, |\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k) \\
&\quad \geq \left(1 - \frac{\varepsilon}{4}\right) \cdot \sum_{k=0}^{h} \mathbf{P}(|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V, A_k) \\
&\quad = \left(1 - \frac{\varepsilon}{4}\right) \cdot \mathbf{P}(|\Phi^R(h, w)| < U, \mathbf{m}(\Theta^R(h, w)) > V).
\end{aligned}$$

Then (3.31) holds, as desired. □

All the above arguments together lead to the proof of Case 2.

3.3. *Proof of Case 3*

In this subsection, we shall prove that the block conditions hold if ξ^0 survives within some 'slab' with positive probability. Choose fixed $K \in \mathbb{N}$ such that

$$\mathbf{P}(\xi^0 \text{ survives within } \lceil -K, K + \infty\mathrm{i} \rfloor) = c > 0. \tag{3.33}$$

For any $x \in \mathbb{H}$, $m, n \in \mathbb{N}$, and $t > 0$, denote by $A(x, t, m, n)$ the event that $x \times t$ is joined to $\lceil x + m + n\mathrm{i} + 2K + 2K\mathrm{i}, x + m + n\mathrm{i} + 4K + 4K\mathrm{i} \rfloor \times [t, \infty)$ within $\{x\} \cup \lceil x + \mathrm{i} + K, x - K + 2K\mathrm{i} + n\mathrm{i} \rfloor \cup \lceil x - K + 2K\mathrm{i} + n\mathrm{i}, x + m + n\mathrm{i} + 4K + 4K\mathrm{i} \rfloor$. See Fig. 2 for intuition. Then,

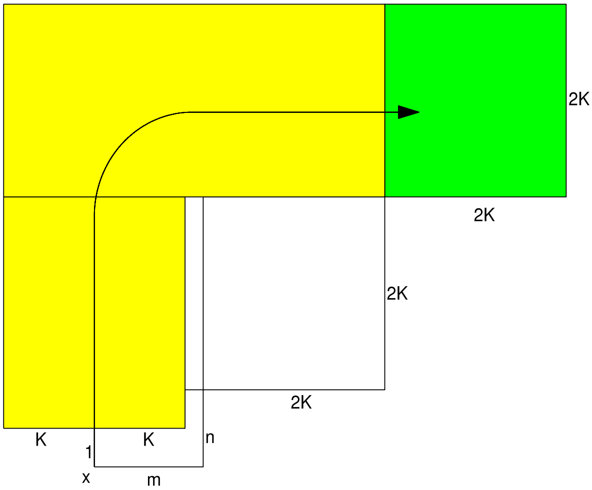

Fig. 2. Description of $A(x, t, m, n)$.

for $m \in \mathbb{N}$, $t > 0$,and $x \in \mathbb{H}$ with $\Im(x) > K$, define

$$T(x, t, m) := \inf\{s \geq t : x \times t \text{ is joined to } \lceil x - K\mathrm{i} + m, x + K\mathrm{i} + m\rfloor \times s \text{ within } \{x\} \cup \lceil x + 1 - K\mathrm{i}, x + K\mathrm{i} + m\rfloor\}.$$

And similarly, define

$$T(x, t, m\mathrm{i}) := \inf\{s \geq t : x \times t \text{ is joined to } \lceil x - K + m\mathrm{i}, x + K + m\mathrm{i}\rfloor \times s \text{ within } \{x\} \cup \lceil x + \mathrm{i} - K, x + K + m\mathrm{i}\rfloor\}$$

for $x \in \mathbb{H}$, $m \in \mathbb{N}$ and $t > 0$. See Fig. 3 for intuition.

We then have the following lemma, which is essential to the proof of Case 3.

Lemma 3.3. *There exists $\alpha > 0$ which is independent of x, t, m, and n, such that*

$$\mathbf{P}(A(x, t, m, n)) > \alpha. \tag{3.34}$$

Proof. For $x \in \mathbb{H}$ and $t > 0$, denote by $C(x, t)$ the event that $x \times t$ is joined to $(x + 3K + 3K\mathrm{i}) \times (t + 1)$ within $\lceil x, x + 3K\mathrm{i}\rfloor \cup \lceil x + 3K\mathrm{i}, x + 3K + 3K\mathrm{i}\rfloor$. By translation invariance we have that $\mathbf{P}(C(x, t)) = \mathbf{P}(C(0, 0))$ for any $x \in \mathbb{H}$ and $t > 0$. We next prove that

$$\alpha := \frac{c^2}{4} \cdot \mathbf{P}(C(0, 0))$$

satisfies (3.34), where c is the positive constant as defined in (3.33). By (3.33) and translation invariance, we have, for any $x \in \mathbb{H}$, $m \in \mathbb{N}$, and $t > 0$,

$$\begin{aligned}\mathbf{P}(T(x, t, m\mathrm{i}) < \infty) &= \mathbf{P}(T(0, 0, h\mathrm{i}) < \infty)\\ &\geq \mathbf{P}(\xi^0 \text{ survives within } \lceil -K, K + \infty\mathrm{i}\rfloor) = c.\end{aligned} \tag{3.35}$$

Furthermore, by rotation invariance, for any $m \in \mathbb{N}$, $t > 0$, and $x \in \mathbb{H}$ with $\Im(x) > K$,

$$\mathbf{P}(T(x, t, m) < \infty) = \mathbf{P}(T(x, t, m\mathrm{i}) < \infty) \geq c. \tag{3.36}$$

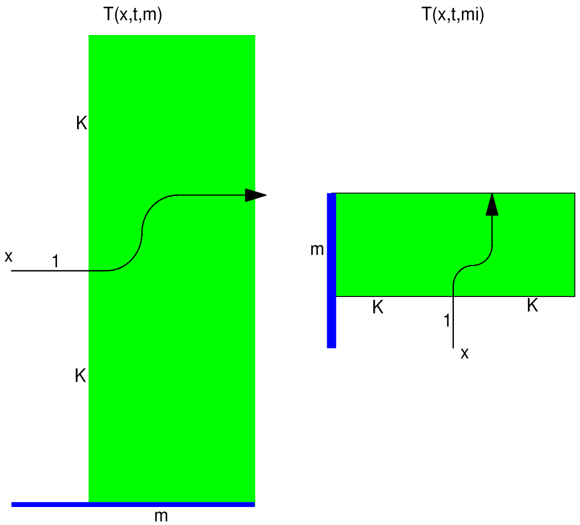

Fig. 3. Description of $T(x,t,m)$ and $T(x,t,m\mathrm{i})$.

Next, if $T(x,t,n\mathrm{i}) < \infty$, then let $X(x,t,n\mathrm{i})$ be the corresponding infected site. For $x \in \mathbb{H}$, $m,n \in \mathbb{N}$, and $t > 0$, define

$$D(x,t,m,n) := \{T(x,t,n\mathrm{i}) < \infty\} \cap C(X(x,t,n\mathrm{i}), T(x,t,n\mathrm{i})) \\ \cap \{T(X(x,t,n\mathrm{i}) + 3K + 3K\mathrm{i}, T(x,t,n\mathrm{i}) + 1, m) < \infty\}.$$

Obviously, for any $x \in \mathbb{H}$, $t > 0$, and $m,n \in \mathbb{N}$, we have

$$D(x,t,m,n) \subseteq A(x,t,m,n). \tag{3.37}$$

Let $\mathscr{F}$ denote the σ-field generated by the graphical representation within $\{x\} \cup \lceil x - K + \mathrm{i}, x + K + n\mathrm{i}\rfloor$. Then $X(x,t,n\mathrm{i})$ and $T(x,t,n\mathrm{i})$ are measurable with respect to $\mathscr{F}$. So, by (3.35)–(3.37), we have

$$\begin{aligned}
\mathbf{P}(A(x,t,m,n)) &\geq \mathbf{P}(D(x,t,m,n)) \\
&= \mathbf{E}(\mathbf{P}(T(x,t,n\mathrm{i}) < \infty, C(X(x,t,n\mathrm{i}), T(x,t,n\mathrm{i})), \\
&\quad T(X(x,t,n\mathrm{i}) + 3K + 3K\mathrm{i}, T(x,t,n\mathrm{i}) + 1, m) < \infty \mid \mathscr{F})) \\
&= \mathbf{E}(\mathbf{P}(s < \infty, C(y,s), T(y + 3K + 3K\mathrm{i}, s+1, m) < \infty \mid \mathscr{F}) \\
&\quad |_{y = X(x,t,n\mathrm{i}), s = T(x,t,n\mathrm{i})}) \\
&= \mathbf{E}(\mathbf{1}_{\{s<\infty\}} \cdot \mathbf{P}(C(y,s)) \cdot \mathbf{P}(T(y + 3K + 3K\mathrm{i}, s+1, m) < \infty) \\
&\quad |_{y = X(x,t,n\mathrm{i}), s = T(x,t,n\mathrm{i})}) \\
&= \mathbf{P}(T(x,t,n\mathrm{i}) < \infty) \cdot \mathbf{P}(C(0,0)) \cdot \mathbf{P}(T(0,0,m) < \infty) \\
&\geq c^2 \cdot \mathbf{P}(C(0,0)) > \alpha.
\end{aligned}$$

We next explain the third equality in detail. By definition, $X(x,t,n\mathrm{i})$ takes a value in $\lceil x - K + n\mathrm{i}, x + K + n\mathrm{i}\rfloor$. For any fixed $y \in \lceil x - K + n\mathrm{i}, x + K + n\mathrm{i}\rfloor$ and $s > 0$, the event $C(y,s)$ is measurable with respect to the σ-field generated by the graphical representation within $G_1 = \lceil y, y + 3K\mathrm{i}\rfloor \cup \lceil y + 3K\mathrm{i}, y + 3K + 3K\mathrm{i}\rfloor$, while the event $\{T(y + 3K + 3K\mathrm{i}, s+1, m) < \infty\}$ is measurable with respect to the σ-field generated by the graphical representation within

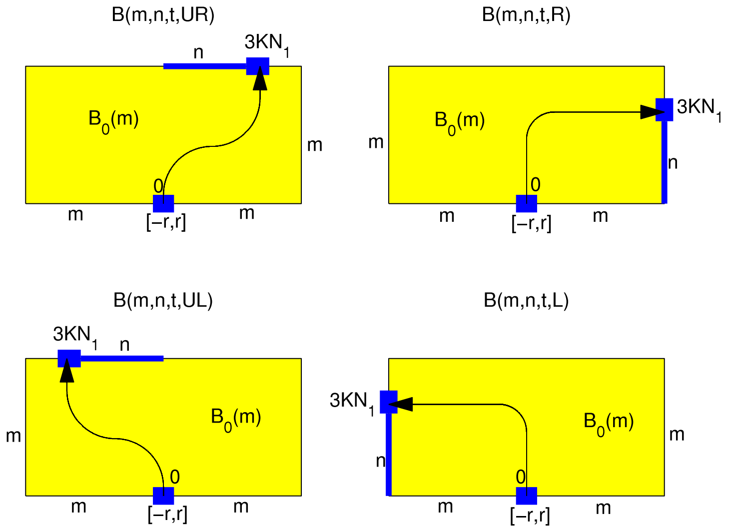

Fig. 4. Description of $B(m,n,t,UR)$, $B(m,n,t,R)$, $B(m,n,t,UL)$, and $B(m,n,t,L)$.

$G_2 = \{y + 3K + 3K\mathrm{i}\} \cup \lceil y + 3K + 1 + 2K\mathrm{i}, y + 3K + m + 4K\mathrm{i}\rfloor$. Note that G_1 and G_2 are disjoint with $\{x\} \cup \lceil x - K + \mathrm{i}, x + K + n\mathrm{i}\rfloor$, respectively. As a result, the events $C(y, s)$ and $\{T(y + 3K + 3K\mathrm{i}, s + 1, m) < \infty\}$ are independent of $\mathscr{F}$, respectively. Furthermore, the two events are independent since G_1 and G_2 are disjoint edge sets which share no common edges.

From the above arguments, we get the inequality in (3.34). Therefore, we have completed the proof of Lemma 3.3. $\square$

Proof of Case 3 Fix $N \in \mathbb{N}$ and $\varepsilon > 0$ sufficiently small. Let N_1 be large enough to ensure that, in N_1 or more independent trials of an experiment with success probability α, the probability of obtaining at least N success exceeds $1 - \varepsilon/2$. Here, α is the positive constant in Lemma 3.3. By Lemma 3.1, there exists r such that $\xi^{\lceil -r,r\rfloor}$ survives with probability greater than $1 - \varepsilon/4$. For $m, n \in \mathbb{N}$ and $t > 0$, define

$$\begin{aligned}
B(m,n,t,UR) &:= \{\lceil -r, r\rfloor \times 0 \text{ is joined to } z \times t \\
&\qquad \times \text{ for all } z \in \lceil n + m\mathrm{i}, n + m\mathrm{i} + 3KN_1\rfloor \text{ within } B_0(m)\},\\
B(m,n,t,R) &:= \{\lceil -r, r\rfloor \times 0 \text{ is joined to } z \times t \\
&\qquad \times \text{ for all } z \in \lceil m + n\mathrm{i}, m + n\mathrm{i} + 3KN_1\mathrm{i}\rfloor \text{ within } B_0(m)\},\\
B(m,n,t,UL) &:= \{\lceil -r, r\rfloor \times 0 \text{ is joined to } z \times t \\
&\qquad \times \text{ for all } z \in \lceil -n + m\mathrm{i}, -n + m\mathrm{i} - 3KN_1\rfloor \text{ within } B_0(m)\},\\
B(m,n,t,L) &:= \{\lceil -r, r\rfloor \times 0 \text{ is joined to } z \times t \\
&\qquad \times \text{ for all } z \in \lceil -m + n\mathrm{i}, -m + n\mathrm{i} + 3KN_1\mathrm{i}\rfloor \text{ within } B_0(m)\}.
\end{aligned}$$

See Fig. 4 for intuition. Then, define

$$\rho := \inf\{m \in \mathbb{N} : \exists t > 0, n \in \mathbb{N}, \text{ and } D \in \{UR, R, UL, L\} \text{ s.t. } B(m,n,t,D) \text{ occurs}\}.$$

We next prove that

$$\mathbf{P}(\rho < \infty \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1. \tag{3.38}$$

Define

$$p := \mathbf{P}(\exists 0 < t < \infty, \text{s.t.} 0 \times 0 \text{ is joined to } z \times t \text{ for all } z \in \lceil 0, 3KN\mathrm{i}\rfloor \text{ within } \lceil 0, 3KN\mathrm{i}\rfloor).$$

Then $p > 0$. For any $m \in \mathbb{N}$, denote by Z_m the first infected site in

$$\partial B_0(m) := \lceil -m, -m + m\mathrm{i}\rfloor \cup \lceil -m + m\mathrm{i}, m + m\mathrm{i}\rfloor \cup \lceil m + m\mathrm{i}, m\rfloor,$$

and by τ_m the corresponding infected time. Furthermore, denote by $\mathscr{G}_m$ the σ-field generated by the graphical representation within $\lceil -m, m + m\mathrm{i}\rfloor \setminus \partial B_0(m)$. If $\xi^{\lceil -r,r\rfloor}$ survives, then $\tau_m < \infty$ and $\tau_m, Z_m \in \mathscr{G}_m$ for any $m \in \mathbb{N}$. Next, for any $m \geq 8KN_1$, we divide $\partial B_0(m)$ into four parts as follows:

$$\partial B_0^L(m) := \lceil -m, -m + m\mathrm{i}\rfloor, \qquad \partial B_0^{UL}(m) := \lceil -m + m\mathrm{i}, m\mathrm{i}\rfloor,$$
$$\partial B_0^{UR}(m) := \lceil m\mathrm{i}, m + m\mathrm{i}\rfloor, \qquad \partial B_0^R(m) := \lceil m + m\mathrm{i}, m\rfloor.$$

Since $m \geq 8KN_1$, no matter which part of $D \in \{L, UL, UR, R\}$ that Z_m lies in, we can find a seed S_m with length $3KN_1$ such that S_m lies entirely in the same part as Z_m and one endpoint of S_m is Z_m. For $m \geq 8KN_1$, denote

$$A_m := \{\exists \tau_m < t < \infty, \text{ s.t. } Z_m \times \tau_m \text{ is joined to } z \times t \text{ for all } z \in S_m \text{ within } S_m\}.$$

Then, by translation and rotation invariance, if $\xi^{\lceil -r,r\rfloor}$ survives, then

$$\mathbf{P}(\rho < \infty \mid \mathscr{G}_m) \geq \mathbf{P}(A_m \mid \mathscr{G}_m) = p > 0$$

for any $m \geq 8KN_1$. That is,

$$\mathbf{P}(\mathbf{P}(\rho < \infty \mid \mathscr{G}_m) \geq p > 0 \text{ for all } m \geq 8KN_1 \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1. \tag{3.39}$$

Furthermore, using the martingale convergence theorem, we can get that

$$\mathbf{P}(\rho < \infty \mid \mathscr{G}_n) \to \mathbf{1}_{\{\rho<\infty\}} \quad \text{a.s.}$$

as n tends to infinity. So, by (3.39), we get

$$\mathbf{P}(\rho < \infty \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) = 1.$$

Therefore, (3.38) holds.

From (3.38), we can get that there exists a positive integer $N_2 > 100r$ such that

$$\mathbf{P}(\rho < N_2 \mid \xi^{\lceil -r,r\rfloor} \text{ survives}) > 1 - \frac{\varepsilon}{4}. \tag{3.40}$$

Set

$$h := 4KN_1 + N_2 \quad \text{and} \quad w := 4h.$$

We next prove that this choice of r, h, and w satisfies (3.3). If $\rho < N_2$, let

$$\tau := \inf\{t > 0 : \exists n \in \mathbb{N} \text{ and } D \in \{L, UL, UR, R\} \text{ s.t. } B(\rho, n, t, D) \text{ occurs}\}.$$

Obviously, if $\rho < N_2$, then $\tau < \infty$. Let

$$\gamma := \inf\{n \in \mathbb{N} : \exists D \in \{L, UL, UR, R\} \text{ s.t. } B(\rho, n, \tau, D) \text{ occurs}\}.$$

We divide our problem into four cases. (I): $B(\rho, \gamma, \tau, UR)$ occurs. (II): $B(\rho, \gamma, \tau, R)$ occurs. (III): $B(\rho, \gamma, \tau, LR)$ occurs. (IV): $B(\rho, \gamma, \tau, L)$ occurs. We only prove Cases (I) and (II), since Cases (III) and (IV) can be easily achieved by symmetry.

Case (I). Suppose that $B(\rho,\gamma,\tau,UR)$ occurs. Then $\lceil -r,r\rfloor\times 0$ is joined to $z\times t$ for all $z\in\lceil \gamma+\rho\mathrm{i},\gamma+\rho\mathrm{i}+3KN_1\rfloor$ within $B_0(\rho)$. For $0\le j\le N_1-1$, let

$$x_j := \gamma+\rho\mathrm{i}+K+3Kj \quad \text{and} \quad n_j = 4KN_1-3Kj.$$

Then

$$\{x_j\}\cup\lceil x_j+\mathrm{i}+K, x_j-K+n_j\mathrm{i}+2K\mathrm{i}\rfloor \\ \cup\lceil x_j-K+n_j\mathrm{i}+2K\mathrm{i}, x+w+n_j\mathrm{i}+4K+4K\mathrm{i}\rfloor,\quad 0\le j\le N_1-1$$

are disjoint. By the assumption of (3.34) and the definition of N_1, with probability greater than $1-\frac{\varepsilon}{2}$ there are at least N events in $\{A(x_j,\tau,w,n_j), 0\le j\le N_1-1\}$ occur. We can see that, if $A(x_j,\tau,w,n_j)$ occurs, then $\lceil -r,r\rfloor\times 0$ is joined to $\lceil w+\Im(x_j)\mathrm{i}+n_j\mathrm{i}+2K\mathrm{i}, w+\Im(x_j)\mathrm{i}+n_j\mathrm{i}+4K\mathrm{i}\rfloor\times[0,\infty)$ within

$$B_0(\rho)\cup\lceil x_j+\mathrm{i}+K, x_j-K+n_j\mathrm{i}+2K\mathrm{i}\rfloor \\ \cup\lceil x_j-K+n_j\mathrm{i}+2K\mathrm{i}, w+\Im(x_j)\mathrm{i}+n_j\mathrm{i}+4K\mathrm{i}\rfloor\subset\lceil -w, w+h\mathrm{i}\rfloor.$$

Therefore, conditioned on $B(\rho,\gamma,\tau,UR)$ occurs, the probability of $|\Phi^R(h,w)|>N$ is greater than $1-\frac{\varepsilon}{2}$.

Case (II). Suppose that $B(\rho,\gamma,\tau,R)$ occurs. Then $\lceil -r,r\rfloor\times 0$ is joined to $z\times t$ for all $z\in\lceil\gamma+\rho\mathrm{i},\gamma+\rho\mathrm{i}+3KN_1\mathrm{i}\rfloor$ within $B_0(\rho)$. For $0\le j\le N_1-1$, let

$$x_j := \gamma+\rho\mathrm{i}+K\mathrm{i}+3Kj\mathrm{i}.$$

Then

$$\{x_j\}\cup\lceil x_j+1-K\mathrm{i}, x_j+w+K\mathrm{i}\rfloor,\quad 0\le j\le N_1-1$$

are disjoint. By the assumption of (3.34) and the definition of N_1, with probability greater than $1-\frac{\varepsilon}{2}$ there are at least N events in $\{A(x_j,\tau,w,n_j), 0\le j\le N_1-1\}$ occur. We can see that if $A(x_j,\tau,w,n_j)$ occurs, then $\lceil -r,r\rfloor\times 0$ is joined to $\lceil w+\Im(x_j)\mathrm{i}-K\mathrm{i}, w+\Im(x_j)\mathrm{i}+K\mathrm{i}\rfloor\times[0,\infty)$ within

$$B_0(\rho)\cup\lceil x_j+1-K\mathrm{i}, x_j+w+K\mathrm{i}\rfloor\subset\lceil -w, w+h\mathrm{i}\rfloor.$$

Therefore, conditioned on $B(\rho,\gamma,\tau,R)$ occurs, the probability of $|\Phi^R(h,w)|>N$ is greater than $1-\frac{\varepsilon}{2}$.

By the above analysis, we have that, conditioned on $\rho<N_2$, the probability of $|\Phi^R(h,w)|>N$ is greater than $1-\frac{\varepsilon}{2}$. Together with (3.40), we get

$$\begin{aligned}\mathbf{P}(|\Phi^R(h,w)|>N) \ge{}& \mathbf{P}(|\Phi^R(h,w)| \\ &> N \mid \rho<N_2)\cdot\mathbf{P}(\rho<N_2\mid \xi^{\lceil -r,r\rfloor}\text{ survives})\cdot\mathbf{P}(\xi^{\lceil -r,r\rfloor}\text{ survives}) \\ &> \left(1-\frac{\varepsilon}{2}\right)\left(1-\frac{\varepsilon}{4}\right)\left(1-\frac{\varepsilon}{4}\right)>1-\varepsilon.\end{aligned}$$

Similarly, we can prove that $\mathbf{P}(|\Phi^R(h,2w)|>N)>1-\varepsilon$. So we have proved Case 3. □

The three subsections above give the whole proof of the 'block conditions', Proposition 3.1. Next, we make further analysis. Let G be the event that 0×0 is joined to every site of $\lceil -r+4r\mathrm{i}, r+4r\mathrm{i}\rfloor\times 1$ within $\langle -r, r+4r\mathrm{i}\rangle$. Fix $N\ge\frac{20r\log\varepsilon}{\log(1-\mathbf{P}(G))}+1$ which is large enough to ensure that, in $[N/20r]$ or more independent trials of an experiment with success probability $\mathbf{P}(G)$, the probability of obtaining at least one success exceeds $1-\varepsilon$. We then have the next lemma.

Lemma 3.4. *Suppose that* $\mathbf{P}(|\Phi^R(h, w)| > N) > 1 - \varepsilon$. *Then, with* $\mathbf{P}$*-probability greater than* $1 - 2\varepsilon$*, there exist* $x \in \lceil w + 4r, w + 4r + h\mathrm{i}\rfloor$ *and* $t > 0$*, such that the horizontal seed* $(0 \times 0)_r$ *is joined to the vertical seed* $(x \times t)_r$ *within* $\langle -w - 1, w + 4r + h\mathrm{i}\rangle$.

Proof. Let t_1 be the first time that some site in $\lceil w + 2r\mathrm{i}, w + (h - 2r)\mathrm{i}\rfloor$ is infected. That is,

$$t_1 := \inf\{t : \lceil -r, r\rfloor \times 0 \text{ is joined to } \lceil w + 2r\mathrm{i}, w + (h - 2r)\mathrm{i}\rfloor \times t \text{ within } \lceil -w, w + h\mathrm{i}\rfloor\}.$$

If $t_1 < \infty$, then with probability 1 there exists a unique infected site $x_1 \in \lceil w+2r\mathrm{i}, w+(h-2r)\mathrm{i}\rfloor$ such that $\lceil -r, r\rfloor \times 0$ is joined to $x_1 \times t_1$ within $\lceil -w, w + h\mathrm{i}\rfloor$. Generally, let t_k be the first time that some site in $\lceil w + 2r\mathrm{i}, w + (h - 2r)\mathrm{i}\rfloor \setminus (\cup_{i=1}^{k-1} \lceil x_i - 3r\mathrm{i}, x_i + 3r\mathrm{i}\rfloor)$ is infected, and let x_k be the corresponding infected site if $t_k < \infty$. Denote by G_k the event that $x_k \times t_k$ is joined to every site of $\lceil x_k + 4r - r\mathrm{i}, x_k + 4r + r\mathrm{i}\rfloor \times (t_k + 1)$ within $\langle x_k - r\mathrm{i}, x_k + 4r + r\mathrm{i}\rangle$. If G_k occurs, then the horizontal seed $(0 \times 0)_r$ is joined to the vertical seed $(x_k \times t_k)_r$ within $\langle -w - 1, w + 4r + h\mathrm{i}\rangle$. By transitivity and rotation invariance of the space, we know that $(\mathbf{1}_{G_k} | t_k < \infty)$ has the same distribution as $\mathbf{1}_G$. Let

$$Y_k = \begin{cases} \mathbf{1}_{G_k}, & \text{if } t_k < \infty, \\ \text{an independent random variable with the same distribution as } \mathbf{1}_G, & \text{if } t_k = \infty. \end{cases}$$

Then $\mathbf{P}(Y_k = 1) = 1 - \mathbf{P}(Y_k = 0) = \mathbf{P}(G)$.

Note that $Y_1, Y_2, \ldots$ are independent with respect to $\mathbf{P}$, since they are measurable with respect to the σ-fields generated by the graphical representations within mutually disjoint edge sets. Also, there exists $t_1 < \cdots < t_{[N/20r]} < \infty$ almost surely if $|\Phi^R(h, w)| > N$. Therefore,

$$\begin{aligned} \mathbf{P}(\text{some } G_k \text{ occurs}) &\geq \mathbf{P}\left(|\Phi^R(h, w)| > N, \sum_{k=1}^{[N/20r]} Y_k \geq 1\right) \\ &\geq \mathbf{P}(|\Phi^R(h, w)| > N) + \mathbf{P}\left(\sum_{k=1}^{[N/20r]} Y_k \geq 1\right) - 1 \\ &\geq 1 - 2\varepsilon. \end{aligned}$$

So, there exist $x \in \lceil w + 4r + 2r\mathrm{i}, w + 4r + (h - 2r)\mathrm{i}\rfloor$ and $t > 0$, such that the horizontal seed $(0 \times 0)_r$ is joined to the vertical seed $(x \times t)_r$ within $\langle -w - 1, w + 4r + h\mathrm{i}\rangle$ with $\mathbf{P}$-probability greater than $1 - 2\varepsilon$. $\square$

Remark. A similar conclusion holds for Φ^L, Φ^{UR}, and Φ^{UL}.

Now, we give the following proposition, which is essential to the analysis in the following sections. See Fig. 5 for intuition.

Proposition 3.2. *Suppose that* $\mathbf{P}(\xi^0 \text{survives}) > 0$. *Then, for any* $\varepsilon > 0$ *sufficiently small, there exist* $r \geq 1$ *and* $h \geq 100r$ *such that the following three assertions hold with* $\mathbf{P}$*-probability greater than* $1 - \varepsilon$.

(i) *The horizontal seed* $(0 \times 0)_r$ *is joined to a vertical seed* $(x \times t)_r$ *within* $\langle -4h - 1, w + h\mathrm{i}\rangle$ *for some* $4h + 4r \leq w < 4.0001h$, $\Re(x) = w$, *and* $t > 0$.

(ii) *The horizontal seed* $(0 \times 0)_r$ *is joined to a vertical seed* $(x \times t)_r$ *within* $\langle -8h - 1, w + h\mathrm{i}\rangle$ *for some* $8h + 4r \leq w < 8.0001h$, $\Re(x) = w$, *and* $t > 0$.

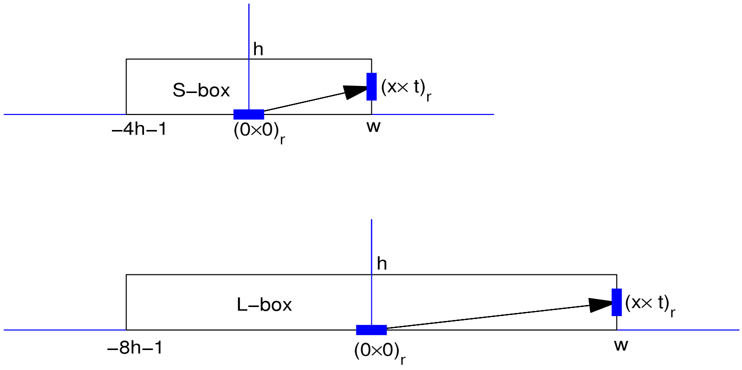

Fig. 5. Construction of blocks.

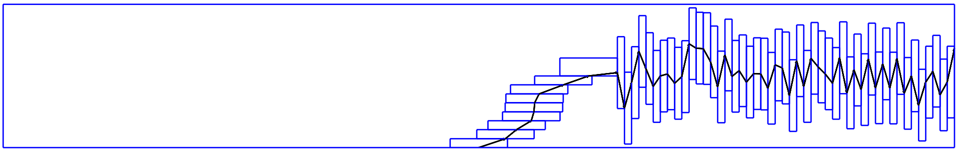

Fig. 6. Construction of (1) through (2).

(iii) *The horizontal seed* $(0\times 0)_r$ *is joined to a vertical seed* $(x_1\times t_1)_r$ *within* $\langle -8h-1, w_1+h\rangle$ *for some* $8h+4r\leq w_1<8.0001h$ *and* $t_1>0$*; and the horizontal seed* $(0\times 0)_r$ *is joined to a vertical seed* $(x_2\times t_2)_r$ *within* $\langle -w_2+h, 8h+1\rangle$ *for some* $8h+4r\leq w_2<8.0001h$ *and* $t_2>0$.

Proof. When $\mathbf{P}(\xi^0\text{survives})>0$, either (1) or (2) of Proposition 3.1 is true. If (1) is true, then, by Lemma 3.4, (i) and (ii) hold. If (2) is true, we can prove the first two conclusions by iterating Lemma 3.4; see Fig. 6. Furthermore, by (ii) together with the symmetric property and the FKG inequality, we can get (iii) in both cases. So we have completed the proof of the proposition. □

4. Dynamic renormalization

From now on, for simplicity, we call the two kinds of edge sets displayed in Fig. 5 *S-boxes* and *L-boxes*, respectively. ('S' stands for 'short'; 'L' stands for 'long'). Rigorously, S-boxes are edge sets having the same shape as $\langle -4h-1, w+h\rangle$ $(4h+4r\leq w<4.0001h, \Re(x)=w)$ described in Part (i) of Proposition 3.2, while L-boxes are edge sets having the same shape as $\langle -8h-1, w+h\rangle$ $(8h+4r\leq w<8.0001h, \Re(x)=w)$ described in Part (ii) of Proposition 3.2. The ratio of the width to the height in an S-box is nearly 8:1, while the ratio of the width to the height in an L-box is nearly 16:1. Translations and rotations are allowed. These edge sets are called 'boxes' since the endpoints of each edge box form a rectangle on $\mathbb{H}$. From Proposition 3.2, we are able to find some S-boxes and L-boxes such that, with large $\mathbf{P}$-probability, a horizontal seed on the bottom of each box is joined within the box to a vertical seed on the right. Fig. 5 gives an intuition for it.

Next, we use these S-boxes and L-boxes to construct a route so that, with large probability, a seed in a fixed square is joined through the route to some seeds in the other two fixed squares

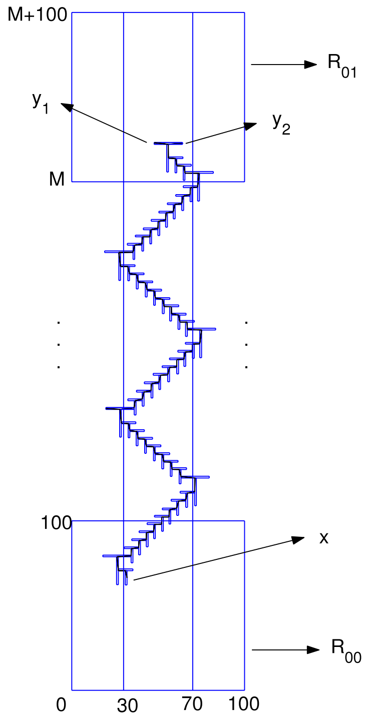

Fig. 7. Producing new seeds in $R_{0,1}(x)$.

(one above, the other on the right). The rigorous arguments are as follows. Set $\mathrm{M} = 10^7$ from now on. For any $\varepsilon > 0$ sufficiently small, fix $r = r(\varepsilon)$ and $h = h(\varepsilon)$ satisfying Proposition 3.2 henceforth. Next, for $x \in \mathbb{H}$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}^+$, define

$$R_{m,n}(x) := \lceil a + m\mathrm{M}h + n\mathrm{M}h\mathrm{i}, b + m\mathrm{M}h + n\mathrm{M}h\mathrm{i}\rfloor = \lceil a, b\rfloor + \mathrm{M}h(m + n\mathrm{i}),$$

where $a = 100h[\Re(x)/100h] + 100h[\Im(x)/100h]\mathrm{i}$ and $b = a + 100(1 + \mathrm{i})$. Then $R_{m,n}(x)$ is a square and $x \in R_{0,0}(x)$.

Suppose that $(x \times s)_r$ is a seed (no matter whether it is horizontal or vertical). We next construct a route by which this seed is joined to two vertical seeds in $R_{0,1}(x)$ with large probability in the following way (see Fig. 7 for intuition). Use S-boxes (horizontal and vertical boxes alternatively) to let the seed spread in the northwest ('$\nwarrow$') direction. If the infection surpasses the line $\{y : \Re(y) = \Re(a) + 30h\}$, then use two L-boxes to change the spread into the northeast ('$\nearrow$') direction. If the infection surpasses the line $\{y : \Re(y) = \Re(a) + 70h\}$, then use two L-boxes to change the spread into the northwest direction. Iterate the procedure until the infection reaches $R_{0,1}(x)$. Then use an extra L-box to get the two infected seeds we want. As a result, by the route described above, the initial vertical seed $(x \times s)_r$ may be joined to two vertical seeds $(y_1 \times t_1)_r$ and $(y_2 \times t_2)_r$, where $y_1, y_2 \in R_{0,1}(x)$. The vertical seed $(y_1 \times t_1)_r$ (centering at y_1 and being generated at time t_1) will be used to make the next route in the 'above' direction, while the vertical seed $(y_2 \times t_2)_r$ (centering at y_2 and being generated at time t_2) will be used to make the next route in the 'right' direction. See Fig. 7 for the precise positions of y_1 and y_2. Note that the route lies entirely in $\lceil a, b + \mathrm{M}h\rfloor$.

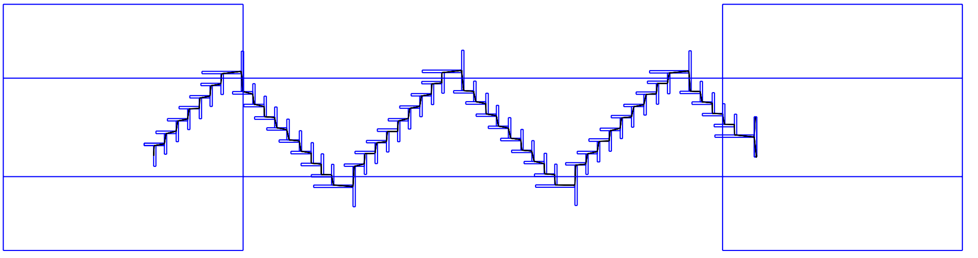

Fig. 8. Producing new seeds in $R_{1,0}(x)$.

The number of steps in the above procedure is no more than M. So, by Proposition 3.2 together with the fact that the events are independent if they are measurable with respect to σ-fields generated by graphical representations within disjoint subgraphs (this has been used several times in Section 3; for details readers can refer to Lemmas 3.5 and 3.6 of [4]), we can get $t_1 + t_2 < \infty$ with large probability. If $t_1 + t_2 < \infty$, then the above procedure generates two seeds as required. Similarly, we can construct a route by which the seed $(x \times s)_r$ is joined to two horizontal seeds in $R_{1,0}(x)$ with large probability. See Fig. 8 for intuition.

Next, we iterate the above procedure many times in both directions (to the right and to above). See Fig. 9 for intuition. For any $n \in \mathbb{N}$, we can construct a route from this iteration in order to get some $y, z \in R_{n,n}$ and $t, u < \infty$ through the route, such that the seed $(x \times s)_r$ is joined to the seeds $(y \times t)_r$ and $(z \times u)_r$ within $\lceil a, b + n\mathrm{M}h(1 + \mathrm{i})\rfloor$.

For any valid sample (that is, a route can be successfully found), we can let the route be unique in some manner. For example, if both the seed in $R_{i-1,j}(x)$ and the seed in $R_{i,j-1}(x)$ can generate new seeds in $R_{i,j}(x)$ in finite time, then we choose the route from $R_{i-1,j}(x)$ to $R_{i,j}(x)$. That is, we put priority to the 'left neighbor'. See Fig. 10 for intuition. From this, we can get that there exist $y, z \in R_{n,n}$, such that the seed $(x \times s)_r$ is joined to two seeds $(y \times t_1^{(n)})_r$ and $(z \times t_2^{(n)})_r$ within $\lceil a, b + n\mathrm{M}h(1 + \mathrm{i})\rfloor$. Furthermore, $t_1^{(n)} + t_2^{(n)} < \infty$ with large probability (depending on n). Denote

$$F_1(s, x, n, 1 + \mathrm{i}) := t_1^{(n)} \quad \text{and} \quad F_2(s, x, n, 1 + \mathrm{i}) := t_2^{(n)},$$

where $1 + \mathrm{i}$ indicates that the orientation of infection is northeast.

Similarly, we can define $F_1(s, x, n, o)$ and $F_2(s, x, n, o)$ for other orientations $o \in \{1-\mathrm{i}, -1+\mathrm{i}, -1-\mathrm{i}\}$. If $F_1(s, x, n, o) + F_2(s, x, n, o) < \infty$, then there exist $x_1, x_2 \in \lceil a, b\rfloor + n\mathrm{M}ho$, such that the seed $(x \times s)_r$ is joined to two seeds $(x_1 \times F_1(s, x, n, o))_r$ and $(x_2 \times F_2(s, x, n, o))_r$, and x, x_1, x_2 are arranged *clockwise*.

Having made the above preparations, we can now state the main proposition in this section.

Proposition 4.1. *Suppose that* $\mathbf{P}(\xi^0$ *survives*$) > 0$. *Let* $x = x(\varepsilon) \in \mathbb{H}$ *with* $\Im(x) > 10h$, *and let* $(x \times 0)_r$ *be a horizontal seed. Then there exists* $\overline{W} > 0$ *which depends only on* ε *and* λ, *such that*

$$\lim_{\varepsilon \to 0+} \liminf_{n \to \infty} \mathbf{P}\left(\frac{7\overline{W}}{6}n < F_1(0, x, n, 1 + \mathrm{i}) < \frac{11\overline{W}}{6}n\right) = 1$$

and

$$\lim_{\varepsilon \to 0+} \liminf_{n \to \infty} \mathbf{P}\left(\frac{7\overline{W}}{6}n < F_2(0, x, n, 1 + \mathrm{i}) < \frac{11\overline{W}}{6}n\right) = 1,$$

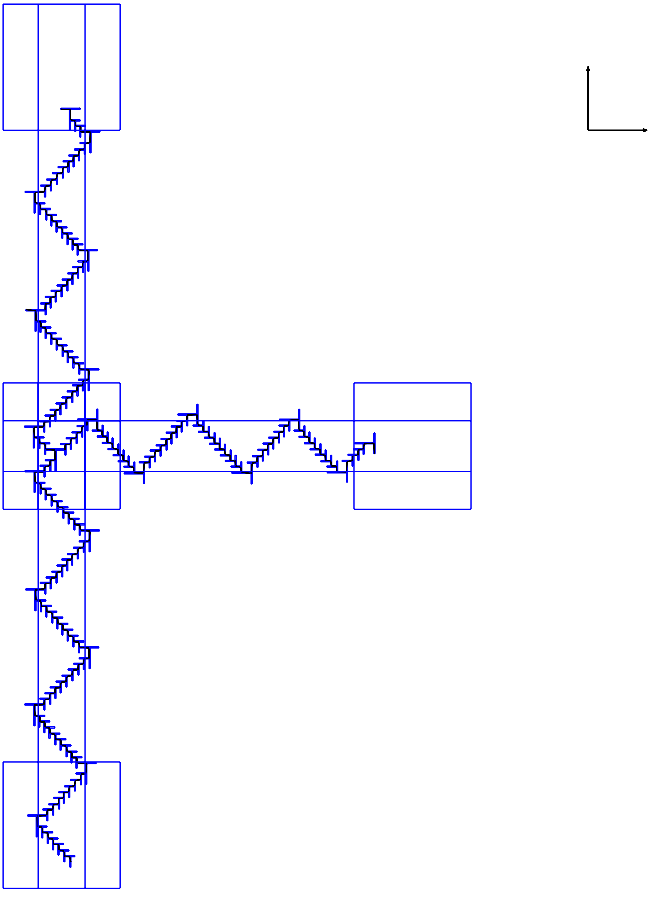

Fig. 9. All S-boxes and L-boxes are disjoint.

where $F_1(0, x, n, 1 + \mathrm{i})$ *and* $F_2(0, x, n, 1 + \mathrm{i})$ *are the time points that generate the two seeds in* $R_{n,n}(x)$ *from the original seed* $(x \times 0)_r$*, respectively, as defined above.*

The proof of Proposition 4.1 is quite similar to the proof of Proposition 4.1 in Chen and Yao [4]. So we omit the formal proof here. Readers can refer to Appendix 2 in Chen and Yao [4] for details. We only state the idea here. We have got a route by which a seed in $R_{m,n}(x)$ is joined to other seeds in $R_{m+1,n}(x)$ and $R_{m,n+1}(x)$ with large probability. As a result, we use the 'dynamic renormalization' method and consider each $R_{m,n}(x)$ as one site. Declare $R_{0,0}(x)$ open if $x \in R_{0,0}(x)$ and $(x \times 0)_r$ is a seed. For $m + n \geq 1$, declare $R_{m,n}(x)$ open if and only if one of the following holds.

(i) $R_{m-1,n}(x)$ is open and the seed in $R_{m-1,n}(x)$ is joined to two seeds in $R_{m,n}(x)$.

(ii) $R_{m-1,n}(x)$ is closed, $R_{m,n-1}(x)$ is open, and the seed in $R_{m,n-1}(x)$ is joined to two seeds in $R_{m,n}(x)$. Refer to Fig. 10 for intuition. The process $(R_{m,n}(x))_{m\in\mathbb{Z},n\in\mathbb{Z}^+}$ is thus an oriented site percolation. Refer to Durrett [5] and Grimmett [7] for more detailed introductions. We can then find a unique open path from $R_{0,0}(x)$ to $R_{n,n}$ with large probability. Furthermore, we can find the unique route constructed by S-boxes and L-boxes, within which the seed in $R_{0,0}(x)$ is joined to another two seeds in $R_{n,n}$. This implies that $F_1(s, x, n, 1 + \mathrm{i})$ is the sum of the times

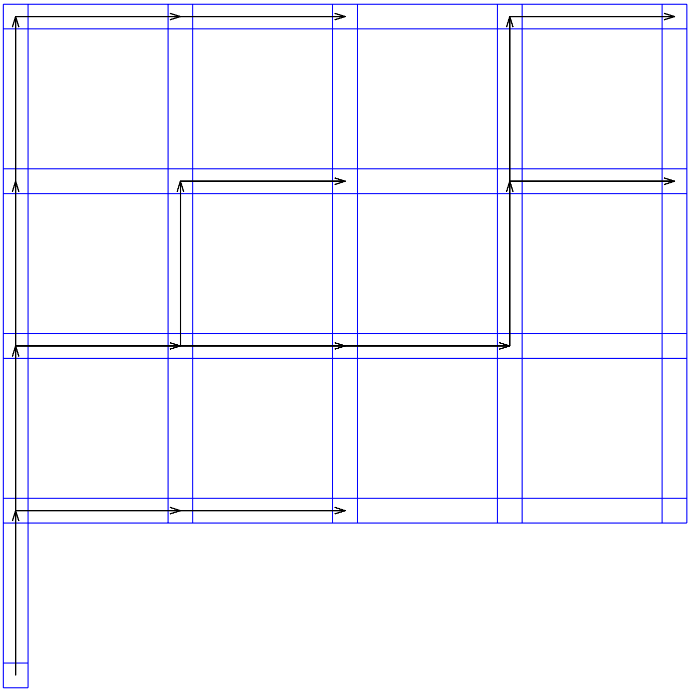

Fig. 10. Dynamic renormalization ($n = 4$).

spent in each box. And $F_2(s, x, n, 1 + \mathrm{i})$ also. Fig. 9 indicates that all S-boxes and L-boxes are disjoint. So the times spent in each box are independent under certain conditions (this has been used several times in Section 3; for details, readers can refer to Lemmas 3.5 and 3.6 of [4]). Through rigorous calculation, we get that the total number of S-boxes on the route is between $2nj_{lower}$ and $2nj_{upper}$. Then, by the law of large numbers, with large probability, the time spent in these S-boxes is between $\frac{7}{6}\overline{S}n$ and $\frac{11}{6}\overline{S}n$. We can deduce that with large probability, the time spent in these L-boxes is between $\frac{7}{6}\overline{L}n$ and $\frac{11}{6}\overline{L}n$, too. Hence with large probability, the total time $F_1(s, x, n, 1 + \mathrm{i})$ is between $\frac{7}{6}\overline{W}n$ and $\frac{11}{6}\overline{W}n$. And $F_2(s, x, n, 1 + \mathrm{i})$ also. Here j_{lower} and j_{upper} are two constants which satisfy $1 \le j_{upper}/j_{lower} < \frac{11}{6}$, and $\overline{S}$, $\overline{L}$ and $\overline{W}$ depend only on λ and ε.

5. The complete convergence theorem

Having established the dynamic renormalization construction, we are now in a position to prove the complete convergence theorem, Theorem 1.1. By Theorem 1.12 of Liggett [10], to prove Theorem 1.1 it suffices to prove that there exists $\Omega_0 \subseteq \Omega_1$ with $\mathbf{P}^{\mu}(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, the next two assertions hold.

(a) $\mathbf{P}_\lambda(x \in \limsup_{t\to\infty} \xi_t^A(\lambda)) = \mathbf{P}_\lambda(\xi^A(\lambda)$ survives) for all $x \in \mathbb{H}$ and $A \subset \mathbb{H}$.

(b) $\lim_{l\to\infty} \liminf_{t\to\infty} \mathbf{P}_\lambda(\xi_t^{B_x(l)}(\lambda) \cap B_x(l) \neq \emptyset) = 1$ for all $x \in \mathbb{H}$.

We will prove (a) and (b) rigorously in Sections 5.1 and 5.2, respectively. The intuitive idea is as follows. We iterate the construction posed in Proposition 4.1 four times to get that, with

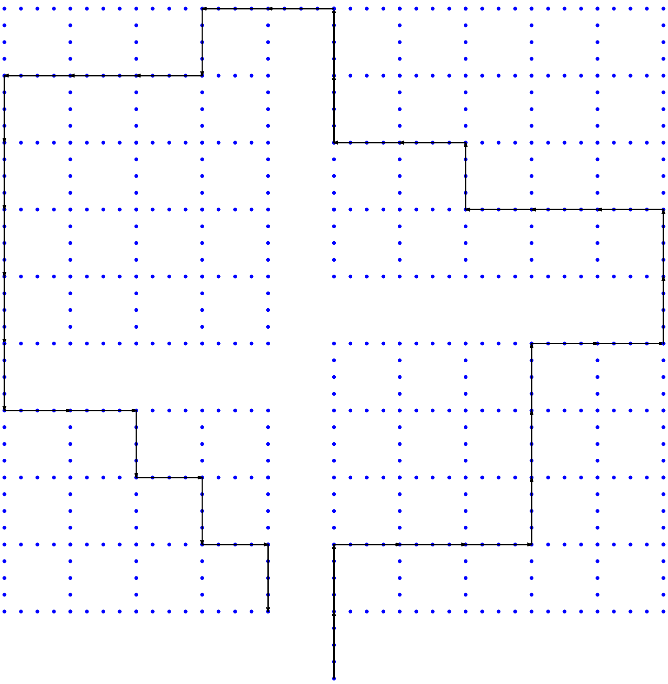

Fig. 11. Description of (a) ($m = 5$).

large probability, a seed in $\lceil a, b \rfloor \times 0$ is joined to another seed in $\lceil e, f \rfloor \times [3\overline{W}n, \infty)$. See Fig. 11 for intuition. From this, we get (a). Extra tricks are needed to check (b). We will prove that, for each n, with large probability, a seed in $\lceil \widetilde{a}, \widetilde{b} \rfloor \times [0, \widetilde{W}]$ is joined to another seed in $\lceil \widetilde{e}, \widetilde{f} \rfloor \times [(n-1)\widetilde{W}, (n+1)\widetilde{W}]$. Together with the fact that every remote site cannot be infected in a short time, we get (b).

5.1. *Proof of (a)*

Without loss of generality, we suppose that $\mathbf{P}(\xi^A \text{ survives}) > 0$, since otherwise both sides in (a) are equal to 0 and (a) holds trivially. We first prove the case when A is a nonempty *finite* subset of $\mathbb{H}$. Let x_0 be any element of A, and let $\sigma_0 = 0$. Hence x_0 is infected at time σ_0 for the process ξ^A. Then define δ_k, τ_k, Y_k, σ_{k+1}, and x_{k+1} inductively for $k \geq 0$ as follows. (See Fig. 12 for intuition.) Let

$$\begin{aligned}\delta_k := \sup\{t > \sigma_k : & \; x_k \times \sigma_k \text{ is joined within } \langle x_k - r - 1, x_k + r + 1 + 2000h\rangle \\ & \text{to } \lceil x_k - r - 1, x_k + r + 1 + 2000h\rfloor \times t\}\end{aligned}$$

be the death time for the contact process starting with single infection x_k at time σ_k and evolving within $\langle x_k - r - 1, x_k + r + 1 + 2000h\rangle$. Then $\delta_k < \infty$ almost surely on $\{\sigma_k < \infty\}$. Let

$$\begin{aligned}\tau_k := \min\{t > \sigma_k : & \; x_k \times \sigma_k \text{ is joined within } \langle x_k - r - 1, x_k + r + 1 + 2000h\rangle \\ & \text{to } z \times t \text{ for all } z \in \lceil x_k - r + 2000h, x_k + r + 2000h\rfloor\} - \sigma_k\end{aligned}$$

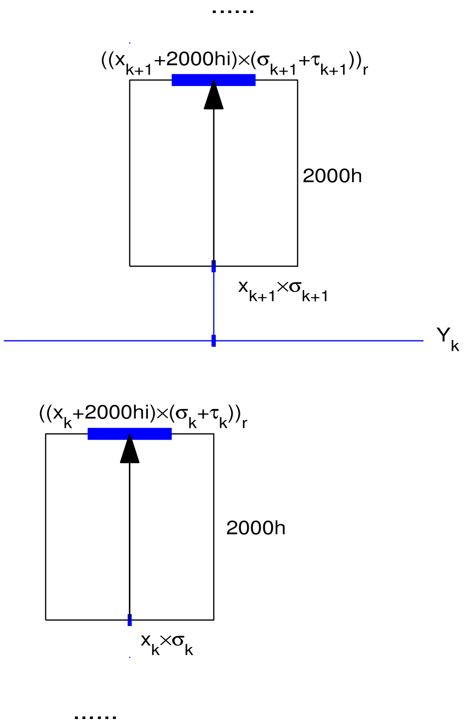

Fig. 12. Inductive definitions.

be the waiting time until the first seed on the top appears. Let

$$Y_k := \sup\left\{\Im(x) : x \in \bigcup_{t \le \delta_k} \xi_t^A\right\}.$$

Then $Y_k < \infty$ almost surely on $\{\sigma_k < \infty\}$. Furthermore, let

$$\sigma_{k+1} := \inf\{t > \delta_k : \exists x \in \xi_t^A, \text{ s.t. } \Im(x) = Y_k + 2\},$$

and let x_{k+1} be the corresponding infected site. Note that, for any $k = 0, 1, 2, \ldots$, if $\tau_k < \infty$, then $\sigma_k + \tau_k < \delta_k$.

Define $K := \min\{k : \tau_k < \infty\}$, and denote $p := \mathbf{P}(\tau_0 < \infty) > 0$. For $t > 0$, we use $\mathscr{A}_t$ to denote the σ-fields generated by the graphical representation for the contact process until time t. Therefore, by translation invariance and the fact that σ_k is a stopping time for all $k \in \mathbb{N}$, we get that, if $\sigma_k < \infty$ for all k and $\sigma_k \uparrow \infty$, then

$$\mathbf{P}(K < \infty \mid \mathscr{A}_{\sigma_k}) \ge \mathbf{P}(\tau_k < \infty \mid \mathscr{A}_{\sigma_k}) = \mathbf{P}(\tau_0 < \infty) = p > 0$$

for any $k = 0, 1, 2, \ldots$. That is,

$$\mathbf{P}(\mathbf{P}(K < \infty \mid \mathscr{A}_{\sigma_k}) \ge p > 0 \text{ for all } k = 0, 1, 2, \ldots \mid \sigma_k < \infty \text{ for all } k, \sigma_k \uparrow \infty) \\ = 1. \tag{5.1}$$

Furthermore, using the martingale convergence theorem, we can get that

$$\mathbf{P}(K < \infty \mid \mathscr{A}_{\sigma_k}) \to \mathbf{1}_{\{K<\infty\}} \quad \text{a.s. on } \{\sigma_k < \infty \text{ for all } k, \sigma_k \uparrow \infty\}$$

as k tends to infinity. So, by (5.1), we get

$$\mathbf{P}(K < \infty \mid \sigma_k < \infty \text{ for all } k, \sigma_k \uparrow \infty) = 1. \tag{5.2}$$

Also, note that

$$\mathbf{P}(\sigma_k < \infty \text{ for all } k, \sigma_k \uparrow \infty \mid \xi^A \text{ survives}) = 1. \tag{5.3}$$

By (5.2) and (5.3), together with our assumption that $\mathbf{P}(\xi^A \text{ survives}) > 0$, we get

$$\mathbf{P}(K < \infty \mid \xi^A \text{ survives}) = 1. \tag{5.4}$$

If $K < \infty$, then let $y_1 := x_K + 2000h\mathrm{i}$, and let $t_1 = \sigma_k + \tau_k$. Therefore, $(y_1 \times t_1)_r$ is a horizontal seed. Let

$$\zeta = F_1(F_1(F_1(F_1(t_1, y_1, m, 1+\mathrm{i}), y_2, m, -1+\mathrm{i}), y_3, m, -1-\mathrm{i}), y_4, m-1, 1-\mathrm{i}),$$

and let $(\vartheta \times \zeta)_r$ be the corresponding seed if $\zeta < \infty$. Here, F_1 is defined as in Section 4, and y_2, y_3, y_4 are the centers of corresponding seeds in each step. Therefore,

$$\vartheta \in R_{-1,1}(y_1) \subset B_{y_1}(2\mathrm{M}h).$$

See Fig. 11 for intuition. Note that ζ is the sum of the times spent in each of the four orientations, as shown in Fig. 11. These times are independent under certain conditions (this has been used several times in Section 3; for details readers can refer to Lemmas 3.5 and 3.6 of [4]). Together with Proposition 4.1, we get

$$\lim_{\varepsilon\to 0+} \liminf_{m\to\infty} \mathbf{P}(3\overline{W}m \le \zeta < \infty \mid K < \infty) = 1,$$

which implies that

$$\lim_{\varepsilon\to 0+} \liminf_{m\to\infty} \mathbf{P}(\exists t \ge 3\overline{W}m, \text{s.t. } \xi_t^A \cap B_{y_1}(2\mathrm{M}h) \ne \emptyset \mid K < \infty) = 1.$$

By the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0+} \mathbf{P}\left(\limsup_{t\to\infty} \xi_t^A \cap B_{y_1}(2\mathrm{M}h) \ne \emptyset \mid K < \infty\right) = 1.$$

Furthermore,

$$\lim_{\varepsilon\to 0+} \mathbf{P}\left(\limsup_{t\to\infty} \xi_t^A \ne \emptyset \mid K < \infty\right) = 1. \tag{5.5}$$

By (5.4) and (5.5), we have

$$\mathbf{P}\left(\limsup_{t\to\infty} \xi_t^A \ne \emptyset \mid \xi^A \text{ survives}\right) = 1.$$

Turning to the quenched law, there exists $\Omega_A \subseteq \Omega_1$ with $\mathbf{P}^\mu(\Omega_A) = 1$, such that, for all $\omega \in \Omega_A$,

$$\mathbf{P}_\lambda\left(\xi^A(\lambda) \text{ survives}, \limsup_{t\to\infty} \xi_t^A(\lambda) = \emptyset\right) = 0. \tag{5.6}$$

That is, $\xi^A(\lambda)$ survives strongly if it survives. See page 42 of Liggett [10] for the definition of ‘strong survival’.

Fix $\omega \in \Omega_A$. For any $y, z \in \mathbb{H}$, we have

$$\mathbf{P}_\lambda(z \in \xi_1^y(\lambda)) > 0.$$

We can construct an appropriate sequence of stopping times and use the strong Markov property under the quenched law to get

$$z \in \limsup_{t\to\infty} \xi_t^A(\lambda) \quad \text{a.s. on } \{y \in \limsup_{t\to\infty} \xi_t^A(\lambda)\}.$$

That is,

$$\mathbf{P}_\lambda\left(z \notin \limsup_{t\to\infty} \xi_t^A(\lambda),\, y \in \limsup_{t\to\infty} \xi_t^A(\lambda)\right) = 0$$

for any $y, z \in \mathbb{H}$. Since

$$\left\{\limsup_{t\to\infty} \xi_t^A(\lambda) \neq \emptyset\right\} = \bigcup_{y\in\mathbb{H}} \left\{y \in \limsup_{t\to\infty} \xi_t^A(\lambda)\right\},$$

we have

$$\mathbf{P}_\lambda\left(z \notin \limsup_{t\to\infty} \xi_t^A(\lambda),\, \limsup_{t\to\infty} \xi_t^A(\lambda) \neq \emptyset\right) = 0. \tag{5.7}$$

From (5.6) and (5.7), together with the fact that $\{z \in \limsup_{t\to\infty} \xi_t^A(\lambda)\} \subseteq \{\xi^A(\lambda) \text{ survives}\}$, we can deduce that, for any finite subset $A \subseteq \mathbb{H}$, $\omega \in \Omega_A$, and $z \in \mathbb{H}$,

$$\mathbf{P}_\lambda(z \in \limsup_{t\to\infty} \xi_t^A(\lambda)) = \mathbf{P}(\xi^A(\lambda) \text{ survives}).$$

Then, let

$$\Omega_0' := \bigcap_{A\subset\mathbb{H}, |A|<\infty} \Omega_A.$$

Then $\mathbf{P}^\mu(\Omega_0') = 1$. Moreover, (a) holds for all $\omega \in \Omega_0'$, $x \in \mathbb{H}$, and $A \subset \mathbb{H}$ with $|A| < \infty$.

Next, we consider the case when $|A| = \infty$. We can get that, for any $n > 0$, there exists m_n such that $\mathbf{P}(\xi^B \text{ survives}) > 1 - 4^{-n}$ for any $B \subset \mathbb{H}$ with $|B| \geq m_n$, for a reason similar to the proof of Lemma 3.1. This implies that

$$\mathbf{P}^\mu(\{\omega \in \Omega_1 : \mathbf{P}_\lambda(\xi^B(\lambda) \text{ survives}) \geq 1 - 2^{-n}\}) \geq 1 - 2^{-n}.$$

Let

$$\Xi_n' := \{\omega \in \Omega_1 : \mathbf{P}_\lambda(\xi^B(\lambda) \text{ survives}) \geq 1 - 2^{-n}\}$$

for any $n \in \mathbb{N}$. Then Ξ_n' decreases as n increases. Set

$$\Omega_0'' := \Omega_0' \cap \left(\bigcap_n \Xi_n'\right).$$

Then $\mathbf{P}^\mu(\Omega_0'') = 1$. If $\omega \in \Omega_0''$, $x \in \mathbb{H}$, $A \subset \mathbb{H}$, and $|A| = \infty$, then let (A_n) be an increasing sequence of finite sets which satisfy $\lim_{n\to\infty} A_n = A$ and $|A_n| > m_n$ for all n. Then, for any

$x \in \mathbb{H}$, we have

$$\begin{aligned}\mathbf{P}_\lambda\left(x \in \limsup_{t\to\infty} \xi_t^A(\lambda)\right) &\geq \lim_{n\to\infty} \mathbf{P}_\lambda\left(x \in \limsup_{t\to\infty} \xi_t^{A_n}(\lambda)\right)\\ &= \lim_{n\to\infty} \mathbf{P}_\lambda(\xi_t^{A_n}(\lambda) \text{ survives})\\ &\geq \lim_{n\to\infty} (1-2^{-n}) = 1.\end{aligned}$$

But $\xi^A(\lambda)$ survives with $\mathbf{P}_\lambda$-probability 1. As a result,

$$\mathbf{P}_\lambda\left(x \in \limsup_{t\to\infty} \xi_t^A(\lambda)\right) = \mathbf{P}_\lambda(\xi_t^A(\lambda) \text{ survives}) = 1.$$

Furthermore, (a) holds for all $\omega \in \Omega_0''$, $x \in \mathbb{H}$, and $A \subset \mathbb{H}$.

5.2. *Proof of (b)*

We begin with the seed $(x \times s)_r$. For convenience, for any $n \in \mathbb{N}$, we use the following algorithm to generate a new seed from $(x \times s)_r$ and record the time used. Recall that, in the algorithm, F_1 and F_2 are as defined in Section 4.

Algorithm
(0) Set $t = s$ and $y = x$.
(1) Set $s' = s - 100\overline{W}n[s/100\overline{W}n]$, $v = 8 \cdot \mathbf{1}_{\{s' \leq 37\overline{W}n\}}$ and $u = 9 - v$.
One can check that

$$s' + (6u + 10v) \cdot \left[\frac{7}{6}\overline{W}n, \frac{11}{6}\overline{W}n\right] \subseteq [100\overline{W}n, 200\overline{W}n).$$

Operate (2)∼(7) u times
(2) $t = F_2(t, n, 1 + \mathrm{i})$;
(3) $t = F_1(t, n, 1 - \mathrm{i})$;
(4) $t = F_1(t, n, 1 + \mathrm{i})$;
(5) $t = F_1(t, n, -1 + \mathrm{i})$;
(6) $t = F_2(t, n - 1, -1 - \mathrm{i})$;
(7) $t = F_2(t, n + 1, -1 + \mathrm{i})$;
Operate (8)∼(17) v times
(8) $t = F_2(t, n, 1 + \mathrm{i})$;
(9) $t = F_1(t, n, 1 - \mathrm{i})$;
(10) $t = F_2(t, n, 1 + \mathrm{i})$;
(11) $t = F_1(t, n, 1 - \mathrm{i})$;
(12) $t = F_1(t, n, 1 + \mathrm{i})$;
(13) $t = F_1(t, n, -1 + \mathrm{i})$;
(14) $t = F_2(t, n - 1, -1 - \mathrm{i})$;
(15) $t = F_1(t, n, -1 + \mathrm{i})$;
(16) $t = F_2(t, n, -1 - \mathrm{i})$;
(17) $t = F_2(t, n + 1, -1 + \mathrm{i})$;
(18) Return t.

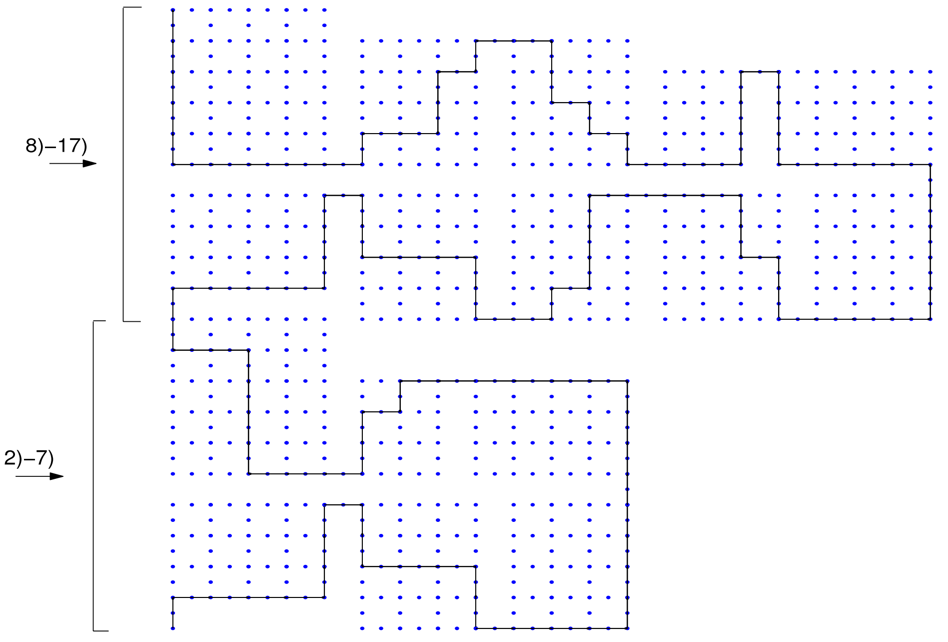

Fig. 13. Description of $G(s, x, n, \mathrm{i})$.

If the output value $t < \infty$, then the corresponding site belongs to $R_{18(n+1),0}(x)$. Moreover, by Proposition 4.1, we know that $t \in [100\overline{W}n, 200\overline{W}n)$ with large probability if $s \in [0, 100\overline{W}n)$. Denote

$$G(s, x, n, \mathrm{i}) := t.$$

See Fig. 13 for intuition. Similarly, we denote $G(s, x, n, 1)$ the corresponding site that belongs to $R_{0,18(n+1)}(x)$ generated in the same way (but in a different direction).

Next, we iterate the above procedure many times in both directions (to the right and above). See Fig. 10 for intuition. For any $m \in \mathbb{N}$, we can construct a route through this iteration in order to get a new seed in $R_{18(n+1)m,18(n+1)m}(x)$. The procedure is similar to the argument before Proposition 4.1, and we can use a similar way (prior to the 'left neighbor') to make the route unique. We denote the time by $L(s, x, n, m, 1 + \mathrm{i})$, which is finite with large probability (depending on n and m). Here, $1 + \mathrm{i}$ indicates that the orientation of infection is northeast.

Similarly, we can define $L(s, x, n, m, o)$ for other orientations $o \in \{1 - \mathrm{i}, -1 + \mathrm{i}, -1 - \mathrm{i}\}$. We then have the following proposition, which is parallel to Proposition 4.1, but it is more accurate.

Proposition 5.1. *Suppose that* $\mathbf{P}(\xi^0$ *survives*$) > 0$. *Let* $x = x(\varepsilon) \in \mathbb{H}$ *with* $\Im(x) > 10h$, *and let* $(x \times 0)_r$ *be a horizontal seed. Then*

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \liminf_{m\to\infty} \mathbf{P}(200\overline{W}nm < L(0, x, n, m, 1 + \mathrm{i}) < 200\overline{W}n(m + 1)) = 1.$$

Proof. By Proposition 4.1 and the FKG inequality, we have that with large probability

$$G(s, x, n, \mathrm{i}) \in [100k\overline{W}n, 100(k + 1)\overline{W}n) \quad \text{and}$$
$$G(s, x, n, \mathrm{i}) \in [100k\overline{W}n, 100(k + 1)\overline{W}n)$$

if $s \in [100(k-1)\overline{W}n, 100k\overline{W}n)$. Similar to the idea of Proposition 4.1, this situation corresponds to a 1-dependent site percolation. Using the result of 1-dependent site percolation (see [5]), we get the conclusion. □

Next, we prove (b). Without loss of generality, we suppose that $\Im(x) \geq 10h$. Suppose that $(x \times 0)_r$ is a horizontal seed. Let

$$\mu := L(L(L(L(0, x, n, m, 1 + \mathrm{i}), x_1, n, m, -1 + \mathrm{i}), x_2, n, m, -1 - \mathrm{i}), x_3, n, m - 1, 1 - \mathrm{i}),$$

and let $(\nu \times \mu)_r$ be the corresponding seed if $\mu < \infty$. Then $\nu \in B_x(40n\mathrm{M}h)$. Here x_1, x_2, and x_3 are the centers of the corresponding seeds in each step. Note that μ is the sum of the times spent in each of the four orientations. These times are independent under certain conditions (this has been used several times in Section 3; for details readers can refer to Lemmas 3.5 and 3.6 of [4]). Together with Proposition 5.1, we get

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \liminf_{m\to\infty} \mathbf{P}(800\overline{W}nm - 200\overline{W}n \leq \mu \leq 800\overline{W}nm + 600\overline{W}n) = 1.$$

That is,

$$\lim_{\varepsilon\to 0+} \liminf_{n\to\infty} \liminf_{m\to\infty} \mathbf{P}(\exists t \in [800\overline{W}n(m-1), 800\overline{W}n(m+1)],$$
$$\text{s.t. } \xi_t^{\lceil x-r, x+r \rfloor} \cap B_x(40n\mathrm{M}h) \neq \emptyset) = 1.$$

We can deduce that, for any $\delta > 0$, there exist $n_0 \in \mathbb{N}$ and $m_0 \geq 2$, such that

$$\mathbf{P}(\exists t \in [800\overline{W}n_0(m_0 - 1), 800\overline{W}n_0(m_0 + 1)], \text{ s.t. } \xi_t^{\lceil x-r, x+r \rfloor} \cap B_x(40n_0\mathrm{M}h) \neq \emptyset)$$
$$> 1 - \delta^2.$$

Turning to the quenched law, denote

$$\Omega_\delta^{(1)} := \{\omega \in \Omega_1 : \mathbf{P}_\lambda(\exists t \in [800\overline{W}n_0(m_0 - 1), 800\overline{W}n_0(m_0 + 1)],$$
$$\text{s.t. } \xi_t^{\lceil x-r, x+r \rfloor} \cap B_x(40n_0\mathrm{M}h)) > 1 - \delta\}.$$

Then $\mathbf{P}^\mu(\Omega_\delta^{(1)}) \geq 1 - \delta$.

On the other hand, consider the Richardson's process (ζ_t) on $\mathbb{H}$ by suppressing all recoveries from (ξ_t), we have

$$\lim_{l\to\infty} \mathbf{P}(\inf\{t > 0 : \zeta_t^A \cap B_x(40n_0\mathrm{M}h) \neq \emptyset\}$$
$$\leq 800\overline{W}n_0(m_0 + 1) + 1 \text{ for some finite subset } A \subseteq \mathbb{H} \setminus B_x(l)) = 0.$$

So, for the above $\delta > 0$, there exists $l_\delta > 40n_0\mathrm{M}h$, such that

$$\mathbf{P}(\text{for some finite subset } A \subseteq \mathbb{H} \setminus B_x(l_\delta),$$
$$\times \text{ there exists } t \in (0, 800\overline{W}n_0(m_0 + 1) + 1] \text{s.t. } \zeta_t^A \cap B_x(40n_0\mathrm{M}h) \neq \emptyset) < \delta^2.$$

Turning to the quenched law, denote

$$\Omega_\delta^{(2)} := \{\omega \in \Omega_1 : \mathbf{P}_\lambda(\text{for some finite subset } A \subseteq \mathbb{H} \setminus B_x(l_\delta),$$
$$\text{there exists } t \in (0, 800\overline{W}n_0(m_0 + 1) + 1]$$
$$\text{s.t. } \zeta_t^A \cap B_x(40n_0\mathrm{M}h) \neq \emptyset) < \delta\}.$$

Then $\mathbf{P}^\mu(\Omega_\delta^{(2)}) \geq 1 - \delta$. So $\mathbf{P}^\mu(\Omega_\delta^{(1)} \cap \Omega_\delta^{(2)}) \geq 1 - 2\delta$.

Next, fix $\omega \in \Omega_\delta^{(1)} \cap \Omega_\delta^{(2)}$. For any $s \geq 1$, set

$$\tau_s := \inf\{u \geq s-1 : \xi_u^{\lceil x-r,x+r \rfloor} \cap B_x(l_\delta) = \emptyset\}.$$

Then τ_s is a stopping time. Using the strong Markov property under the quenched law, together with the facts that $\xi_t^A \subseteq \zeta_t^A$ for any t and ζ_t^A increases as t increases, we can get that, for any finite subset $A \subseteq \mathbb{H} \setminus B_x(l_\delta)$,

$$\begin{aligned}
&\mathbf{P}_\lambda(\exists 0 < t \\
&\quad \leq 800\overline{W}n_0(m_0+1),\ \text{s.t. } \xi_{t+s}^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \neq \emptyset \mid \xi_{\tau_s}^{\lceil x-r,x+r \rfloor}(\lambda) = A) \\
&\quad \leq \mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1)+1,\ \text{s.t. } \zeta_t^A \cap B_x(40n_0\mathrm{M}h) \neq \emptyset) \leq \delta.
\end{aligned}$$

Then we use the strong Markov property under the quenched law again to get

$$\begin{aligned}
&\mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1),\ \text{s.t. } \xi_{t+s}^{\lceil x-r,x+r \rfloor}(\lambda) \\
&\quad \cap B_x(40n_0\mathrm{M}h) \neq \emptyset, \xi_u^{\lceil x-r,x+r \rfloor} \cap B_0(l_\delta) = \emptyset \text{ for some } u \in [s-1,s]) \\
&\quad = \mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1),\ \text{s.t. } \xi_{t+s}^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \neq \emptyset, \tau_s \leq s) \\
&\quad = \mathbf{P}_\lambda(\mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1), \\
&\qquad \times \text{s.t. } \xi_t^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \neq \emptyset \mid \mathscr{F}_{\tau_s}); \tau_s \leq s) \\
&\quad = \mathbf{P}_\lambda(\mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1), \\
&\qquad \times \text{s.t. } \xi_t^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \neq \emptyset \mid \xi_{\tau_s}^{\lceil x-r,x+r \rfloor}(\lambda)); \tau_s \leq s) \\
&\quad \leq \delta \cdot \mathbf{P}_\lambda(\tau_s \leq s) \leq \delta
\end{aligned}$$

for any $s \geq 1$. Therefore, for any $s \geq 1$, we have

$$\begin{aligned}
&\mathbf{P}_\lambda(\xi_u^{\lceil x-r,x+r \rfloor} \cap B_x(l_\delta) \neq \emptyset \text{ for all } u \in [s-1,s]) \\
&\quad \geq \mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1),\ \text{s.t. } \xi_{t+s}^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \neq \emptyset) \\
&\qquad - \mathbf{P}_\lambda(\exists 0 < t \leq 800\overline{W}n_0(m_0+1),\ \text{s.t. } \xi_{t+s}^{\lceil x-r,x+r \rfloor}(\lambda) \cap B_x(40n_0\mathrm{M}h) \\
&\quad \neq \emptyset, \xi_u^{\lceil x-r,x+r \rfloor} \cap B_x(l_\delta) = \emptyset \text{ for some } u \in [s-1,s]) \\
&\quad \geq 1 - 2\delta.
\end{aligned}$$

Since $\xi_t^{A_1} \subseteq \xi_t^{A_2}$ for all $t \geq 0$ if $A_1 \subseteq A_2$, we have, for any $\omega \in \Omega_\delta^{(1)} \cap \Omega_\delta^{(2)}$,

$$\mathbf{P}_\lambda(\xi_s^{B_x(l_\delta)} \cap B_x(l_\delta) \neq \emptyset \text{ for all } s \in [t,t+1]) \geq 1 - 2\delta$$

for any $t \geq 0$. So, if we denote

$$\Omega_\delta := \{\omega \in \Omega_1 : \mathbf{P}_\lambda(\xi_s^{B_x(l_\delta)} \cap B_x(l_\delta) \neq \emptyset \text{ for all } s \in [t,t+1] \geq 1 - 2\delta)\},$$

then

$$\mathbf{P}^\mu(\Omega_\delta) \geq \mathbf{P}^\mu(\Omega_\delta^{(1)} \cap \Omega_\delta^{(2)}) \geq 1 - 2\delta.$$

And furthermore, there exists $l_n \uparrow \infty$ such that, for any $n \in \mathbb{N}$ and $t \geq 0$,

$$\mathbf{P}^{\mu}(\Omega_{n,t}) \geq 1 - 2^{-n-t-1},$$

where we set

$$\Omega_{n,t} := \{\omega \in \Omega_1 : \mathbf{P}_\lambda(\xi_s^{B_x(l_n)} \cap B_x(l_n) \neq \emptyset \text{ for all } s \in [t, t+1] \geq 1 - 2^{-n-t-1})\}$$

for any $n \in \mathbb{N}$ and $t \geq 0$. Next, set

$$\Omega_n := \bigcap_{k=0}^{\infty} \Omega_{n,k}$$

for any $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have $\mathbf{P}(\Omega_n) \geq 1 - 2^{-n}$, and, on Ω_n,

$$\begin{aligned}\liminf_{t\to\infty} \mathbf{P}_\lambda(\xi_t^{B_x(l_n)} \cap B_x(l_n) \neq \emptyset) &\geq \mathbf{P}(\forall t \geq 0, \xi_t^{B_x(l_n)} \cap B_x(l_n) \neq \emptyset)\\ &= \mathbf{P}_\lambda\left(\bigcap_{k=0}^{\infty}\{\xi_s^{B_x(l_n)} \cap B_x(l_n) \neq \emptyset \text{ for all } s \in [k, k+1]\}\right) \geq 1 - 2^{-n}.\end{aligned}$$

Note that Ω_n increases as n increases. So, if we set

$$\Omega_0''' := \bigcup_{n=1}^{\infty} \Omega_n,$$

then $\mathbf{P}^{\mu}(\Omega_0''') = 1$, and, on Ω_0''',

$$\lim_{n\to\infty} \liminf_{t\to\infty} \mathbf{P}_\lambda(\xi_t^{B_x(l_n)} \cap B_x(l_n) \neq \emptyset) = 1.$$

That is, (b) holds for all $\omega \in \Omega_0'''$.

Finally, set $\Omega_0 := \Omega_0'' \cap \Omega_0'''$. As a result, (a) and (b) hold for all $\omega \in \Omega_0$. So, we have proved the complete convergence theorem, Theorem 1.1.

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