Statistics and Probability Letters 103 (2015) 46–56

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Weak convergence of equity derivatives pricing with default risk[☆]

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ARTICLE INFO

Article history:
Received 5 March 2013
Received in revised form 2 April 2015
Accepted 12 April 2015
Available online 23 April 2015

MSC:
60B10
91G80

Keywords:
Default risk
Hazard process
Weak convergence

ABSTRACT

This paper presents a discrete-time equity derivatives pricing model with default risk in a no-arbitrage framework. Using the equity-credit reduced form approach where default intensity mainly depends on the firm's equity value, we deduce the Arrow–Debreu state prices and the explicit pricing result in discrete time after embedding default risk in the pricing model. We prove that the discrete-time defaultable equity derivatives pricing has convergence stability, and it converges weakly to the continuous-time pricing results.

1. Introduction

Default risk is the risk that the agents cannot fulfill their obligations in the contracts. The reduced form approach has become a standard tool for modeling default risk. It considers the default to be an exogenously specified jump process, and derives the default probability as the instantaneous likelihood of default, see, for example, Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Lando (1998). The default time is usually defined as the first jump time of a Cox process with a given intensity (hazard rate). Hence, these models are frequently called *intensity models*.

Recently, an alternative model named *equity-credit market approach* has emerged. It assumes that the default intensity depends on the firm's equity value (stock prices) and allows the stock price to jump to zero at the time of default. It has both reduced form and structural features. Default risk is incorporated in this equity modeling approach by assuming that the stock price S_t at time t can jump to zero with an intensity, which is assumed to be a function of S_t. The models described above are all continuous-time models, they are widely used to model default risk.

However, continuous-time models are often too complicated to handle, it is necessary to deduce discrete-time models and show that the pricing processes converge to the continuous-time models. This is not a trivial job, since weak convergence, by its nature, is not tied to a single probability space. Some authors have presented different discrete-time models for derivatives pricing and have established some weak convergence results. See, for example, Cox and Rubinstein (1979), He (1989), Duffie and Protter (1992) and Nieuwenhuis and Vellekoop (2004), etc.

[☆] This work is partially supported by NSFC grants (Nos. 71271173, 71301132, 71073129, 11426187, 11126236 and 11201150), China Postdoctoral Science Foundation (No. 2013M541526), JBK.120211, the Fundamental Research Funds for Central Universities (No. 239201278210075), and the 111 Project (B14019).

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http://dx.doi.org/10.1016/j.spl.2015.04.015
0167-7152/

In this paper, our aim is to present a discrete-time equity derivatives pricing model with default risk in a no-arbitrage framework, and prove that the pricing in discrete-time converges weakly to the continuous-time pricing results. In comparison, our method is different from Nieuwenhuis and Vellekoop (2004). Following the discrete framework of He (1989) and equity-credit market approach presented in Bielecki et al. (2009), we describe the discrete-time pricing model in a no-arbitrage framework. After embedding default risk, we deduce the Arrow–Debreu state prices and the explicit pricing result in discrete time. In order to prove the weak convergence of pricing processes, several auxiliary results are presented.

The paper is organized as follows: in Section 2, we introduce the continuous-time model using equity-credit reduced-form approach; In Section 3, we illustrate a discrete-time model of the equity derivatives pricing with default risk; In Section 4, weak convergence of equity derivatives pricing with default risk from discrete-time to continuous-time pricing is proved; Finally, in Section 5, we summarize the article and make concluding remarks.

2. The continuous-time model

We first recall the continuous-time defaultable contingent claims pricing model. Given a probability space $(\Omega, \mathcal{F}, P)$, T is a strictly positive real number which represents the final date, $(\omega_t)_{0\le t\le T}$ is a Brownian motion. Let $\mathcal{F}_t = \sigma(\omega_s, s \le t)$ for $t \ge 0$. We suppose $\mathcal{F}_t \subset \mathcal{F}$ for all t, and P is the real-world probability. Furthermore, we denote by "$\Rightarrow$" weak convergence from now on.

A default event occurs at a random time τ, where τ is a non-negative random variable. The default process is defined as $N_t \triangleq \mathbf{1}_{\{\tau\le t\}}$, and $\mathcal{H}_t = \sigma(N_s, s \le t)$, the filtration $\mathcal{H}$ is used to describe the information about default time, where $\mathcal{H} = \bigcup_{0\le t\le T} \mathcal{H}_t$. At any time t, the agent's information on the securities prices and default time is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathcal{G}$ stopping time where $\mathcal{G} = \bigcup_{0\le t\le T} \mathcal{G}_t$. In fact, $\mathcal{G}$ is the smallest filtration which contains $\mathcal{F}$ and allows τ to be a stopping time. Assume that the pre-default stock price S_t has the following dynamics

$$dS_t = (b(S_t) + \lambda(S_t, t)S_t)dt + \sigma(S_t, t)S_t d\omega_t, \quad S_0 > 0. \tag{2.1}$$

Here we assume that $b(x)$ is continuous, $\sigma(S, t)$ is a positively bounded and nonsingular Borel-measurable function. In particular we have that $\sigma(S, t) \ge \sigma$ for some positive constant σ, $\lambda(S, t)$ is a nonnegative, bounded, continuous, $\mathcal{F}-$progressively measurable and integrable function. The functions $b(S)$, $\lambda(S, t)S$ and $\sigma(S, t)S$ are Lipschitz continuous in S, uniformly in t.

The bond price B_t satisfies $dB_t = B_t r(S_t)dt$ and $B_0 = 1$, where $r(x)$ is a nonnegative continuous function, representing the riskless interest rate. Suppose there exists a constant $K > 0$ such that $|x^2 r(x)| \le K(1 + x^2)$.

There exists a $\mathcal{G}$ equivalent martingale measure Q^* which is defined as $dQ^*|_{\mathcal{F}_t} = \xi_t dP|_{\mathcal{F}_t}$, where ξ_t is the Radon–Nikodým density satisfying

$$d\xi_t = \xi_t\theta(S_t)d\omega_t, \quad \xi_0 = 1. \tag{2.2}$$

Here $\theta(x) = -\sigma(x)^{-1}(b(x) - r(x)x)$. Define W_t via $dW_t = d\omega_t - \theta(S_t)dt$, then W_t is a Brownian motion with respect to $\mathcal{F}$, and under the changed measure

$$dS_t = S_t[(r(S_t) + \lambda(S_t, t))dt + \sigma(S_t, t)dW_t], \quad S_0 > 0. \tag{2.3}$$

Define $G_t \triangleq Q^*(\tau > t \mid \mathcal{F}_t)$, $\Gamma_t \triangleq -\ln G_t$. We call Γ_t the $\mathcal{F}$ *hazard process of* τ. For the detailed properties, one can refer to Bielecki and Rutkowski (2002).

Let $g(\cdot) : \mathbb{R} \to \mathbb{R}$ be a square integrable and measurable function, the equity derivatives are defined to be securities that pay $g(S_T)$ dollars on the final date. This formulation subsumes all of the usual examples, such as the European options, convertible bonds and so on. The prices of equity derivatives at time t are

$$V(S_t, t) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{Q^*}\left[\frac{B_t e^{\Gamma_t}}{B_T e^{\Gamma_T}} g(S_T) \,\middle|\, \mathcal{F}_t\right]. \tag{2.4}$$

Poisson process with stochastic intensity is called *Cox process*. Given $\lambda(S_u, u)$, denote by $\{\overline{C}_t\}$ the Poisson process with intensity $C_t = \int_0^t \lambda(S_u, u)du$. Then $\{\overline{C}_t\}$ is a Cox process. Following the equity-credit market models, the canonical construction of default time τ under the Cox process $\{\overline{C}_t\}$ is defined as $\tau = \inf\{t \ge 0 : C_t \ge \Theta\}$, where $\Theta \sim Exp(1)$ and is independent of $\mathcal{F}$ under Q^*. Then

$$Q^*(\tau > t \mid \mathcal{F}_t) = Q^*(\Theta > C_t \mid \mathcal{F}_t) = e^{-C_t}.$$

It is easy to see that under this condition, the default time is the first jump time of the Cox process, the $\mathcal{F}$ hazard process of τ satisfies

$$\Gamma_t = -\ln Q^*(\tau > t \mid \mathcal{F}_t) = -\ln Q^*(\Theta > C_t \mid \mathcal{F}_t) = C_t.$$

Let Δ denote the bankruptcy state when the firm defaults at time τ. Then we can also write the dynamics for the stock price subject to bankruptcy S_t^Δ as follows:

$$dS_t^\Delta = S_t^\Delta[r(S_t)dt + \sigma(S_t, t)dW_t - dM_t],$$

where $M_t = N_t - \int_0^{t\wedge\tau} \lambda(S_u, u)du$, and M_t is a martingale. Moreover, referred to Hypothesis (H) in Blanchet-Scalliet and Jeanblanc (2004): all $\mathcal{F}$-martingales are $\mathcal{G}$-martingales. It implies that the $\mathcal{F}$-Brownian motion W_t remains a Brownian motion under the extended probability measure Q^* and with respect to the enlarged filtration $\mathcal{G}$ and is independent of M_t.

Then we have

$$V(S_t, t) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{Q^*}\left[e^{-\int_t^T (r_s+\lambda_s)ds} g(S_T) \,\middle|\, \mathcal{F}_t\right].$$

Here we write $\sigma_t = \sigma(S_t), r_t = r(S_t), \lambda_t = \lambda(S_t, t)$ for simplicity. We can obtain the following result.

Lemma 2.1. *Let $Y(S_t, t) = \mathbb{E}_{Q^*}\left[e^{-\int_t^T (r_s+\lambda_s)ds} g(S_T) \,\middle|\, \mathcal{F}_t\right]$. Then it satisfies*

$$\frac{\partial Y}{\partial t} + \frac{\sigma_t^2 S_t^2}{2}\frac{\partial^2 Y}{\partial S^2} + (r_t + \lambda_t)S_t\frac{\partial Y}{\partial S} - (r_t + \lambda_t)Y = 0. \tag{2.5}$$

Proof. Let $\hat{Y}(S_t, t) = E_{Q^*}\left[\frac{G_T g(S_T)}{B_T} \,\middle|\, \mathcal{F}_t\right]$. It can be regarded as the discount price of contingent claim $G_T g(S_T)$ at time t, then it is $\mathcal{F}$ martingale. By Itô's formula,

$$\frac{\partial \hat{Y}}{\partial t} + \frac{\sigma_t^2 S_t^2}{2}\frac{\partial^2 \hat{Y}}{\partial S^2} + (r_t + \lambda_t)S_t\frac{\partial \hat{Y}}{\partial S} = 0.$$

Since $\hat{Y}(S_t, t) = e^{-\int_0^t (r_s+\lambda_s)ds} Y(S_t, t)$, (2.5) is proved after using Itô's formula again. □

3. Discrete-time model in the defaultable market

For simplicity, the time horizon is assumed to be [0, 1] and divided into n steps, the length of every step is $1/n$. For $k = 1, 2, \ldots, n$, let ε^k be a random variable on the probability space $\bar{\Omega} = \{\omega_1, \omega_2\}$. For example, set $\varepsilon^k(\omega_1) = 1, \varepsilon^k(\omega_2) = -1$, and $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$.

Let $\bar{\Omega}_n = \overbrace{\bar{\Omega} \times \bar{\Omega} \times \cdots \times \bar{\Omega}}^{n} = \{\omega_1, \omega_2\}^n$, $P_n = \overbrace{P \times P \times \cdots \times P}^{n}$. Then P_n is the probability measure defined on $\bar{\Omega}_n$, representing the real-world probability, $\mathcal{F}_n$ is the filtration generated by ε^k, $k = 1, \ldots, n$, $\{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n\}$ is a sequence of independent and identically distributed random vectors defined on $\{\bar{\Omega}_n, \mathcal{F}_n, P_n\}$.

There are two financial assets in the market: stock and bond. Since the increment of Brownian motion can be approximated by a sequence of independent and identically distributed random variables, the pre-default stock prices and bond prices can be written as

$$S_{k+1}^n = S_n^k + \frac{b(S_k^n) + \lambda(S_k^n, \frac{k}{n})S_n^k}{n} + \frac{\sigma(S_k^n, \frac{k}{n})S_n^k}{\sqrt{n}}\varepsilon^{k+1}, \quad S_0^n = S_0.$$

$$B_{k+1}^n = B_k^n\left(1 + \frac{r(S_k^n)}{n}\right), \quad B_0^n = 1.$$

Here S_k^n, B_k^n denote the stock prices and bond prices at time $\frac{k}{n}$ respectively. Let $\tilde{S}_t^n = S_{[nt]}^n, \tilde{B}_t^n = B_{[nt]}^n$. Then $\tilde{S}_t^n, \tilde{B}_t^n$ are Markov processes and have jumps only at time $\frac{k}{n}$. Moreover $\tilde{S}_t^n$ can be expressed as follows

$$\begin{aligned}\tilde{S}_t^n &= S_0 + \sum_{i=0}^{[nt]-1}\frac{b(S_i^n) + \lambda(S_i^n, \frac{i}{n})S_i^n}{n} + \sum_{i=0}^{[nt]-1}\frac{\sigma(S_i^n, \frac{i}{n})S_i^n}{\sqrt{n}}\varepsilon^{i+1} \\ &= S_0 + \int_0^{\frac{[nt]}{n}}(b(\tilde{S}_u^n, u) + \lambda(\tilde{S}_u^n, u)\tilde{S}_u^n)du + \sum_{i=0}^{[nt]-1}\frac{\sigma(S_i^n, \frac{i}{n})S_i^n}{\sqrt{n}}\varepsilon^{i+1}.\end{aligned}$$

Similarly,

$$\tilde{B}_t^n = B_0 + \sum_{i=0}^{[nt]-1}\frac{r(S_i^n)B_i^n}{n} = B_0 + \int_0^{\frac{[nt]}{n}} r(\tilde{S}_u^n)\tilde{B}_u^n du.$$

The above discrete framework is employed in He (1989) where he assumes the stock prices satisfying $\pi(\omega_1; S_k^n)S_{k+1}^n(\omega_1) + \pi(\omega_2; S_k^n)S_{k+1}^n(\omega_2) = S_k^n$, where $\pi(\omega_s; S_k^n)(s = 1, 2)$ are considered as the Arrow–Debreu state prices and the discount stock prices are martingale. He (1989) gives the result that there exists unique equivalent martingale measure Q_n in discrete-time defaultable market, and $dQ_n = \xi_n^n dP_n$, where $\xi_k^n = 2^k\pi_k^n B_k^n$, π_k^n is defined as the product of Arrow–Debreu state prices from 0 to k, the default-free discrete-time market is complete. Let $\tilde{\xi}_t^n = \xi_{[nt]}^n$, then $\tilde{\xi}_t^n = \xi_0 + \sum_{i=0}^{[nt]-1}\frac{\theta(S_i^n)\xi_i^n}{\sqrt{n}}\varepsilon^{i+1}$.

Suppose default occurs at random time τ_n, where τ_n is a non-negative random variable. The default process is defined as $N_k^n = 1_{\{\tau_n \le \frac{k}{n}\}}$, σ filtration $\mathcal{H}_k^n = \sigma(N_i^n, 0 \le i \le k)$ and $\mathcal{H}^n$ is used to describe the information about default time. At time $\frac{k}{n}$, the agent's information on the prices and on the default time is $\mathcal{G}_k^n = \mathcal{F}_k^n \vee \mathcal{H}_k^n$. Hence, the default time τ_n is a $\mathcal{G}^n$ stopping time where $\mathcal{G}^n = \{\mathcal{G}_k^n, 0 \le k \le n\}$.

From Blanchet-Scalliet and Jeanblanc (2004), if the default-free market is complete and arbitrage-free, the defaultable market is arbitrage-free, then there exists an equivalent martingale measure Q_n^* in $\mathcal{G}^n$-market.

Definition 3.1. For any $0 \le k \le n$, let $Q_n^*(\tau_n = 0) = 0$, $Q_n^*(\tau_n > \frac{k}{n}) > 0$. We write $F_k^n = Q_n^*(\tau_n \le \frac{k}{n} \mid \mathcal{F}_k^n)$, $G_k^n = 1 - F_k^n = Q_n^*(\tau_n > \frac{k}{n} \mid \mathcal{F}_k^n)$. Suppose $F_k^n < 1$, let $\Gamma_k^n \triangleq -\ln G_k^n = -\ln(1 - F_k^n)$ be called the $\mathcal{F}^n$ hazard process of τ_n under Q_n^*.

Several properties in the continuous-time model still hold here, such as (F_k^n) is nonnegative bounded submartingale, Γ_k^n is increasing, and $L_k^n = 1_{\{\tau_n > \frac{k}{n}\}} e^{\Gamma_k^n}$ is a martingale, the detailed properties can refer to Blanchet-Scalliet and Jeanblanc (2004).

Assume the Cox process is defined via intensity $\tilde{C}_t \triangleq \int_0^t \lambda(\tilde{S}_u^n, u)du$, Θ is the random variable with an exponential law of parameter 1, which is independent of $\mathcal{F}^n$ under Q_n^*, then we can give the canonical construction of τ_n, $\tau_n : \bar{\Omega}_n \to [0, T]$, $\tau_n = \inf\left\{\frac{[nt]}{n} \ge 0; \tilde{C}_{\frac{[nt]}{n}} \ge \Theta\right\}$. Therefore,

$$Q_n^*\left(\tau_n > \frac{[nt]}{n} \,\middle|\, \mathcal{F}_{[nt]}^n\right) = Q_n^*\left(\Theta > \tilde{C}_{\frac{[nt]}{n}} \,\middle|\, \mathcal{F}_{[nt]}^n\right) = e^{-\tilde{C}_{\frac{[nt]}{n}}}.$$

The $\mathcal{F}^n$ hazard process τ_n satisfies

$$\Gamma_k^n = -\ln Q_n^*\left(\tau_n > \frac{k}{n} \,\middle|\, \mathcal{F}_k^n\right) = -\ln Q_n^*\left(\Theta > \tilde{C}_{\frac{k}{n}} \,\middle|\, \mathcal{F}_k^n\right) = \tilde{C}_{\frac{k}{n}}.$$

Moreover,

$$\Gamma_{k+1}^n = \Gamma_k^n + \int_{\frac{k}{n}}^{\frac{k+1}{n}} \lambda(\tilde{S}_u^n, u)du = \Gamma_k^n + \frac{\lambda\left(S_k^n, \frac{k}{n}\right)}{n}.$$

Define $\tilde{\Gamma}_t^n = \Gamma_{[nt]}^n$, then $\tilde{\Gamma}_t^n$ is a sequence of Markov process on probability space$(\bar{\Omega}_n, \mathcal{F}^n, Q_n^*)$ with sample path in $D_{\mathbb{R}}[0, 1]$ and

$$\tilde{\Gamma}_t^n = \sum_{i=0}^{[nt]-1} \frac{\lambda\left(S_i^n, \frac{i}{n}\right)}{n} = \int_0^{\frac{[nt]}{n}} \lambda(\tilde{S}_u^n, u)du, \quad \tilde{\Gamma}_0^n = 0.$$

Although τ_n and τ are defined in different probability spaces, the canonical construction provides us a feasible way to prove the weak convergence of the default process which will be shown in the next section.

Define auxiliary discount process

$$\beta_k^n = B_k^n e^{\Gamma_k^n} \triangleq \prod_{i=0}^{k-1}\left(1 + \frac{\tilde{r}_i}{n}\right),$$

then we have $\frac{\beta_{k+1}^n}{\beta_k^n} = \frac{B_{k+1}^n e^{\Gamma_{k+1}^n}}{B_k^n e^{\Gamma_k^n}} = \left(1 + \frac{r(S_k^n)}{n}\right)\exp\left(\frac{\lambda(S_k^n)}{n}\right)$, where $\tilde{r}_k = (r_k + \lambda_k) + \frac{\lambda_k^2 + 2r_k\lambda_k}{2n} + o\left(\frac{1}{n}\right)$. Here we write $r_k = r(S_k^n)$ and $\lambda_k = \lambda(S_k^n)$ for simplicity. Suppose the equity prices and bond prices satisfy

$$\tilde{\pi}(\omega_1; S_k^n)S_{k+1}^n(\omega_1) + \tilde{\pi}(\omega_2; S_k^n)S_{k+1}^n(\omega_2) = S_k^n, \qquad \tilde{\pi}(\omega_1; S_k^n)B_{k+1}^n(\omega_1) + \tilde{\pi}(\omega_2; S_k^n)B_{k+1}^n(\omega_2) = B_k^n.$$

Solve the above two equations, we get $\tilde{\pi}(\omega_s; S_k^n) = \frac{1}{2}\left(1 + \frac{\theta(S_k^n)}{\sqrt{n}}\varepsilon^{k+1}\right)\left(1 + \frac{\tilde{r}_k}{n}\right)^{-1}$. Set

$$\tilde{\pi}_k^n = \tilde{\pi}(\cdot; S_{k-1}^n)\tilde{\pi}(\cdot; S_{k-2}^n)\cdots\tilde{\pi}(\cdot; S_0^n), \quad k = 1, 2, \ldots, n \quad \tilde{\pi}_0^n = 1.$$

Since θ is bounded, for n large enough, $\tilde{\pi}$ is non-negative. Then

$$\tilde{\pi}(\omega_1; S_k^n) + \tilde{\pi}(\omega_2; S_k^n) = \left(1 + \frac{\tilde{r}_k}{n}\right)^{-1}, \qquad \xi_k^n = 2^n\tilde{\pi}_k^n\beta_k^n = 2^n\pi_k^n B_k^n.$$

The price of defaultable contingent claims $g(S_n^n)$ at time $\frac{k}{n}$ is

$$V^n\left(S_k^n, \frac{k}{n}\right) = 1_{\{\tau_n > \frac{k}{n}\}}\mathbb{E}_{Q_n}\left[\frac{B_k^n e^{\Gamma_k^n}}{B_n^n e^{\Gamma_n^n}} g(S_n^n) \,\middle|\, \mathcal{F}_k^n\right]. \tag{3.1}$$

Lemma 3.1. *Let* $Y_n\left(S_k^n, \frac{k}{n}\right) = \mathbb{E}_{Q_n}\left[\frac{\beta_k^n}{\beta_n^n} g(S_n^n) \,\middle|\, \mathcal{F}_k^n\right]$. *Then the following equation holds:*

$$Y_n\left(S_k^n, \frac{k}{n}\right) = \tilde{\pi}(\omega_1; S_k^n)Y_n\left(S_{k+1}^n(\omega_1), \frac{k}{n}\right) + \tilde{\pi}(\omega_2; S_k^n)Y_n\left(S_{k+1}^n(\omega_2), \frac{k}{n}\right). \tag{3.2}$$

Proof. Since $Y_n\left(S_k^n, \frac{k}{n}\right)$ is a $\mathcal{F}^n$ martingale, then we have

$$\begin{aligned} Y_n\left(S_k^n, \frac{k}{n}\right) &= \mathbb{E}_{Q_n}\left[Y_n\left(S_{k+1}^n, \frac{k+1}{n}\right)\left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \,\middle|\, \mathcal{F}_k^n\right] \\ &= \mathbb{E}_{P_n}\left[\frac{\xi_{k+1}^n}{\xi_k^n} Y_n\left(S_{k+1}^n, \frac{k+1}{n}\right)\left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \,\middle|\, \mathcal{F}_k^n\right] = \mathbb{E}_{P_n}\left[\frac{2\pi_{k+1}^n B_{k+1}^n}{\pi_k^n B_k^n} Y_n\left(S_{k+1}^n, \frac{k+1}{n}\right)\left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \,\middle|\, \mathcal{F}_k^n\right] \\ &= \frac{1}{2}\left(1+\frac{\theta(S_k^n)}{\sqrt{n}}\right)\left(1+\frac{\tilde{r}_k}{n}\right)^{-1} Y_n\left(S_{k+1}^n(\omega_1), \frac{k+1}{n}\right) + \frac{1}{2}\left(1-\frac{\theta(S_k^n)}{\sqrt{n}}\right)\left(1+\frac{\tilde{r}_k}{n}\right)^{-1} Y_n\left(S_{k+1}^n(\omega_2), \frac{k+1}{n}\right) \\ &= \tilde{\pi}(\omega_1; S_k^n) Y_n\left(S_{k+1}^n(\omega_1), \frac{k+1}{n}\right) + \tilde{\pi}(\omega_2; S_k^n) Y_n\left(S_{k+1}^n(\omega_2), \frac{k+1}{n}\right). \end{aligned}$$

Therefore, (3.2) is proved. □

We can conclude that $\tilde{\pi}(\cdot; S_k^n)$ can be regarded as the discrete-time Arrow–Debreu state prices in the defaultable market.

4. The main result: weak convergence

In this section, we will prove the weak convergence of pricing process for defaultable equity derivatives under the above model. Firstly, we introduce infinite dimensional multiplicative probability space $\bar{\Omega}_{\mathbb{N}} \triangleq \overbrace{\bar{\Omega} \times \bar{\Omega} \times \cdots \times \bar{\Omega}}^{\infty}$, then $\bar{\Omega}_n$ is a subspace of $\bar{\Omega}_{\mathbb{N}}$. From the infinite multiply probability existence theorem, there exists a unique probability measure $\hat{P}$ satisfying condition: $\hat{P}(A \times \bar{\Omega}_{\mathbb{N}\setminus I_n}) = Q_n(A)$, where $A \in \bar{\Omega}_n$, $I_n = \{1, 2, \ldots, n\}$.

He (1989) proves the weak convergence of Markov process vector including equity prices, bond prices, Radon–Nikodým density. Now we extend this result in the defaultable market. Combining Martingale central limit theorem developed by Ethier and Kurtz (1986) (Page 354), we get a similar result.

Lemma 4.1. *For any $t \in [0, 1]$, let $\tilde{Z}_t^n = (\tilde{S}_t^n, \tilde{B}_t^n, \tilde{\xi}_t^n, \tilde{\Gamma}_t^n)$, $Z_t = (S_t, B_t, \xi_t, \Gamma_t)$. Then*

$$\tilde{Z}_\cdot^n \Rightarrow Z_\cdot \quad \text{as } n \to \infty.$$

Recall the definition of τ_n and τ, the following conclusion holds.

Lemma 4.2. *For $t \in [0, 1]$, let $X_n(t) = 1_{\{\tau_n > \frac{[nt]}{n}\}}$, $X(t) = 1_{\{\tau > t\}}$. Then $X_n(\cdot) \Rightarrow X(\cdot)$.*

Proof. For any $t \in [0, 1]$, we have $\sup_n \mathbb{E}_{\hat{P}}[|X_n(t)|] = \sup_n \mathbb{E}_{Q_n}[|X_n(t)|] \le 1 < \infty$. Therefore,

$$\lim_{C\to\infty} \limsup_n \hat{P}(|X_n(t)| > C) \le \lim_{C\to\infty} \frac{\limsup_n \mathbb{E}_{\hat{P}}[|X_n(t)|]}{C} = 0.$$

So $\{X_n(t)\}$ is tight for any $t \in [0, 1]$.

Choose sequences $\{\alpha_n\}$ and $\{\delta_n\}$ satisfying the following: for all n, α_n is a stopping time with respect to the σ filtration which is generated by the process $\{X_n(t) : 0 \le t \le 1\}$, and α_n has only finite value; δ_n is a constant and $0 \le \delta_n \le 1$. Moreover $\delta_n \to 0$, as $n \to \infty$.

$$\begin{aligned} \hat{P}(|X_n(\alpha_n+\delta_n) - X_n(\alpha_n)| > \epsilon) &= Q_n(|X_n(\alpha_n+\delta_n) - X_n(\alpha_n)| > \epsilon) \\ &\le \frac{1}{\epsilon}\mathbb{E}_{Q_n}[|X_n(\alpha_n+\delta_n) - X_n(\alpha_n)|] \le \frac{1}{\epsilon}\mathbb{E}_{Q_n}\left[1_{\{\tau_n > \frac{[n\alpha_n]}{n}\}} - 1_{\{\tau_n > \frac{[n\alpha_n+n\delta_n]}{n}\}}\right] \\ &= \frac{1}{\epsilon}E_{Q_n}\left[\mathbb{E}_{Q_n}\left[1_{\{\tau_n > \frac{[n\alpha_n]}{n}\}} \,\middle|\, \mathcal{F}_{[n\alpha_n]}^n\right] - \mathbb{E}_{Q_n}\left[1_{\{\tau_n > \frac{[n\alpha_n+n\delta_n]}{n}\}} \,\middle|\, \mathcal{F}_{[n\alpha_n+n\delta_n]}^n\right]\right] \\ &= \frac{1}{\epsilon}\mathbb{E}_{Q_n}\left[\exp(-\Gamma_{[n\alpha_n]}^n) - \exp(-\Gamma_{[n\alpha_n+n\delta_n]}^n)\right] \\ &= \frac{1}{\epsilon}\mathbb{E}_{Q_n}\left[\exp\left(-\sum_{i=0}^{[n\alpha_n]-1}\frac{\lambda(S_i^n)}{n}\right)\left(1-\exp\left(-\sum_{i=[n\alpha_n]}^{[n\alpha_n+n\delta_n]-1}\frac{\lambda(S_i^n)}{n}\right)\right)\right]. \end{aligned}$$

Since $\lambda(S, t)$ is a nonnegative *bounded* continuous function, we have $\sum_{i=[n\alpha_n]}^{[n\alpha_n+n\delta_n]-1} \frac{\lambda(S_i^n)}{n} \le C\delta_n \to 0$ as $n \to \infty$, where C is a fixed constant. Together with the fact that $\exp\left(-\sum_{i=0}^{[n\alpha_n]-1} \frac{\lambda(S_i^n)}{n}\right) \le 1$, we have $\hat{P}(|X_n(\alpha_n+\delta_n) - X_n(\alpha_n)| > \epsilon) \to 0$, that is, $X_n(\alpha_n+\delta_n) - X_n(\alpha_n) \xrightarrow{\hat{P}} 0$ as $n \to \infty$. By the criterion of Aldous (1978) (Page 1), $\{X_n(\cdot)\}$ is tight in $D_{\mathbb{R}}[0, 1]$.

We have $X_n(t) = \mathbf{1}_{\{\tau_n > \frac{[nt]}{n}\}} = \mathbf{1}_{\{\Theta > \tilde{C}_{\frac{[nt]}{n}}\}} = \mathbf{1}_{\{\Theta > \tilde{\Gamma}^n_t\}}$ and $X(t) = \mathbf{1}_{\{\tau > t\}} = \mathbf{1}_{\{\Theta > C_t\}} = \mathbf{1}_{\{\Theta > \Gamma_t\}}$ for any $t \in [0, 1]$. Since $\tilde{\Gamma}^n \Rightarrow \Gamma$ as n tends to infinity, we have $\mathbb{E}_{Q_n}[e^{iuX_n(t)}] \to \mathbb{E}_Q[e^{iuX(t)}]$ as n tends to infinity. According to the dominated convergence theorem, for any $t_1, t_2, \ldots, t_m \in [0, 1], u_1, \ldots, u_m \in \mathbb{R}$,

$$\mathbb{E}_{Q_n}\left[e^{i\sum_{j=1}^m u_j X_n(t_j)}\right] \to \mathbb{E}_Q\left[e^{i\sum_{j=1}^m u_j X(t_j)}\right], \quad n \to \infty.$$

Therefore, $\{X_n\}$ is tight, and their finite dimensional distribution converges. From Ethier and Kurtz (1986) (Page 131), $\{X_n\}$ converges weakly to X. □

Lemma 4.3. *For any integers $l, m, k \geq 0$, $l \leq k \leq n$, there exists a constant $C > 0$, depending on m (large enough), such that $\mathbb{E}_{Q_n}\left[[S_k^n]^{2m} \mid \mathcal{F}_l^n\right] \leq C(1 + [S_0]^{2m})$.*

Next we prove the weak convergence of the second part in the equation of defaultable contingent claims prices, following a similar argument to the main theorem of He (1989).

Lemma 4.4. *For any $t \in [0, 1]$, let*

$$Y(t) \triangleq Y(S_t, t) = \mathbb{E}_Q\left[\frac{B_t e^{\Gamma_t}}{B_T e^{\Gamma_T}} g(S_T) \,\middle|\, \mathcal{F}_t\right], \qquad Y_n(t) \triangleq Y_n\left(S^n_{[nt]}, \frac{[nt]}{n}\right) = \mathbb{E}_{Q_n}\left[\frac{B^n_{[nt]} e^{\Gamma^n_{[nt]}}}{B^n_n e^{\Gamma^n_n}} g(S^n_n) \,\middle|\, \mathcal{F}^n_{[nt]}\right].$$

Suppose that Y is continuously differentiable up to the third order and that Y and all of its derivatives up to the third order satisfy a polynomial growth condition. Then $Y_n(\cdot) \Rightarrow Y(\cdot)$ as n tends to infinity.

Remark. We get the idea of the proofs of Lemmas 4.3 and 4.4 from He (1989), but the results in our paper are rather different from them. In Lemma 4.3, we give the inequality for a more general case. In Lemma 4.4, we prove that $\tilde{e}^n_t$ converges to zero in the sense of almost everywhere.

Then combine Lemmas 4.2 and 4.4, we obtain the main result.

Theorem 4.1. *Suppose $g(\cdot)$ is $\mathbb{R} \to \mathbb{R}$ square integrable measurable function. Let $\tilde{V}^n\left(\tilde{S}^n_t, t\right) = V^n\left(S^n_{[nt]}, \frac{[nt]}{n}\right)$, $V(S_t, t)$ and $\tilde{V}^n\left(\tilde{S}^n_t, \frac{[nt]}{n}\right)$ satisfy (2.5) and (3.2) respectively. Then we have*

$$\tilde{V}^n\left(\tilde{S}^n_\cdot, \cdot\right) \Rightarrow V(S_\cdot, \cdot), \quad \text{as } n \to \infty.$$

Proof. Clearly, we have $\tilde{V}^n\left(\tilde{S}^n_t, \frac{[nt]}{n}\right) = X_n(t)Y_n(t)$ and $V(S_t, t) = X(t)Y(t)$. By Lemma 4.4, Y_n converges weakly to Y, then Y_n is relatively tight. Since $\mathbb{R}$ is separable and $(\mathbb{R}, d)$ is complete, then $D_{\mathbb{R}}[0, 1]$ is separable, it follows that $\{Y_n\}$ is tight.

By Lemma 4.2, $\{X_n\}$ is tight, together with the fact that $Y(t) = Y(S_t, t)$ is continuous with respect to t, then $\{(X_n, Y_n)\}$ is tight according to Jacod and Shiryaev (1987) (Page 353).

Next, we only need to prove the convergence of their finite dimension distribution. That is, for any $u_1, \ldots, u_m \in \mathbb{R}$, $v_1, \ldots, v_m \in \mathbb{R}$,

$$\mathbb{E}_{Q_n}\left[e^{i\sum_{j=1}^m (u_j X_n(t_j) + v_j Y_n(t_j))}\right] \to \mathbb{E}_Q\left[e^{i\sum_{j=1}^m (u_j X(t_j) + v_j Y(t_j))}\right], \quad n \to \infty. \tag{4.1}$$

From Lemmas 4.2 and 4.4, we can obtain the convergence of the finite dimension distribution of $\{X_n\}$, $\{Y_n\}$. Moreover, $X_n(t)$ and $Y_n(t)$ are measurable with respect to $\mathcal{H}_t$ and $\mathcal{F}_t$ respectively, and $X_n(t), Y_n(t)$ are independent, (4.1) holds clearly. Then $(X_n, Y_n) \Rightarrow (X, Y)$ as $n \to \infty$.

Let $f(X_n(t)Y_n(t)) = X_n(t)Y_n(t)$, where f is a continuous function that maps $X_n(t), Y_n(t)$ from $D_{\mathbb{R}}[0, 1] \times D_{\mathbb{R}}[0, 1]$ to $D_{\mathbb{R}}[0, 1]$. By continuous mapping theorem ((Ethier and Kurtz, 1986), p. 354), $X_n(\cdot)Y_n(\cdot) \Rightarrow X(\cdot)Y(\cdot)$ as $n \to \infty$. □

In the following, we give the details of proofs of Lemmas 4.1, 4.3 and 4.4. For simplicity, we write $b(\tilde{S}^n_u) = b_u$, $\lambda(\tilde{S}^n_u, u) = \lambda_u$, $\sigma(\tilde{S}^n_u) = \sigma_u$, $\theta(\tilde{S}^n_u) = \theta_u$ in the proof.

Proof of Lemma 4.1. $d\Gamma_t = \lambda(S_t, t)dt$. Since $b(\cdot)$, $\lambda(\cdot)S$, $\sigma(\cdot)S$ satisfy the Lipschitz condition, (2.1) has a unique solution, which implies that (2.2), (2.3) also have unique solutions respectively. Since $\{\tilde{S}^n\}$, $\{\tilde{B}^n\}$, $\{\tilde{\xi}^n\}$ and $\{\tilde{\Gamma}^n\}$ are processes that are right continuous with left limits, $\{Z^n\}$ is a sequence of Markov process vectors with sample path in $D_{\mathbb{R}^4}[0, 1]$, where $D_{\mathbb{R}^4}[0, 1]$

is the space of functions from [0, 1] to $\mathbb{R}^4$, right continuous with left limits. Denote L_t^n and A_t^n by

$$L_t^n = \begin{pmatrix} \int_0^{\frac{[nt]}{n}} (b_u + \lambda_u \tilde{S}_u^n) du \\ \int_0^{\frac{[nt]}{n}} r \tilde{B}_u^n du \\ 0 \\ \int_0^{\frac{[nt]}{n}} \lambda_u du \end{pmatrix}, \qquad A_t^n = \begin{pmatrix} \int_0^{\frac{[nt]}{n}} \sigma_u^2 S^2 du & 0 & \int_0^{\frac{[nt]}{n}} \sigma_u S \theta_u \tilde{\xi}_u^n du & 0 \\ 0 & 0 & 0 & 0 \\ \int_0^{\frac{[nt]}{n}} \sigma_u S \theta_u \tilde{\xi}_u^n du & 0 & \int_0^{\frac{[nt]}{n}} (\theta_u \tilde{\xi}_u^n)^2 du & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\{L^n\}$, $\{A^n\}$ are 4×1 and 4×4 (symmetric) matrix valued processes respectively, and each of their elements has a sample path in $D_{\mathbb{R}}[0, 1]$. Moreover, $A_t^n - A_s^n$ is non-negative definite for $t > s \geq 0$. Define

$$\tau_n^q := \inf\{t \leq T : |Z_t^n| \geq q \text{ or } |Z_{t-}^n| \geq q\}.$$

Next, we prove the four conditions for martingale central limit theorem hold.

(a) It is directly from $Z_0^n = Z_0 = (S_0, B_0, \xi_0, 0)$.

$$\text{(b) } M_t^n = \tilde{Z}^n - L_t^n = \begin{pmatrix} S_0 + \sum_{i=0}^{[nt]-1} \frac{\sigma(S_i^n) S_i^n}{\sqrt{n}} \varepsilon^{i+1} \\ B_0 \\ \xi_0 + \sum_{i=0}^{[nt]-1} \frac{\theta(S_i^n) \xi_i^n}{\sqrt{n}} \varepsilon^{i+1} \\ 0 \end{pmatrix}.$$

Let $N_k = S_0 + \sum_{i=0}^{k-1} \frac{\sigma(S_i^n) S_i^n}{\sqrt{n}} \varepsilon^{i+1}$. Clearly,

$$\mathbb{E}_{Q_n}[N_{k+1} - N_k \mid S_k^n] = \mathbb{E}_{Q_n}\left[\frac{\sigma(S_k^n) S_k^n}{\sqrt{n}} \varepsilon^{k+1} \,\middle|\, S_k^n \right] = \frac{\sigma(S_k^n) S_k^n}{\sqrt{n}} \mathbb{E}_{Q_n}[\varepsilon^{k+1} \mid S_k^n] = 0.$$

So N_k are martingales. By the same arguments, M_t^n and $M_t^n (M_t^n)^T - A_t^n$ are also martingales.

$$\text{(c) } \tilde{Z}_t^n - \tilde{Z}_{t-}^n = \begin{cases} \begin{pmatrix} \frac{b(S_{k-1}^n) + \lambda(S_{k-1}^n) S_{k-1}^n}{n} + \frac{\sigma(S_{k-1}^n) S_{k-1}^n}{\sqrt{n}} \varepsilon^k \\ \frac{r(S_{k-1}^n) B_{k-1}^n}{n} \\ \frac{\theta(S_{k-1}^n) \xi_{k-1}^n}{\sqrt{n}} \varepsilon^k \\ \frac{\lambda(S_{k-1}^n)}{n} \end{pmatrix} & t = \frac{k}{n}, \\ 0 & \frac{k}{n} < t < \frac{k+1}{n}. \end{cases}$$

By the definition of τ_n^q, when $t \leq \tau_n^q$, $|Z_t^n - Z_{t-}^n| \leq 2q$, and $|Z_t^n - Z_{t-}^n|^2$ is of order $\frac{1}{n}$. By the dominated convergence theorem, we have $\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{t \leq \tau_n^q} |Z_t^n - Z_{t-}^n|^2\right] = 0$. Then using the same argument, we can get $\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{t \leq \tau_n^q} |L_t^n - L_{t-}^n|^2\right] = 0$ and $\lim_{n\to\infty} \mathbb{E}_n\left[\sup_{t \leq \tau_n^q} |A_t^n - A_{t-}^n|\right] = 0$.

(d) For all $q > 0$, we have

$$L_t^n - \begin{pmatrix} \int_0^t (b(\tilde{S}_u^n) + \lambda(\tilde{S}_u^n) \tilde{S}_u^n) du \\ \int_0^t r(\tilde{S}_u^n) \tilde{B}_u^n du \\ 0 \\ \int_0^t \lambda(\tilde{S}_u^n) du \end{pmatrix} = - \begin{pmatrix} (b(\tilde{S}_t^n) + \lambda(\tilde{S}_t^n) \tilde{S}_t^n) \left(t - \frac{[nt]}{n}\right) \\ r(\tilde{S}_t^n) \tilde{B}_t^n \left(t - \frac{[nt]}{n}\right) \\ 0 \\ \lambda(\tilde{S}_t^n) \left(t - \frac{[nt]}{n}\right) \end{pmatrix},$$

where $\frac{k}{n} < t < \frac{k+1}{n}$, and the above equation equals zero when $t = \frac{k}{n}$. Therefore,

$$Q_n\left[\sup_{t\le\tau_n^q}\left|L_t^n - \int_0^t b(X_s^n)ds\right| \ge \epsilon\right] \le \frac{\mathbb{E}_{Q_n}\left[\sup_{t\le\tau_n^q}\left|L_t^n - \int_0^t b(X_s^n)ds\right|\right]}{\epsilon}.$$

We can easily prove that $\lim_{n\to\infty}\hat{P}\left[\sup_{t\le\tau_n^q}\left|L_t^n - \int_0^t b(X_s^n)ds\right| \ge \epsilon\right] = 0$ as n tends to infinity, as desired. □

Proof of Lemma 4.3. Applying Taylor's expansion to the function x^{2m}, we obtain

$$\begin{aligned}[S_{k+1}^n]^{2m} &= [S_k^n]^{2m} + 2m[S_k^n]^{2m-1}(S_{k+1}^n - S_k^n) + m(2m-1)[\bar{S}_k^n]^{2m-2}(S_{k+1}^n - S_k^n)^2 \\ &= [S_k^n]^{2m} + 2m[S_k^n]^{2m-1}\left(\frac{b+\lambda S}{n} + \frac{\sigma S}{\sqrt{n}}\varepsilon^{k+1}\right) + m(2m-1)[\bar{S}_k^n]^{2m-2}\left(\frac{b+\lambda S}{n} + \frac{\sigma S}{\sqrt{n}}\varepsilon^{k+1}\right)^2,\end{aligned}$$

where

$$\bar{S}_k^n = S_k^n + \beta\left(\frac{b(S_k^n)}{n} + \frac{\sigma(S_k^n)}{\sqrt{n}}\varepsilon^{k+1}\right), \quad \beta \in [0,1].$$

Moreover,

$$\begin{aligned}\mathbb{E}_{Q_n}[\varepsilon^{k+1} \mid \mathcal{F}_k^n] &= \mathbb{E}_{P_n}\left[\frac{2\tilde{\pi}_{k+1}^n\beta_{k+1}^n}{\tilde{\pi}_k^n\beta_k^n}\varepsilon^{k+1} \,\middle|\, \mathcal{F}_k^n\right] \\ &= \mathbb{E}_{P_n}\left[\left(1 + \frac{\theta(S_k^n)}{\sqrt{n}}\varepsilon^{k+1}\right)\varepsilon^{k+1} \,\middle|\, \mathcal{F}_k^n\right] \\ &= \frac{1}{2}\left(1 + \frac{\theta(S_k^n)}{\sqrt{n}}\right) - \frac{1}{2}\left(1 - \frac{\theta(S_k^n)}{\sqrt{n}}\right) = \frac{\theta(S_k^n)}{\sqrt{n}}.\end{aligned}$$

Then

$$\frac{b+\lambda S}{n} + \frac{\sigma S}{\sqrt{n}}\mathbb{E}_{P_n}\left[\varepsilon^{k+1} \mid \mathcal{F}_k^n\right] = \frac{b+\lambda S}{n} + \frac{\sigma S\theta}{n} = \frac{(r+\lambda)S}{n}.$$

Notice that

$$|\bar{S}_k^n| \le |S_k^n| + |b| + |\lambda S| + |\sigma S|, \qquad |\varepsilon^{k+1}| = 1,$$

and

$$x^{2m-2} \le 1 + x^{2m}, \qquad (x+y)^m \le 2^m(x^m + y^m), \qquad x^2 r(x) \le K(1+x^2)$$

when $x, y > 0$. Taking the conditional expectation with respect to $\mathcal{F}_k^n$ under Q_n, we have

$$\begin{aligned}\mathbb{E}_{Q_n}\left[[S_{k+1}^n]^{2m} \mid \mathcal{F}_k^n\right] \le\ & [S_k^n]^{2m} + 2m[S_k^n]^{2m-1}\left(\frac{|r+\lambda||S|}{n}\right) \\ &+ \frac{m(2m-1)}{n}\left(|S_k^n| + |b| + |\lambda S| + |\sigma S|\right)^{2m-2}(|b| + |\lambda S| + |\sigma S|)^2.\end{aligned}$$

Given the conditions on $b, \lambda S, \sigma S$, we can find a constant $K' > 0$, such that for any $x \in R$,

$$|b(x)| \le K'(1+|x|), \qquad |\lambda(x)x| \le K'(1+|x|), \qquad |\sigma(x)x| \le K'(1+|x|), \qquad |b(x)|^2 \le K'(1+x^2),$$
$$|\lambda(x)x|^2 \le K'(1+x^2), \qquad |\sigma(x)x|^2 \le K'(1+x^2), \quad \text{and} \quad |x^2 r(x)| \le K'(1+x^2).$$

Hence we can obtain

$$\begin{aligned}&\mathbb{E}_{Q_n}\left[[S_{k+1}^n]^{2m} \mid \mathcal{F}_k^n\right] \\ &\le [S_k^n]^{2m} + \frac{2mK'}{n}(1 + 2[S_k^n]^{2m}) + \frac{9K'^2 m(2m-1)}{n}(3K' + (1+3K')[S_k^n])^{2m-2}(1 + [S_k^n]^2) \\ &\le [S_k^n]^{2m} + \frac{2mK'}{n}(1 + 2[S_k^n]^{2m}) + \frac{9(2+6K')^{2m-2}m(2m-1)}{n}(1 + [S_k^n]^{2m-2})(1 + [S_k^n]^2) \\ &\le K/n + (1 + K/n)(1 + [S_k^n]^{2m}),\end{aligned}$$

and furthermore, $\mathbb{E}_{Q_n}[[S_k^n]^{2m} \mid \mathcal{F}_l^n] \le (1+K/n)^{k-l}(1 + [S_l^n]^{2m}) \le A(1 + [S_l^n]^{2m})$, where K depends on K' and m, $0 \le l \le k$, and $A = \sup_n(1+K/n)^{k-l}$.

Since $\mathbb{E}_{Q_n}\left[S_{k+1}^n \mid \mathcal{F}_k^n\right] \geq \mathbb{E}_{Q_n}\left[S_{k+1}^n \frac{B_k^n}{B_{k+1}^n} \,\middle|\, \mathcal{F}_k^n\right] = S_k^n$, we can get that (S_k^n) is a submartingale. Moreover, $\varphi(x) = x^{2m}$ is a convex and increasing function in $\mathbb{R}^+$ and (S_k^n) is nonnegative. By Jensen's inequality, we have $\mathbb{E}_{Q_n}[(S_{k+1}^n)^{2m} \mid \mathcal{F}_k^n] \geq \mathbb{E}_{Q_n}[S_{k+1}^n \mid \mathcal{F}_k^n]^{2m} \geq (S_k^n)^{2m}$. It is easy to see that $((S_k^n)^{2m})$ is a submartingale. By submartingale inequality, we have

$$\begin{aligned}\mathbb{E}_{Q_n}[[S_k^n]^{2m} \mid \mathcal{F}_l^n] &\leq A(1+\mathbb{E}_{Q_n}[\sup_{0\leq t\leq T}|\tilde{S}_t^n|^{2m}]) \leq A(1+\left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}_{Q_n}[(\tilde{S}_T^n)^{2m}]) \\ &\leq A(1+\left(\frac{2m}{2m-1}\right)^{2m}A(1+[(S_0^n)^{2m})]) \leq C(1+(S_0)^{2m}),\end{aligned}$$

where C is large enough, and $\tilde{S}_T^n = S_n^n$. $\square$

Proof of Lemma 4.4. By Lemma 4.1, $\tilde{Z}^n$ converges weakly to Z, Y is a continuous function of $\tilde{Z}^n$. Applying continuous mapping theorem, we get

$$Y\left(\tilde{S}_\cdot^n, \frac{[n\cdot]}{n}\right) \Rightarrow Y(S_\cdot, \cdot), \quad n \to \infty. \tag{4.2}$$

Since $Y_n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) = Y\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - \tilde{e}_t^n$, where $\tilde{e}_t^n = Y\left(\tilde{S}_t^n, \frac{[nt]}{n}\right) - Y_n\left(\tilde{S}_t^n, \frac{[nt]}{n}\right)$, we need only prove that the stochastic process $\tilde{e}_\cdot^n$ converges weakly to zero.

Let "+" and "−" denote the states $\varepsilon^{k+1} = 1$ and $\varepsilon^{k+1} = -1$ respectively, and define $S_{k+1}^{n+} = S_{k+1}^n(\omega_1)$, $S_{k+1}^{n-} = S_{k+1}^n(\omega_2)$. We define two functions as follows.

$$f_+^{k,n}(t) = Y(S_k^n + t(S_{k+1}^{n+} - S_k^n), t_k^n + t(t_{k+1}^n - t_k^n)), \qquad f_-^{k,n}(t) = Y(S_k^n + t(S_{k+1}^{n-} - S_k^n).t_k^n + t(t_{k+1}^n - t_k^n)).$$

Let

$$\frac{\partial Y}{\partial S} = Y_S, \quad \frac{\partial Y}{\partial t} = Y_t, \quad \frac{\partial^2 Y}{\partial S^2} = Y_{SS}, \quad \frac{\partial^2 Y}{\partial t^2} = Y_{tt}, \quad \frac{\partial^2 Y}{\partial S \partial t} = Y_{St}, \quad Y\left(S_k^n, \frac{k}{n}\right) = Y_k.$$

Then by Taylor's expansion,

$$\begin{aligned}f_+(1) &= f_+(0) + f'_+(0) + \frac{1}{2}f''_+(0) + R_k^n \\ &= Y_k + Y_S(S_{k+1}^{n+} - S_k^n) + \frac{1}{n}Y_t + \frac{1}{2}Y_{SS}(S_{k+1}^{n+} - S_k^n)^2 + \frac{1}{2n^2}Y_{tt} + Y_{St}\frac{1}{n}(S_{k+1}^{n+} - S_k^n) + R_k^n,\end{aligned}$$

where $R_k^n = \frac{1}{2}\int_0^1 (1-s)^2 f_+^{(3)}(s)ds$. The expression of $f_-(1)$ is similar to $f_+(1)$ with S_{k+1}^{n+} replaced by S_{k+1}^{n-}. By denoting the remaining terms by Q_k^n, we have

$$\begin{aligned}&\tilde{\pi}(+; S_k^n)f_+(1) + \tilde{\pi}(-; S_k^n)f_-(1) \\ &= \left(1+\frac{\tilde{r}_k}{n}\right)^{-1}\left[Y_k + \frac{(r_k+\lambda_k)S_kY_S + Y_t}{n} + \left(\frac{(b_k+\lambda_kS_k)^2 + 2(b_k+\lambda_kS_k)\sigma_k\theta_kS_k}{2n^2} + \frac{\sigma_k^2S_k^2}{2n}\right)Y_{SS}\right. \\ &\quad \left. + \frac{1}{2n^2}Y_{tt} + \frac{(r_k+\lambda_k)S_k}{n^2}Y_{St}\right] - \gamma_k^n.\end{aligned}$$

By Lemma 2.1, the above equation is equal to

$$\begin{aligned}&\left(1+\frac{\tilde{r}_k}{n}\right)^{-1}\left[\left(1+\frac{r_k+\lambda_k}{n}\right)Y_k + \left(\frac{(b_k+\lambda_kS_k)^2 + 2(b_k+\lambda_kS_k)\sigma_k\theta_kS_k}{2n^2} + \frac{Y_{tt}}{2n^2} + \frac{(r_k+\lambda_k)S_k^n}{n^2}Y_{St}\right)\right] - \gamma_k^n \\ &= Y_k - \left(1+\frac{\tilde{r}_k}{n}\right)^{-1}\frac{1}{n^2}m\left(S_k^n, \frac{k}{n}\right) - \gamma_k^n,\end{aligned}$$

where $\gamma_k^n = -\tilde{\pi}(+; S_k^n)R_k^n - \tilde{\pi}(-; S_k^n)Q_k^n$, and

$$m\left(S_k^n, \frac{k}{n}\right) = \left(\frac{(b_k+\lambda_kS_k)^2 + 2(b_k+\lambda_kS_k)\sigma_k\theta_kS_k}{2n^2} + \frac{1}{2n^2}Y_{tt} + \frac{(r_k+\lambda_k)S_k^n}{n^2}Y_{St}\right) + Y_k\left(\frac{\lambda_k^2 + 2r_k\lambda_k}{2} + o\left(\frac{1}{n}\right)\right).$$

Hence we obtain the following recurrent equation for e_k^n,

$$\begin{aligned} e_k^n &= \tilde{\pi}(+; S_k^n) Y\left(S_{k+1}^{n+}, \frac{k+1}{n}\right) + \tilde{\pi}(-; S_k^n) Y\left(S_{k+1}^{n-}, \frac{k+1}{n}\right) + \left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2} m\left(S_k^n, \frac{k}{n}\right) \\ &\quad + \gamma_k^n - \tilde{\pi}(+; S_k^n) Y_n\left(S_{k+1}^{n+}, \frac{k+1}{n}\right) - \tilde{\pi}(-; S_k^n) Y_n\left(S_{k+1}^{n+}, \frac{k+1}{n}\right) \\ &= \tilde{\pi}(+; S_k^n) e_{k+1}^{n+} + \tilde{\pi}(-; S_k^n) e_{k+1}^{n-} + \left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2} m\left(S_k^n, \frac{k}{n}\right) + \gamma_k^n. \end{aligned}$$

By the definition of $\tilde{\pi}(\cdot; S_k^n)$, we obtain

$$e_k^n = \mathbb{E}_{Q_n}\left[e_{k+1}^n \left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \,\middle|\, \mathcal{F}_k^n \right] + \left(1+\frac{\tilde{r}_k}{n}\right)^{-1} \frac{1}{n^2} m\left(S_k^n, \frac{k}{n}\right) + \gamma_k^n.$$

Since $e_n^n = Y(S_n^n, 1) - Y_n(S_n^n, 1) = g(S_n^n) - g(S_n^n) = 0$, we get

$$e_k^n = \mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \frac{1}{n^2} m\left(S_i^n, \frac{i}{n}\right) \frac{\beta_k^n}{\beta_{i+1}^n} + \gamma_i^n \frac{\beta_k^n}{\beta_i^n} \,\middle|\, \mathcal{F}_k^n \right].$$

By the assumption that Y and its derivative satisfy linear increasing condition, there exist constants $C_1 > 0$ and q, such that $\left| m\left(S_i^n, \frac{i}{n}\right)\right| \le C_1(1 + |S_i^n|^{2q})$.

By Lemma 4.3, for $k \le i \le n$ there exists constant $C > 0$ large enough such that $E_{Q_n}[|S_i^n|^{2q} \mid \mathcal{F}_k^n] \le C(1 + |S_0|^{2q})$. Therefore,

$$\begin{aligned} &\mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \left| \frac{1}{n^2} m\left(S_i^n, \frac{i}{n}\right) \frac{\beta_k^n}{\beta_{i+1}^n} \right| \,\middle|\, \mathcal{F}_k^n \right] \le \frac{1}{n^2} \sum_{i=k}^{n-1} \mathbb{E}_{Q_n}\left[\left| m\left(S_i^n, \frac{i}{n}\right)\right| \,\middle|\, \mathcal{F}_k^n \right] \\ &\quad \le \frac{1}{n^2} \sum_{i=k}^{n-1} \mathbb{E}_{Q_n}[C_1(1 + |S_i^n|^{2q}) \mid \mathcal{F}_k^n] \le \frac{1}{n^2} \sum_{i=k}^{n-1} C_1(1 + C(1 + |S_0|^{2q})). \end{aligned}$$

For the second part we can also write out the expressions of $f_+^{(3)}(s), f_-^{(3)}(s)$, they are of order $n^{-\frac{3}{2}}$, by analogous argument we can choose q large enough and constant $D > 0$ satisfying

$$\mathbb{E}_{Q_n}\left[\sum_{i=k}^{n-1} \left| \gamma_i^n \frac{\beta_k^n}{\beta_i^n} \right| \,\middle|\, S_k^n \right] \le \frac{D}{\sqrt{n}}(1 + |S_k^n|^{2q}).$$

Then we can choose $\tilde{C}$ large enough which depends on q, k and n, such that $|e_k^n| \le \frac{\tilde{C}}{\sqrt{n}}(1 + |S_0|^{2q})$. So

$$\hat{P}\left(\sup_{0\le t\le 1} |\tilde{e}_t^n| \ge \epsilon \right) \le \hat{P}\left(\sup_{0\le t\le 1} \frac{\tilde{C}(1 + |\tilde{S}_0|^{2q})}{\sqrt{n}} \ge \epsilon \right) \le \frac{\tilde{C}\left(1 + \sup\limits_{0\le t\le 1} |\tilde{S}_0|^{2q}\right)}{\epsilon\sqrt{n}} \to 0$$

as $n \to \infty$. Therefore, $\sup_{0\le t\le 1} |\tilde{e}_t^n| \to 0$ as $n \to \infty$, which means that $\tilde{e}_t^n$ converges to zero almost surely, that is, $\tilde{e}_\cdot^n \Rightarrow 0$. Combined with (4.2), we get the conclusion. □

5. Conclusion

In this paper, the weak convergence of discrete-time equity derivatives pricing model with default risk is proved in a no-arbitrage framework. Our results present a mathematical foundation for derivative pricing with default risk using numerical method. It remains to study the convergence for the hedging strategy.

Acknowledgments

We would like to thank the anonymous referee for the useful suggestions, which are a great help to improve the manuscript. We thank Xiaobin Li for his careful reading of the manuscript and useful comments. The first author would like to thank Professors Zenghu Li and Wenming Hong for their invaluable comments and encouragement during her stay in Beijing Normal University.

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