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ABSTRACT

In this paper, we mainly investigate the long-time asymptotic behavior of the solution for coupled dispersive AB systems with weighted Sobolev initial data, which allows soliton solutions via the Dbar steepest descent method. Based on the spectral analysis of Lax pairs, the Cauchy problem of coupled dispersive AB systems is transformed into a Riemann–Hilbert problem, and the existence and uniqueness of its solution is proved by the vanishing lemma. The stationary phase points play an important role in determining the long-time asymptotic behavior of these solutions. We demonstrate that in any fixed time cone \( \mathcal{C}(x_1, x_2, v_1, v_2) = \{(x, t) \in \mathbb{R}^2 \mid x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2]\} \), the long-time asymptotic behavior of the solution for coupled dispersive AB systems can be expressed by \( \mathcal{O}(t^{-1/2}) \) on the continuous spectrum, the leading order term \( \mathcal{O}(t^{-1/4}) \) on the discrete spectrum, and the allowable residual \( \mathcal{O}(t^{-3/4}) \).

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I. INTRODUCTION

In the past few decades, many solutions of integrable equations have been studied, including soliton solutions, breather solutions, rogue wave solutions, and so on. However, for continuous spectra or reflection, it is necessary to analyze the solution’s asymptotic behavior. In fact, as early as 1973, Manakov et al. first studied the long-time asymptotic behavior of the solution of the fast decay initial value problem for nonlinear integrable systems by using the inverse scattering method. In 1976, Zakharov and Manakov obtained the first asymptotic expression of the solution, which was about the initial value problem of the nonlinear Schrodinger (NLS) equation and explicitly dependent on the initial value. In 1981, the Soviet mathematician Its used the single value deformation theory to transform the long-time behavior problem of the initial value problem of the NLS equation into the local Riemann–Hilbert problem (RHP) in the small neighborhood of the stationary phase point, which provided a set of feasible and strict ways to analyze the long-time behavior of the integrable equation. In 1993, Deift and Zhou developed the nonlinear descent method for solving oscillatory RHP based on the classical descent method and studied the asymptotic properties of the solution for the mKdV equation with decaying initial value. Since then, more and more scholars have given attention to the nonlinear descent method to study the long-time asymptotic behavior of the solution for the initial value problem of integrable systems, and many equations have been studied by this method. Later, McLaughlin and Miller extended the classical Deift–Zhou steepest descent method to the Dbar steepest descent method, which was successfully used to study the asymptotic stability of NLS multiple soliton solutions and the long-time proximity of KdV equation and NLS equation. Instead of analyzing the asymptotic characteristics of orthogonal polynomials in singular integrals on the jump line, the Dbar steepest descent approach transforms the discontinuous component of the jump line into a Dbar problem. Later, it proved to be a more powerful tool in the study of the long-time asymptotic behavior of solutions to integrable equations, such as the NLS equation, the Hirota equation, the Fokas–Lenells equation, the modified Camassa–Holm equation, the short-pulse equation, and so on.
The coupled dispersion AB system given by

\[ \begin{align*}
A_{xt} - \alpha A + \beta AB &= 0, \\
B_x + \gamma \left( |A|^2 \right)_t &= 0
\end{align*} \]  \hspace{1cm} (1)

describes the evolution of micro-stable or unstable wave packets in baroclinic shear flow through a quasi-geostrophic two-layer model on a beta-plane, where \( A \equiv A(x,t) \) is the complex function representing the amplitude of the data packet, the real function \( B \equiv B(x,t) \) is the measure of the average flow change caused by the baroclinic wave packet, \( \alpha \) represents the critical situation of shear, and the real parameters \( \beta, \gamma \) represent the corresponding nonlinear coefficients. Moreover, system (1) fulfills the following compatibility condition:

\[ \frac{\gamma}{\beta} |A|^2 + B^2 + \frac{2\alpha}{\beta} B = f(t), \]

where \( f(t) \) is an integral function of time. If \( A(x,t) = 1/\sqrt{\beta} \psi(x,t) \) and \( B(x,t) = \pm [\cos(\psi(x,t)) - \alpha]/\beta \) with \( \beta \gamma > 0 \), system (1) can reduce to \( \psi_{xt} = \pm \sin \psi \). If \( A(x,t) = 1/\sqrt{\beta} \psi(x,t) \) and \( B(x,t) = \pm [\cos(\psi(x,t)) - \alpha]/\beta \) with \( \beta \gamma < 0 \), system (1) can reduce to \( \psi_{xt} = \pm \sinh \psi \).

In recent years, many studies have focused on system (1).40–42 For example, the soliton solutions of system (1) are obtained by the inverse scattering method.41 Kamchatnov and Pavlov found the periodic wave solution of system (1).41 Guo et al. have studied its classical Darboux transformation (DT) and N-flof DT and obtained breather solutions and multi-soliton solutions.43–46 The rogue wave solution of system (1) was also obtained by generalized DT under the condition \( \alpha = 0, \beta = \gamma = 1, f(t) = 1 \).47 In addition, the high-order wave and modulation instability of system (1) were further obtained via generalized DT.48 Recently, Chen and Yan studied the long-time asymptotic behavior of solutions with initial values belonging to Schwarz space through the classical Deift–Zhou steepest descent method.49

In this paper, we mainly study the long-time asymptotic behavior of the solution of system (1) satisfying the following initial value problem:

\[ A(x,0) = A_0(x) \in H^{1,1}(\mathbb{R}), B(x,0) = B_0(x) \in H^{1,1}(\mathbb{R}), \]

where \( H^{1,1}(\mathbb{R}) \) is the weighted Sobolev space, given by

\[ H^{1,1}(\mathbb{R}) = \{ f(x) \in L^2(\mathbb{R}) : f'(x), sf(x) \in L^2(\mathbb{R}) \}. \]

The main tool used here is the Dbar steepest descent method. Compared with Deift–Zhou steepest descent method, the Dbar steepest descent method avoids estimating the Cauchy integral operator in \( L^p \) space and rewrites the discontinuous part of RHP into a Dbar problem that can be solved by integral equations. Compared with previous studies,50 we consider the initial values in a wider space instead of in the Schwarz space and allow the existence of soliton solutions.

This paper is organized as follows:

In Sec. II, we analyze the spectrum of system (1) based on Lax pairs. The analyticity, symmetry, and asymptotic behavior of the characteristic function and scattering matrix are studied. In terms of asymptotic behavior, two singular points, \( k = 0 \) and \( k = \infty \), need to be considered. We also prove that when the initial value belongs to a weighted Sobolev space, \( r(k) \) belongs to \( H^{1,1}(\mathbb{R}) \).

In Sec. III, the RHP of system (1) is constructed by the piecewise smooth property of matrix \( M(x,t) \).

In Sec. IV, we prove the existence and uniqueness of the solution of RHP by using the vanishing lemma.

In Sec. V, by introducing the function \( T(k) \), we change \( M(x,k) \) into \( M^{(1)}(x,k) \) and get a new RHP about \( M^{(1)}(x,k) \). This is mainly to allow the jump matrix near the phase points to have two triangular decompositions.

In Sec. VI, the contour is deformed near the phase points, and a mixed \( \bar{\partial} \)-RHP is obtained by defining \( R^{(1)} \).

In Sec. VII, we decompose the mixed \( \bar{\partial} \)-RHP into a model RHP of \( M_{\text{app}}(k) \) and the pure \( \bar{\partial} \) problem of \( M^{(3)}(k) \).

In Sec. VIII, the RHP about \( M_{\text{app}}(k) \) is studied and the long-time behavior of the soliton solution is analyzed. It is seen that the soliton solution can be represented by a ring region, and the applicability near the phase points is also explained. Secondly, the error function is calculated with RHP of a small norm.

In Sec. IX, the pure \( \bar{\partial} \) problem for \( M^{(3)}(k) \) is analyzed.

In Sec. X, using the above deformation and results, Eq. (103) is obtained and then the long-time behavior of the solution of the coupled dispersion AB system (1) in the case of a weighted Sobolev initial value is obtained and presented in the form of a theorem.
II. THE SPECTRAL ANALYSIS

System (1) has a Lax pair, with \( k \)

\[
\phi_{x} = \dot{\phi} = \begin{pmatrix} -ik & \frac{\sqrt{\beta} y}{2} \\ -\frac{\sqrt{\beta} y}{2} A^* & ik \end{pmatrix} \phi, \\
\phi_{t} = \dot{\psi} = \begin{pmatrix} (i(\alpha + \beta)) & -i\frac{\sqrt{\beta} y}{4k} \\ i\frac{\sqrt{\beta} y}{4k} A^* & i(\alpha + \beta) \end{pmatrix} \psi.
\]

(2)

In order to better facilitate the presentation of the results, we mainly consider the case of \( \beta y = 1 \). At this time, the Lax pair of Eq. (1) can be simplified to

\[
\Phi_{x} = X \Phi, \quad \Phi_{t} = T \Phi,
\]

where

\[
X = -ik\sigma_{3} + U, \quad T = i\alpha \sigma_{3} + T_{1}
\]

(4)

with

\[
\sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U(x, t) = \frac{1}{2} \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}, \quad T_{1}(x, t, k) = \frac{i}{4k} \left( \beta B - A_{0} \right).
\]

(5)

and \( k \in \mathbb{C} \) is a spectral parameter. Under the rapidly decaying initial condition

\[
A_{0}(x), B_{0}(x) \to 0 \quad \text{as} \quad x \to \infty,
\]

we make the following correction transformation:

\[
\Phi = \Psi e^{-ik\sigma_{3}t + \frac{i}{4k} t\sigma_{0}},
\]

(6)

the modified Lax pair can be obtained as

\[
\Psi_{x} + i\left[ \sigma_{3}, \Psi \right] = U \Psi, \\
\Psi_{t} - \frac{i\alpha}{4k} \left[ \sigma_{3}, \Psi \right] = T_{1} \Psi,
\]

(7)

which can be written in full derivative form as

\[
d\left( e^{i(kx + \frac{\beta y}{4k} t)} \Psi(x, t, k) \right) = e^{i(kx + \frac{\beta y}{4k} t)} \left[ U \Psi + T_{1} \Psi \right].
\]

From the above Lax pair (7), we can see that it has two singular points \( k = 0 \) and \( k = \infty \). Therefore, we need to investigate the different expansion forms of the characteristic function at two singular points. After simple calculation, the following form is obtained:

\[
\Psi(x, t, k) = I + \frac{\psi_{1}}{k} + \mathcal{O}(k^{-2}), \quad k \to \infty,
\]

\[
\Psi(x, t, k) = \psi_{0} + \psi_{1} k + \mathcal{O}(k^{2}), \quad k \to 0,
\]

where

\[
\psi_{0}(x, t) = \begin{pmatrix} e^{i \int_{-\infty}^{x} \frac{\alpha x}{\beta} dx} & e^{i \int_{-\infty}^{x} \frac{\alpha x + \gamma}{\beta} dx} \\ e^{i \int_{-\infty}^{x} \frac{\alpha x + \gamma}{\beta} dx} & e^{i \int_{-\infty}^{x} \frac{\alpha x}{\beta} dx} \end{pmatrix}, \quad \psi_{12} = -\frac{i}{4} A, \quad \psi_{21} = -\frac{i}{4} A^*.
\]

(8)

The asymptotic solution \( \Psi(x, t, k) \) satisfies

\[
\psi_{\pm}(x, t, k) \to I, \quad x \to \pm \infty,
\]

and the modified solution satisfies the following integral equation:
\( \Psi_{\pm}(x, t, k) = 1 + \int_{-\infty}^{+\infty} e^{-ik(x-y)t} U(y, t) \Psi_{\pm}(y, t, k) e^{ik(x-y)t} \, dy. \) (9)

Dividing \( \Psi_{\pm} \) into columns as \( \Psi_{\pm} = (\Psi_{\pm}^{(1)}, \Psi_{\pm}^{(2)}) \), due to the structure of the potential \( U \) and Volterra integral Eq. (9), we have the following:

**Proposition II.1.** For \( A(x), B(x) \in H_{1,1}(\mathbb{R}) \) and \( t \in \mathbb{R}^+ \), there exist unique eigenfunctions \( \Psi_{\pm} \) that satisfy Eq. (9), respectively, and we have the following properties:

- \( \det \Psi_{\pm}(x, t, k) = 1, \quad x, t \in \mathbb{R}; \)
- \( [\Psi_-], [\Psi_+] \) are analytic in \( \mathbb{C}^+ \) and continuous in \( \mathbb{C}^+ \cup \mathbb{R} \); and
- \( [\Psi_+], [\Psi_-] \) are analytic in \( \mathbb{C}^- \) and continuous in \( \mathbb{C}^- \cup \mathbb{R} \).

The partition of \( \mathbb{C}^\pm \) is shown in Fig. 1. It can be seen from the equation that \( \Psi_- e^{-ikx} e^{\frac{\pi i}{2\alpha} t} \) and \( \Psi_+ e^{-ikx} e^{\frac{\pi i}{2\alpha} t} \) are the solutions of Lax Eq. (9), so they are linearly related, i.e.,

\[ \Psi_- e^{-ikx} e^{\frac{\pi i}{2\alpha} t} = \Psi_+ e^{-ikx} S(k), \quad k \in \mathbb{R}, \]

where

\[ S(k) = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \]

**Proposition II.2.** \( \Psi(x, t, k) \) and \( S(t, k) \) satisfy the symmetry relation

\[ \Psi(x, t, k) = \sigma_0 \Psi^*(x, t, k^*) \sigma_0^{-1}, \quad S(k) = \sigma_0 S^*(k^*) \sigma_0^{-1}, \]

where

\[ \sigma_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

**Proof.** The symmetry of \( \Phi \) can be easily obtained from the Lax equation,

\[ \Phi(x, t, k) = \sigma_0 \Phi^*(x, t, k^*) \sigma_0^{-1}, \]

followed by contact transformation (6), which is easy to verify,

\[ \Psi(x, t, k) = \sigma_0 \Psi^*(x, t, k^*) \sigma_0^{-1}. \]

Then, according to the relationship (10), we have

\[ S(k) = \sigma_0 S^*(k^*) \sigma_0^{-1}. \]

\[ \square \]

**FIG. 1.** Definition of the \( \mathbb{C}^+=\{k \mid \text{Im}k > 0\} \), \( \mathbb{C}^- = \{k \mid \text{Im}k < 0\} \).
Based on the symmetry condition of \( S(k) \), we get the following relations:
\[
s_{11}(k) = s_{22}^*(k^*), s_{12}(k) = -s_{21}^*(k^*).
\]

\( S(t,k) \) is called the scattering matrix and \( s_{ij}(i,j=1,2) \) are usually called scattering data. Moreover, Eq. (10) leads to
\[
s_{11}(k) = \det([\Psi^\pm]_1, [\Psi^\pm]_2), \quad s_{12}(k) = e^{2ikx+i\frac{\pi}{2}t} \det([\Psi^\pm]_2, [\Psi^\pm]_3).
\]

On the basis of the properties of \( \Psi \) in Proposition II.1, we get the following:

**Proposition II.3.** The scattering matrix \( S(t,k) \) and scattering data \( s_{11}(k), s_{12}(k) \) satisfy the following conditions:

- \( \det S(k) = 1, \quad x, t \in \mathbb{R}; \)
- \( |s_{11}(k)|^2 + |s_{12}(k)|^2 = 1 \) for \( k \in \mathbb{R}; \)
- \( s_{11}(k) \) is analytic in \( \mathbb{C}^+ \) and continuous in \( \mathbb{C} \cup \mathbb{R}; \)
- \( s_{12}(k) \) is continuous on the real \( k; \)
- \( s_{11}(k) = 1 + \mathcal{O}(k^{-1}) \) as \( k \to \infty; \)
- \( s_{12}(k) = \mathcal{O}(k^{-1}) \) as \( k \to \infty; \) and
- \( s_{11}(k) = \varphi_0 + \mathcal{O}(k^2) \) as \( k \to 0; \)
- \( s_{12}(k) = \mathcal{O}(k^2) \) as \( k \to 0, \)

where \( \varphi_0 = e^{\int_{-\infty}^{\infty} \frac{\delta\alpha_0}{2\pi} \, dx} - e^{\int_{-\infty}^{\infty} \frac{\delta\alpha_2}{2\pi} \, dx}. \)

The reflection coefficient is described below, which is usually defined as
\[
r(k) = \frac{s_{12}(k)}{s_{11}(k)}.
\]

Furthermore, using the first two properties in Proposition II.3, we have
\[
1 + |r(k)|^2 = \frac{1}{|s_{11}(k)|^2}.
\]

To cope with the following derivation, we assume that the initial data meets the following assumption.

**Assumption II.4.** The initial data \( A, B \in H^{1,1}(\mathbb{R}) \) and they generate generic scattering data with the following properties:

- \( s_{11}(k) \) has no zeros on \( \mathbb{R}; \)
- \( s_{11}(k) \) only has finite number of simple zeros.

Based on the above analysis, we get the following theorem for \( r(k) \).

**Theorem II.5.** For any given initial value \( A, B \in H^{1,1}(\mathbb{R}) \), we have \( r(k) \in H^{1,1}(\mathbb{R}). \)

**Proof.** According to the definition of \( r(k) \), we have and the asymptotic behavior of \( s_{11} \) and \( s_{12} \) at singular points in Proposition II.3 and we can obtain
\[
r(k) = \mathcal{O}(k^{-1}), \quad k \to \infty, \quad r(k) = \mathcal{O}(k^2), \quad k \to 0.
\]

From Lemma X.2 in Appendix A,
\[
s_{11}(k), s_{12}(k) \in L^2(\mathbb{R}),
\]
which leads to
\[
r(k) \in L^2(\mathbb{R}).
\]

In addition, it can also be seen that \( s_{11}'(k), s_{12}'(k) \in L^2(\mathbb{R}). \) Therefore,
\[
r'(k) = \frac{s_{12}(k)s_{11}(k) - s_{11}'(k)s_{12}(k)}{s_{12}^2(k)} \in L^2(\mathbb{R}).
\]

Finally, we come to the conclusion that the map \( A_0, B_0 \to r(k) \) is Lipschitz continuous from \( H^{1,1}(\mathbb{R}) \) into \( H^{1,1}(\mathbb{R}). \) \( \blacksquare \)
III. THE CONSTRUCTION OF A RHP

Suppose \( s_{11}(k) \) has \( N \) simple zeros \( \mathcal{K} = \{ k_j \}_{j=1}^N \subset \mathbb{C}^+ \), reviewing the symmetry of \( S \), we can see that there are also \( N \) simple zeros \( \mathcal{K}^* = \{ k_j^* \}_{j=1}^N \subset \mathbb{C}^- \). Then, we define a meromorphic function \( M(x, t, k) \) as

\[
M(x, t, k) = \begin{cases}
\begin{bmatrix}
\Psi_1^- & \Psi_2^- \\
\bar{s}_{11}(k) & \bar{s}_{22}(k)
\end{bmatrix}, & k \in \mathbb{C}^+, \\
\begin{bmatrix}
\Psi_1^+ & \Psi_2^+ \\
\bar{s}_{11}(k) & \bar{s}_{22}(k)
\end{bmatrix}, & k \in \mathbb{C}^-,
\end{cases}
\]

which solves the following RHP.

Riemann–Hilbert Problem III.1. Find a matrix-valued function \( M(x, t, k) \) that satisfies the following conditions:

- Analyticity: \( M(x, t, k) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \) and has single poles.
- Symmetry: \( M(x, t, k) = \sigma_0 M^*(x, t, k^*) \sigma_0^{-1} \).
- Jump condition: \( M(x, t, k) \) has continuous boundary values \( M\pm(x, t, k) \) on \( \mathbb{R} \) and

\[
M_+(x, t, k) = M_-(x, t, k) V(k), \quad k \in \mathbb{R},
\]

where

\[
V(k) = \begin{pmatrix}
1 + |r(k)|^2 & r^*(k) e^{-2i\theta(k)} \\
r(k) e^{2i\theta(k)} & 1
\end{pmatrix},
\]

- Asymptotic behavior:

\[
M(x, t, k) = I + O(k^{-1}), \quad k \to \infty.
\]
- Residue conditions: \( M \) has simple poles at each point in \( k_j \in \mathbb{C}^+ \) and \( k_j^* \in \mathbb{C}^- \) with

\[
\text{Res}_{k=k_j} M(z) = \lim_{k \to k_j} M(k) \begin{pmatrix} 0 & 0 \\ -c_j^* e^{-2i\theta(k_j^*)} & 0 \end{pmatrix},
\]

\[
\text{Res}_{k=k_j^*} M(z) = \lim_{k \to k_j^*} M(k) \begin{pmatrix} 0 & 0 \\ c_j e^{2i\theta(k_j)} & 0 \end{pmatrix},
\]

where \( \theta = k_j^* - \frac{\pi}{4} \), \( c_j = \frac{s_{12}(k_j)}{s_{11}(k_j)} \).

The solution of the coupled dispersion AB system (1) can be expressed by

\[
A = 4i \lim_{k \to \infty} (kM)_{12}(x, t, k), \quad B = -4i \lim_{k \to \infty} \frac{d}{dt} (kM)_{11}.
\]

IV. EXISTENCE AND UNIQUENESS OF SOLUTION OF THE RH PROBLEM

In this section, we will mainly focus on the existence and uniqueness of the solution of the RH problem constructed above. Our aim here is to use the vanishing lemma to show that the integral equation has only zero solution to the homogeneous equation and to then show that the equation has a unique solution. However, it should be noted that the coupled dispersion AB system (1) is hyperbolic and that the proposed initial value problem is characteristic—that is, the initial data are given along the boundary of the light cone, which is similar to the Maxwell–Bloch system. This situation requires more attention than a non-characteristic problem and some causality. Otherwise, the solution cannot be guaranteed to be unique and the solution may quickly leave the function space and be occupied by the initial data. The causality we added here is to require \( A \) and \( B \) to disappear outside the light cone in order to obtain the unique solution through inverse scattering.

In order to facilitate the application of the subsequent lemma, we focus all the residue conditions in RHP III.1 on the circle \( \{ k_j, k_j^* \} \) centered on the pole \( \{ k_j \in \mathbb{C}^+, k_j^* \in \mathbb{C}^- \} \). As long as these circles are small enough, they will not intersect, as shown Fig. 2 in detail. Let

\[
\Sigma^{(1)} = \mathbb{R} \cup \{ k_j \}_{j=1}^N \cup \{ k_j^* \}_{j=1}^N.
\]
Now, rewrite RHP III.1 as follows:

**Riemann–Hilbert Problem IV.1.** Find a matrix-valued function \( M(x, t, k) \) satisfying the following conditions:

- **Analyticity:** \( M(x, t, k) \) is analytic in \( \mathbb{C} \setminus \Sigma^{(1)} \).
- **Symmetry:**
  \[
  M(x, t, k) = \sigma_0 M^*(x, t, k^*) \sigma_0^{-1}.
  \]
- **Jump condition:** \( M(x, t, k) \) has continuous boundary values \( M_\pm(x, t, k) \) on \( \Sigma^{(1)} \) and
  \[
  M_+(x, t, k) - M_-(x, t, k) = V'(k), \quad k \in \mathbb{R},
  \]
  where
  \[
  V'(k) = \begin{cases}
  \begin{pmatrix}
  1 + |r(k)|^2 & \rho(k^*) e^{-2ikx} \\
  r(k)e^{2ikx} & 1
  \end{pmatrix}, & k \in \mathbb{R}, \\
  \begin{pmatrix}
  1 & 0 \\
  \frac{c}{k - k_j} & 1
  \end{pmatrix}, & k \in \kappa_j, \\
  \begin{pmatrix}
  1 & \frac{c^*}{k^* - k_j} \\
  0 & 1
  \end{pmatrix}, & k \in \kappa_j^*.
  \end{cases}
  \]

- **Normalization:***
  \[
  (a) \quad M(k, x) = I + \mathcal{O}(k^{-1}), \quad k \to \infty,
  \]
  \[
  (b) \quad M(k, x) = \mathcal{O}(k^{-1}), \quad k \to \infty.
  \]

Here, we deliberately turn \( t \) into a gray variable, mainly to better show the importance of \( x \) in the proof process. In normalization, part (a) is mainly used to reconstruct the potential \( A, B \). The role of (b) is to illustrate the existence and uniqueness of solutions to the RHP of type (a).

As we all know, \( V'(k) \) has the following decomposition:

\[
V'(k) = (I - \Lambda^-)^{-1}(I + \Lambda^+).
\]

After substituting it into (19) and defining \( \omega \) as

\[
\omega = M_+(I + \Lambda^+)^{-1} = M_-(I - \Lambda^-)^{-1},
\]

the above formula can be rewritten as

\[
M_+ - M_- = \omega(\Lambda^+ + \Lambda^-).
\]
Using the Plancherel formula, it is easy to give the solution of solvable RHP (IV.1) equation with regularized boundary (a) in the form of

\[ M(x, k) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\omega(k^+ + k^-)}{s - k} ds, \]  

(22)

with the solution of the equation with boundary condition (b) given by

\[ M(x, k) = \frac{1}{2\pi i} \int_{\Sigma} \frac{\omega(k^+ + \Lambda^-)}{s - k} ds. \]  

(23)

The main task in the following is to prove that the solution with boundary condition (b) tends to 0, that is, Eq. (23) tends to 0, which is also obvious. Let us write the expansion for \( M(x, k) \) as

\[ M(x, k) = \frac{1}{2\pi i} \left( \int_{R} \frac{\omega(k^+ + \Lambda^-)}{s - k} ds + \int_{\Sigma \setminus R} \frac{\omega(k^+ + \Lambda^-)}{s - k} ds \right) \]  

(24)

Here, we define

\[ \mathcal{N}(k) = M(k)M(k^*)^H, \]

where "\( H \)" represents the conjugate transpose of the matrix and we only need to prove

\[ \int_{\Sigma} \mathcal{N}_+(k) dk = 0 \]  

(25)

below. Obviously, according to Schwartz’s reflection principle, \( \mathcal{N} \) is analytical in \( \mathbb{C} \setminus \Sigma(1) \).

For \( k \in \mathbb{R} \), we have

\[ \mathcal{N}_+(k) = M_+(k)M_-(k)^H = M_-(k)V'(k)M_-(k)^H, \]

\[ \mathcal{N}_-(k) = M_-(k)M_+(k)^H = M_-(k)V'(k)^H M_-(k)^H. \]

Therefore, if Eq. (25) is satisfied, we have

\[ \int_{\mathbb{R}} M_-(k)(V'(k) + V'(k)^H)M_-(k)^H = 0; \]

since \( k \in \mathbb{R} \), we have \( V'(k) = V'(k)^H \). Then, we know that \( V'(k) + V'(k)^H \) is also Hermite and positive definite. Therefore, if the above formula holds, there must be \( M_-(k) = 0 \) on \( \mathbb{R} \). So,

\[ M_+(k) = M_-(k)V'(k) = 0, \quad k \in \mathbb{R}. \]

According to Morera’s theorem, we know that \( M(k) \) is analytic in the neighborhood of each point on \( \mathbb{R} \) and \( M(k) \) tends to disappear.

For \( k \in \Sigma(1) \setminus \mathbb{R} \), we have

\[ \mathcal{N}_+(k) = M_+(k)M_-(k^*)^H = M_-(k)V'(k)\left( \mathcal{N}_-(k^*)^{-1} \right)^HM_+(k^*)^H \]

\[ = M_-(k)M_+(k^*)^H = \mathcal{N}_-(k). \]

Similarly, it can be seen from Morera’s theorem that \( \mathcal{N}(k) \) is analytic in \( \Sigma(1) \setminus \mathbb{R} \). Moreover, \( M_+(x, \cdot) \in L^2(\mathbb{R}) \), we know that \( \mathcal{N}(k) \) is integrable. According to the integral form of Eq. (24),

\[ M(x, k) = \frac{1}{2\pi i} \int_{\Sigma(1) \setminus \mathbb{R}} \frac{\omega(k^+ + \Lambda^-)}{s - k} ds \]  

\[ = \frac{1}{2\pi i} \int_{\Sigma(1) \setminus \mathbb{R}} \omega(x, s)(\Lambda^+ + \Lambda^-) ds - \frac{1}{2\pi i} \int_{\Sigma(1) \setminus \mathbb{R}} \omega(x, s)(\Lambda^+ + \Lambda^-) ds, \]

the Taylor expansion of \( k \) shows that \( M(x, k) \sim \mathcal{O}(k^{-1}), k \in \mathbb{C} \setminus \mathbb{R} \). Moreover, we know that \( \mathcal{N}(k) \sim \mathcal{O}(k^{-2}), k \in \mathbb{C} \setminus \mathbb{R} \). Thus, Eq. (25) can be obtained by Cauchy’s integral theorem and Jordan’s theorem. We can apply the same argument \( k \in \mathbb{R} \) to \( k \in \Sigma(1) \setminus \mathbb{R} \), and we can also get \( M(x, k) = 0 \). Therefore, it is concluded that \( M(x, k) = 0 \) is true on the whole complex plane.

Thus, the following proposition can be obtained by using the vanishing lemma and Fredholm alternative theorem.
Proposition IV.2. Assume that the initial data \( A, B \in H^{1,1}(\mathbb{R}) \), then the RHP(IV.1) with normalization boundary condition (a) has a unique solution \( M(x, k) \).

V. CONJUGATION

Through the form of the jump matrix and the residual condition, it can be found that the long-term asymptotic of RHP III.1 is affected by the growth and attenuation of the exponential function \( e^{\pm 2it\theta(k)} \). Therefore, it is necessary to deal with the jump matrix in Eq. (16) and the oscillation term in the residual condition and decompose the jump matrix according to the sign change graph of \( \text{Re}(i\theta) \) to ensure that any jump matrix is bounded in a given area. We further rewrite \( \theta \) in the form

\[
\theta = \frac{ak}{4k_0^2} - \frac{\alpha}{4k} = \frac{\alpha}{4} \left( \frac{k}{k_0} + \frac{1}{k} \right),
\]

which leads to

\[
\text{Re}(i\theta) = \frac{Im(k)\alpha}{4} \left( \frac{1}{k_0^2} - \frac{1}{|k|^2} \right)
\]

For convenience, we mainly consider the case of \( \alpha x < 0 \) and \( t > 0 \), which can be treated similarly for \( t < 0 \). Without losing generality, it can be assumed that \( \alpha < 0 \) and \( x > 0 \). Based on this, we find that the two phase points of function \( \theta(k) \) are \( \pm k_0 \), where \( k_0 = \sqrt{-\frac{\alpha}{4x}} \).

According to the sign change of \( \text{Re}(i\theta) \), the attenuation region of oscillation term \( e^{\pm 2it\theta} \) can be obtained when \( t \to \infty \), which is shown in Fig. 3.

The jump matrix \( V(k) \) in RHP III.1 has the following decomposition:

\[
V(k) = \begin{pmatrix}
1 & r^*(k^*)e^{-2it\theta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\( |k| > k_0 \)

and

\[
V(k) = \begin{pmatrix}
1 & 0 \\
0 & 1 + |r(k)|^2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & r^*(k^*)e^{-2it\theta} \\
0 & 1 + |r(k)|^2
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\( |k| < k_0 \).

In order to remove the diagonal matrix in the second decomposition, we introduce a scalar RHP about \( \delta(k) \) by using the method in Ref. 36.

Riemann–Hilbert Problem V.1. Find a scalar function \( \delta(k) \) satisfying the following conditions:

- \( \delta(k) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \).
- \( \delta_+(k) = \delta_-(k) \left( 1 + |r(k)|^2 \right) \) for \( |k| < k_0 \) and \( \delta_+(k) = \delta_-(k) \) for \( |k| > k_0 \).
- \( \delta(k) \to 1 \) as \( k \to \infty \).

Riemann–Hilbert Problem V.1.

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With the help of the Plemelj formula, it is easy to write the unique solution of the above RHP V.1 as

$$\delta(k) = \exp\left[i \int_{-k_0}^{k_0} \frac{v(s)}{s-k} \, ds\right],$$

(27)

where $v(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2)$. The trace formulas can be given directly according to the following calculation.

**Proposition V.2.** The trace formulas are expressed as

$$s_{11}(k) = \prod_{j=1}^{N} \frac{k - k_j^*}{k - k_j^*} \exp\left[-i \int_{-\infty}^{\infty} \frac{v(z)}{z-k} \, dz\right],$$

$$s_{22}(k) = \prod_{j=1}^{N} \frac{k - k_j^*}{k - k_j^*} \exp\left[i \int_{-\infty}^{\infty} \frac{v(z)}{z-k} \, dz\right].$$

(28)

(29)

**Proof.** From the above, we know that $s_{11}(k)$ and $s_{22}(k)$ are analytic on $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, and according to the Assumption II.4, the discrete zeros of $s_{11}(k)$ and $s_{22}(k)$ are $k_j$ and $k_j^*$, respectively. So, let

$$\beta^+(k) = s_{11}(k) \prod_{j=1}^{N} \frac{k - k_j^*}{k - k_j^*}, \quad \beta^-(k) = s_{22}(k) \prod_{j=1}^{N} \frac{k - k_j^*}{k - k_j^*},$$

(30)

they are still analytic on $\mathbb{C}^+$, $\mathbb{C}^-$, but they no longer have zeros. Then, we have $\beta^+(k)\beta^-(k) = s_{11}(k)s_{22}(k)k \in \mathbb{R}$, and one obtains

$$\beta^+(k)\beta^-(k) = s_{11}s_{22} = \frac{1}{1 + |r(k)|^2}, \quad k \in \mathbb{R}.$$

In addition, we know that

$$\det S(k) = s_{11}(k)s_{22}(k) - s_{21}(k)s_{12}(k) = 1,$$

which leads to

$$\log \beta^+(k) - (-\log \beta^-(k)) = -\log(1 + |r(k)|^2), \quad k \in \mathbb{R}.$$

By using the Plemelj formula, we get

$$\beta^+(k) = \exp\left(-i \int_{-\infty}^{\infty} \frac{v(z)}{z-k} \, dz\right), \quad k \in \mathbb{C}^+.$$

(31)

Therefore, the trace formulas can be obtained by substituting Eq. (31) into (30).

For convenience, we introduce some notations,

$$\Delta_{k_0}^+ = \{ j \in \{1, \ldots, N \} \mid |k_j| > k_0 \},$$

$$\Delta_{k_0}^- = \{ j \in \{1, \ldots, N \} \mid |k_j| < k_0 \}.$$  

(32)

In addition, the following function is introduced:

$$T(k) = \prod_{j \in \Delta_{k_0}^-} \frac{k - k_j^*}{k - k_j^*} \delta(k).$$

(33)

**Proposition V.3.** The function $T(k)$ defined by Eq. (33) has the following properties:

- $T(k)$ is a meromorphic function in $\mathbb{C}\backslash(-k_0, k_0)$. For $j \in \Delta_{k_0}^-$, it has a simple pole at $k_j$ and a simple zero at $k_j^*$. In other places, it is nonzero and analytic.
- $T^+(k^*)T(k) = 1,$ for $k \in \mathbb{C}\backslash(-k_0, k_0)$.
- For $k \in (-k_0, k_0)$, the boundary values $T_+(k)$ satisfy

$$T_+(k) = T_-(k)(1 + |r(k)|^2), \quad k \in (-k_0, k_0).$$
• When $|k| \to \infty$, $|\arg(k)| \leq c < \pi$; thus,

$$T(k) = 1 + \frac{i}{k} \left[ 2 \sum_{j \in \Delta_0} \text{Im} \ k_j - \int_{k}^{k} v(s) ds \right] + O(k^{-2}).$$

• $k \to \pm k_0$, along any ray $\pm k_0 + e^{i \theta} \mathbb{R}_+$ with $|\theta| \leq c < \pi$,

$$\left| T(k) - T_0(\pm k_0)(k \mp k_0)^{i \nu(k_0)} \right| \leq c|k \mp k_0|^\frac{1}{2},$$

where

$$T_0(\pm k_0) = \prod_{j \in \Delta_0} \frac{\pm k_0 - k_j^*}{\mp k_0 - k_j} e^{\theta(k \pm k_0)},$$

$$\beta(k, -k_0) = \left( i \int_{-k_0}^{-k_0+1} \frac{v(-k_0)}{z-k} dz + i \int_{-k_0}^{k_0} \frac{v(z) - \chi_1(z)v(-k_0)}{z-k} dz \right),$$

$$\beta(k, k_0) = \left( i \int_{k_0}^{k_0+1} \frac{v(k_0)}{z-k} dz + i \int_{k_0}^{k_0} \frac{v(z) - \chi_2(z)v(k_0)}{z-k} dz \right),$$

$$\chi_0(k) = \begin{cases} 1, & -k_0 < k < -k_0 + 1, \\ 0, & \text{elsewhere}, \end{cases} \quad \chi_\pi(k) = \begin{cases} 1, & k_0 - 1 < k < k_0, \\ 0, & \text{elsewhere}. \end{cases}$$

With this definition of $T(k)$, the following transformation can be carried out to eliminate the diagonal matrix in the decomposition of the jump matrix in the interval $[-k_0, k_0]$:

$$M^{(1)}(k) = M(k) T(k)^{-\sigma_0},$$

which satisfies the following RHP:

**Riemann–Hilbert Problem V.4.** Find a matrix $M^{(1)}(k)$ satisfying the following properties:

• $M^{(1)}$ is analytic within $\mathbb{C} \setminus \mathbb{R} \cup \mathcal{K} \cup \mathcal{K}^*$;

• $M^{(1)}(k) = \sigma_0 (M^{(1)})^*(k) \sigma_0^{-1}$;

• $M^{(1)}(k) = I + O(k^{-1})$ for $k \to \infty$;

• $M^{(1)}(k) = M^- (k) V^{(1)}(k), \quad k \in \mathbb{R}_+$,

where

$$V^{(1)}(k) = \begin{pmatrix} 1 & 0 \frac{1}{1 + |r|^2} T^2 e^{-2i\theta(k)} & \frac{1}{1 + |r|^2} T^2 e^{-2i\theta(k)} \end{pmatrix}, \quad |k| < k_0,$$

$$V^{(1)}(k) = \begin{pmatrix} 1 & 0 \frac{1}{1 + |r|^2} T^2 e^{-2i\theta(k)} & \frac{1}{1 + |r|^2} T^2 e^{-2i\theta(k)} \end{pmatrix}, \quad |k| > k_0;$$

• $M^{(1)}(k)$ has simple poles at each point in $k_j \in \mathbb{C}^+$ and $k_j^* \in \mathbb{C}^-$, with

$$\text{Res} M^{(1)}(k) = \begin{cases} \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & \frac{1}{2} \left( k_j \right)^{'} e^{-2i\theta(k)} \end{pmatrix}, \quad j \in \Delta_0^-; \\ \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & \frac{1}{2} \left( k_j \right)^{'} e^{-2i\theta(k)} \end{pmatrix}, \quad j \in \Delta_0^+, \\ \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & \frac{1}{2} \left( k_j \right)^{'} e^{-2i\theta(k)} \end{pmatrix}, \quad j \in \Delta_0^-; \\ \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & \frac{1}{2} \left( k_j \right)^{'} e^{-2i\theta(k)} \end{pmatrix}, \quad j \in \Delta_0^+; \end{cases}$$

(35)
In addition, since $T(k)^{-\infty} \rightarrow I$, as $k \rightarrow \infty$, the solution of the coupled dispersion AB system can be expressed as

$$A = 4i \lim_{k \rightarrow \infty} (kM^{(1)}(k))_{12}, \quad B = \frac{4i}{\beta} \lim_{k \rightarrow \infty} d dt (kM^{(1)}(k))_{11}. \quad (36)$$

VI. CONTOUR DEFORMATION

Next, we eliminate the jump on the real axis and open it at a small angle at the steady-state phase point to deform the contour of RHP V.4. According to the number of steady-state phase points and spectral singularity points, the following contours can be considered:

$$\Sigma^{(2)} = \Sigma_1 \cup \Sigma_2^+ \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6 \cup \Sigma_7 \cup \Sigma_8, \quad (37)$$

which are shown in Fig. 4.

In order to facilitate later applications, new tokens are introduced,

$$\rho = \frac{1}{2} \min_{k \in \text{spectra}} |k - \mu|. \quad (38)$$

Because of the symmetry, and the pole is not on the real axis; then, for any $k_j = u_j + iv_j \in \mathbb{C}$, it is obvious that $k_j = u_j - iv_j \in \mathbb{C}^*$. According to the above representation, there must be $|k_j - k^*| = 2|v_j| > \rho$. So, it must be true that dis$(k, \mathbb{R}) = |v| \geq \rho > 0$ because of the arbitrariness of $j$.

In order to keep the residual condition unchanged during contour deformation, an eigenfunction is defined near the discrete spectrum as

$$Y_{k_i}(k) = \begin{cases} 1, & \text{dis}(k, \mathbb{C} \cup \mathbb{C}^*) < \rho/3, \\ 0, & \text{dis}(k, \mathbb{C} \cup \mathbb{C}^*) > 2\rho/3. \end{cases} \quad (39)$$

Now, we find a matrix $R_1 \rightarrow \mathbb{C}$ with the following boundary conditions:

$$R_1(k) = \begin{cases} r(k)T^{-2}(k), & k \in (-\infty, -k_0), \\ r(-k_0)T_1^{-2}(-k_0)(k + k_0)^{-2iv(-k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_1, \end{cases} \quad (40)$$

$$R_2(k) = \begin{cases} r^*(k^*)T_2^2(k) \frac{1}{1 + |r(k)|^2}, & k \in (-k_0, 0), \\ r^*(-k_0)T_2^2(-k_0)(k + k_0)^{-2iv(-k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_2^+, \\ 0, & k \in \Sigma_2^- \end{cases} \quad (41)$$

$$R_3(k) = \begin{cases} r^*(k^*)T_3^2(k) \frac{1}{1 + |r(k)|^2}, & k \in (-k_0, 0), \\ r^*(-k_0)T_3^2(-k_0)(k - k_0)^{2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_3^+, \\ 0, & k \in \Sigma_3^- \end{cases} \quad (42)$$

$$R_4(k) = \begin{cases} r(k)T^{-2}(k), & k \in (k_0, \infty), \\ r(k_0)T_4^{-2}(k)(k - k_0)^{-2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_4, \end{cases} \quad (43)$$

$$R_5(k) = \begin{cases} r(k)T^{-2}(k), & k \in (k_0, \infty), \\ r(k_0)T_5^{-2}(k)(k - k_0)^{-2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_5, \end{cases} \quad (44)$$

$$R_6(k) = \begin{cases} r(k)T^{-2}(k), & k \in (k_0, \infty), \\ r(k_0)T_6^{-2}(k)(k - k_0)^{-2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_6, \end{cases} \quad (45)$$

$$R_7(k) = \begin{cases} r(k)T^{-2}(k), & k \in (k_0, \infty), \\ r(k_0)T_7^{-2}(k)(k - k_0)^{-2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_7, \end{cases} \quad (46)$$

$$R_8(k) = \begin{cases} r(k)T^{-2}(k), & k \in (k_0, \infty), \\ r(k_0)T_8^{-2}(k)(k - k_0)^{-2iv(k_0)}(1 - Y_{k_i}(k)), & k \in \Sigma_8, \end{cases} \quad (47)$$

FIG. 4. Deformation from $\mathbb{R}$ to $\Sigma^{(2)}$. 
\[ R_6(k) = \begin{cases} r^*(k^*) T^2(k), & k \in (k_0, \infty), \\ r^*(k_0) T_0^2(k_0) (k - k_0)^{-2i(k_0)} (1 - Y_\epsilon(k)), & k \in \Sigma_z, \end{cases} \]  

\[ R_7(k) = \begin{cases} r(k) T^{-2}(k) \frac{1 + |r(k)|^2}{1 + |r(k)|^2}, & k \in (-k_0, k_0), \\ r(k_0) T_0^{-2}(k_0) \frac{(k - k_0)^{-2i(k_0)} (1 - Y_\epsilon(k))}{1 + |r(k)|^2}, & k \in \Sigma^+, \\ 0, & k \in \Sigma^-_6, \end{cases} \]  

\[ R_8(k) = \begin{cases} r(k) T^{-2}(k) \frac{1 + |r(k)|^2}{1 + |r(k)|^2}, & k \in (-k_0, k_0), \\ r(-k_0) T_0^{-2}(-k_0) (k + k_0)^{-2i(-k_0)} (1 - Y_\epsilon(k)), & k \in \Sigma^-, \\ 0, & k \in \Sigma^-_7, \end{cases} \]  

\[ R_9(k) = \begin{cases} r^*(k) T^2(k), & k \in (-\infty, -k_0), \\ r^*(-k_0) T_0^2(-k_0) (k + k_0)^{-2i(-k_0)} (1 - Y_\epsilon(k)), & k \in \Sigma^x. \end{cases} \]  

Moreover, the above matrices meet the following estimates:

\[ |R_j| \lesssim \sin^2 \left( \arg(k \pm k_0) \right) + \langle \text{Re}(k) \rangle^{-\frac{1}{2}}, \quad j = 1, 3, \ldots, 9. \]  

Thus, the original jump in \( \pm k_0 \) will look like what is shown in Figs. 5 and 6.

In order to facilitate future calculations and estimations, the region needs to be divided appropriately, as shown in Fig. 7 below.
We define a few new marks as follows:

\[ p_1(k) = p_5(k) = r(k), \quad p_6(k) = p_9(k) = r^*(k), \]
\[ p_3(k) = p_4(k) = r^*(k) \frac{1}{1 + |r(k)|^2}, \quad p_7(k) = p_8(k) = r(k) \frac{1}{1 + |r(k)|^2}. \]

Further estimates of \( \bar{\partial}R_j \) in different regions can be obtained as follows:

**Lemma VI.1.** Suppose \( r \in H^{1,1}(\mathbb{R}) \), \( \bar{\partial}R_j \) defined by (39)–(46) satisfy

\[ |\partial R_j(k)| \lesssim |\bar{\partial}Y_k(k)| + |p_j'(\text{Re}(k))| + |k \pm k_0|^{-\frac{3}{2}} \tag{48} \]

on \( D_1, D_5, D_6, D_9 \), and \( D_5^+, D_6^+, D_9^+, D_8^+ \). \( \bar{\partial}R_j \) satisfy

\[ |\partial R_j(k)| \lesssim |\bar{\partial}Y_k(k)| + |p_j'(\text{Re}(k))| + |k|^{-\frac{3}{2}} \tag{49} \]

on \( D_3, D_7, D_7^+, D_8^+ \).

Correspondingly, the jump line is transformed into \( \Sigma^{(3)} \). Here, two jump lines are added, with the following form:

\[ \tilde{v} = \begin{cases} 
1, & k \in \left( \frac{-ik_0}{2}, \frac{\pi}{24}, \frac{ik_0}{2}, \frac{\pi}{24} \right), \\
\left( \begin{array}{c} 1 \\
\left( R_2^+ - R_2^- \right) e^{2i\theta} \\
0 \\
1 \\
\left( R_7^+ - R_7^- \right) e^{2i\theta} \\
0 \\
1 \end{array} \right), & k \in \left( \frac{-ik_0}{2}, \frac{\pi}{24}, \frac{k_0}{2}, \frac{ik_0}{2}, \frac{k_0}{2}, \frac{ik_0}{2} \right), \\
\left( \begin{array}{c} 1 \\
\left( R_5^+ - R_5^- \right) e^{2i\theta} \\
0 \\
1 \\
\left( R_8^+ - R_8^- \right) e^{2i\theta} \end{array} \right), & k \in \left( \frac{-ik_0}{2}, \frac{\pi}{24}, \frac{k_0}{2}, \frac{ik_0}{2}, \frac{k_0}{2}, \frac{ik_0}{2} \right). 
\end{cases} \tag{50} \]

The jump line \( \Sigma^{(3)} \) is shown in Fig. 8.

At the same time, similar to Ref. 51, we can get \( \tilde{v} \) subjects to the following estimate;

\[ |\tilde{v} - I| \lesssim e^{-t}. \]

Therefore, the following changes can be made:
\[ M^{(2)}(k) = M^{(1)}(k) R^{(2)}, \]  

(51)

where

\[
R^{(2)}(k) = \begin{cases} 
\begin{bmatrix} 1 & 0 \\ -m_7 R_j(k) e^{2it \theta} & 1 \end{bmatrix}, & k \in D_j, \ j = 1, 5, 7, 8, \\
\begin{bmatrix} 1 & 0 \\ (-1)^{m_7} R_j(k) e^{-2it \theta} & 1 \end{bmatrix}, & k \in D_j, \ j = 3, 4, 6, 9, \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & k \in D_2 \cup D_{10},
\end{cases}
\]

where \( m_1 = m_3 = m_4 = m_5 = 1, m_6 = m_7 = m_8 = m_9 = 0 \). Matrix \( M^{(2)}(k) \) satisfies the following RHP:

**Riemann–Hilbert Problem VI.2.** The \( M^{(2)}(k) \) obtained from \( M^{(1)}(k) \) and \( R^{(2)}(k) \) above is expected to satisfy the following properties:

- \( M^{(2)} \) is analytic within \( C \setminus \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^* \).
- \( M^{(2)}(k) = \sigma_0 (M^{(2)}(k^*))^* \sigma_0^{-1} \).
- \( M^{(2)}(k) = I + \mathcal{O}(k^{-1}) \) as \( k \to \infty \).
- \( M^{(2)}_+(x) = M^{(2)}_-(x) V^{(2)}(x) \) as \( x \in \Sigma^{(2)} \), where

\[
\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{K} = \{ k_1, k_2 \}, \quad \mathcal{K}^* = \{ k_1^*, k_2^* \}.
\]
\[
V^{(2)}(k) = \begin{cases} 
1 \quad & k \in \Sigma_1, \\
1 e^{2i\theta} & k \in \Sigma_2, \\
1 e^{-2i\theta} & k \in \Sigma_3, \\
1 R_2(k) e^{2i\theta} & k \in \Sigma_4, \\
1 R_6(k) e^{-2i\theta} & k \in \Sigma_5, \\
1 R_8(k) e^{2i\theta} & k \in \Sigma_6, \\
0 & k \in \Sigma_7, \\
1 & k \in \Sigma_8.
\end{cases}
\]

- \(M^{(2)}(k)\) has simple poles at each point in \(\mathcal{K} \cup \mathcal{K}^*\) with

\[
\text{Res} M^{(2)}(k) = \begin{cases} 
\lim_{k \to k_j} M^{(2)}(k) \begin{pmatrix} 0 & \zeta_j^{-1} \left( \frac{1}{2} \right) (k_j)^2 e^{-2i\theta(k_j)} \\ 0 & 0 \end{pmatrix}, & j \in \Delta_{k_j}, \\
\lim_{k \to k_j'} M^{(2)}(k) \begin{pmatrix} 0 & 0 \\ \zeta_j T(k_j) e^{2i\theta(k_j)} & 0 \end{pmatrix}, & j \in \Delta_{k_j},
\end{cases}
\]

\(\text{(52)}\)

- For \(\mathbb{C} \setminus \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^*\), we have the \(\bar{\partial}\) derivative

\[
\bar{\partial} M^{(2)}(k) = M^{(2)}(k) \bar{\partial} \mathcal{R}^{(2)}(k),
\]

\(\text{(53)}\)

where

\[
\mathcal{R}^{(2)}(k) = \begin{cases} 
1 & k \in D_j, \quad j = 1, 5, 7, 8, \\
(-1)^m \partial R_j(k) e^{2i\theta} & k \in D_j, \quad j = 3, 4, 6, 9, \\
0 & k \in D_2 \cup D_{10},
\end{cases}
\]
where \( n_1 = n_3 = n_4 = n_5 = 1, n_6 = n_7 = n_8 = n_9 = 0 \).

The relationship between the solution of the coupled dispersion AB system and \( M^{(2)}(k) \) is

\[
A(x, t) = 4i \lim_{k \to \infty} (kM^{(2)}(k))_{12}, \quad B(x, t) = -\frac{4i}{\partial k} \lim_{k \to \infty} \frac{d}{dt} (kM^{(2)}(k))_{11}.
\]

(54)

The \( \partial \) derivative appears in the above RHP, so it is also called mixed \( \partial \)-RHP.

### VII. Decomposition of the RHP VI.2

In this section, we mainly discuss the classification of RHP VI.2. For the case of \( \overline{\mathcal{R}}^{(2)}(k) = 0 \), it is called a pure RH problem, and for the case of \( \overline{\mathcal{R}}^{(2)}(k) \neq 0 \), it is called a pure \( \partial \) problem. In the process of classification, consider the transformation \( M^{(2)}(k) = M^{(3)}(k)M^{(2)}_{\text{rhp}}(k) \).

If \( \overline{\mathcal{R}}^{(2)}(k) = 0 \), it corresponds to \( M^{(2)}_{\text{rhp}}(k) \); if \( \overline{\mathcal{R}}^{(2)}(k) \neq 0 \), it corresponds to \( M^{(3)}(k) = M^{(2)}(k)(M^{(2)}_{\text{rhp}}(k))^{-1} \). For \( M^{(2)}_{\text{rhp}}(k) \), its jump is the same as \( M^{(2)}(k) \), and its other properties are summarized in the following problem.

**Riemann–Hilbert Problem VII.1.** Find a matrix-valued function \( M^{(2)}_{\text{rhp}}(k) \) with the following properties:

- \( M^{(2)}_{\text{rhp}}(k) \) is analytic within \( \mathbb{C} \setminus \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^* \).
- Symmetry: \( M^{(2)}_{\text{rhp}}(k) = \sigma_0 (M^{(2)}_{\text{rhp}})^*(k^*) \sigma_0^{-1} \).
- Analytic behavior:
  \[
  M^{(2)}_{\text{rhp}}(k) = I + O(k^{-1}), \quad k \to \infty.
  \]
  (55)
- Jump condition:
  \[
  M^{(2)}_{\text{rhp}}(k) = M^{(2)}_{\text{rhp}}(k) V^{(2)}(k), \quad k \in \Sigma^{(2)}.
  \]
  (56)
- Residue conditions: \( M^{(2)}_{\text{rhp}}(k) \) has simple poles at each point in \( \mathcal{K} \cup \mathcal{K}^* \) with
  \[
  \text{Res} M^{(2)}_{\text{rhp}}(k) = \begin{cases}
  \lim_{k \to k_j} M^{(2)}_{\text{rhp}}(k) \begin{pmatrix}
    0 & -c_j^{-1}(1/T)\begin{pmatrix} k_j^{-2}e^{-2i\theta(k_j)} \\
    0
  \end{pmatrix}
  \\
    0 & 0
  \end{pmatrix}, & j \in \Delta_k^-, \\
  \lim_{k \to k_j} M^{(2)}_{\text{rhp}}(k) \begin{pmatrix}
    0 & 0
  \\
    -c_j T(k_j^{-2}e^{-2i\theta(k_j)}) & 0
  \end{pmatrix}, & j \in \Delta_k^+.
\end{cases}
\]

\[
(57)
\]
- \( \partial \)-Derivative: \( \partial \mathcal{R}^{(2)}(k) = 0 \) for \( k \in \mathbb{C} \).

When \( \overline{\mathcal{R}}^{(2)}(k) \neq 0 \), we use the above \( M^{(2)}_{\text{rhp}}(k) \) to construct a transformation: \( M^{(3)}(k) = M^{(2)}(k)(M^{(2)}_{\text{rhp}}(k))^{-1} \), which is a pure \( \partial \) problem. For \( M^{(3)}(k) \), we have the following properties:

**Riemann–Hilbert Problem VII.2.** Find a matrix-valued function \( M^{(3)}(k) \) with the following properties:

- \( M^{(3)}(k) \) is continuous in \( \mathbb{C} \) with continuous first partial derivatives in \( \mathbb{C} \setminus \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^* \).
- Symmetry: \( M^{(3)}(k) = \sigma_0 (M^{(3)}(k^*))^* \sigma_0^{-1} \).
- Jump condition: \( M^{(3)}_{\text{rhp}}(k) = M^{(3)}(k) \) for \( k \in \Sigma^{(2)} \).
- Analytic behavior: \( M^{(3)}(k) = I + O(k^{-1}) \) for \( k \to \infty \).
- \( \partial \)-Derivative: \( \partial M^{(3)}(k) = M^{(3)}(k)M^{(2)}_{\text{rhp}}(k) \partial \mathcal{R}^{(2)}(k)(M^{(2)}_{\text{rhp}}(k))^{-1} \).

For the above proof, we can refer to Ref. 35. Here, to avoid repetition, we do not present it again.
Next, we focus on finding $M_{\text{rhp}}^{(2)}(k)$. First, two open intervals are defined as follows:

$$A_1 = \left\{ k : |k + k_0| \leq \min \left( \frac{k_0}{2}, \frac{\rho}{3} \right) \leq \varepsilon \right\},$$

$$A_2 = \left\{ k : |k - k_0| \leq \min \left( \frac{k_0}{2}, \frac{\rho}{3} \right) \leq \varepsilon \right\}.$$ (58)

Then, $M_{\text{rhp}}^{(2)}(k)$ is divided into three parts,

$$M_{\text{rhp}}^{(2)}(k) = \begin{cases} M_{\text{err}}^{\text{err}}(k)M_{\text{err}}^{\text{err}}(k), & k \notin \{A_1 \cup A_2\}, \\ M_{\text{err}}^{\text{err}}(k)M_{(k_0)}^{(k_0)}(k), & k \in A_1, \\ M_{\text{err}}^{\text{err}}(k)M_{(k_0)}^{(k_0)}(k), & k \in A_2. \end{cases}$$ (59)

Since $\text{dist}(\mathcal{K} \cup \mathcal{K}^*, \mathbb{R}) > \rho$, $M_{\text{rhp}}^{(2)}(k)$, $M_{(k_0)}^{(k_0)}(k)$, and $M_{(k_0)}^{(k_0)}(k)$ have no poles in $A_1$ and $A_2$. The matrix $M_{\text{rhp}}^{(2)}(k)$ is divided into three parts by decomposition: One part can be called the external model RH problem, represented by $M_{\text{err}}^{(k_0)}(k)$, which can be solved directly by considering the standard RH problem without reflection potential. The other two parts are near the phase points $M_{(k_0)}^{(k_0)}(k)$ and $M_{(k_0)}^{(k_0)}(k)$, which can be matched to the known PC model, viz., the parabolic cylinder model in $A_1$ and $A_2$, to be solved in Sec. VIII D. In addition, matrix $M_{\text{err}}^{(k_0)}(k)$ is an error function, which can be solved by the small norm RH problem in Sec. VIII E.

Let us define $L_\varepsilon$ for a fixed $\varepsilon$,

$$L_\varepsilon = \left\{ k : k = k_0 + Ak_0 e^{i\pi}, \varepsilon \leq A \leq \frac{1}{\sqrt{2}} \right\}$$

$$\cup \left\{ k : k = k_0 + Ak_0 e^{i\pi}, \varepsilon \leq A \leq \infty \right\}$$

$$\cup \left\{ k : k = -k_0 + Ak_0 e^{i\pi}, \varepsilon \leq A \leq \frac{1}{\sqrt{2}} \right\}$$

$$\cup \left\{ k : k = -k_0 + Ak_0 e^{i\pi}, \varepsilon \leq A \leq \infty \right\}.$$ 

Through the above definition, we can write the estimation of $V^{(2)}(k)$ as follows:

$$\left\| V^{(2)}(k) - I \right\|_{L^\infty(\Sigma^{(2)} \setminus (A_1 \cup A_2))} = O\left(e^{\frac{2\rho}{\varepsilon} |k|^2} \left| k_0^2 - |k|^2 \right|^\varepsilon \right),$$

$$\left\| V^{(2)}(k) - I \right\|_{L^\infty(\Sigma^{(2)} \setminus (A_1 \cup A_2))} = O\left(e^{\frac{2\rho}{\varepsilon} |k|^2} \left| k_0^2 - |k|^2 \right|^\varepsilon \right),$$

$$\left\| V^{(2)}(k) - I \right\|_{L^\infty(\Sigma^{(2)} \setminus (A_1 \cup A_2))} = O(e^{-2\varepsilon}).$$ (60)

where the contours are defined by

$$\Sigma^{(2)}_\varepsilon = \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6,$$

$$\Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_7 \cup \Sigma_8.$$ 

This implies that the jump matrix $V^{(2)}(k)$ goes to $I$ on both $\Sigma^{(2)} \setminus (A_1 \cup A_2)$.

$M_{\text{err}}^{(k_0)}(k)$ is the solution on the soliton region of $M^{(2)}(k)$, which is defined as no jump on $\mathbb{C}$ and only discrete spectral points. At the same time, it is analytical in $A_1 \cup A_2$ and outside discrete spectral points; that is, we have the following RHP:

Riemann–Hilbert Problem VII.3. Find a matrix-valued function $M_{\text{err}}^{(k)}(k)$ with the following properties:

- Analyticity: $M_{\text{err}}^{(k)}(k)$ is analytical in $\mathbb{C} \setminus \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^*$.
- Symmetry: $M_{\text{err}}^{(k)}(k) = \phi_0 \left(M_{\text{err}}^{(k)}(k)^* \right) \phi_0^{-1}$.
- Analytic behavior: $M_{\text{err}}^{(k)}(k) = I + \mathcal{O}(k^{-1})$, $k \to \infty$.
- Residue conditions: $M_{\text{err}}^{(k)}(k)$ has simple poles at each point in $\mathcal{K} \cup \mathcal{K}^*$ satisfying the same residue relations with (57) of $M_{\text{rhp}}^{(2)}(k)$.

VIII. PURE RHP AND ITS ASYMPTOTIC BEHAVIOR

In this section, we will probe into the asymptotic behavior of the external soliton region and the internal non-soliton region.
A. External soliton solution region

As we all know, solitons are generated when the reflection data are equal to 0, that is, \( r(k) = 0 \). At this time, the trace formulas are simplified to

\[
\begin{align*}
\sigma_{11}(k) &= \prod_{j=1}^{N} \frac{k-k_j}{k-k_j^*}, \\
\sigma_{22}(k) &= \prod_{j=1}^{N} \frac{k-k_j^*}{k-k_j}.
\end{align*}
\]

Moreover, the jump matrix \( V(k) = I \), then RHP III.1 can be simplified as follows:

\textbf{Riemann–Hilbert Problem VIII.1.} For a given set of scattering data \( \mathcal{P} = \{(k_j, c_j)\}_{j=1}^{N} \), a matrix \( M(k|\mathcal{P}) \) can be found to satisfy the following properties:

- Analyticity: \( M(k|\mathcal{P}) \) is analytical in \( \mathbb{C}\setminus\left(\Sigma^{(1)} \cup \mathcal{K} \cup \mathcal{K}^*\right) \).
- Symmetry: \( M(k|\mathcal{P}) = \sigma_0 M^*(k'|\mathcal{P}) \sigma_0^{-1} \).
- Asymptotic behavior: \( M(k|\mathcal{P}) = I + \mathcal{O}(k^{-1}) \), \( k \to \infty \).
- Residue conditions: \( M(k|\mathcal{P}) \) has simple poles at each point in \( k_j \in \mathbb{C}^+ \) and \( k_j^* \in \mathbb{C}^- \).

\[
\begin{align*}
\text{Res} M(k|\mathcal{P}) &= \lim_{k \to k_j} M(k|\mathcal{P})
\begin{pmatrix}
0 & \frac{1}{c_j} e^{2i\theta(k_j)} \\
\frac{1}{c_j} e^{-2i\theta(k_j)} & 0
\end{pmatrix}, \\
\text{Res} M(k|\mathcal{P}) &= \lim_{k \to k_j^*} M(k|\mathcal{P})
\begin{pmatrix}
0 & -\frac{1}{c_j^*} e^{-2i\theta(k_j^*)} \\
-\frac{1}{c_j^*} e^{2i\theta(k_j^*)} & 0
\end{pmatrix}.
\end{align*}
\]

The uniqueness of RHP VIII.1 solution can be easily proved by using Liouville theorem.

In order to facilitate future research, we divide the scattering data into two parts. Note \( \nabla \subseteq \{1, \ldots, N\} \), and define

\[
\sigma_{11}(\nabla(k)) = \prod_{j=1}^{N} \frac{k-k_j}{k-k_j^*}.
\]

Next, we make a modified transformation of \( M(k|\mathcal{P}) \) defined above as follows:

\[
M_{\nabla}(k|\mathcal{D}) = M(k|\mathcal{P}) \sigma_{11}(\nabla(k))^N,
\]

where the scattering data are given by

\[
\mathcal{D} = \{(k_j, c_j')\}_{j=1}^{N}, \quad c_j' = c_j \sigma_{11}(\nabla(k))^2.
\]

Therefore, \( M_{\nabla}(k|\mathcal{D}) \) satisfies the following modified discrete RHP:

\textbf{Riemann–Hilbert Problem VIII.2.} For a given scattering data \( \mathcal{D} = \{(k_j, c_j')\}_{j=1}^{N} \), a matrix \( M_{\nabla}(k|\mathcal{D}) \) can be found to satisfy the following properties:

- Analyticity: \( M_{\nabla}(k|\mathcal{D}) \) is analytical in \( \mathbb{C}\setminus\left(\Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^*\right) \).
- Symmetry: \( M_{\nabla}(k|\mathcal{D}) = \sigma_0 M_{\nabla}^*(k'|\mathcal{D}) \sigma_0^{-1} \).
- Asymptotic behavior: \( M_{\nabla}(k|\mathcal{D}) = I + \mathcal{O}(k^{-1}) \), \( k \to \infty \).
- Residue conditions: \( M_{\nabla}(k|\mathcal{D}) \) has simple poles at each point in \( k_j \in \mathbb{C}^+ \) and \( k_j^* \in \mathbb{C}^- \) with

\[
\begin{align*}
\text{Res} M_{\nabla}(k|\mathcal{D}) &= \lim_{k \to k_j} M_{\nabla}(k|\mathcal{D})
\begin{pmatrix}
0 & \tilde{\omega}_{1,1}(\nabla(k))^2(k_j) \\
\tilde{\omega}_{1,1}(\nabla(k))^2(k_j)^* & 0
\end{pmatrix}, \\
\text{Res} M_{\nabla}(k|\mathcal{D}) &= \lim_{k \to k_j^*} M_{\nabla}(k|\mathcal{D})
\begin{pmatrix}
0 & -\tilde{\omega}_{1,1}(\nabla(k))^2(k_j^*) \\
\tilde{\omega}_{1,1}(\nabla(k))^2(k_j^*) & 0
\end{pmatrix}.
\end{align*}
\]
where \( \omega_j = c_j e^{2\pi i (k_j)} \).

Proposition VIII.3. If \( A_{\text{out}}(x,t) = A_{\text{out}}(x,t|D), B_{\text{out}}(x,t) = B_{\text{out}}(x,t|D) \) denote the N-soliton solution of system (1), for the scattering data without reflection \( D = \{(k_j, c'_j)\}_{j=1}^N \), RHP VIII.2 has a unique solution and

\[
A_{\text{out}}(x,t|D) = 4i \lim_{k \to \infty} [k M_D(k|D)]_{12} = 4i \lim_{k \to \infty} [k M(k|P)]_{12} = A_{\text{out}}(x,t|P),
\]

\[
B_{\text{out}}(x,t|D) = -\frac{4i}{\beta} \lim_{k \to \infty} \frac{d}{dt} [k M_D(k|D)]_{11} = -\frac{4i}{\beta} \lim_{k \to \infty} \frac{d}{dt} [k M(k|P)]_{11} = B_{\text{out}}(x,t|P).
\]

B. Existence and uniqueness of solutions for external RH problems

We see that \( M_{\text{out}}(k) \) is a reflection soliton solution and the reflection mainly comes from \( T(k) \). One way to connect \( M_{\text{out}}(x,t,k) \) with the case of non-reflection scattering \( D = \{(k_j, c'_j)\}_{j=1}^N \) is to let \( \nabla = \Delta_{k_j} \) in Eq. (63). So, we have

\[
T(k) = \prod_{j=1}^N \frac{k - k_j^*}{k - k_j} \exp \left( i \int_{k_j}^{k_j^*} \frac{v(x)}{s - k} dk \right) = s_{11,\Delta k_j} \delta(k),
\]

\[
T(k_j)^2 = s_{11,\Delta k_j}(k_j)^2 \delta(k_j)^2, \quad \left( \frac{1}{T} \right)(k_j)^2 = s'_{11,\Delta k_j}(k_j)^2 \delta(k_j)^2.
\]

The scattering data can be written as

\[
\mathcal{D} = \{(k_j, c_j)\}_{j=1}^N, \quad c_j = \begin{cases} c_j^{-1} s'_{11,\Delta k_j}(k_j)^2 \delta(k_j)^2, & j \in \Delta_{k_j}^- \\ c_j s_{11,\Delta k_j}(k_j)^2 \delta(k_j)^2, & j \notin \Delta_{k_j}^- \end{cases}
\]

(65)

Therefore, RHP VIII.2 can be rewritten as follows:

**Riemann–Hilbert Problem VIII.4.** For a set of given scattering data \( \mathcal{D} = \{(k_j, c_j)\}_{j=1}^N \), a matrix \( M_{\Delta k_j}(k|\mathcal{D}) \) can be found to satisfy the following properties:

- \( M_{\Delta k_j}(k|\mathcal{D}) \) is analytical in \( \mathbb{C} \backslash (\mathcal{K} \cup \mathcal{K}^*) \).
- \( M_{\Delta k_j}(k|\mathcal{D}) = \sigma_0 M_{\Delta k_j}^-(k^*|\mathcal{D}) \sigma_0^{-1} \).
- \( M_{\Delta k_j}(k|\mathcal{D}) = 1 + O(k^{-1}), \quad k \to \infty \).
- \( M_{\Delta k_j}(k|\mathcal{D}) \) has simple poles at each point in \( k_j \in \mathbb{C}^+ \) and \( k_j^* \in \mathbb{C}^- \) with

\[
\text{Res}_{k=k_j} M_{\Delta k_j}(k|\mathcal{D}) = \begin{cases} \lim_{k \to k_j} M_{\Delta k_j}(k|\mathcal{D}) & \begin{pmatrix} 0 & \Delta_j^{-1} s_{11,\Delta k_j}(k_j) \\ \Delta_j s_{11,\Delta k_j}(k_j) & 0 \end{pmatrix}, & j \in \Delta_{k_j}^- \\ \lim_{k \to k_j} M_{\Delta k_j}(k|\mathcal{D}) & \begin{pmatrix} 0 & 0 \\ \Delta_j & 0 \end{pmatrix}, & j \notin \Delta_{k_j}^- \end{cases}
\]

(66)

\[
\text{Res}_{k=k_j^*} M_{\Delta k_j}(k|\mathcal{D}) = \begin{cases} \lim_{k \to k_j^*} M_{\Delta k_j}(k|\mathcal{D}) & \begin{pmatrix} 0 & 0 \\ \Delta_j^{-1} s_{11,\Delta k_j}(k_j^*) & 0 \end{pmatrix}, & j \in \Delta_{k_j}^- \\ \lim_{k \to k_j^*} M_{\Delta k_j}(k|\mathcal{D}) & \begin{pmatrix} 0 & 0 \\ -\Delta_j s_{11,\Delta k_j}(k_j^*) & 0 \end{pmatrix}, & j \notin \Delta_{k_j}^- \end{cases}
\]

where \( \Lambda_j = c_j \delta^{-2} e^{2\pi i (k_j)} \).
So, we have the following corollary:

**Corollary VIII.5.** There exists a unique solution for the RHP VII.3; moreover,

\[ M^\text{out} (x, t, k) = M_{\bar{\chi}}(x, t, k | \hat{D}), \]

where the scattering data \( \hat{D} \) are given by (65) and

\[ A_{\text{sol}}(x, t | D) = 4i \lim_{k \to \infty} [kM^\text{out} (x, t, k | D)]_{12} \]

\[ = 4i \lim_{k \to \infty} [kM_{\bar{\chi}}(x, t, k | \hat{D})]_{12} = A_{\text{sol}}(x, t | \hat{D}), \]

\[ B_{\text{sol}}(x, t | D) = -\frac{4i}{k} \lim_{k \to \infty} \frac{d}{dt} [kM^\text{out} (x, t, k | D)]_{11} \]

\[ = -\frac{4i}{k} \lim_{k \to \infty} \frac{d}{dt} [kM_{\bar{\chi}}(x, t, k | \hat{D})]_{11} = B_{\text{sol}}(x, t | \hat{D}). \]

**C. The long-time behavior of soliton solution**

This section mainly considers the asymptotic behavior of soliton solutions. On the basis of the residue condition (66), let \( N = 1, k = \zeta + i\eta \), under the condition of no scattering; the soliton solution of the coupled dispersion AB system is

\[ A(x, t) = 4\eta \text{sech}\left( 2\eta \left( x + \frac{\alpha}{4|k|^2} t \right) \right) e^{\left( -2\eta + \frac{\alpha}{4|k|^2} t \right)}, \]

\[ B(x, t) = -\frac{2\alpha}{|k|^2} \text{sech}\left( \eta \left( 2x + \frac{\alpha}{2|k|^2} t \right) \right)^2. \]

(67)

It can be seen that the velocity of the soliton solution is \( v = -\frac{\alpha}{4|k|^2} \). Suppose \( x_1 < x_2 \) with \( x_1, x_2 \in \mathbb{R} \) and \( v_1 < v_2 \) with \( v_1, v_2 \in \mathbb{R}^- \), a conical region can be defined as

\[ C(x_1, x_2, v_1, v_2) = \{(x, t) \in \mathbb{R}^2 \mid x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2]\}, \]

which is shown in Fig. 9. Then, we make the following marks

\[ \mathcal{I} = \{k : f(v_2) < |k| < f(v_1)\}, \quad f(v) = \left( -\frac{\alpha}{4v} \right)^2, \]

\[ K(\mathcal{I}) = \{k_j \in K : k_j \in \mathcal{I}\}, \quad N(\mathcal{I}) = |K(\mathcal{I})|, \]

\[ K^+(\mathcal{I}) = \{k_j \in K : |k_j| > f(v_1)\}, \quad K^- (\mathcal{I}) = \{k_j \in K : |k_j| < f(v_2)\}, \]

\[ c_j^n (\mathcal{I}) = c_j \prod_{k_j \in K \setminus K^n} \left( k_j - k_{n_j} \right)^2 \exp \left[ \pm \frac{1}{n_j} \int_{k_j} \log \left[ 1 + r(c)(c^n) \frac{r(c^n)}{c - k_j} \right] dc \right]. \]

(68)

**Proposition VIII.6.** Let \( \nabla = \Delta \) in \( C(x_1, x_2, v_1, v_2) \) on RHP VIII.2 and ensure the following estimation when \( t \to \infty \):

\[ \left\| \Theta_j^n \right\| = \begin{cases} \mathcal{O}(1), & k_j \in K(\mathcal{I}), \\ \mathcal{O}(e^{-2\mu t}), & k_j \in K \setminus K(\mathcal{I}), \end{cases} \quad t \to \infty, \]

(69)

where

\[ \mu = \min_{k_j \in K \setminus K(\mathcal{I})} \left\{ \text{Im}k_j \text{dist}(v_k - v) \right\}, \quad v_j = -\frac{\alpha}{4|k_j|^2}. \]

Here, \( v_j - v \) represents the difference between the speed of a soliton and the speed corresponding to \( k \).

**Proof.** Here, we take \( \nabla = \Delta \) as an example to prove the above estimation. When \( k = k_j \in K^- (\mathcal{I}) \) and \( (x, t) \in C(x_1, x_2, t_1, t_2) \), then for Eq. (64), we have
\[
|\tilde{\omega}(x_0 + vt, k_j)| \leq ce^{-2i\theta(k_j)t}.
\]

Next, notice the index part
\[
-2i\theta(k_j)t = -2i\left(\frac{k_j \cdot x}{t} + \frac{\alpha}{4k_j}t\right) = -2i\left(\frac{x_0 + vt}{t} - \frac{\alpha}{4k_j}t\right).
\]

Its real part is
\[
\text{Re}(-2i\theta(k_j)t) = 2I\kappa k_j x_0 - 2I\kappa k_j t(\nu - \nu);
\]
thus,
\[
|e^{-2i\theta(k_j)t}| = e^{2I\kappa k_j x_0 - 2I\kappa k_j t(\nu - \nu)} \leq ce^{-2\mu t}.
\]

For the discrete spectrum \(k_j \in \mathcal{K} \setminus \mathcal{K}(I)\), we make a disk \(D_k\) with a sufficiently small radius at each point such that they do not intersect each other and define the function
\[
\Gamma(k) = \begin{cases} 
I - \frac{1}{k - k_j} \Theta^{\delta +}_{\Delta k}, & k \in D_k, \\
I - \frac{1}{k - k_j} \sigma_2 \Theta^{\delta +}_{\Delta k} \sigma_2, & k \in \bar{D}_k, \\
I, & \text{elsewhere},
\end{cases} \tag{70}
\]
where
\[
\Theta^{\delta +}_{\Delta k} = \begin{cases} 
\begin{pmatrix} 0 & \omega_j^{-1}(s^t\nu)^{-1}(k_j) \\
0 & 0 \\
\omega_j s^t\nu(k_j) & 0 \\
\omega_j s^t\nu(k_j) & 0 \\
\end{pmatrix}, & j \in \nabla, \\
\begin{pmatrix} 0 & 0 \\
0 & 0 \\
\omega_j s^t\nu(k_j) & 0 \\
\end{pmatrix}, & j \notin \nabla.
\end{cases}
\]

The following transformation can be made:
\[
\tilde{M}_{\Delta k} \left( k \mid \bar{D}_{\Delta k} \right) = M_{\Delta k} \left( k \mid \bar{D}_{\Delta k} \right) \Gamma(k),
\]
where \(\tilde{M}_{\Delta k} \left( k \mid \bar{D}_{\Delta k} \right)\) satisfies the following relationship:
\[
\tilde{M}_{\Delta k} \left( k \mid \bar{D}_{\Delta k} \right) = M_{\Delta k} \left( k \mid \bar{D}_{\Delta k} \right) \bar{V}(k), k \in \Sigma = \bigcup_{k \in \mathcal{K}(I)} \partial D_k \cup \partial \bar{D}_k. \tag{71}
\]

The jump matrix \(\bar{V}(k) = \Gamma(k)\) is known from Proposition VIII.6,
\[
\left\| \bar{V}(k) - I \right\|_{L^{\infty}(\Sigma)} = O(e^{-2\mu t}). \tag{72}
\]

In addition, let
\[ M_0(k) = \tilde{M}_{\Sigma_0} \left( k \mid \tilde{D}(\mathcal{I}) \right) \left[ M_{\Sigma_0} \left( k \mid \tilde{D}(\mathcal{I}) \right) \right]^{-1}, \]  
(73)

where \( \tilde{D}(\mathcal{I}) = \{ k_0, c_j(\mathcal{I}) \} \), \( c_j(\mathcal{I}) = c_j \prod_{k \in \mathcal{K} \setminus \Sigma_j} \left( \frac{k-k_j}{k_j-k} \right)^2 \); it is obvious that \( \tilde{M}_{\Sigma_0} \left( k \mid \tilde{D} \right) \) and \( M_{\Sigma_0} \left( k \mid \bar{D} \right) \) have the same residue condition, so \( M_0(k) \) has no poles, its jump is

\[ M_0^\star(k) = M_0^0(k) V_{\Sigma_0}(k), \quad k \in \Sigma. \]  
(74)

Furthermore, the jump matrix \( V_{\Sigma_0}(k) \) has the form

\[ V_{\Sigma_0}(k) = M_{\Sigma_0} \left( k \mid \tilde{D}(\mathcal{I}) \right) \tilde{V}(k) \left( k \mid \tilde{D}(\mathcal{I}) \right)^{-1}; \]  
(75)

according to Eq. (72), we have

\[ \| V_{\Sigma_0}(k) - I \|_{L^\infty(\mathcal{D})} = \| \tilde{V}(k) - I \|_{L^\infty(\mathcal{D})} = O\left( e^{-2 \mu t} \right), \quad t \to \infty. \]  
(76)

Due to the small norm RH property, it is known that \( M_0(k) \) exists and

\[ M_0(k) = I + O\left( e^{-2 \mu t} \right), \quad t \to \infty. \]

So, we have

\[ M_{\Sigma_0} \left( k \mid \tilde{D} \right) = \left( I + O\left( e^{-2 \mu t} \right) \right) M_{\Sigma_0} \left( k \mid \tilde{D}(\mathcal{I}) \right). \]  
(77)

**Corollary VIII.7.** \( \tilde{D} \) represents the scattering data of Eq. (1), corresponding to the \( N \)-soliton solution of the coupled dispersion \( AB \) system, and \( \tilde{D}(\mathcal{I}) \) represents the scattering data on \( \mathcal{K}(\mathcal{I}) \), when \( x, t \in \mathcal{C}(x_1, x_2, v_1, v_2) \),

\[ A_{\text{sol}}(x,t;\tilde{D}) = 4i \lim_{k \to \infty} \left[ k M_{\Sigma_0}(x,t,k \mid \tilde{D}) \right]_{12} \]

\[ = 4i \lim_{k \to \infty} \left[ k M_{\Sigma_0}(x,t,k \mid \tilde{D}(\mathcal{I})) \right]_{12} + O\left( e^{-2 \mu t} \right), \quad t \to \infty, \]

\[ B_{\text{sol}}(x,t;\tilde{D}) = -\frac{4i}{\beta} \lim_{k \to \infty} \frac{d}{dt} \left[ k M_{\Sigma_0}(x,t,k \mid \tilde{D}) \right]_{11} \]

\[ = -\frac{4i}{\beta} \lim_{k \to \infty} \frac{d}{dt} \left[ k M_{\Sigma_0}(x,t,k \mid \tilde{D}(\mathcal{I})) \right]_{11} + O\left( e^{-2 \mu t} \right), \quad t \to \infty. \]

**D. Solvable RHP near phase points**

From Eq. (60), we can see that \( \| V(k) - I \| \) has no consistent small jump in the neighborhood \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) when \( t \to \infty \), so we need to establish a local model \( M^{\alpha\beta}(k) \), which completely matches the jump of \( M_{\text{rhs}}(k) \) on \( (\Sigma^{(2)} \cap \mathcal{A}_1) \cup (\Sigma^{(2)} \cap \mathcal{A}_2) \) of the error function \( M^{\alpha\beta}(k) \), so as to achieve a consistent estimation of the attenuation of the transition.

**Riemann–Hilbert Problem VIII.8.** For a matrix \( M^{AB}(k) \), the following properties are satisfied:

- \( M^{AB}(k) \) is analytic in \( \mathbb{C} \cap \Sigma^{AB}, \Sigma^{AB} = \cup_j \Sigma_j, j = 1, 2^*, 3^*, 4, 5, 6^*, 7^*, 8. \)
- \( M^{AB}(k) = I + O(k^{-1}), \quad k \to \infty. \)
- \( M^{AB}(k) \) has continuous boundary values \( M^{AB}_+(k) \) on \( \Sigma^{AB} \) and

\[ M^{AB}_+(k) = M^{AB}_-(k) V^{AB}_{\mathcal{A}_1 \cup \mathcal{A}_2}(k), \quad k \in \Sigma^{AB}, \]
where

\[
V^{A^0}_{A_1 \cup A_0}(k) = \begin{cases}
1 & k \in \Sigma_1,
\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{2i\theta} 0 \right) & k \in \Sigma_2,
\left( 1 + |r(-k_0)|^2 \right)\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{-2i\theta} 0 \right) & k \in \Sigma_3,
\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{2i\theta} 0 \right) & k \in \Sigma_4,
\left( 1 + |r(-k_0)|^2 \right)\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{-2i\theta} 0 \right) & k \in \Sigma_5,
\left( 1 + |r(-k_0)|^2 \right)\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{2i\theta} 0 \right) & k \in \Sigma_6,
\left( 1 + |r(-k_0)|^2 \right)\left( r(-k_0) \delta^{2}(-k_0)(k + k_0)^{-2i\nu(k_0)} e^{-2i\theta} 0 \right) & k \in \Sigma_7.
\end{cases}
\]

This is mainly because when there is no discrete spectrum, \(T_{0}(\pm k_0)\) can be reduced to \(\delta(\pm k_0)\) and \(1 - Y_{\pm}(k) = 1\).

Here, we need to consider the well-known PC model near the two phase points, as shown in Fig. 10. First, expand \(\theta(k)\) at \(k_0\),

\[
\theta(k) = -\frac{1}{2} \frac{\alpha}{k_0} - \frac{1}{4} \frac{\alpha}{k_0^2} (k - k_0)^2 + \frac{1}{4} \frac{\alpha}{k_0^3} (k - k_0)^3. \tag{78}
\]

Then, define the following transformation:

\[
T : f(k) \rightarrow (T f)(k) = f \left( \frac{-k_0^2}{\alpha} \zeta + k_0 \right). \tag{79}
\]

so, on \(\Sigma_4\), we have

\[
r(k_0) \delta^2(k_0)(k - k_0)^{-2i\nu(k_0)} e^{2i\theta} = r(k_0) \delta^2(k_0) \left( \frac{-k_0^2}{\alpha \zeta} \right) e^{-2i\nu(k_0)} e^{2i\theta} \left( \frac{-i}{\nu(k_0)} \right)^3 e^{-\frac{2i\theta}{3}} e^{-\frac{2i\eta}{3}} \left( k - k_0 \right)^3.
\]

\[
FIG. 10. The jump contour jump profile of the local RHP near phase points \(\pm k_0\).\]
We set
\[ r_k = r(k_0)\delta^2(k_0) e^{i\nu(k_0)(\ln(k_0^2) - \ln(-\alpha))} e^{-\text{int} \left[ \frac{1}{2} - \frac{1}{4} (k-k_0)^2 \right]}, \]
(80)
in the same way, and we can calculate the values of \( \Sigma_1', \Sigma_2', \Sigma_4', \Sigma_5' \) and \( \Sigma_6 \). Therefore, the jump matrix of \( V^{AB} \) in \( A_1 \cup A_2 \) can be rewritten as
\[
V^{AB}_{A_1 \cup A_2} (k) = \begin{cases} 
1 & \text{if } k \in \Sigma_1, \\
1 & \text{if } k \in \Sigma_4, \\
1 & \text{if } k \in \Sigma_5, \\
1 & \text{if } k \in \Sigma_6, \\
1 & \text{if } k \in \Sigma_7, \\
1 & \text{if } k \in \Sigma_8, \\
1 & \text{if } k \in \Sigma_9.
\end{cases}
\]

Let us consider the PC model in \( k_0 \) to be expanded in the following form:
\[
M^{\text{PC}}(k_0) = I + \frac{M^{\text{PC}}(k_0)}{i\zeta} + \mathcal{O}(\zeta^{-2}),
\]
where
\[
M^{\text{PC}}(k_0) = \begin{pmatrix} 0 & \Xi_{12}(k_0) \\ -\Xi_{12}(k_0) & 0 \end{pmatrix},
\]
\[
\Xi_{12}(k_0) = \frac{2\pi e^{-i\frac{\pi}{4}} e^{-\nu(k_0)/2}}{r_1 T(-iv(k_0))},
\]
\[
\Xi_{12}(k_0) = -\frac{2\pi e^{-i\frac{\pi}{4}} e^{-\nu(k_0)/2}}{r_1 T(iv(k_0))}.
\]

Then, the same is true for \(-k_0\). According to the transformation Eq. (79), the relationship between \( M^{AB} \) and \( M^{PC} \) is
\[
M^{AB} = I + \frac{1}{i\zeta} \left( M^{PC}(k_0) + M^{PC}(-k_0) \right) + \mathcal{O}(\zeta^{-2}),
\]
furthermore, \( M^{AB} \) can be written as
\[
M^{AB}(x, t, k) = I + \sqrt{-k_0^2 M^{PC}(k_0)} \frac{1}{at} - \frac{1}{k - k_0} + \sqrt{-k_0^2 M^{PC}(-k_0)} \frac{1}{at} - \frac{1}{k + k_0} + \mathcal{O}(\zeta^{-2}).
\]
(81)
From this, we get a consistent estimate

$$|M^{AB} - I| \leq O\left(\frac{1}{r^2}\right).$$

(82)

Near the local circle of $\pm k_0$, we have

$$\left|\frac{1}{k + k_0}\right| < \epsilon,$$

where $\epsilon$ is a constant. Finally, we use $M^{AB}$ to define a local model in $\pm k_0$,

$$M^{(m)}(k) = M^{\text{out}}(k)M^{AB}(k);$$

it has the same jump matrix as $M_{\text{RHP}}$ and is a bounded function in $A_1$ and $A_2$.

E. Small norm RH problem of $M^{err}(k)$

In the disk $A_1 \cup A_2$, $M^{PC}(k)$ is consistent with the jump matrix of $M_{\text{RHP}}^{(2)}(k)$, but $M^{\text{out}}(k)$ does not have a jump. Therefore, the matrix $M^{err}(k)$ defined by (59) erases the jump of $M_{\text{RHP}}^{(2)}(k)$ inside the disk $A_1 \cup A_2$, and there is still a jump from $M_{\text{RHP}}^{(2)}(k)$ outside the disk $\mathbb{C}\setminus A_1 \cup A_2$. Therefore, the jump path of $M^{err}(k)$ is

$$\Sigma^{err} = \cup_{n=1}^2 \partial A_n \cup \left(\Sigma^{(2)} \setminus \cup_{n=1}^2 A_n\right),$$

where $\partial A_n$ is clockwise; see Fig. 11. $M^{err}(k)$ can be directly verified to meet the following RHP:

Riemann–Hilbert Problem VIII.9. For a matrix $M^{err}(k)$, the following properties are satisfied:

- $M^{err}(k)$ is analytical in $\mathbb{C}\setminus \Sigma^{err}$.
- $M^{err}(k) = \sigma_0(M^{err}(k^*))^* \sigma_0^{-1}$.
- $M^{err}(k) = I + O(k^{-1})$ as $k \to \infty$.
- $M^{err}(k)$ has continuous boundary values $M^{err}_c(k)$ on $\Sigma^{err}$ and

$$M^{err}_c(k) = M^{err}_+(k)V^{err}_+(k), \quad k \in \Sigma^{err},$$

where

$$V^{err}_+(k) = \begin{cases} M^{\text{out}}(k) V^{(2)}(k) M^{\text{out}}(k)^{-1}, & k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n, \\
M^{\text{out}}(k) M^{AB}(k) M^{\text{out}}(k)^{-1}, & k \in \cup_{n=1}^2 \partial A_n. \end{cases}$$

Proof. Here, we mainly prove the form of $V^{err}_+(k)$. When $k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n$, $M^{err}(k) = M_{\text{RHP}}^{(2)}(k)(M^{\text{out}}(k))^{-1}$ and $M^{\text{out}}(k)$ does not jump in (59), then

$$M^{err}_+(k) = M_{\text{RHP}}^{(2)}(k)(M^{\text{out}}(k))^{-1} M^{(2)}(k) V^{(2)}(k)(M^{\text{out}}(k))^{-1}$$

$$= M^{err}_+(k) V^{err}_+(k) = M_{\text{RHP}}^{(2)}(k) M^{\text{out}}(k) M^{err}_+(k)^{-1} V^{err}_+(k);$$

obviously,

$$V^{err} = M^{\text{out}}(k) V^{(2)}(M^{\text{out}}(k))^{-1}, \quad k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n.$$

When $k \in \cup_{n=1}^2 A_n$, we have

$$M^{err}_+ = M_{\text{RHP}}^{(2)}(k)(M^{(m)}(k))^{-1} = M_{\text{RHP}}^{(2)}(k) (M^{AB}(k))^{-1} (M^{\text{out}}(k))^{-1},$$

where $\partial A_n$ is clockwise; see Fig. 11. $M^{err}(k)$ can be directly verified to meet the following RHP:

Riemann–Hilbert Problem VIII.9. For a matrix $M^{err}(k)$, the following properties are satisfied:

- $M^{err}(k)$ is analytical in $\mathbb{C}\setminus \Sigma^{err}$.
- $M^{err}(k) = \sigma_0(M^{err}(k^*))^* \sigma_0^{-1}$.
- $M^{err}(k) = I + O(k^{-1})$ as $k \to \infty$.
- $M^{err}(k)$ has continuous boundary values $M^{err}_c(k)$ on $\Sigma^{err}$ and

$$M^{err}_c(k) = M^{err}_+(k)V^{err}_+(k), \quad k \in \Sigma^{err},$$

where

$$V^{err}_+(k) = \begin{cases} M^{\text{out}}(k) V^{(2)}(k) M^{\text{out}}(k)^{-1}, & k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n, \\
M^{\text{out}}(k) M^{AB}(k) M^{\text{out}}(k)^{-1}, & k \in \cup_{n=1}^2 \partial A_n. \end{cases}$$

Proof. Here, we mainly prove the form of $V^{err}_+(k)$. When $k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n$, $M^{err}(k) = M_{\text{RHP}}^{(2)}(k)(M^{\text{out}}(k))^{-1}$ and $M^{\text{out}}(k)$ does not jump in (59), then

$$M^{err}_+(k) = M_{\text{RHP}}^{(2)}(k)(M^{\text{out}}(k))^{-1} M^{(2)}(k) V^{(2)}(k)(M^{\text{out}}(k))^{-1}$$

$$= M^{err}_+(k) V^{err}_+(k) = M_{\text{RHP}}^{(2)}(k) M^{\text{out}}(k) M^{err}_+(k)^{-1} V^{err}_+(k);$$

obviously,

$$V^{err} = M^{\text{out}}(k) V^{(2)}(M^{\text{out}}(k))^{-1}, \quad k \in \Sigma^{(2)} \setminus \cup_{n=1}^2 A_n.$$

When $k \in \cup_{n=1}^2 A_n$, we have

$$M^{err}_+ = M_{\text{RHP}}^{(2)}(k)(M^{(m)}(k))^{-1} = M_{\text{RHP}}^{(2)}(k) (M^{AB}(k))^{-1} (M^{\text{out}}(k))^{-1},$$

\[ \begin{array}{c}
\text{FIG. 11. The jump contour } \Sigma^{err} \text{ for } M^{err}(k). \\
\end{array} \]
and $M_{\text{shift}}^{(2)}(k)$ does not jump in $\mathbb{R}^2 \setminus \partial A_0$; then,

$$M_{\text{err}}^{\text{in}}(k) = M_{\text{shift}}^{(2)}(k)(M_{AB}^{\text{out}}(k))^{-1}(M_{\text{out}}^{\text{out}}(k))^{-1}.$$ 

For $k \in \mathbb{R}^2 \setminus \partial A_0$,

$$M_{\text{err}}^{\text{in}}(k) = M_{\text{shift}}^{(2)}(k)(M_{\text{out}}^{\text{out}}(k))^{-1}.$$ 

So, from

$$M_{\text{err}}^{\text{in}}(k) = M_{\text{err}}^{\text{out}}(k) V_{\text{err}}^{\text{out}}(k), \quad k \in \mathbb{R}^2 \setminus \partial A_0,$$

we get

$$V_{\text{err}}^{\text{out}}(k) = M_{\text{out}}^{\text{out}}(k) M_{AB}^{\text{out}}(k)(M_{\text{out}}^{\text{out}}(k))^{-1}, \quad k \in \mathbb{R}^2 \setminus \partial A_0.$$

Next, we will show that the small norm RH problem can be well solved by the error function $M^{\text{err}}(k)$ for large time. It is known from Eqs. (26) and (60) that the jump matrix satisfies the following estimation:

$$\|V_{\text{err}}^{\text{out}} - I\| \leq O(e^{-2t}), \quad k \in \Sigma^{(2)} \setminus (A_1 \cup A_2).$$

(83)

In addition, it is known from Eq. (82) that when $k \in \mathbb{R}^2 \setminus \partial A_0$, $V_{\text{err}}^{\text{out}}(k)$ satisfies the following estimation:

$$\|V_{\text{err}}^{\text{out}}(k) - I\| = |M_{\text{out}}^{\text{out}}(k) - (M_{AB}^{\text{out}}(k) - I)M_{\text{out}}^{\text{out}}(k)| = O\left(t^{-\frac{1}{2}}\right).$$

(84)

**Proposition VIII.10.** RHP VIII.2 has a unique solution in the form of

$$M^{\text{err}}(k) = I + \frac{1}{2\pi i} \int_{\Sigma^{\text{err}}} \frac{\mu^{\text{err}}(s)(V_{\text{err}}^{\text{out}}(s) - I)}{s - k} ds,$$

(85)

where $\mu^{\text{err}} \in L^2(\Sigma^{\text{err}})$, meet $(1 - C_{\text{err}})\mu^{\text{err}} = I$, $C_{\text{err}}$ is the Cauchy projection operator, defined as

$$C_{\text{err}} f(z) = \lim_{k' \to \infty} \frac{1}{2\pi i} \int_{\Sigma^{\text{err}}} \frac{f(s)}{s - k'} ds.$$

**Proof.** It can be seen from Eq. (84) that the operator is bounded. It can be seen from Ref. 36 that

$$\|\mu\|_{L^2(\Sigma^{\text{err}})} \leq \frac{\|C_{\text{err}}\|}{1 - \|C_{\text{err}}\|} \leq t^{-\frac{1}{2}};$$

(86)

obviously, $1 - C_{\text{err}}$ is reversible, so $M^{\text{err}}$ is unique. □

**Proposition VIII.11.** Consider the asymptotic expansion of $M^{\text{err}}$ when $k \to \infty$,

$$M^{\text{err}}(k) = I + \frac{M_{1}^{\text{err}}}{k} + O\left(\frac{1}{k^2}\right),$$

(87)

where

$$M_{1}^{\text{err}} = \frac{1}{i} \sqrt{-\frac{k_0^2}{a^2}} M_{\text{out}}^{\text{out}}(k_0) M_A^{\text{pc}}(k_0)(M_{\text{out}}^{\text{out}})^{-1}(k_0)$$

$$- \frac{1}{i} \sqrt{-\frac{k_0^2}{a^2}} M_{\text{out}}^{\text{out}}(-k_0) M_A^{\text{pc}}(-k_0)(M_{\text{out}}^{\text{out}})^{-1}(-k_0) + O(t^{-1}).$$

**Proof.** From Proposition VIII.10,

$$M_{1}^{\text{err}} = -\frac{1}{2\pi i} \int_{\Sigma^{\text{err}}} \mu^{\text{err}}(s)(V_{\text{err}}^{\text{out}}(s) - I) ds.$$
Then, we have
\[
M_1^{err} = -\frac{1}{2\pi i} \oint_{\Sigma^c} (V^{err} - I) ds - \frac{1}{2\pi i} \int_{\Sigma^c} (\mu^{err} (s) - I) (V^{err} - I) ds
\]
\[
= -\frac{1}{2\pi i} \oint_{\Sigma^c} (V^{err} - I) ds - \frac{1}{2\pi i} \int_{\Sigma^c} (\mu^{err} (s) - I) (V^{err} - I) ds
\]
\[
= -\frac{1}{2\pi i} \int_{\Sigma^c} (\mu^{err} (s) - I) (V^{err} - I) ds.
\]
(88)

Using Eqs. (83) and (86), \(M_1^{err}\) can be written as
\[
M_1^{err} = -\frac{1}{2\pi i} \oint_{\Sigma^c} (V^{err} (s) - I) ds + O(r^{-1}).
\]

From Eqs. (81) and (84) and residue theorem, it follows that
\[
M_1^{err} = 1 \sqrt{-\frac{k_o^2}{at}} M^{out}(k_0) M^{\text{PC}}(k_0) (M^{out})^{-1} (k_0)
\]
\[
- \frac{1}{i} \sqrt{-\frac{k_o^2}{at}} M^{out}(-k_0) M^{\text{PC}}(-k_0) (M^{out})^{-1} (-k_0) + O(r^{-1}).
\]

\[\square\]

IX. ASYMPTOTIC ANALYSIS ON THE PURE \(\partial \bar{\partial}\) PROBLEM

In this section, we mainly solve the part of \(\partial R^{(2)}(k) \neq 0\) in \(M^{(3)}(k)\), that is, RHP VII.2. Its solution can be expressed in the following form:
\[
M^{(3)}(k) = I - \frac{1}{\pi} \int_{c} \frac{M^{(3)} W^{(3)}}{s - k} \, dA(s),
\]
(89)

where \(W^{(3)} = M^{(2)}_{\text{lp}}(k) \partial R^{(2)} M^{(2)}_{\text{lp}}^{-1}\), and \(dA(s)\) is the Lebesgue measure on the real plane. In fact, Eq. (89) can also be written in the using operators,
\[
(I - \Sigma) M^{(3)}(k) = I,
\]
(90)

where \(\Sigma\) is the Cauchy–Green operator,
\[
\Sigma[f](k) = -\frac{1}{\pi} \int_{c} \frac{f(s) W(s)}{s - k} \, dA(s).
\]

If the operator \((I - \Sigma)^{-1}\) exists, then the above equation has a solution. Next, we will prove the existence of operator \((I - \Sigma)^{-1}\) in regions \(D_k\) and \(D_k^\times\), which can be proved by similar methods in other regions. Before proving the existence of operator \((I - \Sigma)^{-1}\), we prove the following lemma:

**Lemma IX.1.**
\[
|\partial R_2^{(2)} e^{-2i\theta t}| \leq \left| \frac{\partial Y_k (s)}{\partial k} \right| + \left| \frac{p_k' (\text{Re}(k))}{\partial k} \right| + \frac{1}{|k|^{\gamma \theta}} e^{-c_1 |u| |v| t}, \quad k \in D_k,
\]
\[
|\partial R_2^{(3)} e^{-2i\theta t}| \leq \left| \frac{\partial Y_0 (s)}{\partial k} \right| + \frac{1}{|k|^{\gamma \theta}} e^{-c_1 |u| |v| t}, \quad k \in D_k^\times,
\]
\[
\left( \left| \frac{\partial Y_0 (s)}{\partial k} \right| + |k - k_0|^{-1} \right) e^{-c_1 |u| |v| t}, \quad k \in D_k^\times.
\]
(91)

**Proof.** Recalling the above condition \(\alpha < 0\), for ease of analysis, the exponential part can be written as \(- \frac{\alpha t}{2} \left( \frac{1}{(u + k_0)^2 + v^2} - \frac{1}{k_0^2} \right)\), which is a decreasing function of \(u\). For the index part in region \(D_k\), we have
\[
- \frac{\alpha t}{2} \left( \frac{1}{(u + k_0)^2 + v^2} - \frac{1}{k_0^2} \right) = - \frac{\alpha t}{2} \left( \frac{1}{(u + k_0)^2 + v^2} - \frac{1}{k_0^2} \right)
\]
\[
= - \frac{\alpha t}{2} \left( \frac{-u^2 - 2k_0 u - v^2}{((u + k_0)^2 + v^2)(k_0^2)} \right)
\]
\[
\leq -c_1 |u| |v| t.
\]
A similar analysis holds in region $D_i^*$, where $k = u + iv$ and $0 \leq v < u$, the index part is

$$\frac{avt}{2} \left( \frac{1}{u^2 + v^2} - \frac{1}{k_0^2} \right) \leq \frac{avt}{2} \left( \frac{1}{u^2 + v^2} - \frac{1}{k_0^2} \right) \leq -c_2\nu t. \tag{91}$$

The overall conclusion is summarized as Eq. (91).

Therefore, our next goal is to prove the existence of $(I - S)^{-1}$.

**Proposition IX.2.** For sufficiently large $t$, operator $S$ has a small norm with

$$\|S\|_{L^\infty \rightarrow L^\infty} \leq ct^{-\frac{1}{3}}, \tag{92}$$

therefore, $(I - S)^{-1}$ exists.

**Proof.** This is discussed in detail in region $D_i$, and the results for other regions can be obtained similarly. Let $s = u + k_0 + iv$ for any $f \in L^\infty$, we have

$$\left\| S(f) \right\| \leq \frac{1}{\pi} \int_{k_0}^{\infty} \frac{|M^{(2)}_n| |M^{(2)}_{n-1}|}{|s - k|} dA(s) \leq \frac{1}{\pi} \int_{k_0}^{\infty} \left( \int_{k_0}^{\infty} \frac{|M^{(2)}_n|}{|s - k|} d\nu \right) d\nu \leq c(\Omega_1 + \Omega_2 + \Omega_3), \tag{93}$$

where

$$\Omega_1 = \int_{k_0}^{\infty} \int_{k_0}^{\infty} \frac{|\partial \tilde{Y}_k(s)| e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)}}{|s - k|} d\nu d\sigma,$$

$$\Omega_2 = \int_{k_0}^{\infty} \int_{k_0}^{\infty} \frac{|\partial \tilde{Y}_k(s)| e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)}}{|s - k|} d\nu d\sigma,$$

$$\Omega_3 = \int_{k_0}^{\infty} \int_{k_0}^{\infty} \frac{|s - k|^{-1} e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)}}{|s - k|} d\nu d\sigma.$$

Next, we wish to estimate $\Omega_j, j = 1, 2, 3$. An important inequality to know in advance is

$$\left\| \frac{1}{s - k} \right\|_{L^2(\infty k_0, \infty)}^2 = \int_{-\infty}^{\infty} \frac{1}{|s - k|} du \leq \int_{-\infty}^{\infty} \frac{1}{|s - k|} du = \int_{-\infty}^{\infty} \frac{1}{|u - \xi|^2 + (v - \eta)^2} du = \frac{\pi}{|v - \eta|}, \tag{94}$$

where $y = \frac{\pi - \xi}{\sqrt{\eta}}$.

For $\Omega_1$, from a direct calculation using Eq. (94), it follows that

$$\Omega_1 \leq c_3 \int_{k_0}^{\infty} e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)} d\nu \leq c_3 \int_{k_0}^{\infty} e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)} d\nu \leq c_3 \int_{k_0}^{\infty} e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)} d\nu \leq c_3 \int_{k_0}^{\infty} e^{-\frac{1}{2} \left( \frac{1}{(s - k)^2} \right)} d\nu \leq ct^{-1}. \tag{95}$$
The same method can be used to estimate $\Omega_2$. For the estimation of $\Omega_3$, we need the help of Hölder’s inequality with $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Moreover,

\[
\|s - k\|_{L^p(s+k,v,\infty)} = \left(\int_{s+k}^{\infty} |u - k + tv|^{-\frac{1}{p}} du\right)^{\frac{1}{p}} = \left(\int_v^{\infty} |u + iv|^{-\frac{1}{p}} du\right)^{\frac{1}{p}} = v^{\frac{1}{p} - 1} \left(\int_1^{\infty} (1 + x^2)^{-\frac{1}{2}} dx\right)^{\frac{1}{p}} \lesssim c v^{\frac{1}{p} - 1}.
\]

(96)

Similar estimates can be proved,

\[
\left\| \frac{1}{s - k} \right\|_{L^p(v,\infty)} \lesssim \left| v^\frac{1}{q} - \eta \right|^{\frac{1}{p} - 1}, \quad \frac{1}{q} + \frac{1}{p} = 1.
\]

Then, we have

\[
\begin{align*}
\Omega_3 & \leq c \int_0^{\infty} \left\| s - k \right\|_{L^p(s+k,v,\infty)} \left\| \frac{1}{s - k} \right\|_{L^p(v,\infty)} e^{-\frac{v^2}{\eta^2}} e^{\frac{\alpha}{(s-k)^2 + \eta^2}} dv \\
& \leq c \int_0^{\infty} \left\| s - k \right\|_{L^p(s+k,v,\infty)} \left\| \frac{1}{s - k} \right\|_{L^p(v,\infty)} e^{\frac{\alpha}{\eta^2}} dv \\
& \leq c \int_0^{\infty} \left\| s - k \right\|_{L^p(s+k,v,\infty)} \left\| \frac{1}{s - k} \right\|_{L^p(v,\infty)} e^{-\frac{1}{\eta^2}} dv \\
& \leq c \int_0^{\infty} v^{\frac{1}{p} - 1} |v - \eta|^\frac{1}{q} e^{-\frac{1}{\eta^2}} dv + \int_\eta^{\infty} v^{\frac{1}{p} - 1} |v - \eta|^\frac{1}{q} e^{-\frac{1}{\eta^2}} dv \\
& \lesssim H_1 + H_2.
\end{align*}
\]

The following is the estimation of the two integrals. For $H_1$, since $0 \leq v < \eta$, the following estimates are obtained by replacing $v = w\eta$ with variables:

\[
H_1 = \int_0^{\eta} v^{\frac{1}{p} - 1} (\eta - v)^\frac{1}{q} e^{-\frac{1}{\eta^2}} dv \\
= \int_0^{\eta} \eta^\frac{1}{q} w^{\frac{1}{p} - 1} (1 - w)^\frac{1}{q} e^{-\frac{1}{\eta^2}} dw \\
\lesssim ct^{\frac{1}{q} - 1}.
\]

For $H_2$, we let $w = v - \eta$, then

\[
H_2 = \int_\eta^{\infty} v^{\frac{1}{p} - 1} (v - \eta)^\frac{1}{q} e^{-\frac{1}{\eta^2}} dv \\
= \int_0^{\infty} (w + \eta)^\frac{1}{p} w^{\frac{1}{q} - 1} e^{-\frac{1}{\eta^2}} (w + w\eta)^\frac{1}{q} dw \\
\lesssim ct^{\frac{1}{q} - 1} \int_0^{\infty} w^{\frac{1}{q} - 1} e^{-\frac{1}{\eta^2}} dw \\
\lesssim ct^{\frac{1}{q} - 1}.
\]

So, combining the estimates of $H_1$ and $H_2$, we get $\Omega_3 \lesssim ct^{\frac{1}{q} - 1}$.

In region $D_4$, the estimation is still divided into three parts; moreover, $s = u + iv$. Thus,

\[
\begin{align*}
\left| S(f) \right| & \leq \frac{1}{\pi} \iint_{D_4^*} \left| f \mathcal{M}_R^{(2)} \mathcal{R}^{(2)}_M \mathcal{M}_R^{(2)} \right| |s - k| dA(s) \\
& \leq \frac{1}{\pi} \left\| f \right\|_{L^p} \left\| \mathcal{M}_R^{(2)} \right\|_{L^q} \left\| \mathcal{M}_R^{(2)} \right\|_{L^r} \int_{D_4^*} \left| \mathcal{R}_M \right| e^{\frac{\alpha}{(s-k)^2 + \eta^2}} dA(s) \\
& \lesssim c \left( \Omega_1^* + \Omega_2^* + \Omega_3^* \right),
\end{align*}
\]

(98)
where

\[
\Omega' = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |\partial Y(x)| e^{\frac{1}{\alpha_1} \left( \frac{1}{s-k} - \frac{1}{\alpha_1} \right)} \, du \, dv,
\]

\[
\Omega'_1 = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |r' \left( \text{Re}(s) \right)| e^{\frac{1}{\alpha_1} \left( \frac{1}{s-k} - \frac{1}{\alpha_1} \right)} \, du \, dv,
\]

\[
\Omega'_2 = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |s| e^{\frac{1}{\alpha_1} \left( \frac{1}{s-k} - \frac{1}{\alpha_1} \right)} \, du \, dv.
\]

Combined with lemma (91) and Eq. (94), we can estimate each integral. For \( \Omega'_1 \),

\[
\Omega'_1 \leq c \int_0^{\frac{\pi}{2}} \frac{1}{s-k} \left| e^{-c|\alpha_1|t} \right| \, du \, dv
\]

\[
\leq c \int_0^{\frac{\pi}{2}} \frac{1}{s-k} \left| e^{-c|\alpha_1|t} \right| \, dv
\]

\[
\leq c \int_0^\infty \frac{1}{\sqrt{|\gamma - v|}} \, dv
\]

\[
\leq ct^{-\frac{1}{2}}.
\]

The estimation of \( \Omega'_2 \) is similar to that of \( \Omega'_1 \), the estimation method of \( \Omega'_3 \) is similar to that of \( \Omega_3 \), and we get

\[
\Omega'_j \leq ct^{-\frac{1}{2}}, \quad j = 2, 3.
\]

Therefore, this proves that all regions satisfy (92). \( \Box \)

Next, consider the expansion of \( M^{(3)} \),

\[
M^{(3)}(k) = I + \frac{M^{(3)}(x, t)}{k} + O(k^{-2}), \quad k \to \infty,
\]

from Eq. (89); it is easy to see that

\[
M^{(3)}(x, t) = \frac{1}{\pi} \int_D M^{(3)}(s, t) W^{(3)}(s) \, dA(s).
\]

Furthermore, we can prove the following proposition:

**Proposition IX.3.** For large values of \( t \), we have

\[
\left| M^{(3)}_1 \right| \leq ct^{-\frac{1}{2}}.
\]

**Proof.** Since \( M^{(2)}_{\text{rhp}} \) is bounded outside the pole, we have

\[
\left| M^{(3)}_1 \right| \leq \frac{1}{\pi} \int_D \left| M^{(3)} \right| M^{(2)}_{\text{rhp}} \, dA(s)
\]

\[
\leq \frac{1}{\pi} \left\| M^{(3)} \right\|_{L^\infty} \left\| M^{(2)}_{\text{rhp}} \right\|_{L^\infty} \left\| \left( M^{(2)}_{\text{rhp}} \right)^{-1} \right\|_{L^\infty} \int_D |\partial R e^{2\theta}| \, dA(s)
\]

\[
\leq c \int_D \left| \partial Y(x) \right| e^{-\frac{1}{\alpha_1} \left( \frac{1}{s-k} - \frac{1}{\alpha_1} \right)} \, dA(s) + \int_D |p_5(u)| e^{-\frac{1}{\alpha_1} \left( \frac{1}{s-k} - \frac{1}{\alpha_1} \right)} \, dA(s)
\]

\[
\leq c \left( \Omega_4 + \Omega_5 + \Omega_6 \right).
\]
We constrain $\Omega_4$ by using the Cauchy–Schwarz inequality,

$$|\Omega_4| \leq \int_0^\infty \left\| \mathbf{Y} \right\|_{L^2(v \to \infty)} \left( \int_0^\infty e^{-c_1^2|v|^2} \, du \right)^{\frac{1}{2}} \, dv$$

$$\leq ct^{-\frac{1}{2}} \int_0^\infty \sqrt{v} e^{-4tv^2} \, dv$$

$$\leq ct^{-\frac{1}{2}} \int_0^\infty e^{-4tw^2} \, dw \leq ct^{-\frac{1}{2}}.$$  

Similar constraints can be used for $\Omega_5$. For $\Omega_6$, we follow the method used for $\Omega_3$ and use Hölder’s inequality and Eq. (96) to obtain

$$I_3 = \int_0^\infty \int_0^\infty \left( (u - k_0)^2 + v^2 \right)^{-\frac{1}{2}} e^{-\frac{c_1^2}{4} \left( \frac{1}{v^2} + \frac{k_0^2}{v^2} \right)} \, du \, dv$$

$$\leq \int_0^\infty \left\| \left( (u - k_0)^2 + v^2 \right)^{-\frac{1}{2}} \left( \int_{k_0^2}^\infty e^{-c_1^2|v|^2} \, du \right)^{\frac{1}{2}} \, dv \right\|_{L^2}$$

$$\leq \int_0^\infty \frac{1}{2} \left( \int_{k_0^2}^\infty e^{-c_1^2|v|^2} \, du \right)^{\frac{1}{2}} \, dv$$

$$\leq ct^{-\frac{1}{2}} \int_0^\infty \frac{1}{2} e^{-c_1^2v^2} \, dv,$$

$$\leq ct^{-\frac{1}{2}} \int_0^\infty e^{-c_1^2v^2} \, dv \leq ct^{-\frac{1}{2}}.$$  

Here, we replace the variable $v = wt^{-\frac{1}{2}}$. Notice that here $2 < p < 4$ and $-1 < \frac{2}{p} - \frac{1}{2} < -\frac{1}{2}$.

## X. Long-time asymptotic behavior of soliton solution region for the coupled dispersive AB System

After many deformations, we now begin to construct the long-time asymptotic properties of the coupled dispersive AB system (1). Looking back at the previous transformation, we have

$$M(k) = M^{(3)}(k)M^{(0)}(k)M^{(1)}(k)R^{(1)}(k)^{-1}T(k)^0, \quad k \in \mathbb{C}\setminus A_1 \cup A_2.$$  (103)

In particular, in the vertical direction $k \in D_2, D_{10}$, we have $R^{(2)} = I$. Therefore we consider $k \to \infty$ in this region, which gives

$$M = \left( I + \frac{M^{(3)}(k)}{k} + \cdots \right) \left( I + \frac{M^{(0)}(k)}{k} + \cdots \right) \left( I + \frac{T_1^0}{k} + \cdots \right).$$

In order to recover the potential, the coefficient of $k^{-1}$ needs to be collected,

$$M_1 = \frac{M^{(0)}(k)}{k} + \frac{M^{(3)}(k)}{k} + T_1^0.$$  

So, we have

$$A = 4i(M_1^{out} + M_1^{int})_{12} + O(t^{-\frac{3}{2}}),$$

$$B = -\frac{4i}{\beta} \frac{d}{dt} (M_1^{out} + M_1^{int})_{11} + O(t^{-\frac{3}{2}}).$$  (104)

Based on the above formula, the specific asymptotic state can be written in the form of the following theorem.

**Theorem X.1.** Suppose $A_0, B_0 \in H^{1,1}(\mathbb{R})$ have general scattering data, and $A(x, t)$ and $B(x, t)$ are the solutions of system (1). For fixed $x_1 < x_2$ with $x_1, x_2 \in \mathbb{R}$ and $v_1 < v_2$ with $v_1, v_2 \in \mathbb{R}$, a conical region can be defined as
Then, we define two regions for the spectral parameters as follows:

\[ \mathcal{I} = \{ k : f(v_2 < |k| < f(v_1) \}, \quad f(v) \doteq \left( \frac{\alpha}{4v} \right)^{1/2}, \]

as shown in Fig. 12. We use \( A_{sol}(x, t, k \hat{D}) \) and \( B_{sol}(x, t, k \hat{D}) \) for the coupled dispersive AB system corresponding to \( N(\mathcal{I}) \leq N \) modulated non-reflection scattering data

\[ \hat{D}(\mathcal{I}) = \{ k_0, c(\mathcal{I}) \}, \quad c(\mathcal{I}) = c_j \prod_{k_{\text{ex}}(\mathcal{I})} \left( \frac{k_0 - k_0}{k_0 - k_0} \right)^2. \]

Then, when \( t \to \infty \) and \( (x, t) \in \mathcal{C}(x_1, x_2, v_1, v_2) \), we have

\[
A(x, t) = A_{sol}(x, t, \hat{D}(\mathcal{I})) + t^{-\frac{3}{2}} (g_1 + g_2) + \mathcal{O}(t^{-\frac{1}{2}}),
\]

\[
B(x, t) = B_{sol}(x, t, \hat{D}(\mathcal{I})) + t^{-\frac{3}{2}} (h_1 + h_2 + f_1 + f_2) + \mathcal{O}(t^{-\frac{1}{2}}),
\]

where

\[
M_{sol}^{\text{out}}(\pm k_0) = \begin{pmatrix}
m_{11}(\pm k_0) & m_{12}(\pm k_0) \\
m_{21}(\pm k_0) & m_{22}(\pm k_0)
\end{pmatrix}, \quad j = 1, 2.
\]

\[
g_1 = 4 \sqrt{-\frac{k_0^3}{\alpha}} (m_{12}(-k_0)\Xi_{21}(-k_0) + m_{11}(-k_0)\Xi_{12}(-k_0)),
\]

\[
g_2 = 4 \sqrt{-\frac{k_0^3}{\alpha}} (m_{12}(k_0)\Xi_{21}(k_0) + m_{11}(k_0)\Xi_{12}(k_0)),
\]

\[
h_1 = 4 \frac{\alpha}{\beta} \sqrt{-\frac{k_0^3}{\alpha}} \frac{d}{dt} \left[ m_{12}(-k_0)m_{22}(-k_0)\Xi_{21}(-k_0) + m_{11}(-k_0)m_{23}(-k_0)\Xi_{12}(-k_0) \right],
\]

\[
h_2 = 4 \frac{\alpha}{\beta} \sqrt{-\frac{k_0^3}{\alpha}} \frac{d}{dt} \left[ m_{12}(k_0)m_{22}(k_0)\Xi_{21}(k_0) + m_{11}(k_0)m_{23}(k_0)\Xi_{12}(k_0) \right],
\]

\[
f_1 = 2 \frac{\alpha}{\beta} \sqrt{-\frac{k_0^3}{\alpha}} m_{12}(-k_0)m_{23}(-k_0)\Xi_{21}(-k_0) + m_{11}(-k_0)m_{21}(-k_0)\Xi_{12}(-k_0),
\]

\[
f_2 = 2 \frac{\alpha}{\beta} \sqrt{-\frac{k_0^3}{\alpha}} m_{12}(k_0)m_{23}(k_0)\Xi_{21}(k_0) + m_{11}(k_0)m_{21}(k_0)\Xi_{12}(k_0).
\]

**FIG. 12.** Distribution of discrete spectral points: two pairs inside the cone \( C \) and two pairs outside the cone \( C \).
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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jin-Yan Zhu: Writing – original draft (equal); Writing – review & editing (equal). Yong Chen: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX A: ESTIMATION OF THE CHARACTERISTIC FUNCTION

In Assumption II.4, there is no zero point of $s^{11}$ on $\mathbb{R}$, so there is no pole of $r(k)$ on $\mathbb{R}$. From the Volterra integral formula (9),

$$
\psi_{11}^\pm(x, 0, k) = 1 + \int_{\pm\infty}^x \frac{1}{2} A(y, k) \psi_{21}^\pm dy,
$$

$$
\psi_{21}^\pm(x, 0, k) = \int_{\pm\infty}^x -\frac{1}{2} A^*(y, k) \psi_{11}^\pm e^{2ik(x-y)} dy.
$$

Let

$$
\Psi_1 = (\psi_{11}^-, \psi_{21}^-)^T.
$$

For $k \in \mathbb{R}$, we introduce an operator mapping

$$
\mathcal{R}_0 f = \int_{-\infty}^x T_0 f(y) dy
$$

and

$$
T_0 = \begin{pmatrix}
0 & 1 \frac{1}{2} A \\
-\frac{1}{2} A^* e^{2ik(x-y)} & 0
\end{pmatrix}.
$$

It can be seen from this that

$$
|\mathcal{R}_0 f(x)| \leq \int_{-\infty}^x \frac{1}{2} |A(y)| dy \| f \|_{L^\infty(-\infty, 0]},
$$

which means that $\mathcal{R}_0$ is a bounded linear operator on $L^\infty(-\infty, 0]$. Furthermore, it can be obtained by mathematical induction that

$$
|\mathcal{R}_0^n f(x)| \leq \frac{1}{n!} \left( \int_{-\infty}^x \frac{1}{2} |A(y)| dy \right)^n \| f \|_{L^\infty(-\infty, 0]}.
$$

Therefore, the following series is uniformly convergent on $x \in (-\infty, 0]$:

$$
\Psi_1(x, k) = \sum_{n=0}^{\infty} \mathcal{R}_0^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

(A1)

So, for $k \in \mathbb{R}$ and $x \in (-\infty, 0]$, \n
$$
|\Psi_1(x, k)| \leq e^{\frac{1}{2} |A|_1(\mathbb{R})}.$$
Similarly, we can get
\[
\begin{align*}
\Psi_\pm(x, k) & \leq e^{k|A_x|L^1(\mathbb{R})} x \in (-\infty, 0], \\
\Psi_0(x, k) & \leq e^{k|A_x|L^1(\mathbb{R})} x \in [0, \infty), \\
\Psi_\mp(x, k) & \leq e^{k|A_x|L^1(\mathbb{R})} x \in [0, \infty).
\end{align*}
\]

Let us look at the derivative of the characteristic function with respect to \( k \). According to the uniform convergence property of series (A1), we have
\[
\Psi_{k, x}(x) = \sum_{n=0}^{\infty} \partial_k \mathcal{R}_0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \mathcal{R}_0^m \mathcal{R}_0 \mathcal{R}_0^{n-m-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]
where the form of \( \mathcal{R}_0 \) is
\[
\mathcal{R}_0 f(x) = \int_{-\infty}^{x} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{i 2k (x-y)} \left( \begin{array}{c} \frac{1}{2} A(y) f_2 \\ -i (x-y) A^{*} f_1 \end{array} \right) dy.
\]
Similarly, \( \mathcal{R}_0 \) is a bounded linear operator on \( x \in (-\infty, 0] \) with
\[
\| \mathcal{R}_0 \|_{L^\infty(-\infty, 0]} \leq |A|_{L^1(\mathbb{R})}.
\]
So, we have
\[
\begin{align*}
\Psi_{k, x}(x, k) & \leq \sum_{n=1}^{\infty} |\partial_k \mathcal{R}_0^m \left( \begin{array}{c} 1 \\ 0 \end{array} \right)| \leq \sum_{n=1}^{\infty} \frac{1}{2} |A|_{L^1(\mathbb{R})}^{n-1} |A|_{L^1(\mathbb{R})} \\
& \leq \| A \|_{L^1(\mathbb{R})} e^{k |A_x|L^1(\mathbb{R})}
\end{align*}
\]
for \( k \in \mathbb{R} \) and \( x \in (-\infty, 0] \).

In a similar manner, we can also get
\[
\begin{align*}
\Psi_+^{\pm}(x, k) & \leq |A|_{L^1(\mathbb{R})} e^{\pm k |A_x|L^1(\mathbb{R})}, \\
\Psi_-^{\pm}(x, k) & \leq |A|_{L^1(\mathbb{R})} e^{\pm k |A_x|L^1(\mathbb{R})}, \\
\Psi_{0}^{\pm}(x, k) & \leq |A|_{L^1(\mathbb{R})} e^{\pm k |A_x|L^1(\mathbb{R})}.
\end{align*}
\]

Next, we will estimate the characteristic function \( (\Psi^*)_j(x, k) \), \( (i, j = 1, 2) \), respectively. In fact, the following lemma holds:

**Lemma X.2.** For \( k \in \mathbb{R} \), we have the following estimates:
\[
\begin{align*}
\| \Psi_{21}^-(x, 0, k) \|_{C^0(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{21}^+(x, 0, k) \|_{L^1(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{22}^-(x, 0, k) \|_{C^0(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{22}^+(x, 0, k) \|_{L^1(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{23}^-(x, 0, k) \|_{C^0(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{23}^+(x, 0, k) \|_{L^1(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{24}^-(x, 0, k) \|_{C^0(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}, \\
\| \Psi_{24}^+(x, 0, k) \|_{L^1(\mathbb{R} \times -L^1(\mathbb{R}))} & \leq |A|_{H^{1,1}}.
\end{align*}
\]

**Proof.** From the above integral form, it follows that
\[
\Psi_{21}^-(x, 0, k) = \int_{-\infty}^{x} -\frac{1}{2} A^{*}(y, k) \Psi_{22}^+ e^{2i k (x-y)} dy.
\]
Then, for \( k \in \mathbb{R} \), \( \forall \sigma \in C_0(\mathbb{R}) \), we compute

\[
\left\| \int_{-\infty}^{x} -\frac{1}{2} A^* \psi_{11}^- e^{2ik(x-y)} \, dy \right\|_{L^2(I_k)}^2 = \sup_{|\sigma|<1} \int_{I_k} \sigma(k) \int_{-\infty}^{x} -\frac{1}{2} A^* \psi_{11}^- e^{2ik(x-y)} \, dy \leq ce^{2\|A\|_{L^2}} \|A\|_{L^2}^2;
\]

this estimate is direct. In addition,

\[
\int_{-\infty}^{x} \int_{-\infty}^{x} -\frac{1}{2} A^* \psi_{11}^- e^{2ik(x-y)} \, dy \, dk \, dx \leq \int_{-\infty}^{x} \int_{-\infty}^{x} -\frac{1}{2} A^* \psi_{11}^- e^{2ik(x-y)} \, dy \, dx \leq \|A\|_{L^1},
\]

which implies that for \( k \in \mathbb{R} \),

\[
\left\| \psi_{11}^-(x,0,k) \right\|_{C^0(\mathbb{R},-L^2(\mathbb{R}))} \leq \|A\|_{H^1},
\]

\[
\left\| \psi_{11}^-(x,0,k) \right\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq \|A\|_{H^1}.
\]

Similarly, other estimates can also be obtained.

\( \square \)

Its derivative form for \( k \) is

\[
\psi_{11}^-(x,0,k) = \int_{-\infty}^{x} -\frac{1}{2} A^* 2i(x-y) \psi_{11}^- e^{2ik(x-y)} \, dy + \int_{-\infty}^{x} -\frac{1}{2} A^* \psi_{11}^- e^{2ik(x-y)} \, dy.
\]

Similarly, the following estimates can be obtained:

\[
\left\| \psi_{11}^-(x,0,k) \right\|_{C^0(\mathbb{R},-L^2(\mathbb{R}))} \leq \|A\|_{H^1},
\]

\[
\left\| \psi_{11}^-(x,0,k) \right\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq \|A\|_{H^1}.
\]

According to Eq. (12),

\[
s_{11}(k) - 1 = (\psi_{11}^- - 1)(\psi_{22}^- - 1) + (\psi_{22}^- - 1) - \psi_{21}^- \psi_{12}^-,
\]

\[
s_{12}(k) = e^{2iky}(\psi_{12}^- \psi_{22}^- - \psi_{22}^- \psi_{12}^-).
\]

Let us examine \( \psi_{11}^- - 1 \) and \( \psi_{22}^- - 1 \). Thus,

\[
\psi_{11}^- - 1 = \int_{-\infty}^{x} -\frac{1}{2} A(y) \psi_{21}^- dy,
\]

\[
\psi_{22}^- - 1 = \int_{-\infty}^{x} -\frac{1}{2} A(y) \psi_{12}^- dy;
\]

it follows that \( \psi_{11}^- - 1 \) is bounded and can be obtained using Lemma X.2,

\[
\left\| \psi_{11}^-(x,0,k) - 1 \right\|_{C^0(\mathbb{R},-L^2(\mathbb{R}))} \leq \|A\|_{H^1},
\]

\[
\left\| \psi_{11}^-(x,0,k) - 1 \right\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq \|A\|_{H^1}.
\]

The estimation for \( \psi_{12}^- - 1 \) can be obtained similarly. In addition, we know that

\[
s_{11,k} = \psi_{11}^- \psi_{22}^- + \psi_{11}^- \psi_{22,k}^- - \psi_{21}^- \psi_{12}^- - \psi_{21}^- \psi_{12,k}^-,
\]

where

\[
\psi_{11,k}^- = \int_{-\infty}^{x} -\frac{1}{2} A(y,k) \psi_{21,k}^- dy, \quad \psi_{22,k}^- = \int_{-\infty}^{x} -\frac{1}{2} A(y,k) \psi_{12,k}^- dy.
\]

A similar operation is performed on \( s_{12}(k) \). Tracing back to the above estimates, when \( k \in \mathbb{R} \) and initial value \( A \in H^{1,1} \), we have

\[
s_{11}(k) \in L^2(\mathbb{R}), \quad s_{11,k}(k) \in L^2(\mathbb{R}),
\]

\[
s_{12}(k) \in L^2(\mathbb{R}), \quad s_{12,k}(k) \in L^2(\mathbb{R}).
\]
APPENDIX B: SOLVABLE PARABOLIC CYLINDER MODEL

Here, we mainly describe the solution of the parabolic cylinder model introduced above. For the cAB equation studied in this paper, since there are two stationary phase points, we need two parabolic cylinders to describe it. Their expansion forms are the same, and they all use the following model.

For \( r_0 \in \mathbb{R} \), if \( \nu = -\frac{1}{2\pi} \log(1 + |r_0|^2) \), define the contour \( \Sigma_{PC} = \bigcup_{j=1}^{4} \Sigma_j \),

\[
\Sigma_j = \{ \xi \in \mathbb{C} : \arg\xi = (2j-1)\pi/4 \}, \quad j = 1, 2, 3, 4.
\]

These four contours divide the plane into six areas, \( D_j, j = 1, \ldots, 6 \), as shown in Fig. 13.

Therefore, we can consider the RHP corresponding to the following parabolic cylinder model.

**Riemann–Hilbert Problem X.3.** Find a matrix-valued function \( M_{PC}(x, t, k) \) satisfying the following properties:

- **Analyticity:** \( M(x, t, k) \) is analytic in \( \mathbb{C} \setminus \Sigma_{PC} \).
- **Asymptotic behavior:** \( M_{PC}(k) = I + \frac{N_{PC}}{k} + O(k^{-2}) \), \( k \to \infty \).
- **Jump condition:** \( M_{PC}(x, t, k) \) has continuous boundary values \( M_{PC}^\pm(x, t, k) \) on \( \mathbb{R} \) and

\[
M_{PC}^+(x, t, k) = M_{PC}^-(x, t, k) V_{PC}(k), \quad k \in \mathbb{R},
\]

where

\[
V_{PC}(k) = \begin{cases}
    \begin{pmatrix}
    1 & 0 \\
    \frac{r_0}{1 + |r_0|^2} k^{2i\nu} e^{-\frac{u^2}{2}} & 1
    \end{pmatrix}, & \text{if } k \in \Sigma_1, \\
    \begin{pmatrix}
    1 & \frac{r_0}{1 + |r_0|^2} k^{2i\nu} e^{-\frac{u^2}{2}} \\
    0 & 1
    \end{pmatrix}, & \text{if } k \in \Sigma_2, \\
    \begin{pmatrix}
    1 & 0 \\
    \frac{r_0}{1 + |r_0|^2} k^{2i\nu} e^{-\frac{u^2}{2}} & 1
    \end{pmatrix}, & \text{if } k \in \Sigma_3, \\
    \begin{pmatrix}
    1 & \frac{r_0}{1 + |r_0|^2} k^{2i\nu} e^{-\frac{u^2}{2}} \\
    0 & 1
    \end{pmatrix}, & \text{if } k \in \Sigma_4.
\end{cases}
\]

Let

\[
M_{PC} = \mathcal{F} k^{-i\nu_0} e^{\frac{u^2}{2}},
\]

FIG. 13. The jump contour \( V_{PC} \) for \( M_{PC}(k) \).
where
\[
\mathcal{P}(k) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ -r_0 & 1 \end{pmatrix}, & k \in D_1, \\
\begin{pmatrix} 1 & 0 \\ 1 + |r_0|^2 & 1 \end{pmatrix}, & k \in D_3, \\
\begin{pmatrix} 1 & r_0 \\ 0 & 1 \end{pmatrix}, & k \in D_4, \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \in D_6, \\
\begin{pmatrix} 1 & 0 \\ -\bar{r}_0 & 0 \end{pmatrix}, & k \in D_2 \cup D_5.
\end{cases}
\]

Then, we can get a standard RHP with jump only in \( k = 0 \).

**Riemann–Hilbert Problem X.4.** Find a matrix-valued function \( \vartheta(x, t, k) \) satisfying the following properties:

- **Analyticity:** \( \vartheta(x, t, k) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \).
- **Asymptotic behavior:** \( \vartheta(k) e^{ik\frac{2}{4}\sigma_3 k - i\nu k} = I + O(k^{-1}), \quad k \to \infty \).
- **Jump condition:** \( \vartheta^+(x, t, k) = \vartheta^-(x, t, k) v(0) \).

The above RH problem can be reduced to the Weber equation,
\[
\frac{\partial^2}{\partial x^2} D(z) + \left[ \frac{1}{2} - \frac{z^2}{4} + a \right] D(z) = 0,
\]
where
\[
z = e^{-\frac{3\pi}{4}i} k, \quad a = \nu \Xi_{12} \Xi_{21} = i\nu,
\]
\[
\Xi_{12} = \frac{\sqrt{2\pi}e^{-\frac{3\pi}{4}k}}{r_0 \Gamma(-a)}, \quad \Xi_{21} = \frac{\sqrt{2\pi}e^{-\frac{3\pi}{4}k}}{r_0 \Gamma(a)} = \frac{\nu}{\Xi_{12}}.
\]

The explicit form of the parabolic cylindrical explicit solution \( \vartheta(k) \) can be given by parabolic a cylindrical function. From Eq. (B3) it follows that
\[
M_{PC}(k) = I + \frac{M_{4PC}}{ik} + O(k^{-2}),
\]
where
\[
M_{4PC} = \begin{pmatrix} 0 & \Xi_{12} \\ -\Xi_{21} & 0 \end{pmatrix}.
\]

**REFERENCES**

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