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Mécanique des fluides/*Fluid Mechanics*

On second harmonic generation and the onset of simultaneous capillary-gravity solitary waves (simultons or quadratic solitons)

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Abstract. We show that using a water channel as a wave guide the second harmonic generation (SHG) of capillary-gravity waves is possible. We also provide the appropriate parameter and geometry values and cascading process permitting the simultaneous solitonic behavior of both the fundamental and second harmonic waves (simultons or quadratic solitons). © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

nonlinear hydrodynamic waves / second harmonic resonance / quadratic solitons / simultons

Sur la génération d'harmoniques d'ordre deux et des simultons ou solitons quadratiques

Résumé. *Nous montrons sous quelles conditions la génération d'harmoniques d'ordre deux (SHG) est possible dans un canal d'eau pris comme guide des ondes capillaro-gravitationnelles. Nous montrons aussi comment un choix adéquat des valeurs des paramétres et de la géometrie du canal permet l'apparition simultanée, comme des solitons, du mode fondamental et du second harmonique d'où l'appelation solitons quadratiques ou simultons. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

ondes non linéaires / résonance de second harmonique / solitons quadratiques / simultons

1. Introduction

Second harmonic generation (SHG), a degenerate case of three-wave resonance, wave modulation instabilities, solitary waves or hydraulic jumps and eventually solitons in the surface of a water channel are fascinating phenomena [1]. SHG in a water layer with surface tension was first investigated by Simmons [2], McGoldrick [3,4] and Nayfeh [5]. McGoldrick [4] pointed out that in Wilton's theory of ripples [6] the appearance of singularities in the Stokes expansion for large-amplitude capillary-gravity waves corresponds to a SHG and some higher-order resonances.

In existing SHG theory of liquid layers of finite depth, dispersion has not been considered and hence the corresponding coupled amplitude equations obtained for the envelopes of fundamental and second

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harmonic waves only include first-order time- and space derivatives. Such approach is only valid for fairly long wavetrains, thus precluding a cascading process between two wave modes. Here we show that a new type of nonlinear surface wave excitations, i.e., simultaneous solitary waves (quadratic solitons or simultons), may result, when the dispersion of the system plays a significant role, in a SHG process. Simultons occur through cascading when two simultaneously soliton-like wave components interact with themselves through repeated three-wave interaction processes; e.g., the fundamental wave is first upconverted to the second harmonic wave and then downconverted.

SHG and simultons have also been studied in nonlinear optical media [7–10]. In the original SHG theory in optical media possessing, e.g., quadratic nonlinearity, neither dispersion nor diffraction were considered in deriving coupled amplitude equations for fundamental and second harmonic waves and hence theory was only valid for long pulses or wide beams with no cascading process being possible between two wave modes. Consideration of dispersion and diffraction has permitted the study of the cascading process leading to simultons [7–10]. Stationary self supported two-component localized beams (spatial simultons) and propagating self-supported two-component localized pulses (temporal simultons) have been observed [9,10].

Clearly, these phenomena, and hence simultons, are expected to appear in other realms of science and engineering when wave guides are endowed with dispersion, diffraction, eventually dissipation and indeed, appropriate nonlinearity.

2. Basic equations and phase-matching conditions for SHG and the onset of hydrodynamic simultons

We consider the irrotational motion of an incompressible inviscid liquid of density ρ in a gravitational field. The fluid at rest fills a horizontal rectangular channel to the depth d with $-d < z < 0$, where z is the vertical coordinate, b is the width along the transverse coordinate y, and of infinite extent along the other transverse coordinate, x. The surface of the liquid is open to ambient air, deformable and with nonvanishing surface tension, α. Air is considered hydrodynamically passive, as its (dynamic) shear viscosity is two orders of magnitude below, say, that of water. The velocity potential ϕ of the fluid satisfies the Laplace equation:

$$\nabla^2\phi = 0 \quad \text{for } -d < z < \zeta(x,y,t) \tag{1}$$

with the boundary conditions $\phi_y = 0$ on two sides $y = 0, b$ and $\phi_z = 0$ on the bottom of the channel $z = -d$. The boundary conditions on the free surface, $z = \zeta(x,y,t)$, are:

$$\zeta_t + \phi_x\zeta_x + \phi_y\zeta_y = \phi_z \tag{2}$$

$$\phi_t + g\zeta + \frac{1}{2}(\nabla\phi)^2 = \sigma\frac{\zeta_{xx}(1+\zeta_y^2) + \zeta_{yy}(1+\zeta_x^2) - 2\zeta_x\zeta_y\zeta_{xy}}{(1+\zeta_x^2+\zeta_y^2)^{3/2}} \tag{3}$$

with $\sigma = \alpha/\rho$.

Then for capillary-gravity waves the linear dispersion relation is $\omega^2 = k(g+\sigma k^2)\tanh(kd)$, where k and ω are the wavenumber and frequency, respectively. $k^2 = k_x^2 + k_y^2$, where k_x is an arbitrary real number and $k_y = n\pi/b$, n is an integer, hence $\omega = \omega(k_x, k_y) = \omega(k_x, n)$. Consequently, there is multiplicity of dispersion branches denoted by the index n. For the nth branch there exists a lower cutoff frequency $\omega_{nc} = \omega(0,n)$. Nonpropagating solitary waves and hydraulic jumps related to the (0, 1)-mode (i.e., $k_x = 0$ and $n = 1$) have been discovered in the wave guide which is the water channel [11,12].

Now we look for possible hydrodynamic simulton excitations in the system. Accordingly, we consider two wave modes for which a SHG can occur. A necessary kinematic condition for the SHG is the phase-matching:

$$\boldsymbol{k}_2 = 2\boldsymbol{k}_1, \qquad \omega(\boldsymbol{k}_2) = 2\omega(\boldsymbol{k}_1) \tag{4}$$

where $\boldsymbol{k}_1$ ($\boldsymbol{k}_2$) is the wavevector corresponding to the fundamental (second harmonic) wave. Besides (4) there is an additional necessary condition for exciting the hydrodynamic simultons, i.e., the group-velocity matching:

$$\boldsymbol{v}_g(\boldsymbol{k}_2) = \boldsymbol{v}_g(\boldsymbol{k}_1) \tag{5}$$

where $\boldsymbol{v}_g(\boldsymbol{k}) = \partial\omega/\partial\boldsymbol{k}$ is the group velocity of the corresponding mode. Modes satisfying the condition (5) are the cutoff modes of the system, i.e., $\boldsymbol{k} = (0, n\pi/b)$. These modes have in fact equal (zero) group velocity. To meet the condition (4), we chose, for simplicity, $\boldsymbol{k}_1 = (0, k_1)$ and $\boldsymbol{k}_2 = (0, 2k_1)$ with $k_1 = \pi/b$. Then (4) requires:

$$f(k_1^*) \equiv \tanh^2 k_1^* - \frac{3(k_1^*/k_0)^2}{1+(k_1^*/k_0)^2} = 0$$

where $k_1^* = k_1 d$ and $k_0 = d(g/\sigma)^{1/2}$. The function $f(k_1^*)$ has two zero points. The first one is at $k_1^* = 0$, alien to the SHG. The other is at a nonvanishing value of k_1^* and hence corresponds to the sought SHG. The condition (4) imposes a constraint on the parameters of the system. For example for water at room temperature we have $\rho_{\mathrm{water}} = 1\ \mathrm{g}\cdot\mathrm{cm}^{-3}$ and $\alpha_{\mathrm{water}} = 72.5\ \mathrm{dyne}\cdot\mathrm{cm}^{-1}$. Taking $d = 2\,\mathrm{cm}$ and $g = 980\ \mathrm{cm}\cdot\mathrm{s}^{-2}$, we obtain $k_0 = 7.35$. Then the dimensionless wavenumber corresponding to the SHG is $k_1^* = 5.20$. Thus the realization of the SHG requires that the width parameter b $(= \pi d/k_1^*)$ equals to 1.2083 cm. Note that in the limit of an infinitely deep water layer, we have the exact solution $k_1 = [g/(2\sigma)]^{1/2}$, hence $\omega_1 = [9g^3/(8\sigma)]^{1/4}$, corresponding to $b = (2\sigma/g)^{1/2}\pi = 1.20843\,\mathrm{cm}$. Thus if the channel has a width b near 1.21 cm and depth $d \geqslant 2\,\mathrm{cm}$, the phenomena described here should be experimentally observable.

3. SHG dynamics using the method of multiple scales

Since we are interested in a cascading process with the excitation width smaller than that in the usual SHG case, we scale variables and use a suitable asymptotic expansion to characterize the evolution of the amplitudes of the fundamental and the second harmonic waves. We set:

$$\xi = \epsilon^{1/2}(x - v_g t), \qquad \tau = \epsilon t$$
$$\phi = \epsilon\big(\phi^{(0)} + \epsilon^{1/2}\phi^{(1)} + \epsilon\phi^{(2)} + \cdots\big) \quad \text{and} \quad \zeta = \epsilon\big(\zeta^{(0)} + \epsilon^{1/2}\zeta^{(1)} + \epsilon\zeta^{(2)} + \cdots\big)$$

where ϵ is the smallness parameter measuring the deformation of the wavy surface, v_g is a velocity to be determined in the solution of the problem, $\phi^{(j)}$ and $\zeta^{(j)}$ $(j = 0, 1, 2, \ldots)$ are functions of the fast variables x, y, z and t and the slow variables ξ and τ. Then equations (1)–(3) become:

$$\nabla^2\phi^{(j)} = P^{(j)} \quad \text{for } -d < z < \zeta \tag{6}$$

with $\phi_y^{(j)} = 0$ at $y = 0, b$ and $\phi_z^{(j)} = 0$ at $z = -d$, together with:

$$\zeta_t^{(j)} - \phi_z^{(j)} = Q^{(j)} \tag{7}$$

$$\phi_t^{(j)} + g\zeta^{(j)} - \sigma\big(\zeta_{xx}^{(j)} + \zeta_{yy}^{(j)}\big) = R^{(j)} \tag{8}$$

at $z = 0$, with $j = 0, 1, 2, \ldots$. The explicit expressions for $P^{(j)}$, $Q^{(j)}$ and $R^{(j)}$ are not needed here.

At leading order ($j = 0$) the solution when including only the $(0, 1)$ mode (considered the fundamental wave) and the $(0, 2)$ mode (second harmonic wave) reads:

$$
\begin{aligned}
\phi^{(0)} = &\frac{\cosh k_1(z+d)}{\cosh k_1 d}\cos k_1 y\{A_1(\xi,\tau)\exp(-\mathrm{i}\omega_1 t)+c.c.\}\\
&+\frac{\cosh k_2(z+d)}{\cosh k_2 d}\cos k_2 y\{A_2(\xi,\tau)\exp(-\mathrm{i}\omega_2 t)+c.c.\}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\zeta^{(0)} = &\,\mathrm{i}(k_1T_1/\omega_1)\cos k_1 y\{A_1(\xi,\tau)\exp(-\mathrm{i}\omega_1 t)-c.c.\}\\
&+\mathrm{i}(k_2T_2/\omega_2)\cos k_2 y\{A_2(\xi,\tau)\exp(-\mathrm{i}\omega_2 t)-c.c.\}
\end{aligned}
\tag{10}
$$

with $k_1=\pi/b$, $k_2=2k_1=2\pi/b$, $\omega_l^2=(g+\sigma k_l^2)k_lT_l$ with $T_l=\tanh k_l d$ $(l=1,2)$. $c.c.$ denotes the complex conjugate term. A_l $(l=1,2)$ are amplitude functions corresponding, respectively, to the $(0,1)$ and $(0,2)$ modes yet to be determined. The condition for the SHG requires $\omega_2=2\omega_1$, which corresponds to $f(k_1^*)=0$, discussed above.

At the next order ($j=1$), the solvability condition demands that v_g vanishes, as the modes we have chosen in (9) and (10) are two cutoff modes corresponding to the dispersion branches $\omega(k_x,\pi/b)$ and $\omega(k_x,2\pi/b)$, respectively. At the third order $(j=2)$, we obtain two solvability conditions:

$$
\mathrm{i}\left(\frac{\partial u_1}{\partial t}+v_1\frac{\partial u_1}{\partial x}\right)+\frac{1}{2}\mathit{\Gamma}_1\frac{\partial^2 u_1}{\partial x^2}+\mathrm{i}\Delta_1 u_1^* u_2\exp(\mathrm{i}\delta\omega t)=0 \tag{11}
$$

$$
\mathrm{i}\left(\frac{\partial u_2}{\partial t}+v_2\frac{\partial u_2}{\partial x}\right)+\frac{1}{2}\mathit{\Gamma}_2\frac{\partial^2 u_2}{\partial x^2}-\mathrm{i}\Delta_2 u_1^2\exp(-\mathrm{i}\delta\omega t)=0 \tag{12}
$$

when returning to original variables, where:

$$
u_l=\epsilon A_l,\quad \mathit{\Gamma}_l=\frac{1}{\omega_l}\left\{\frac{\omega_l^2}{2k_l^2T_l}\left[T_l+k_ld(1-T_l^2)\right]+\sigma k_lT_l\right\}\quad (l=1,2)
$$

$$
\Delta_1=\frac{k_1^2}{2}\left[1-\frac{3T_1T_2}{2}+\frac{2T_1+T_2}{2T_1}\right],\quad \Delta_2=\frac{k_2^2}{4}\left[\frac{1-3T_1^2}{4}+\frac{T_1}{T_2}\right]
$$

v_1 and v_2 are, respectively, the group velocities of the fundamental and the second harmonic waves near $k_{x1}=0$, $n=1$ and $k_{x2}=0$, $n=2$. In deriving equations (11) and (12), a small frequency mismatch is allowed in the SHG, i.e., $\omega_2=2\omega_1+\delta\omega$ where $\delta\omega$ is a small quantity of order ϵ. Similar equations have been obtained by Karamzin and Sukhorukov [13] in nonlinear optics. An important difference between equations (11) and (12) and the amplitude equations obtained by McGoldrick [3] and Nayfeh [5] for the usual SHG is that here we have included dispersion. Dispersion is known to drastically affect the evolution of nonlinear water waves. In addition, our amplitude equations (11) and (12) are also valid for infinitely deep water (i.e., for $d\to\infty$ and hence $T_l\to 1$, $l=1,2$).

Let us obtain exact solutions of the amplitude equations (11) and (12). We look for $u_l(x,t)=U_l(\eta)\exp(\mathrm{i}\theta_l)$, where $\eta=Kx-\Omega t$, $\theta_l=K_lx-\Omega_lt+\phi_l$ $(l=1,2)$ with $K_2=2K_1$, $\Omega_2=2\Omega_1+\delta\omega$, $\phi_2=2\phi_1-\pi/2$. K, Ω, K_1, Ω_1 and ϕ_1 are constants yet to be determined. Then equations (11) and (12) are transformed into:

$$
U_{1\eta\eta}+\alpha_1U_1U_2-\beta_1U_1=0 \tag{13}
$$

$$
U_{2\eta\eta}+\alpha_2U_1^2-\beta_2U_2=0 \tag{14}
$$

with

$$
\alpha_l=\frac{2\Delta_l}{\mathit{\Gamma}_lK^2},\quad \beta_l=\frac{\mathit{\Gamma}_lK_l^2-2(\Omega_l-v_lK_l)}{\mathit{\Gamma}_lK^2}\quad (l=1,2)
$$

$$
K_1=\frac{v_2-v_1}{\mathit{\Gamma}_1-2\mathit{\Gamma}_2}\quad\text{and}\quad \Omega=v_1K+\mathit{\Gamma}_1KK_1
$$

A coupled soliton-soliton (i.e., simultaneous solitons for two wave components) solution of equations (13) and (14) reads:

$$U_1 = \frac{6s_1}{\sqrt{\alpha_1\alpha_2}}\mathrm{sech}^2\eta, \qquad U_2 = \frac{6}{\alpha_1}\mathrm{sech}^2\eta \tag{15}$$

with $s_1 = \pm 1$ and:

$$K^2 = \frac{2(v_2 - v_1)K_1 + (2\Gamma_2 - \Gamma_1)K_1^2 - \delta\omega}{2(\Gamma_2 - 2\Gamma_1)}$$

In this case the free surface displacement of the fluid is:

$$\begin{aligned}\zeta(x,y,t) = &-\frac{12s_1}{\sqrt{\alpha_1\alpha_2}}\frac{k_1T_1}{\omega_1}\mathrm{sech}^2(Kx - \Omega t)\cos k_1y\sin\big[K_1x - (\omega_1 + \Omega_1)t + \phi_1\big]\\ &+\frac{12}{\alpha_1}\frac{k_2T_2}{\omega_2}\mathrm{sech}^2(Kx - \Omega t)\cos k_2y\cos\big[K_2x - (\omega_2 + \Omega_2)t + 2\phi_1\big]\end{aligned} \tag{16}$$

where ϕ_1 is a phase constant depending on the initial condition. The first and second part of the right hand side of (16) correspond to the fundamental and second harmonic wave component respectively. We see that each component of the excitation is a standing wave in the y-direction and a bright (above level) envelope soliton in the x-direction, and hence the solution (16) is a hydrodynamic simulton. If k_{x1} (k_{x2}) is exactly zero but with $\delta\omega \neq 0$, one has $v_1 = v_2 = 0$. Consequently, one has $K_1 = K_2 = 0$, $K^2 = \delta\omega/(4\Gamma_1 - 2\Gamma_2)$, $\Omega = 0$, $\Omega_1 = -2\Gamma_1K^2$ and $\Omega_2 = -2\Gamma_2K^2$. Then (16) represents a hydrodynamic *nonpropagating* simulton, in which the oscillating frequencies of both the fundamental and second harmonic waves are smaller than the lower cutoff frequencies of the corresponding linear modes.

Equations (13) and (14) admit also another type of simulton solution with both the fundamental and the second harmonic waves being dark (below level) envelope solitons in the x-direction.

4. Conclusion

We have investigated the conditions for the second harmonic generation (SHG) in nonlinear surface water waves in a long, rectangular channel filled with a liquid to finite depth and open to ambient air, hence taking into consideration the surface tension. We have shown that with wave dispersion included and group-velocity matching condition satisfied a new type of solitonic nonlinear capillary-gravity excitation can be obtained through cascading between two wave-modes.

At variance with the usual solitons in deep water, for which the formation mechanism is self-trapping of one linear plane wave [14], the mechanism studied here is a cascading effect between two wave modes. In this process, the fundamental and the second harmonic waves interact with themselves through repeated three-wave interactions. For example, the energy of the fundamental wave is first upconverted to the second harmonic wave and then downconverted, resulting in a mutual self-trapping of each wave thus leading to the appearance of two simultaneous hydrodynamic solitons and which are hence called quadratic solitons or simultons.

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