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A novel (2+1)-dimensional Sawada-Kotera type system: multisoliton solution and variable separation solution

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Abstract A novel (2+1)-dimensional system of the Sawada-Kotera type is considered. The existence of three-soliton and four-soliton solutions with wave number constraints is confirmed. Other interesting solutions, such as the long-range interaction between a line soliton and a y-periodic soliton, are also presented based on the Hirota formalism. By extending the multilinear variable separation approach to the fifth-order nonlinear evolution equation, various localized excitations are introduced, including solitoff, dromion, and an instanton excited by three resonant dromions. In addition to these localized excitations, the general fusion or fission type N-solitary wave solution is obtained, the Y-shaped resonant soliton and the T-type resonant soliton interaction in shallow water are graphically explored.

Keywords Hirota bilinear form · Multisoliton solution · Multilinear variable separation approach · Localized excitations · Fusion and fission phenomena

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1 Introduction

The following fifth-order Korteweg-de Vries like equation

$$w_t+w_{xxxxx}+15w_x w_{xx}+15ww_{xxx}+45w^2w_x = 0, \quad (1)$$

also known as the Sawada-Kotera (SK) equation, is a system of crucial importance in soliton theory [1,2]. As a member of the Liouville field hierarchy [3], the SK equation has played an important role in conformal field theory, quantum gravity and shallow water flows [4]. It has been widely and profoundly studied. For instance, Satsuma and Kaup derived its Bäcklund transformation and conservation quantities [5], while Oevel established the bi-Hamiltonian structure and recursion formula [6]. A supersymmetric SK equation has been proposed in [7], demonstrating remarkable integrability through the existence of a Lax representation. Recently, some special traveling wave structures, including few-cycle-pulse solitons and soliton molecules, have been constructed by a new traveling wave method related to the bilinear transformation [8,9].

Konopelcheno and Dubrovsky introduced the (2+1)-dimensional generalization of the SK system in the form of [10]

$$w_t + (w_{xxxx} + 15ww_{xx} + 15w^3 + 5w_{xy} +15w \int w_y dx)_x - 5 \int w_{yy} dx = 0. \quad (2)$$

If w is independent of y, then Eq. (2) straightforwardly reduces to the SK equation (1). Many studies have shown that the (2+1)-dimensional SK system (2) has significant integrable properties from various perspectives. Geng [11] provided a rigorous investigation, deriving both Darboux and Bäcklund transformations. Additionally, the exploration of the connection between the singularity manifold equation and corresponding pseudopotential equations was undertaken in [12]. Lou [13] introduced a novel Bäcklund transformation and a related quintic linear equation, resulting in the discovery of a new hierarchy of multiple soliton solutions. Li [14] studied the collisions among lump waves, line waves, and breather waves based on the N-soliton solutions. In particular, it is confirmed that the interaction between a line wave and a breather wave will generate two breathers under certain conditions. Furthermore, the analysis of the transitions and mechanisms of nonlinear waves governed by Eq. (2) was conducted using characteristic line and phase shift analysis by Tian [15].

In this paper, we start with the conjecture that the nonlinear version of the following bilinear equation can be solved by the multilinear variable separation approach [16–22]

$$D_y(D_x^5 + D_t) f \cdot f = 0, \tag{3}$$

where f is an analytic function of (x, y, t) and the Hirota bilinear derivative operator is defined by [23,24]

$$D_x^l D_y^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^l (\partial_y - \partial_{y'})^m (\partial_t - \partial_{t'})^n a(x, y, t) b(x', y', t') \big|_{x=x', y=y', t=t'} .$$

A comparison with the bilinear form of the SK system (1)

$$D_x(D_x^5 + D_t) f \cdot f = 0, \quad w = 2(\ln f)_{xx},$$

it is obvious that we extend the bilinear SK system to its (2+1)-dimensional counterpart (3) only by changing one of the Hirota operators $D_x \to D_y$. However, this seemingly simple extension is worth further consideration, since the Hirota bilinear method is powerful and effective not only for finding soliton solutions, but also for finding new systems with multisoliton solutions or infinitely many generalized symmetries in the classical [25,26] and supersymmetric [27,28] contexts.

The nonlinear version of (3) corresponds to a different (2+1)-dimensional Sawada-Kotera (2DSK) system

$$\begin{aligned} &u_t + u_{xxxxx} + 5(uv_{xx} + 2u_{xx}v + 3uv^2)_x = 0, \\ &u_x - v_y = 0. \end{aligned} \tag{4}$$

Obviously, setting $u = v = w(x, y, t)$ and rescaling $y \to x$ degenerates the 2DSK system (4) into the SK equation (1). As a natural (2+1)-dimensional extension of the SK equation, the 2DSK system serves as a model for an incompressible fluid, where u and v denote the components of the dimensionless velocity. Additionally, it may find applications in conformal field theory, quantum gravity, and acoustics [29,51].

The paper is structured as follows. Section 2 conjectures the existence of an N-soliton solution, based on the confirmation that at least a four-soliton solution exists with wave number constraints. The periodic soliton solution and the interaction between a line soliton and a y-periodic soliton are considered in Sect. 3. In Sect. 4, the multilinear variable separation approach is applied to construct localized excitations such as solitoffs, dromions, and dromion-like instantons. Furthermore, the general fusion or fission type N solitary wave solutions and solitary waves on a moving cnoidal wave background are obtained. Section 5 is devoted to summary and discussion.

2 Multi-soliton solution

Through the following dependent variable transformations

$$u = 2(\ln f)_{xy}, \quad v = 2(\ln f)_{xx}, \tag{5}$$

the 2DSK system (4) can be written in the bilinear form (3). According to the Hirota's bilinear method, one can construct multi-soliton solutions by applying the perturbation procedure. To this end, we expand f into power series of a small parameter ϵ as

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots , \tag{6}$$

where f_i are the analytic functions to be determined. Substituting (6) into (3) and comparing the coefficients of the same powers of ϵ yields the recursion relations for f_i. Up to the order ϵ^4, we have

$$f_{1yt} + f_{1xxxxxy} = 0, \tag{7}$$

$$f_{2yt} + f_{2xxxxxy} = -\frac{1}{2} D_y(D_x^5 + D_t) f_1 \cdot f_1, \tag{8}$$

$$f_{3yt} + f_{3xxxxxy} = -D_y(D_x^5 + D_t) f_1 \cdot f_2, \tag{9}$$

$$f_{4yt} + f_{4xxxxxy} = -\frac{1}{2} D_y(D_x^5 + D_t) f_2 \cdot f_2 - D_y(D_x^5 + D_t) f_1 \cdot f_3. \tag{10}$$

If we choose the solution to the linear differential equation (7) to be e^{ξ_1} and take $f_2 = f_3 = \cdots = 0$, we obtain a single line soliton of the 2DSK system with $\epsilon = 1$

$$u = \frac{k_1 l_1}{2}\,\mathrm{sech}^2\left(\frac{\xi_1}{2}\right), \quad v = \frac{k_1^2}{2}\,\mathrm{sech}^2\left(\frac{\xi_1}{2}\right),$$
$$\xi_1 = k_1 x + l_1 y + \omega_1 t + \xi_{10}, \quad \omega_1 = -k_1^5. \tag{11}$$

It is evident that the u component vanishes as $l_1 \to 0$, indicating that the u field is driven by a ghost soliton and thus suggesting the existence of dromion structures [25].

Similarly, a two-soliton solution of the 2DSK system (4) can be derived by taking

$$f = 1 + e^{\xi_1} + e^{\xi_2} + a_{12}e^{\xi_1+\xi_2}, \tag{12}$$

with

$$\xi_j = k_j x + l_j y + \omega_j t + \xi_{j0}, \omega_j = -k_j^5 (j = 1, 2),$$
$$a_{12} = \frac{(k_1 - k_2)(l_1 - l_2)(k_1^2 - k_1k_2 + k_2^2)}{(k_1 + k_2)(l_1 + l_2)(k_1^2 + k_1k_2 + k_2^2)}. \tag{13}$$

Subsequently, a three-soliton solution for the 2DSK system (4) can be obtained as

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{23}e^{\xi_2+\xi_3} + a_{12}a_{13}a_{23}e^{\xi_1+\xi_2+\xi_3}, \tag{14}$$

with

$$\xi_j = k_j x + l_j y + \omega_j t + \xi_{j0}, \omega_j = -k_j^5 (j = 1, 2, 3),$$
$$a_{ij} = \frac{(k_i - k_j)(l_i - l_j)(k_i^2 - k_ik_j + k_j^2)}{(k_i + k_j)(l_i + l_j)(k_i^2 + k_ik_j + k_j^2)}, (i < j). \tag{15}$$

Additionally, wave numbers need to satisfy the following algebraic equation

$$l_2 l_3 (l_2^2 - l_3^2)k_1^3 - l_1 l_3 (l_1^2 - l_3^2)k_2^3 + l_1 l_2 (l_1^2 - l_2^2)k_3^3 = 0. \tag{16}$$

The plots for the one-soliton, two-soliton, and three-soliton scenarios are shown in Fig. 1. Proceeding further, one can verify that the 2DSK system (4) possesses a four-soliton solution by taking

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_4} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{14}e^{\xi_1+\xi_4} + a_{23}e^{\xi_2+\xi_3} + a_{24}e^{\xi_2+\xi_4} + a_{34}e^{\xi_3+\xi_4} + a_{12}a_{13}a_{23}e^{\xi_1+\xi_2+\xi_3} + a_{12}a_{14}a_{24}e^{\xi_1+\xi_2+\xi_4} + a_{13}a_{14}a_{34}e^{\xi_1+\xi_3+\xi_4} + a_{23}a_{24}a_{34}e^{\xi_2+\xi_3+\xi_4} + a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}e^{\xi_1+\xi_2+\xi_3+\xi_4}, \tag{17}$$

with

$$\xi_j = k_j x + l_j y + \omega_j t + \xi_{j0},$$
$$\omega_j = -k_j^5 (j = 1, 2, 3, 4),$$
$$a_{ij} = \frac{(k_i - k_j)(l_i - l_j)(k_i^2 - k_ik_j + k_j^2)}{(k_i + k_j)(l_i + l_j)(k_i^2 + k_ik_j + k_j^2)} (i < j). \tag{18}$$

Similar to the case of the three-soliton solution, we have the following wave number constraints

$$k_1 = \frac{l_1}{l_3^2 - l_4^2}\left[\frac{l_1^2 - l_4^2}{l_3}k_3^3 - \frac{l_1^2 - l_3^2}{l_4}k_4^3\right]^{\frac{1}{3}},$$
$$k_2 = \frac{l_2}{l_3^2 - l_4^2}\left[\frac{l_2^2 - l_4^2}{l_3}k_3^3 - \frac{l_2^2 - l_3^2}{l_4}k_4^3\right]^{\frac{1}{3}}. \tag{19}$$

On the basis of these results, we can conjecture that the 2DSK system (4) may have an N-soliton solution by assuming that

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \xi_j + \sum_{1\leqslant j<l}^{N} \mu_j \mu_l a_{jl}\right], \tag{20}$$

with

$$\xi_j = k_j x + l_j y + \omega_j t + \xi_{j0}, \quad \omega_j = -k_j^5,$$
$$a_{ij} = \frac{(k_i - k_j)(l_i - l_j)(k_i^2 - k_ik_j + k_j^2)}{(k_i + k_j)(l_i + l_j)(k_i^2 + k_ik_j + k_j^2)}, (i < j), \tag{21}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_N)$, $\mu = 0, 1$ means that each μ_i takes 0 or 1. For $N \geq 3$, the wave numbers need to satisfy

$$k_j = \frac{l_j}{l_{N-1}^2 - l_N^2}\left[\frac{l_j^2 - l_N^2}{l_{N-1}}k_{N-1}^3 - \frac{l_j^2 - l_{N-1}^2}{l_N}k_N^3\right]^{\frac{1}{3}},$$
$$j = 1, 2, \cdots, N-2. \tag{22}$$

3 Interaction between a line soliton and a *y*-periodic soliton

In this subsection, an interaction solution between a line soliton and a y-periodic soliton is investigated. This scenario has been demonstrated in the asymmetric Nizhnik-Novikov-Veselov (ANNV) equation [30]. Although the multi-soliton solution introduced in the

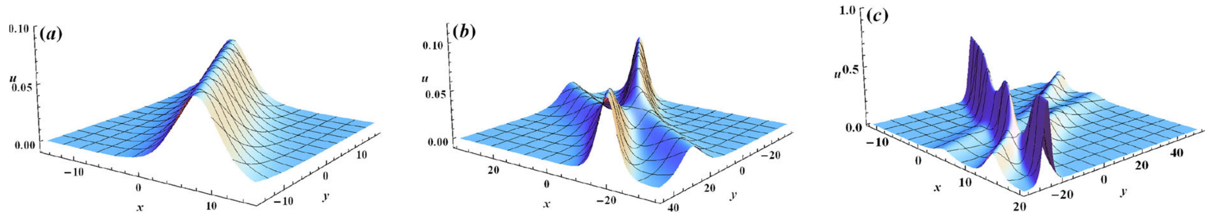

Fig. 1 Plots of soliton solutions of the u field. **a** One-soliton solution with $k_1 = 0.5$, $l_1 = 0.25$ and $x_{10} = 0$; **b** Two-soliton solution with $k_1 = 0.5$, $k_2 = 0.2$, $l_1 = 0.3$, $l_2 = 0.25$ and $x_{10} = x_{20} = 0$; **c** Three-soliton solution with $k_2 = 1.2$, $k_3 = 0.9$, $l_1 = 0.9$, $l_2 = 0.3$, $l_3 = 0.12$, $x_{10} = -x_{20} = 3$, $x_{30} = 0$, and k_1 is determined by equation (16)

previous section is derived for real-valued wave parameters, it remains applicable even when some of them are complex numbers [31–33].

For the two-soliton solution (12)–(13), if we take a special case

$$k_1 = k_2^* = \alpha_1 + i\alpha_2, \quad l_1 = l_2^* = \beta_1 + i\beta_2,$$
$$\omega_1 = \omega_2^* = \Omega_1 + i\Omega_2,$$
$$\xi_{10} = \xi_{20}^* = -\ln(K/2) + \theta_1 + i\theta_2, \tag{23}$$

then f can be simplified as

$$f = \sqrt{K}\cosh(\eta_1) + \cos(\eta_2),$$
$$\eta_j = \alpha_j x + \beta_j y + \Omega_j t + \theta_j, \quad (j = 1, 2). \tag{24}$$

Substituting (24) into Eq. (3) gives

$$\Omega_1 = -\alpha_1(\alpha_1^4 - 10\alpha_1^2\alpha_2^2 + 5\alpha_2^4),$$
$$\Omega_2 = -\alpha_2(\alpha_2^4 - 10\alpha_1^2\alpha_2^2 + 5\alpha_1^4),$$
$$K = \frac{\alpha_2\beta_2(3\alpha_2^2 - \alpha_1^2)}{\alpha_1\beta_1(3\alpha_1^2 - \alpha_2^2)}. \tag{25}$$

As a consequence, a periodic soliton solution for the v field is obtained as

$$v = \frac{2(K\alpha_1^2 - \alpha_2^2) + 2\sqrt{K}[(\alpha_1^2 - \alpha_2^2)\cosh(\eta_1)\cos(\eta_2) + 2\alpha_1\alpha_2\sinh(\eta_1)\sin(\eta_2)]}{[\sqrt{K}\cosh(\eta_1) + \cos(\eta_2)]^2}. \tag{26}$$

Obviously, the existence condition for a non-singular solution (26) is $K > 1$. A typical spatial structure of the solution (26) is shown in Fig. 2a. As can be observed, the periodic solution represents a soliton-like wave structure constructed by an inclined sequence of algebraic solitons. In a special case of $\alpha_2 = \beta_1 = 0$, the solution (26) degenerates to

$$v = \frac{2K_y\alpha_1^2[\cosh(\alpha_1 x + \Omega_1 t + \theta_1)\cos(\beta_2 y + \theta_2) + K_y]}{[K_y\cosh(\alpha_1 x + \Omega_1 t + \theta_1) + \cos(\beta_2 y + \theta_2)]^2}, \tag{27}$$

with the non-singular condition $|K_y| > 1$. As shown in Fig. 2b, the solution (27) represents a wave stationary and periodic in the y direction, while decaying exponentially along the propagation direction.

To search for an interacting solution between a line soliton and a y-periodic soliton, let us assume that

$$f = 1 + P_1 e^{2\zeta_1} + e^{\zeta_2}$$
$$+2(e^{\zeta_1} + P_3 e^{\zeta_1+\zeta_2})\cos(\eta) + Pe^{2\zeta_1+\zeta_2},$$
$$\zeta_j = k_j x + \omega_j t + \zeta_{j0} \quad (j = 1, 2), \quad \eta = \sigma y. \tag{28}$$

The substitution of (28) into Eq. (3) leads to

$$\omega_1 = -k_1^5,$$
$$\omega_2 = -5k_2(k_1^4 + 2k_1^3 k_2 + 2k_1^2 k_2^2 + k_1 k_2^3 + k_2^4)$$
$$-\frac{10k_1 k_2^2(2k_1^2 + k_2^2)}{P_3 - 1},$$
$$P = P_1 P_3^2. \tag{29}$$

Below is an asymptotic analysis for a better interpretation of the interaction behavior between these two waves. Without loss of generality, we assume $k_1 > 0$, $k_2 > 0$ and $\omega_2/k_2 > \omega_1/k_1$. When $t \to \pm\infty$, in the frame comoving with the line soliton as the exponent ζ_2 is finite, we find

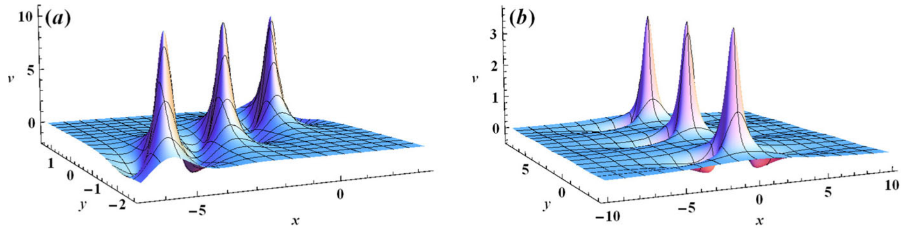

Fig. 2 **a** Plot of the periodic soliton solution (26) with $\alpha_1 = -1$, $\alpha_2 = 1.8$, $\beta_1 = 3$, $\beta_2 = 0.2$, $\theta_1 = \theta_2 = 0$; **b** The y-periodic soliton solution (27) with parameters $\alpha_1 = 0.2$, $\beta_2 = 1.5$, $K_y = 1.1$, $\theta_1 = \theta_2 = 0$

$$v_{l_-} = v\,|_{\zeta_2=c,t\to-\infty} = \frac{k_2^2}{2}\,\mathrm{sech}^2\left[\frac{\zeta_2}{2} + \ln(|P_3|)\right],$$
$$v_{l_+} = v\,|_{\zeta_2=c,t\to+\infty} = \frac{k_2^2}{2}\,\mathrm{sech}^2\left(\frac{\zeta_2}{2}\right), \quad (30)$$

while in the frame comoving with the y-periodic soliton as the exponent ζ_1 is finite, we obtain

$$v_{p_-} = v\,|_{\zeta_1=c,t\to-\infty} = \frac{4k_1^2 e^{\zeta_1}[\cos(\eta) + 2P_1 e^{\zeta_1} + P_1 e^{2\zeta_1}\cos(\eta)]}{(1 + 2e^{\zeta_1}\cos(\eta) + P_1 e^{2\zeta_1})^2},$$
$$v_{p_+} = v\,|_{\zeta_1=c,t\to+\infty} = \frac{4k_1^2 P_3 e^{\zeta_1}[\cos(\eta) + 2P_1 P_3 e^{\zeta_1} + P_1 P_3^2 e^{2\zeta_1}\cos(\eta)]}{(1 + 2P_3 e^{\zeta_1}\cos(\eta) + P_1 P_3^2 e^{2\zeta_1})^2}$$
$$= v_{p-}\left[\zeta_1 + \ln(|P_3|), \eta + \arccos\left(\frac{P_3}{|P_3|}\right)\right]. \quad (31)$$

Eqs. (30)–(31) demonstrate that the phase difference between these waves depends on the coefficient P_3. In detail, the phase shifts of the line soliton and the y-periodic soliton in the x direction are determined by the magnitude of $\ln|P_3|$, and the phase shift of the latter in the transverse direction is determined by the sign of P_3.

Interestingly, with the parameters $k_1 = 0.3$, $k_2 = 0.6$, $\sigma = 0.8$, $P_1 = 1.6$, $\zeta_{10} = \zeta_{20} = 0$, we find that the y-periodic soliton behaves like a periodic soliton as $|P_3| \to 0$, and their interaction undergoes a remarkable long-range repulsive effect. As shown in Fig. 3, the line soliton and the the y-periodic soliton both go to the right, and the line soliton travels faster. When the line soliton catches up with the periodic soliton, the line soliton swallows it, forming bumps on the peak. As a result, the hump of the periodic soliton decreases, while the trough increases and then transforms into the line soliton. During the interaction process, the two waves keep a distance from each other and seem to exchange energy and momentum through wave tails in the propagating direction. Comparing Fig. 3b, e, the y-periodic soliton seems to move backwards due to the phase shift. The comparison between Fig. 3a, f shows that the y-periodic soliton experiences a remarkable phase shift in the transverse direction for a negative P_3. On the other hand, the short-range interaction between a line soliton and a y-periodic soliton is shown in Fig. 4, where the phase shift in the y direction is invisible.

4 Variable separation solution

To employ the multilinear variable separation approach [16–22], we utilize the following Bäcklund transformations:

$$u = 2(\ln f)_{xy} + u_0, \quad v = 2(\ln f)_{xx} + v_0, \quad (32)$$

where u_0 and v_0 are arbitrary seed solutions to the 2DSK system. Substituting the Bäcklund transformation (32) with the specific seed solution $\{u_0 = 0, v_0 = v_0(x,t)\}$ into (4), and subsequently integrating with respect to x, we obtain the following bilinear equation

$$[D_y D_x^5 + 10v_0 D_y D_x^3 + 5(v_{0xx} + 3v_0^2) D_y D_x + D_y D_t + C(y,t)] f \cdot f = 0. \quad (33)$$

To separate the independent variables x and y, it is natural to explore a particular solution in the following form:

$$f = a_0 + a_1 p + a_2 q + a_3 pq. \quad (34)$$

Here a_0, a_1, a_2 and a_3 are arbitrary constants, $p = p(x,t)$ and $q = q(y,t)$ are functions of the indicated

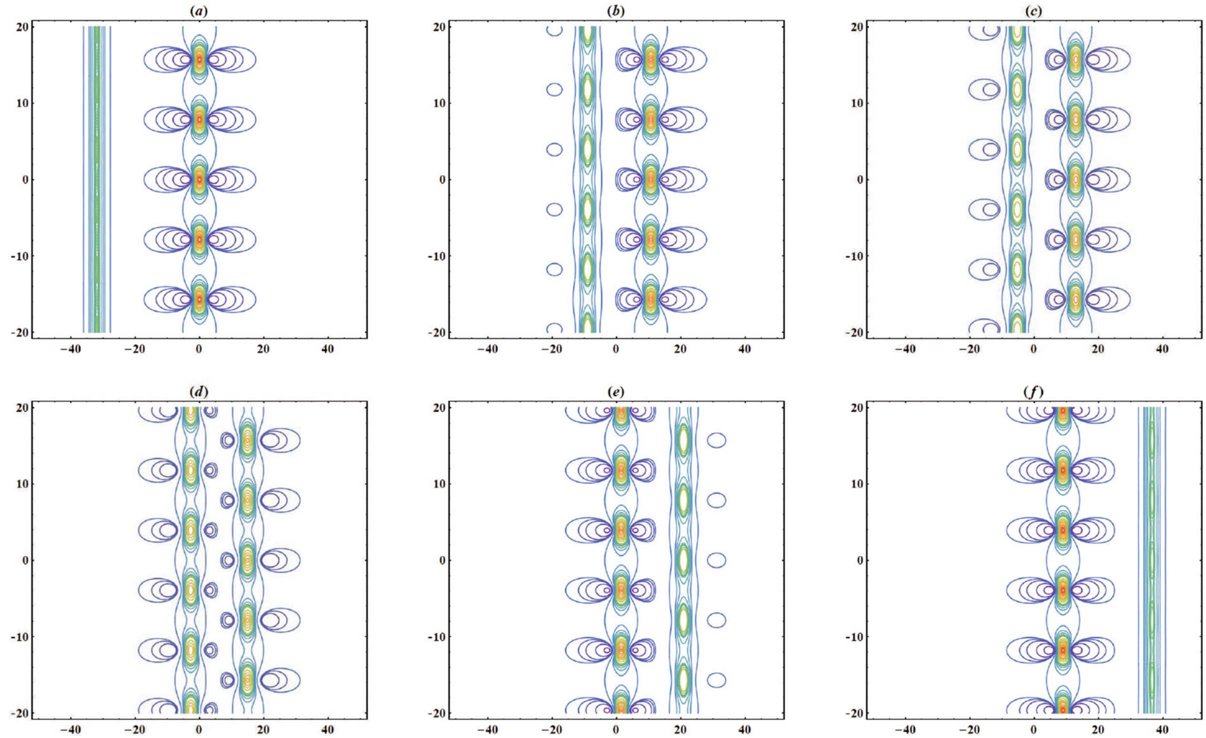

Fig. 3 Contourplot of the long-range interaction between a line soliton and a y-periodic soliton with $P_3 = -0.01$. **a** $t = -1500$; **b** $t = -500$; **c** $t = -270$; **d** $t = -90$; **e** $t = 300$; and **f** $t = 1200$

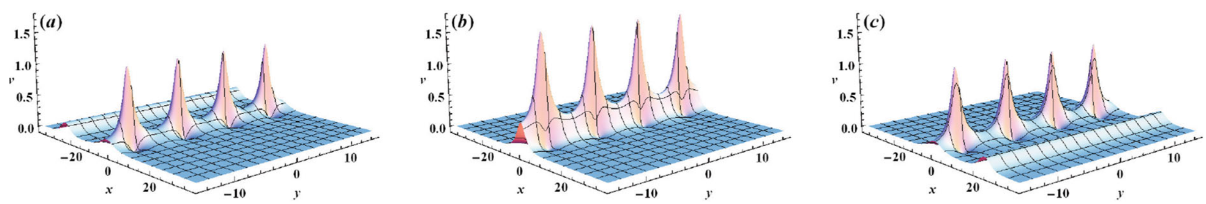

Fig. 4 Space-time evolution of the interaction between a line soliton and a y-periodic soliton with parameters $k_1 = 0.35$, $k_3 = 0.6$, $\sigma = 0.8$, $P_1 = 1.6$, $P_3 = 2$. **a** $t = -10$, **b** $t = 0$, and **c** $t = 10$

variables. A direct substitution of (34) into (33) gives

$$(a_0a_3 - a_1a_2)[p_t + p_{xxxxx} + 10v_0p_{xxx} + 15v_0^2p_x + 5p_xv_{0xx}] - (a_2 + a_3p)^2q_t + q_y^{-1}\Big[(a_2 + a_3p)(a_0 + a_1p + a_2q + a_3pq)q_{yt} + \frac{1}{2}C(y,t)(a_0 + a_1p + a_2q + a_3pq)^2\Big] = 0. \tag{35}$$

By imposing the constraint

$$C(y,t) = -2(a_3^2c_1 - a_2a_3c_2 + a_2^2c_3)q_y, \tag{36}$$

Eq. (35) can be separated into the following two equations:

$$p_t = -p_{xxxxx} - 5p_xv_{0xx} - 10v_0p_{xxx} - 15v_0^2p_x + A_0(c_2p^2 + c_1p + c_0), \tag{37}$$

$$q_t = c_0(a_1 + a_3q)^2 - c_1(a_1 + a_3q)(a_0 + a_2q) + c_2(a_0 + a_2q)^2, \tag{38}$$

where $A_0 = a_0a_3 - a_1a_2$ and $c_i = c_i(t)$ for $i = 0, 1, 2$ are arbitrary functions of t introduced by the variable separation process. The separated equations (37)–(38) are not completely independent. Instead, they are connected by separation functions c_i. The first-order nonlinear equation (38) is known as the Riccati equation, and its solution is given by [17]

$$q = \frac{A_1}{A_3 + F(y)} + A_2, \tag{39}$$

where $A_i = A_i(t)$, $i = 1, 2, 3$ and $F = F(y)$ are considered as arbitrary functions of the indicated variables while c_0, c_1, and c_2 are related to A_1, A_2 and A_3 by

$$c_0 = \frac{a_2^2 A_{2t}}{A_0^2} - \frac{a_2(a_0+a_2A_2)A_{1t}}{A_0^2 A_1} - \frac{(a_0+a_2A_2)^2 A_{3t}}{A_0^2 A_1}, \tag{40}$$

$$c_1 = \frac{2a_2a_3A_{2t}}{A_0^2} - \frac{(a_0a_3+a_1a_2+2a_2a_3A_2)A_{1t}}{A_0^2 A_1} - \frac{2(a_0+a_2A_2)(a_1+a_3A_2)A_{3t}}{A_0^2 A_1}, \tag{41}$$

$$c_2 = \frac{a_3^2 A_{2t}}{A_0^2} - \frac{a_3(a_1+a_3A_2)A_{1t}}{A_0^2 A_1} - \frac{(a_1+a_3A_2)^2 A_{3t}}{A_0^2 A_1}. \tag{42}$$

To conclude, we have the following theorem on solutions to the 2DSK system (4).

Theorem 1 *If $\{p, q\}$ is a solution of the separated Eqs. (37) and (38), then $\{u, v\}$ given by*

$$u = \frac{2A_0 p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \tag{43}$$

$$v = -\frac{2(a_1 + a_2 q)^2 p_x^2}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} + \frac{2(a_1 + a_2 q) p_{xx}}{a_0 + a_1 p + a_2 q + a_3 pq} + v_0. \tag{44}$$

presents a class of exact solutions to the 2DSK system (4).

So far, we have shown that the 2DSK system is solvable in the sense of the multilinear variable separation approach. As expected, the physical field u is exactly expressed by the universal formula applicable to all multilinear variable separable systems. To construct localized excitations of the u field, which is of particular interest, one can take the function p as arbitrary, and Eq. (37) can be satisfied in principle thanks to the arbitrary seed solution v_0. As an alternative, we would like to present here the explicit solutions for both the u and the v fields in some special but important cases.

4.1 Fusion or fission type N-solitary wave solution

Assuming that c_i and v_0 statisfy

$$c_0 = \frac{1}{A_0}\frac{T_{2t}}{T_1}, \quad c_1 = -\frac{1}{A_0}\frac{T_{1t}}{T_1}, \quad c_2 = v_0 = 0,$$

$$T_1 = T_1(t), \quad T_2 = T_2(t). \tag{45}$$

Under these conditions, the solution to the separated equations (37)–(38) can be directly obtained as

$$p = \frac{1}{T_1}\sum_i^N (e^{\zeta_i} + T_2), \quad q = -\frac{a_0T_1 - a_1(Y - T_2)}{a_2T_1 - a_3(Y - T_2)},$$
$$Y = Y(y), \quad \zeta_i = k_i x - k_i^5 t + x_{i0}. \tag{46}$$

Substituting (46) into (43)–(44), the t-dependent functions $\{T_1, T_2\}$ are omitted and the solutions can be rewritten in a more concise manner

$$u = -\frac{2Y_y \sum_i^N k_i e^{\zeta_i}}{(Y + \sum_i^N e^{\zeta_i})^2},$$
$$v = \frac{2Y \sum_i^N k_i^2 e^{\zeta_i} + 2\sum_{1\le i<j}^N (k_i - k_j)^2 e^{\zeta_i+\zeta_j}}{(Y + \sum_i^N e^{\zeta_i})^2}. \tag{47}$$

Obviously, by choosing the appropriate arbitrary function Y, different types of solitary wave structures can be generated. For example, the one-soliton solution (11) can be reconstructed by setting $N = 1$ and $Y = e^{-l_1 y}$. Similarly, if we set $N = 1$ and $Y = 1+e^{-ly}$, the solution (47) will degenerate to

$$u = \frac{2k_1 l e^{\xi_1}}{(1 + e^{ly} + e^{\xi_1})^2},$$
$$v = \frac{2k_1^2 e^{\xi_1}(1 + e^{ly})}{(1 + e^{ly} + e^{\xi_1})^2}, \quad \xi_1 = k_1 x + ly - k_1^5 t + x_{10}. \tag{48}$$

Here, the u component illustrates a solitary solitoff structure [34], as depicted in Fig. 5a, while the v component represents a V-shaped soliton, as shown in Fig. 5b. Additionally, Fig. 5c, d exhibit the periodic soliton structure of the u field and the corresponding spaced-curved line soliton of the v field with $N = 1$ and $Y = e^{-y+\frac{5}{2}\cos(y)}$.

By taking $Y = L + e^{-ly}$, one obtain the general fusion or fission type N-solitary wave solution [35–37]

$$u = \frac{2l \sum_i^N k_i e^{\xi_i}}{(1 + Le^{ly} + \sum_i^N e^{\xi_i})^2},$$
$$v = \frac{2(1 + Le^{ly}) \sum_i^N k_i^2 e^{\xi_i} + 2\sum_{1\le i<j}^N (k_i - k_j)^2 e^{\xi_i+\xi_j}}{(1 + Le^{ly} + \sum_i^N e^{\xi_i})^2},$$
$$\xi_i = k_i x + ly - k_i^5 t + x_{i0}. \tag{49}$$

Fission or fusion phenomena can be observed in both the u and v fields for $N \geqslant 2$. Figure 6 depicts the fission phenomenon in the v field with parameters:

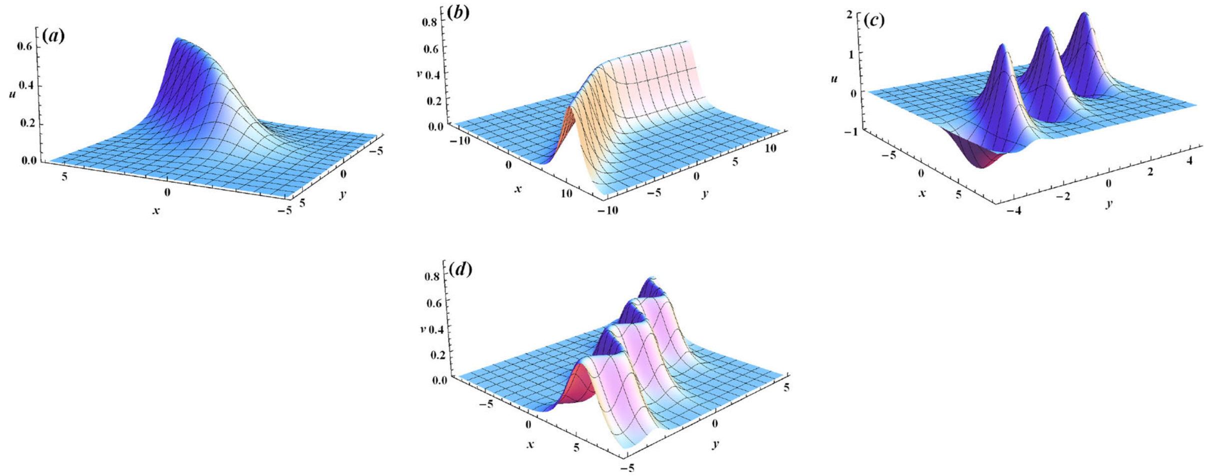

Fig. 5 Plots illustrating localized excitations with different choices of Y. **a** Single solitoff structure; **b** The corresponding V-shaped soliton; **c** Periodic soliton structure; **d** Space-curved line soliton

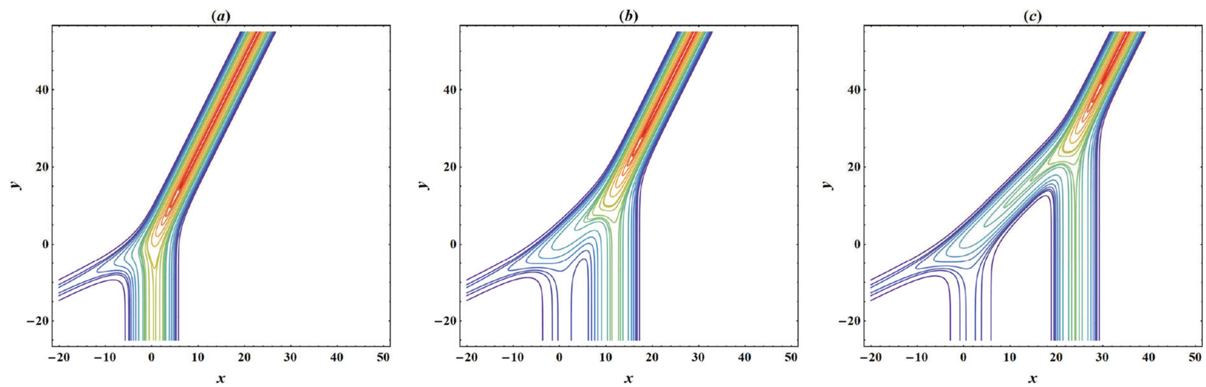

Fig. 6 Contourplot of the fission phenomenon in a Y-shaped soliton. **a** $t = 0$; **b** $t = 3$; **c** $t = 6$

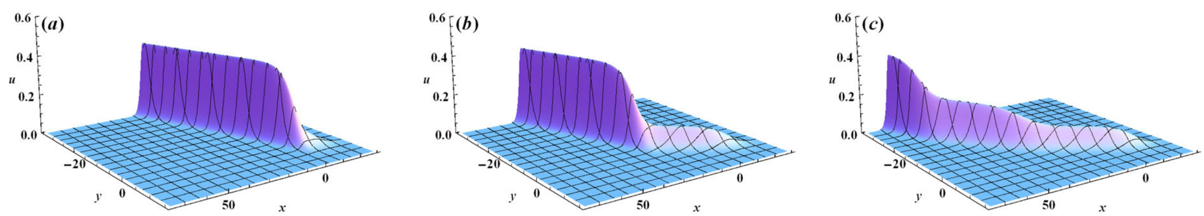

Fig. 7 Plots of the u component for $N = 3$ with parameters are $k_1 = 1$, $k_2 = 0.5$, $k_3 = 0.25$, $l = 0.8$, $x_{10} = 185$, $x_{20} = x_{30} = 0$ and $L = 0.02$. **a** $t = 180$; **b** $t = 200$; **c** $t = 220$

$N = 3$, $k_1 = 1.2$, $k_2 = 0.6$, $k_3 = 0.3$, $l = -0.5$, and $x_{10} = x_{20} = x_{30} = L = 0$. Initially, only one Y-shaped soliton can be observed in Fig. 6a at time $t = 0$. As time evolves, the line solitary wave propagating along the x direction splits into two line waves, as shown in Fig. 6b. Finally, two Y-shaped solitons may be observed in Fig. 6c. The opposite process, the fusion phenomenon, can be generated by setting $k_1 = -1.2$, $k_2 = -0.6$, $k_3 = -0.3$ with other parameters unchanged.

For $L > 0$, the resonant behavior of multisolitoff can be observed. At time $t = 180$, only one solitoff

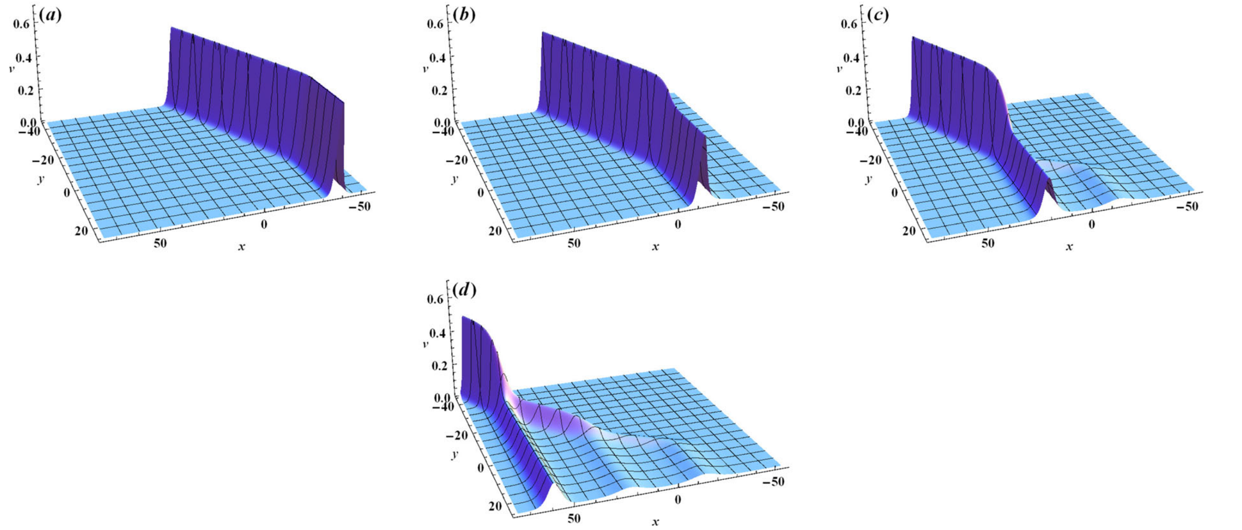

Fig. 8 The corresponding v field. **a** $t = 150$; **b** $t = 175$; **c** $t = 200$; **d** $t = 220$

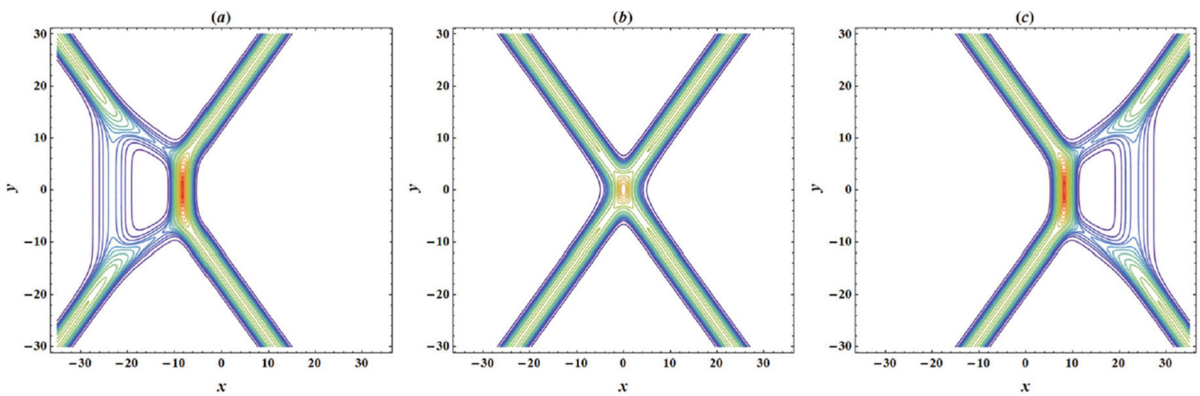

Fig. 9 Contourplots of the space-time evolution of a resonant two-soliton solution for the v field with parameters $(k_1, k_2, k_3, k_4, l) = (1, -1, 0.5, -0.5, 0.8)$ and $x_{i0} = 0$ for $i = 1, 2, 3, 4$. **a** $t = -12$; **b** $t = 0$; **c** $t = 12$

structure, with the end appearing to flatten slightly, is visible in Fig. 7a. As time progresses, a second solitoff, followed by a third, emerges sequentially from the end of the first, as depicted in Fig. 7b, c. This interesting phenomenon can be further elucidated by examining the corresponding v field in Fig. 8.

By setting $N = 4$, $k_1 = -k_2$, $k_3 = -k_4$, and $Y = \cosh(ly)$ in Eq. (47), we derive the resonant two-soliton interaction solution, as illustrated in Fig. 9. The interaction pattern exhibits reflection symmetry $(x, y, t) \to (-x, -y, -t)$ and is commonly referred to as a T-type interaction in shallow water [38,39].

4.2 Dromion structures and instanton-like excitations

To construct the dromion structure that is exponentially localized in all directions in the x–y plane for the u component, we assume

$$Y = \frac{a_0 + a_2 e^{\eta}}{a_1 + a_3 e^{\eta}}, \quad \eta = ly. \tag{50}$$

Substitution of Y into (47), the expression of u becomes

$$u = \frac{2A_0 l e^{\eta} \sum_i^N k_i e^{\zeta_i}}{[a_0 + a_2 e^{\eta} + (a_1 + a_3 e^{\eta}) \sum_i^N e^{\zeta_i}]^2},$$
$$\zeta_i = k_i x - k_i^5 t + x_{i0}, \quad \eta = ly. \tag{51}$$

A single dromion solution can be immediately obtained by taking $N=1$

$$u=\frac{2A_0k_1le^{\zeta_1+\eta}}{(a_0+a_1e^{\zeta_1}+a_2e^{\eta}+a_3e^{\zeta_1+\eta})^2}. \tag{52}$$

Obviously, the single dromion solution (52) will degenerate to a straight line soliton when $a_1=a_2=0$ or $a_0=a_3=0$, and may degenerate to a solitoff structure with one of $\{a_0,a_1,a_2,a_3\}$ equal to zero. To characterize the dromion structure, one can define its maximum amplitude and mass [40]. Letting the partial derivatives u_x and u_y be zero, we find that the tip of the dromion is at the critical point where

$$e^{\zeta_1}=\frac{a_0+a_2\beta}{a_1+a_3\beta},\quad e^{\eta}=\beta,\quad \beta=\sqrt{\frac{a_0a_1}{a_2a_3}}, \tag{53}$$

which result in the maximum amplitude

$$u_{max}=\frac{k_1l}{2A_0}(\sqrt{a_0a_3}-\sqrt{a_1a_2})^2. \tag{54}$$

Fig. 10 exhibits the single dromion structure and the corresponding v field with parameters $a_0=a_3=9$, $a_1=a_2=k_1=1$, $l=2$, and $x_{10}=0$. From Eq. (54), $u_{\max}$ is calculated to be 0.8, which is consistent with the figure.

Taking $N=2$, the solution for the u field becomes

$$u=\frac{2A_0le^{\eta}(k_1e^{\zeta_1}+k_2e^{\zeta_2})}{[a_0+a_2e^{\eta}+(a_1+a_3e^{\eta})(e^{\zeta_1}+e^{\zeta_2})]^2}. \tag{55}$$

Without loss of generality, we carry out the asymptotic analysis of the solution (55) for two different cases. The first case is the spatio-temporal evolution of a single resonant dromion, as shown in Fig. 11. With $k_1>k_2>0$, one can determine the limiting values of u as $t\to\pm\infty$

$$\begin{aligned}
u\big|_{\zeta_1=c,t\to-\infty}&=\frac{2A_0k_1le^{\zeta_1+\eta}}{(a_0+a_1e^{\zeta_1}+a_2e^{\eta}+a_3e^{\zeta_1+\eta})^2},\\
u\big|_{\zeta_1=c,t\to+\infty}&=0,
\end{aligned} \tag{56}$$

$$\begin{aligned}
u\big|_{\zeta_2=c,t\to-\infty}&=0,\quad u\big|_{\zeta_2=c,t\to+\infty}\\
&=\frac{2A_0k_2le^{\zeta_2+\eta}}{(a_0+a_1e^{\zeta_2}+a_2e^{\eta}+a_3e^{\zeta_2+\eta})^2}.
\end{aligned} \tag{57}$$

It can be interpreted from Eqs. (56)–(57) that the solution depicts a single dromion structure moving with velocity k_1^4 when $t\ll 0$, and characterizes a dromion moving with smaller velocity k_2^4 when $t\gg 0$.

The second case is the annihilation process of two dromions, as shown in Fig. 12. For $k_1>0>k_2$ and $k_1<|k_2|$, the limits of u at $t\to\pm\infty$ can be given by

$$\begin{aligned}
u\big|_{\zeta_1=c,t\to-\infty}&=\frac{2A_0k_1le^{\zeta_1+\eta}}{(a_0+a_1e^{\zeta_1}+a_2e^{\eta}+a_3e^{\zeta_1+\eta})^2},\\
u\big|_{\zeta_1=c,t\to+\infty}&=0,
\end{aligned} \tag{58}$$

$$\begin{aligned}
u\big|_{\zeta_2=c,t\to-\infty}&=\frac{2A_0k_2le^{\zeta_2+\eta}}{(a_0+a_1e^{\zeta_2}+a_2e^{\eta}+a_3e^{\zeta_2+\eta})^2},\\
u\big|_{\zeta_2=c,t\to+\infty}&=0.
\end{aligned} \tag{59}$$

Take $k_1>0>k2$, and $k_1>|k_2|$, the opposite process can be observed, namely an up-dromion and a down-dromion are generated on the plane.

For $N=3$, an interesting instanton-like excitation can be observed when $k_1>|k_2|>k_3>0$, and $k_2<0$. Such a process can be readily identified by applying the asymptotic analysis

$$\begin{aligned}
&u\big|_{\zeta_1=c,t\to\pm\infty}=0,\quad u\big|_{\zeta_2=c,t\to\pm\infty}=0,\\
&u\big|_{\zeta_3=c,t\to\pm\infty}=0.
\end{aligned} \tag{60}$$

A schematic diagram of the instanton evolution is shown in Fig. 13a, c for several time intervals. We can see that the instanton initially increases with t, peaking at $t=0$, and then decays rapidly as t continues to increase. The amplitude of the instanton decays tremendously to 10^{-8} at time $t=7$.

4.3 Solitary wave solutions on cnoidal wave background

Recently, some important new results have been obtained in the study of localized excitations on a periodic wave background. In particular, the single soliton standing on the elliptic function background is constructed by the localization of nonlocal symmetry method [41], the consistent Tanh expansion method [42], or the consistent Riccati expansion method [43]. In addition, the rogue waves on the elliptic function background are obtained by combining the Darboux transformation with the nonlinearization of the Lax pair [44–46]. Using the multilinear variable separation approach, it is also possible, under certain conditions, to construct a solitary wave standing on a moving cnoidal wave background.

Under the condition $c_1=c_2=0$ and $c_0=\frac{T_t}{a_3A_0}$, one can easily identify that the following selection of p,q

$$\begin{aligned}
&p=\frac{1}{a_3}(T+e^{\zeta}),\quad q=-\frac{a_1}{a_3}+\frac{A_0}{a_3(Y-a_2-T)},\\
&\zeta=kx+\omega t+x_0,
\end{aligned} \tag{61}$$

where $Y=Y(y)$ and $T=T(t)$, is a solution of Eqs. (37)–(38) if and only if v_0 satisfies the ordinary differ-

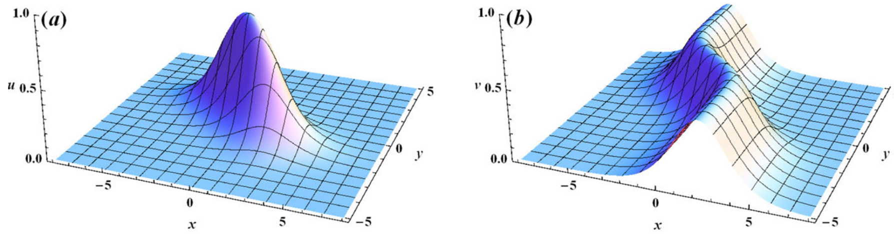

Fig. 10 **a** Dromion structure of the u field with parameters $k_1 = 1$, $l = 2$, $x_{10} = 0$, $a_0 = a_3 = 3$, and $a_1 = a_2 = 1$. **b** The corresponding v field

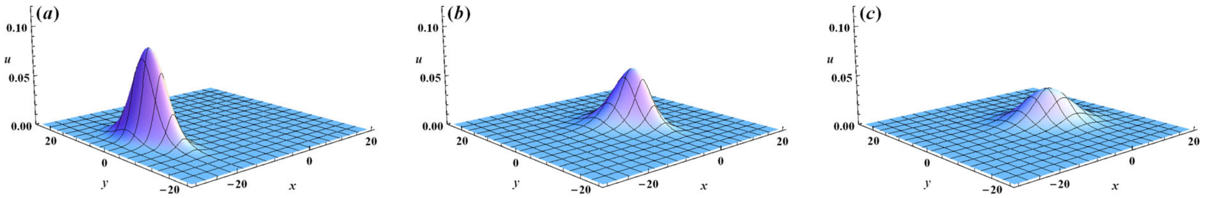

Fig. 11 Space-time evolution of a resonant dromion with parameters $k_1 = 0.9$, $k_2 = 0.35$, $l = 0.4$, $x_{10} = x_{20} = 0$, $a_0 = a_3 = 3$ and $a_1 = a_2 = 1$. **a** $t = -30$, **b** $t = 0$, and **c** $t = 100$

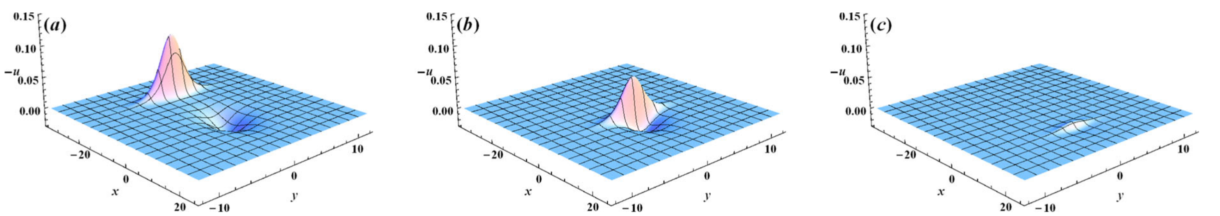

Fig. 12 Annihilation of two dromions with parameters $k_1 = 0.3$, $k_2 = -1.2$, $l = a_1 = a_2 = a_3 = 1$, $a_2 = 2$ and $x_{10} = x_{20} = 0$. **a** $t = -10$; **b** $t = 0$; **c** $t = 6$

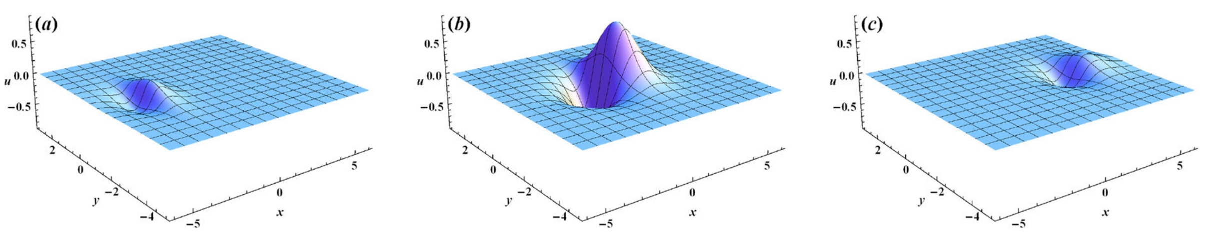

Fig. 13 Space-time evolution of the instanton-like excitation with parameters $k_1 = 2$, $k_2 = -1.5$, $k_3 = a_1 = a_2 = 1$, $l = 2$, $a_0 = a_3 = 9$ and $x_{10} = x_{20} = x_{30} = 0$. **a** $t = -0.3$; **b** $t = 0$; **c** $t = 0.9$

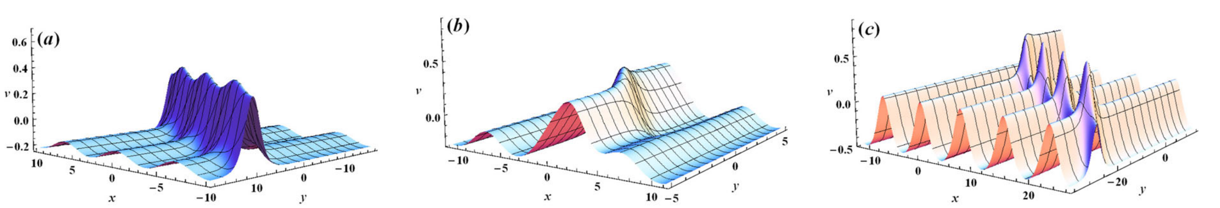

Fig. 14 Different types of solitary waves on a moving cnoidal wave background. **a** A straight line soliton; **b** A twisted line soliton propagating along x-axis; **c** A V-shaped soliton

ential equation

$$v_{0xx} + 3v_0^2 + 2k^2 v_0 + \frac{1}{5}\left(\frac{\omega}{k} + k^4\right) = 0. \quad (62)$$

To determine the solution of the prescribed Eq. (62), we apply the Jacobi elliptic function expansion method [47,48]. Balancing the highest nonlinearity and dispersive terms in the Eq. (62), we assume that its solution has the following form

$$v_0 = \alpha_0 + \alpha_1 \mathrm{sn}(Kx + \Omega t, m) + \alpha_2 \mathrm{sn}^2(Kx + \Omega t, m). \quad (63)$$

Substitution of the assumption (63) into Eq. (62) and setting the coefficients of the different powers of the Jacobi elliptic functions to zero, we obtain a group of overdetermined equations. In solving these overdetermined equations, if we take k, K and m as arbitrary, a nontrivial solution of $\{\omega, \alpha_0, \alpha_1, \alpha_2\}$ can be determined as

$$\alpha_0 = \frac{1}{3}(2m^2K^2 + 2K^2 - k^2),\ \alpha_1 = 0,\ \alpha_2 = -2mK^2,$$
$$\omega = \frac{2}{3}k^5 - \frac{20}{3}kK^4(m^4 - m^2 + 1). \quad (64)$$

By combining the results with Eqs. (43)–(44), we obtain

$$u = -\frac{2kY_y e^{\zeta}}{(Y + e^{\zeta})^2},$$
$$v = \frac{2k^2 Y e^{\zeta}}{(Y + e^{\zeta})^2} + \alpha_0 + \alpha_1 \mathrm{sn}^2(Kx + \Omega t, m). \quad (65)$$

It should be pointed out that the u field has a different dispersion relation from the solution in (47). Regarding the v field, various types of solitary waves on a cnoidal wave background can be constructed by choosing the arbitrary function Y appropriately. For instance, by setting $Y = e^{-y}$, $k = 1$, $x_0 = 0$, $K = 0.5$, and $m = 0.4$, a straight-line soliton on a moving cnoidal wave background is obtained, as shown in Fig. 14a. Furthermore, a twisted line soliton propagating along the x-axis can be observed in Fig. 14b by selecting $Y = (9+e^{2y})/(1+9e^{2y})$, $k = 1$, $x_0 = 0$, $K = 0.5$, and $m = 0.6$. Additionally, a V-shaped soliton on a cnoidal wave background, as depicted in Fig. 14c, can be constructed with $Y = 2/(1 + 2e^{-y})$, $k = -1$, $x_0 = 0$, $K = 0.6$, and $m = 0.9$. Under the ultra-limit condition $m = 0$ and $K = \sqrt{2}|k|/2$, the background wave in the v field disappears and the solution (65) degenerates to (47) with $N = 1$.

5 Summary and discussions

In conclusion, a new (2+1)-dimensional generalization of the SK system is proposed. By employing the Hirota bilinear method, the existence of three-soliton and four-soliton solutions, with constraints on wave numbers $\{k_i, l_i\}$, is confirmed. Furthermore, the multilinear variable separation approach is extended to the 2DSK system to generate a variable separation solution. It is found that the arbitrary function q is determined by a nonlinear Riccati equation, and the physical field u is precisely expressed by the universal formula applicable to all multilinear variable separable systems. Compared to recent results [49–52], several patterns of localized excitations are constructed for the first time using the variable separation approach, such as the T-type soliton interaction in shallow water, the instanton excited by three resonant dromions, and solitary waves on a moving cnoidal wave background.

Based on our study, we can conjecture that the nonlinear version of the following bilinear system

$$D_y(D_x^{2n+1} + D_t) f \cdot f = 0, \quad (n \geq 1)$$

is solvable under the multilinear variable separation approach. It is noted that $n = 1$ and $n = 2$ correspond to the $(2+1)$-dimensional ANNV equation [17] and the 2DSK system, respectively.

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Data Availibility Our manuscript has no associated data.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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