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On Coherency and Other Properties of MAXVAR

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Abstract This paper is concerned with the MAXVAR risk measure on $\mathscr{L}^2$ space. We present an elementary and direct proof of its coherency and averseness. Based on the observation that the MAXVAR measure is a continuous convex combination of the CVaR measure, we provide an explicit formula for the risk envelope of MAXVAR.

Keywords Coherent risk measure · Risk averse · Risk envelope

Mathematics Subject Classification (2010) 49N15 · 91B30

1 Introduction

In Cherny and Madan [2] and Cherny and Orlov [3], a new kind of risk measure—"MAXVAR"—is proposed, which is useful in the analysis of large portfolios. Given a probability

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space $(\Omega, \Sigma, \mathbb{P}_0)$ and a random variable $X \in \mathscr{L}^2(\Omega, \Sigma, \mathbb{P}_0)$, where $\mathscr{L}^2(\Omega, \Sigma, \mathbb{P}_0)$ is the square integrable Lebesgue space ($\mathscr{L}^2$ for short), the MAXVAR is defined as

$$\text{MAXVAR}_n(X) := \mathbb{E}(\max\{X_1, \ldots, X_n\}),$$

where $X_1, \ldots, X_n$ are i.i.d. copies of X. We call $\text{MAXVAR}_n(\cdot)$ the "MAXVAR risk measure."

Note that $\text{MAXVAR}_n(\cdot)$ is always finite on $\mathscr{L}^2$ since $|\text{MAXVAR}_n(X)| \leq n\mathbb{E}(|X|) < +\infty$ for any $X \in \mathscr{L}^2$.

In [2, 3], the name of "MINVAR risk measure" was used. Since we treat risk measures as a nondecreasing function, we use "MAXVAR risk measure" instead. Different from the papers [2, 3], which considered coherency of MINVAR in $\mathscr{L}^\infty$ space, this paper deals with the $\mathscr{L}^2$ space. Our proof of the coherency of MAXVAR risk measure is direct and independent of [2, 3]. Moreover, we show risk averseness of MAXVAR and give an explicit formula for its risk envelope.

In Section 2, we present a simple proof for the coherency of MAXVAR. We show its averseness in Section 3. Section 4 is devoted to the discussion of a continuous representation of MAXVAR and Section 5 provides an explicit formula for its risk envelope.

2 Coherency of MAXVAR

In this section, we show that MAXVAR is a coherent risk measure in basic sense of Rockafellar.

Definition 1 (Rockafellar [5]) A functional $\mathcal{R} : \mathscr{L}^2 \to (-\infty, +\infty]$ is a coherent risk measure in basic sense if it satisfies

(A1) $\mathcal{R}(C) = C$ for all constant C;
(A2) ("convexity") $\mathcal{R}(\lambda X + (1-\lambda)Y) \leq \lambda \cdot \mathcal{R}(X) + (1-\lambda) \cdot \mathcal{R}(Y)$ for any $X, Y \in \mathscr{L}^2$ and any fixed $0 \leq \lambda \leq 1$;
(A3) ("monotonicity") $\mathcal{R}(X) \leq \mathcal{R}(Y)$ for any $X, Y \in \mathscr{L}^2$ satisfying $X \leq Y$;
(A4) ("closedness") If $\|X^k - X\|_2 \to 0$ and $\mathcal{R}(X^k) \leq 0$ for all $k \in \mathbb{N}$, then $\mathcal{R}(X) \leq 0$;
(A5) ("positive homogeneity") $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for any $\lambda > 0$ and $X \in \mathscr{L}^2$.

Theorem 1 $\text{MAXVAR}_n(\cdot)$ *is a coherent risk measure in basic sense.*

Proof **(A1)** is obvious by definition. **(A5)** is also easy to check since if $X_1, \ldots, X_n$ are i.i.d. copies of X and $\lambda > 0$, then $\lambda X_1, \ldots, \lambda X_n$ are i.i.d. copies of λX.

Proof of **(A2)** We only need to show the following subadditive property of MAXVAR

$$\text{MAXVAR}_n(X + Y) \leq \text{MAXVAR}_n(X) + \text{MAXVAR}_n(Y) \quad \forall X, Y. \tag{1}$$

Then, (1) and **(A5)** imply **(A2)**. For any $X, Y \in \mathscr{L}^2$, take $(X_1, Y_1), \ldots, (X_n, Y_n)$ as i.i.d. copies of the two dimensional random vector (X, Y). That is, the random vectors $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and have the same joint distribution as the random vector (X, Y). Then, $X_1, \ldots, X_n$ are i.i.d. copies of X and $Y_1, \ldots, Y_n$ are i.i.d. copies of Y. We next show that $X_1 + Y_1, \ldots, X_n + Y_n$ are i.i.d. copies of $X + Y$.

Since (X_i, Y_i) has the same joint distribution as (X, Y), $i = 1, \dots, n$, it follows that $X_i + Y_i$ has the same distribution as $X + Y$. In order to prove that $X_1 + Y_1, \dots, X_n + Y_n$ are independent, we only need to prove that for any $t_1, \dots, t_n \in \mathbb{R}$,

$$\mathbb{P}_0(X_1 + Y_1 \leq t_1, \dots, X_n + Y_n \leq t_n) = \mathbb{P}_0(X_1 + Y_1 \leq t_1) \cdots \mathbb{P}_0(X_n + Y_n \leq t_n). \quad (2)$$

In fact, since the random vectors $(X_1, Y_1), \dots, (X_n, Y_n)$ are independent, we have

$$\mathbb{P}_0((X_1, Y_1) \in B_1, \dots, (X_n, Y_n) \in B_n) = \mathbb{P}_0((X_1, Y_1) \in B_1) \cdots \mathbb{P}_0((X_n, Y_n) \in B_n) \quad (3)$$

for any Borel sets $B_1, \dots, B_n \subseteq \mathbb{R}^2$. In particular, if we take

$$B_i = \{(x, y) \in \mathbb{R}^2 : \ x + y \leq t_i\}$$

for any $1 \leq i \leq n$ in (3), we can get (2). Therefore, $X_1 + Y_1, \dots, X_n + Y_n$ are independent. Therefore, they are i.i.d. copies of $X + Y$.

Since the definition of MAXVAR does not depend on the choice of the i.i.d. copies, we have

$$\begin{aligned} \mathrm{MAXVAR}_n(X) &= \mathbb{E}(\max\{X_1, \dots, X_n\}), \\ \mathrm{MAXVAR}_n(Y) &= \mathbb{E}(\max\{Y_1, \dots, Y_n\}), \\ \mathrm{MAXVAR}_n(X + Y) &= \mathbb{E}(\max\{X_1 + Y_1, \dots, X_n + Y_n\}). \end{aligned}$$

Furthermore, since

$$\max\{X_1 + Y_1, \dots, X_n + Y_n\} \leq \max\{X_1, \dots, X_n\} + \max\{Y_1, \dots, Y_n\},$$

we get

$$\begin{aligned} \mathrm{MAXVAR}_n(X + Y) &= \mathbb{E}(\max\{X_1 + Y_1, \dots, X_n + Y_n\}) \\ &\leq \mathbb{E}(\max\{X_1, \dots, X_n\}) + \mathbb{E}(\max\{Y_1, \dots, Y_n\}) \\ &= \mathrm{MAXVAR}_n(X) + \mathrm{MAXVAR}_n(Y). \end{aligned}$$

Proof of **(A3)** For any $X, Y \in \mathscr{L}^2$ satisfying $X \leq Y$, suppose $X_1, \dots, X_n$ are i.i.d. copies of X and $Y_1, \dots, Y_n$ are i.i.d. copies of Y. We can see that $\mathbb{P}_0(X \leq t) \geq \mathbb{P}_0(Y \leq t)$ for any $t \in \mathbb{R}$ since $X \leq Y$. Then, we have

$$\begin{aligned} \mathrm{MAXVAR}_n(X) &= \int_{-\infty}^{0} [\mathbb{P}_0(\max\{X_1, \dots, X_n\} > t) - 1]\, dt + \int_0^{+\infty} \mathbb{P}_0(\max\{X_1, \dots, X_n\} > t) dt \\ &= -\int_{-\infty}^{0} (\mathbb{P}_0(X \leq t))^n dt + \int_0^{+\infty} \left[1 - (\mathbb{P}_0(X \leq t))^n\right] dt \\ &\leq -\int_{-\infty}^{0} (\mathbb{P}_0(Y \leq t))^n dt + \int_0^{+\infty} \left[1 - (\mathbb{P}_0(Y \leq t))^n\right] dt \\ &= \int_{-\infty}^{0} [\mathbb{P}_0(\max\{Y_1, \dots, Y_n\} > t) - 1]\, dt + \int_0^{+\infty} \mathbb{P}_0(\max\{Y_1, \dots, Y_n\} > t) dt \\ &= \mathrm{MAXVAR}_n(Y). \end{aligned}$$

The detail of the first equality is as follows. Denote by

$$F(t) = \mathbb{P}_0(\max\{X_1, \dots, X_n\} \leq t)$$

the cumulative distribution function of $\max\{X_1, \ldots, X_n\}$. Then,

$$\begin{aligned}
\mathbb{E}(\max\{X_1, \ldots, X_n\}) &= \int_{-\infty}^{+\infty} x dF(x) \\
&= -\int_{-\infty}^{0} \left[\int_{x}^{0} dt\right] dF(x) + \int_{0}^{+\infty} \left[\int_{0}^{x} dt\right] dF(x) \\
\text{(by Fubini's theorem)} &= -\int_{-\infty}^{0} \left[\int_{-\infty}^{t} dF(x)\right] dt + \int_{0}^{+\infty} \left[\int_{t}^{+\infty} dF(x)\right] dt \\
&= -\int_{-\infty}^{0} F(t)dt + \int_{0}^{+\infty} [1 - F(t)]dt. \qquad (4)
\end{aligned}$$

And the second equality comes from the fact that $F(t) = (\mathbb{P}_0(X \leq t))^n$.

Proof of **(A4)** Suppose X^k $(k = 1, 2, \ldots)$, $X \in \mathscr{L}^2$ and $\|X^k - X\|_2 \to 0$ as k tends to infinity. Then, $X^k \to X$ in distribution. Denote by $F_k(t)$ the distribution function of X^k $(k = 1, 2, \ldots)$ and by $F(t)$ the distribution of X. Then, $\lim_{k\to\infty} F_k(t) = F(t)$ for all continuous points of $F(\cdot)$. It implies that $\lim_{k\to\infty}[F_k(t)]^n = [F(t)]^n$ for all continuous points of $[F(\cdot)]^n$. Note that $[F_k(t)]^n$ is the distribution function of $\max\{X_1^k, \ldots, X_n^k\}$ and $[F(t)]^n$ is the distribution function of $\max\{X_1, \ldots, X_n\}$, where $X_1^k, \ldots, X_n^k$ are i.i.d. copies of X^k $(k = 1, 2, \ldots)$ and $X_1, \ldots, X_n$ are i.i.d. copies of X. Therefore, we have $\max\{X_1^k, \ldots, X_n^k\} \to \max\{X_1, \ldots, X_n\}$ in distribution, and

$$\begin{aligned}
\text{MAXVAR}_n(X^k) &= \mathbb{E}(\max\{X_1^k, \ldots, X_n^k\}) \\
&\to \mathbb{E}(\max\{X_1, \ldots, X_n\}) = \text{MAXVAR}_n(X)
\end{aligned}$$

as k tends to infinity. Thus, if $\text{MAXVAR}_n(X^k) \leq 0$ for all $k = 1, 2, \ldots,$ then $\text{MAXVAR}_n(X) \leq 0$. The proof of the theorem is complete. □

3 Risk-Averseness of MAXVAR

Suppose $\mathcal{R}$ is a functional from $\mathscr{L}^2$ to $(-\infty, +\infty]$. Recall that an *averse* risk measure is defined by axioms (A1), (A2), (A4), (A5) and

(A6) $\mathcal{R}(X) > \mathbb{E}(X)$ for all nonconstant X.

We then have the next theorem.

Theorem 2 *If $n \geq 2$, then* $\text{MAXVAR}_n(\cdot)$ *is averse.*

Föllmer and Schied [4] proved that if $\mathcal{R}$ is a coherent, law-invariant risk measure in $\mathscr{L}^\infty$ (not $\mathscr{L}^2$) other than $\mathbb{E}(\cdot)$, then $\mathcal{R}$ is averse, where "law-invariant" stands for that $\mathcal{R}(X) = \mathcal{R}(Y)$ whenever X and Y have the same distribution under $\mathbb{P}_0$. Since we are now considering the $\mathscr{L}^2$ case, we cannot use the result in Föllmer and Schied [4] directly. We next give a separate proof.

Proof of Theorem 2 On one hand, for any $X \in \mathscr{L}^2$, let $X_1, \ldots, X_n$ be i.i.d. copies of X. Then, we have

$$\text{MAXVAR}_n(X) = \mathbb{E}(\max\{X_1, \ldots, X_n\}) \geq \mathbb{E}(X_1) = \mathbb{E}(X).$$

On the other hand, if $\mathrm{MAXVAR}_n(X) = \mathbb{E}(X) = \mathbb{E}(X_1)$ $(n \geq 2)$, then $\max\{X_1, \ldots, X_n\} = X_1$ almost surely. Similarly, $\max\{X_1, \ldots, X_n\} = X_2$ almost surely. Therefore, $X_1 = X_2$ almost surely. Since X_1 and X_2 are independent, we must have X_1 equals to a constant almost surely, which is equivalent to say X equals a constant almost surely. Therefore, $\mathrm{MAXVAR}_n(X) > \mathbb{E}(X)$ for nonconstant X, which implies that $\mathrm{MAXVAR}_n(\cdot)$ is averse when $n \geq 2$. □

Remark 1 In fact, Theorems 1 and 2 can be obtained as corollaries of Theorem 3 in the next section. See the remark after the proof of Theorem 3 for details. However, we think it is of interest to provide an elementary proof only based on the definition of MAXVAR.

4 MAXVAR as a Continuous Convex Combination of CVaR

An important coherent risk measure in basic sense is the conditional value at risk (CVaR) popularized by Rockafellar and Uryasev [6]. Among several equivalent definitions of CVaR, the most familiar one is probably the following.

$$\mathrm{CVaR}_\alpha(X) = \min_{\beta \in \mathbb{R}} \left\{ \beta + \frac{1}{1-\alpha} \mathbb{E}(X - \beta)_+ \right\}, \tag{5}$$

where $(t)_+ = \max(t, 0)$ and $\alpha \in [0, 1)$. The minimum is attained at $\beta^* = \mathrm{VaR}_\alpha(X)$, and the VaR ("Value-at-Risk") is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{\nu \in \mathbb{R} : \ \mathbb{P}_0(X > \nu) < 1 - \alpha\}. \tag{6}$$

In this section, we show that $\mathrm{MAXVAR}_n(\cdot)$ is certain "continuous convex combination" of the CVaR measure in the sense that

$$\mathrm{MAXVAR}_n(\cdot) = \int_0^1 \mathrm{CVaR}_\alpha(\cdot) w_n(\alpha) d\alpha,$$

where $w_n(\alpha)$ $(\alpha \in [0, 1])$ is the "weight function" which satisfies $w_n(\alpha) \geq 0$ on $[0, 1]$ and $\int_0^1 w_n(\alpha) d\alpha = 1$. Specifically, we have the next theorem.

Theorem 3 *For any $X \in \mathscr{L}^2$, we have*

$$\mathrm{MAXVAR}_n(X) = \int_0^1 \mathrm{CVaR}_\alpha(X) w_n(\alpha) d\alpha,$$

where

$$w_n(\alpha) := n(n-1)(1-\alpha)\alpha^{n-2}, \quad \alpha \in [0, 1]$$

is the weight function.

Remark 2 It can be easily checked that $w_n(\alpha) \geq 0$ on $[0, 1]$, and

$$\int_0^1 w_n(\alpha) d\alpha = n(n-1) \int_0^1 (\alpha^{n-2} - \alpha^{n-1}) d\alpha = n(n-1) \left[\frac{1}{n-1} - \frac{1}{n} \right] = 1.$$

Therefore, $w_n(\alpha)$ is indeed a weight function.

Theorem 3 was mentioned in Cherny and Orlov [3] without details. We now give a detailed proof by using the so-called Choquet integral. First, we need a lemma. For any $\alpha \in [0, 1)$, define $f_\alpha(\cdot) : \Sigma \to [0, 1]$ in the following way,

$$\begin{aligned} f_\alpha(A) : &= \begin{cases} \frac{1}{1-\alpha}\mathbb{P}_0(A) & \text{if } \mathbb{P}_0(A) \le 1-\alpha, \\ 1 & \text{otherwise} \end{cases} \\ &= g_\alpha[\mathbb{P}_0(A)], \end{aligned}$$

where

$$g_\alpha(x) := \begin{cases} \frac{1}{1-\alpha}x & \text{if } x \in [0, 1-\alpha), \\ 1 & \text{if } x \in [1-\alpha, 1]. \end{cases} \tag{7}$$

We then have the following lemma, which implies that the CVaR measure can be written as the "Choquet integral" with respect to $f_\alpha(\cdot)$.

Lemma 1 *For any $X \in \mathscr{L}^2$ and $\alpha \in [0, 1)$, we have*

$$\mathrm{CVaR}_\alpha(X) = \int_{-\infty}^{0} [f_\alpha(X > t) - 1]\, dt + \int_0^{+\infty} f_\alpha(X > t) dt.$$

Proof If $\mathrm{VaR}_\alpha(X) \le 0$, then

$$\begin{aligned} \int_{-\infty}^{0} [f_\alpha(X > t) - 1] dt + \int_0^{+\infty} f_\alpha(X > t) dt &= \int_{\mathrm{VaR}_\alpha(X)}^{0} \left[\frac{1}{1-\alpha}\mathbb{P}_0(X > t) - 1\right] dt \\ &\quad + \int_0^{+\infty} \frac{1}{1-\alpha}\mathbb{P}_0(X > t) dt \\ &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha} \times \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \mathbb{P}_0(X > t) dt \\ &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha} \times \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] \\ &= \mathrm{CVaR}_\alpha(X). \end{aligned}$$

The last step above is due to (5) and (6).

If $\mathrm{VaR}_\alpha(X) > 0$, then

$$\begin{aligned} \int_{-\infty}^{0} [f_\alpha(X > t) - 1] dt + \int_0^{+\infty} f_\alpha(X > t) dt &= \int_0^{\mathrm{VaR}_\alpha(X)} dt + \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \frac{1}{1-\alpha}\mathbb{P}_0(X > t) dt \\ &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha} \times \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} \mathbb{P}_0(X > t) dt \\ &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha} \times \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] \\ &= \mathrm{CVaR}_\alpha(X), \end{aligned}$$

which completes the proof. □

Proof of Theorem 3 Define

$$h(x) := 1 - (1-x)^n, \qquad x \in [0, 1].$$

It is not difficult to check that

$$h(x) = \int_0^1 g_\alpha(x) w_n(\alpha) d\alpha, \qquad x \in [0, 1], \tag{8}$$

where $g_\alpha(x)$ is as defined in (7). By (4), for any $X \in \mathscr{L}^2$ we have

$$\mathrm{MAXVAR}_n(X) = \int_{-\infty}^{0} [h(\mathbb{P}_0(X > t)) - 1]dt + \int_{0}^{+\infty} h(\mathbb{P}_0(X > t))dt. \tag{9}$$

So by (8), (9), and Lemma 1, together with Fubini's theorem and the fact that $\int_0^1 w_n(\alpha)d\alpha = 1$, we get

$$\begin{aligned}\mathrm{MAXVAR}_n(X) &= \int_{-\infty}^{0}\int_{0}^{1} [f_\alpha(X > t) - 1]w_n(\alpha)d\alpha dt \\ &\quad + \int_{0}^{+\infty}\int_{0}^{1} f_\alpha(X > t)w_n(\alpha)d\alpha dt \\ &= \int_0^1 \left[\int_{-\infty}^{0} [f_\alpha(X > t) - 1]dt + \int_0^{+\infty} f_\alpha(X > t)dt\right] w_n(\alpha)d\alpha \\ &= \int_0^1 \mathrm{CVaR}_\alpha(X)w_n(\alpha)d\alpha\end{aligned}$$

for any $X \in \mathscr{L}^2$, as desired. □

Remark 3 Theorem 3 says that $\mathrm{MAXVAR}_n(\cdot)$ is a continuous convex combination of the CVaR measure, its coherency in basic sense follows from Proposition 2.1 of Ang et al. [1], and its averseness follows from the averseness of the CVaR (Proposition 4.4 of Ang et al. [1]) together with the basic property of integral. Therefore, Theorem 3 can actually provide an alternative proof of the coherency and averseness of $\mathrm{MAXVAR}_n(\cdot)$.

5 The Risk Envelope of MAXVAR

Since

$$\mathrm{MAXVAR}_n(\cdot) = \int_0^1 \mathrm{CVaR}_\alpha(\cdot)w_n(\alpha)d\alpha$$

is a coherent risk measure on $\mathscr{L}^2$, by the dual representation theorem (Rockafellar [5]), there exists a unique, nonempty, convex, and closed set $\mathcal{Q}_n \subseteq \mathscr{L}^2$, called "the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$" such that

$$\mathrm{MAXVAR}_n(X) = \sup_{Q \in \mathcal{Q}_n} \mathbb{E}(XQ)$$

for any $X \in \mathscr{L}^2$.

In this section, we aim at characterizing the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$. First, recall the following well-known result for the discrete convex combination of the CVaR measure, which can be found in Rockafellar [5] and whose proof can be found in Ang et al. [1].

Proposition 1 *Let $\mathcal{R}(\cdot) = \sum_{i=1}^n \lambda_i \mathrm{CVaR}_{\alpha_i}(\cdot)$ with positive weights λ_i adding up to 1. Then, $\mathcal{R}$ is a coherent risk measure in the basic sense and its risk envelope is*

$$\left\{\sum_{i=1}^n \lambda_i Q_i : 0 \le Q_i \le \frac{1}{1-\alpha_i},\ \mathbb{E}(Q_i) = 1\ \forall i = 1, 2, \dots, n\right\}.$$

A continuous version of Proposition 1 gives the risk envelope of MAXVAR as follows.

Theorem 4 *The risk envelope of MAXVAR is*

$$\mathcal{Q}_n := \mathrm{cl}\left\{\int_0^1 Q_\alpha w_n(\alpha)d\alpha,\ 0 \le Q_\alpha \le \frac{1}{1-\alpha},\ \mathbb{E}(Q_\alpha) = 1\ \forall \alpha \in [0,1)\right\}, \tag{10}$$

where

$$w_n(\alpha) := n(n-1)(1-\alpha)\alpha^{n-2}, \quad \alpha \in [0,1]$$

is the weight function (0^0 *is defined as* 1), *and* "cl" *stands for the closure in* $\mathcal{L}^2$.

Proof Note that the integration "$\int_0^1 Q_\alpha w_n(\alpha)d\alpha$" in (10) is defined pointwise. That is, $Y = \int_0^1 Q_\alpha w_n(\alpha)d\alpha$ means $Y(\omega) = \int_0^1 Q_\alpha(\omega)w_n(\alpha)d\alpha$ for any $\omega \in \Omega$. Since $0 \le Q_\alpha \le \frac{1}{1-\alpha}$ for any $\alpha \in [0,1)$, we have

$$0 \le \int_0^1 Q_\alpha(\omega)w_n(\alpha)d\alpha \le \int_0^1 n(n-1)\alpha^{n-2}d\alpha = n$$

for any $\omega \in \Omega$. Therefore, $\mathcal{Q}_n \subseteq \mathcal{L}^\infty \subseteq \mathcal{L}^2$. In addition, we can check that

$$\begin{aligned}\mathrm{MAXVAR}_n(X) &= \int_0^1 \mathrm{CVaR}_\alpha(X)w_n(\alpha)d\alpha \\ &= \sup\left\{\mathbb{E}\left(X\int_0^1 Q_\alpha w_n(\alpha)d\alpha\right) : \int_0^1 Q_\alpha w_n(\alpha)d\alpha \in \mathcal{Q}_n\right\}\end{aligned} \tag{11}$$

for any $X \in \mathcal{L}^2$. Furthermore, it is easy to check the convexity of $\mathcal{Q}_n$. Since $\mathcal{Q}_n$ is closed in $\mathcal{L}^2$, it follows from the dual representation theorem that Formula (11) implies that (10) is the risk envelope of $\mathrm{MAXVAR}_n(\cdot)$. □

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