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Fluctuations of the condensate in ideal and interacting Bose gases

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Abstract

We investigate the fluctuations of the condensate in the ideal and weakly interacting Bose gases confined in a box of volume V within a canonical ensemble. A canonical ensemble is developed to describe the behaviour of the fluctuations when different methods of approximation to the weakly interacting Bose gases are used. Research shows that the fluctuations of the condensate exhibit anomalous behaviour for the interacting Bose gas confined in a box.

1. Introduction

The experimental achievement of Bose–Einstein condensation (BEC) in dilute alkali atoms [1], spin-polarized hydrogen [2] and recently in metastable helium [3] has enormously stimulated theoretical research [4] into ultracold bosons. In particular, fluctuations $\langle \delta^2 N_0 \rangle$ of the mean ground state occupation number N_0 have been recently thoroughly investigated in a series of papers. Apart from the intrinsic theoretical interest, it is foreseeable that such fluctuations will become experimentally testable in the near future [5].

It is well known that within the grand canonical ensemble the fluctuations of the condensate are given by $\langle \delta^2 N_0 \rangle = N_0 (N_0 + 1) \sim V^2$, implying that δN_0 becomes of order N when the temperature approaches zero. To avoid these sorts of unphysically large condensate fluctuations, a canonical (or microcanonical) ensemble has to be used to investigate the fluctuations of the condensate. Within microcanonical and canonical ensembles, the fluctuations of the condensate have been studied in a systematic way in the case of the ideal Bose gas [6–13]. Recently, the question of how interatomic interactions affect the fluctuations of the condensate has been the object of several theoretical investigations [14–19]. Giorgini *et al* [14] found the anomalous behaviour of the fluctuations in a weakly interacting Bose gas confined in a box within the traditional particle-number-non-conserving Bogoliubov approach. In [14] the fluctuation of the condensate follow the law $\langle\delta^2 N_0\rangle \sim V^{4/3}$. However, Idziaszek *et al* [15] considered that the fluctuations are proportional to the volume. Recently, Kocharovsky *et al* [18] supported and extended the results of the work of Giorgini *et al* [14] using the particle-number-conserving operator formalism.

Although the correction to the ground state occupation number due to interatomic interaction has been clearly discussed within a grand canonical ensemble [20] and a canonical ensemble [21], the role of the interaction on the condensate fluctuations of the weakly interacting Bose gas is still an open and unsolved problem. Different from the ground state occupation number, different models of describing the weakly interacting Bose gas will lead to vastly different predictions concerning the fluctuations of the condensate.

The purpose of this paper is to present a unified method of calculating the fluctuations of the condensate when different methods of approximation to the weakly interacting Bose gases are used. Within the canonical ensemble we give the distribution function of the ground state occupation number for the ideal and interacting boson system in a box. We obtain the fluctuations of the condensate from the distribution function. In particular, we found that the distribution function is not a Gaussian function in the case of an interacting boson system in a box. The paper is organized as follows. Section 2 is devoted to outlining the canonical ensemble, which is developed to discuss the fluctuations of the condensate for the ideal Bose gas in a box. In section 3 we investigate the fluctuations of the interacting Bose gas based on the lowest-order perturbation theory. In section 4 the fluctuations are calculated based on the Bogoliubov theory. Finally, we give a discussion and summary of the results in section 5.

2. Mean ground state occupation number and fluctuations in the ideal Bose gases

Let us start our investigation on the fluctuations of the ideal Bose gases within the framework of a canonical ensemble. According to the canonical ensemble the partition function of the N non-interacting bosons in a box is given by

$$Z_{\text{ideal}}[N] = \sum_{\sum N_{\boldsymbol{n}}=N} \exp\left[-\beta\left\{\sum N_{\boldsymbol{n}}\varepsilon_{\boldsymbol{n}}\right\}\right] \tag{1}$$

where $N_{\boldsymbol{n}}$ and $\varepsilon_{\boldsymbol{n}}$ are the occupation numbers and energy level of the state $\boldsymbol{n} = \{n_x, n_y, n_z\}$, respectively. $\beta = 1/k_{\text{B}}T$. In (1) the energy level of the system takes the form

$$\varepsilon_{\boldsymbol{n}} = \frac{\pi^2\hbar^2\left(n_x^2+n_y^2+n_z^2\right)}{2mL^2}. \tag{2}$$

Separating out the ground state $\boldsymbol{n} = 0$ from the state $\boldsymbol{n} \neq 0$, we have

$$Z_{\text{ideal}}[N] = \sum_{N_0=0}^{N} \{\exp\left[-\beta N_0\varepsilon_0\right] Z_0\left(N-N_0\right)\} \tag{3}$$

where $Z_0\left(N-N_0\right)$ denotes the partition function of a fictitious system comprising $N-N_0$ non-interacting bosons. Assuming $A_0\left(N-N_0\right)$ denotes the free energy of the fictitious system,

$$A_0\left(N-N_0\right) = -k_{\text{B}}T\ln Z_0\left(N-N_0\right). \tag{4}$$

From (3) and (4) the partition function $Z_{\text{ideal}}[N]$ becomes

$$Z_{\text{ideal}}[N] = \sum_{N_0=0}^{N} \exp\left[q\left(N, N_0\right)\right] \tag{5}$$

where $q(N, N_0) = -\beta N_0\varepsilon_0 - \beta A_0(N - N_0)$. Obviously $\exp[q(N, N_0)]/Z_{\rm ideal}[N]$ represents the probability of finding N_0 atoms in the condensate. We will give the distribution function of the ground state occupation number in the following.

Let us first investigate the largest term in the sum of the partition function $Z_{\rm ideal}[N]$. Assume the number of condensed atoms is N_0^p in the largest term of the partition function $Z_{\rm ideal}[N]$. The largest term $Z_0\left(N - N_0^p\right)$ is determined by the requirement that $\frac{\partial}{\partial N_0} q(N, N_0)\,|_{N_0=N_0^p} = 0$, i.e.

$$-\beta\varepsilon_0 - \beta\frac{\partial}{\partial N_0^p} A_0\left(N - N_0^p\right) = 0. \tag{6}$$

The calculation of the free energy $A_0\left(N - N_0^p\right)$ is non-trivial because there is a requirement that the number of particles should be $N - N_0^p$ in the summation of the partition function $Z_0\left(N - N_0^p\right)$. Using the saddle-point method developed by Darwin and Fowler [22] it is straightforward to obtain the free energy $A_0\left(N - N_0^p\right)$ of the fictitious system.

$$A_0\left(N - N_0^p\right) = \left(N - N_0^p\right) k_{\rm B}T \ln z_0^p - V\frac{k_{\rm B}T}{\lambda^3} g_{5/2}\left(z_0^p\right) \tag{7}$$

where $\lambda = \sqrt{2\pi\beta\hbar^2/m}$ is the thermal wavelength. z_0^p is the fugacity of the $N - N_0^p$ non-interacting bosons and is determined by the equation

$$N - N_0^p = \sum_{n\neq 0} \frac{1}{\exp[\varepsilon_n/k_{\rm B}T]\left(z_0^p\right)^{-1} - 1} = \frac{V}{\lambda^3} g_{3/2}\left(z_0^p\right). \tag{8}$$

From (7) and (8) one finds

$$-\beta\frac{\partial}{\partial N_0^p} A_0\left(N - N_0^p\right) = \ln z_0^p. \tag{9}$$

Combining (6) and (9) one obtains $\ln z_0^p = \beta\varepsilon_0$. Therefore, the most probable value N_0^p is determined by

$$N_0^p = N - \sum_{n\neq 0} \frac{1}{\exp[(\varepsilon_n - \varepsilon_0)/k_{\rm B}T] - 1}. \tag{10}$$

N_0^p is exactly the mean occupation number of the condensate atoms in the frame of the grand canonical ensemble. For sufficiently large N, the sum $\sum_{N_0=0}^{N}$ in (3) may be replaced by the largest term, for the error in doing so will be statistically negligible. In this case, (10) shows the equivalence between the canonical ensemble and grand canonical ensemble for large N. From (10), below the critical temperature, N_0^p is determined by

$$N_0^p = N - \frac{V}{\lambda^3}\zeta\left(\tfrac{3}{2}\right) = N\left(1 - \left(\frac{T}{T_{\rm c}^0}\right)^{3/2}\right) \tag{11}$$

where

$$T_{\rm c}^0 = \frac{2\pi}{\left[\zeta\left(\frac{3}{2}\right)\right]^{2/3}} \frac{\hbar^2}{mk_{\rm B}} \left(\frac{N}{V}\right)^{2/3}$$

is the transition temperature of the ideal Bose gas.

Other terms in (5) will contribute to the fluctuations of the condensate, and lead to the deviation of the mean occupation number $\langle N_0 \rangle$ from the most probable value N_0^p. When $N_0 \neq N_0^p$, $\frac{\partial}{\partial N_0} q(N, N_0) \neq 0$. Assuming

$$\frac{\partial}{\partial N_0} q(N, N_0) = \alpha(N, N_0) \tag{12}$$

and repeating the saddle-point method, one obtains

$$N_0 = N - \sum_{n \neq 0} \frac{1}{\exp[(\varepsilon_n - \varepsilon_0)/k_B T] \exp[-\alpha(N, N_0)] - 1}. \tag{13}$$

From (11) and (13) one obtains

$$\alpha(N, N_0) = -\frac{\lambda^6}{V^2} \frac{\left(N_0 - N_0^p\right)^2}{4\pi} \theta\left(N_0 - N_0^p\right) \tag{14}$$

where $\theta\left(N_0 - N_0^p\right)$ is a Sign function. $\theta\left(N_0 - N_0^p\right) = 1$ when $N_0 > N_0^p$, and $\theta\left(N_0 - N_0^p\right) = -1$ when $N_0 < N_0^p$. To obtain (14) we have used the expansions $g_{3/2}(1 - \delta) \approx \zeta\left(\frac{3}{2}\right) - 2\sqrt{\pi\delta}$ [23] and the approximation $\exp[-\alpha(N, N_0)] \approx 1 - \alpha(N, N_0)$. From (12) and (14) one easily obtains the following result for $q(N, N_0)$:

$$q(N, N_0) = \int_{N_0^p}^{N_0} \alpha(N, N_0)\, dN_0 + q\left(N, N_0^p\right) = -\frac{\lambda^6}{12\pi V^2} \left|N_0 - N_0^p\right|^3 + q\left(N, N_0^p\right). \tag{15}$$

The partition function $Z_{\text{ideal}}[N]$ is thus

$$Z_{\text{ideal}}[N] = \sum_{N_0=0}^{N} \left\{\exp\left[q\left(N, N_0^p\right)\right] G_{\text{ideal}}(N, N_0)\right\} \tag{16}$$

where we have introduced a distribution function $G_{\text{ideal}}(N, N_0)$,

$$G_{\text{ideal}}(N, N_0) = \exp\left[-\frac{\lambda^6}{12\pi V^2} \left|N_0 - N_0^p\right|^3\right]. \tag{17}$$

Assuming $P(N_0|N)$ is the probability of finding N_0 atoms in the condensate, the distribution function $G_{\text{ideal}}(N, N_0)$ represents the ratio $P(N_0|N)/P\left(N_0^p|N\right)$, i.e. the relative probability of finding N_0 atoms in the condensate. From (16) and (17) one obtains the mean occupation number $\langle N_0 \rangle$ and fluctuations $\left\langle \delta^2 N_0 \right\rangle$ within the canonical ensemble,

$$\langle N_0 \rangle = \frac{\sum_{N_0=0}^{N} N_0 G_{\text{ideal}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{ideal}}(N, N_0)} \tag{18}$$

$$\left\langle \delta^2 N_0 \right\rangle = \left\langle N_0^2 \right\rangle - \langle N_0 \rangle^2 = \frac{\sum_{N_0=0}^{N} N_0^2 G_{\text{ideal}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{ideal}}(N, N_0)} - \left(\frac{\sum_{N_0=0}^{N} N_0 G_{\text{ideal}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{ideal}}(N, N_0)}\right)^2. \tag{19}$$

From (11) and (17)–(19) it is easy to obtain $\langle N_0 \rangle$ and $\left\langle \delta^2 N_0 \right\rangle$ of the non-interacting Bose gases in a box. At the critical temperature T_c^0, $N_0^p = 0$. Thus

$$G_{\text{ideal}}\left(T = T_c^0\right) = \exp\left[-\frac{\lambda_0^6}{12\pi V^2} N_0^3\right]$$

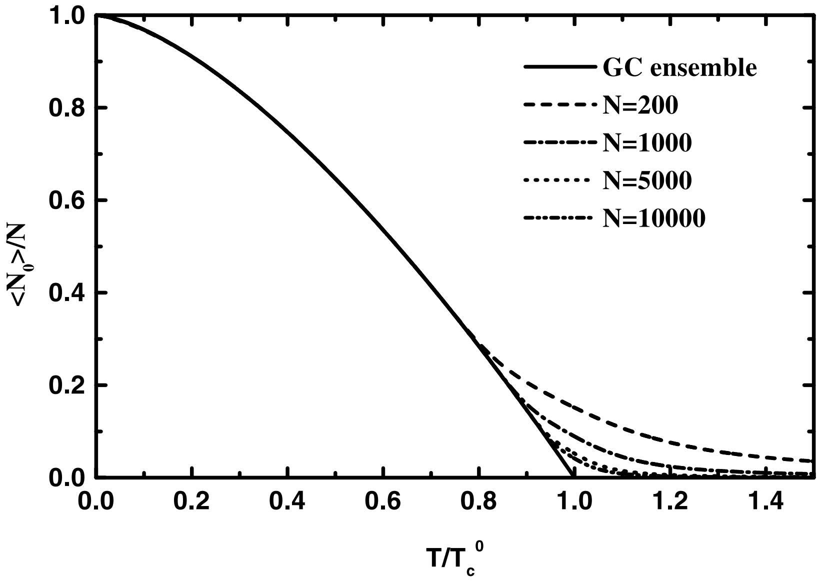

Figure 1. Temperature dependence of the mean ground state occupation number for an ideal Bose gas confined in a box within the canonical ensemble. The full curve shows $\langle N_0\rangle /N$ within the grand canonical ensemble (or N_0^p/N in the canonical ensemble). When $N \to \infty$, the mean ground state occupation number of the canonical ensemble coincides with that of the grand canonical ensemble.

where λ_0 is the thermal wavelength at T_c^0. From (18) and (19) one obtains the analytical result for the condensate fluctuations at T_c^0:

$$\left\langle \delta N_0^2\right\rangle_{T=T_c^0} = \left[\frac{1}{3\Gamma\left(\frac{4}{3}\right)} - \left(\frac{\Gamma\left(\frac{5}{3}\right)}{2\Gamma\left(\frac{4}{3}\right)}\right)^2\right]\left(\frac{12\pi}{\lambda_0^6}\right)^{2/3} V^{4/3} \tag{20}$$

where $\Gamma(n) = \int_0^\infty \mathrm{e}^{-t} t^{n-1}\,\mathrm{d}t$ is the Gamma function. $\Gamma\left(\frac{4}{3}\right) = 0.893$ and $\Gamma\left(\frac{5}{3}\right) = 0.903$. Equation (20) clearly shows that there is anomalous behaviour for the fluctuations of the condensate. When $T \to 0$, from (17) one finds $G_{\mathrm{ideal}}(N, N_0) \to 0$ when $N_0 \neq N$. Therefore, when $T \to 0$ one obtains $\langle N_0\rangle \to N$ and $\langle \delta^2 N_0\rangle \to 0$.

In figure 1 we plot $\langle N_0\rangle /N$ as a function of temperature for the ideal Bose gases in a box. The full curve displays the mean ground state occupation number within the grand canonical ensemble (or N_0^p). When $N > 10^4$, the mean ground state occupation number of the canonical ensemble agrees well with that of the grand canonical ensemble. Obviously, in the case of $N \to \infty$, the mean ground state occupation number of the canonical ensemble coincides with that of the grand canonical ensemble.

In figure 2 we plot numerical result of $\delta N_0 = \sqrt{\left\langle \delta^2 N_0\right\rangle}$ (thick full curve) for the ideal Bose gas with $N = 1000$ atoms confined in a three-dimensional box. The broken line shows the result of [14], where the fluctuations are given by

$$\left\langle \delta^2 N_0\right\rangle = 2A\left(\frac{mk_{\mathrm{B}}T}{\hbar^2}\right)^2 V^{4/3}. \tag{21}$$

The coefficient in (21) is $A = 2/\pi^4 \times \Sigma_{n\neq 0} 1/\boldsymbol{n}^4 = 0.105$. The broken line is larger than our result because of the approximation in [14]. In [14] $\left\langle \delta^2 N_0\right\rangle = \sum_{n\neq 0} f_n^2$, where

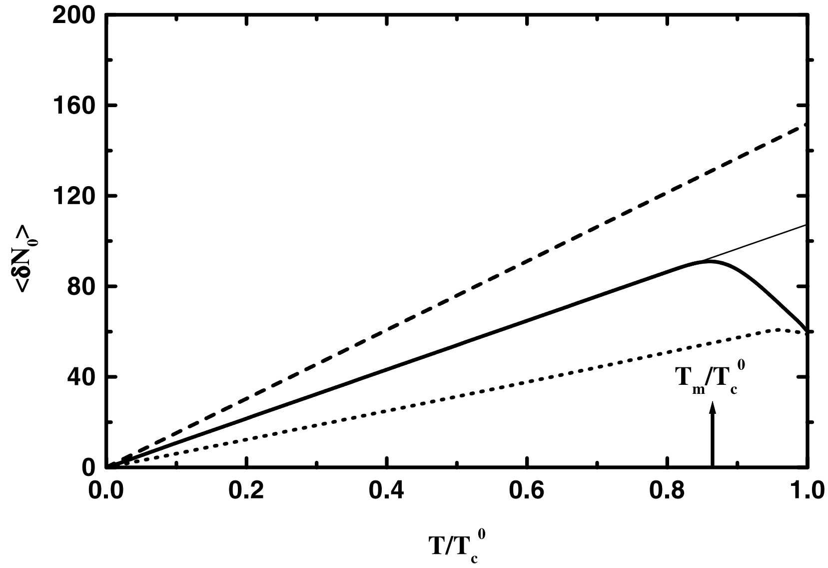

Figure 2. Temperature dependence of δN_0 for the ideal Bose gas confined in a box. The thick full curve displays the numerical result of (19), while the thin full line shows the analytical result (22) for δN_0 below T_m (The arrow marks T_m which corresponds to the maximum condensate fluctuations.) The broken line is obtained from (21) which comes from [14]. The dotted curve displays the numerical result of Wilkens and Weiss [9].

$f_n = (\exp(\varepsilon_n/k_B T) - 1)^{-1}$. For the convenience of calculations, f_n is approximated as $\varepsilon_n/k_B T$ for low-energy atoms. However, this approximation is also used for the atoms whose energy level is larger than $k_B T$. Obviously, this approximation will lead to the fluctuations becoming larger. On the other hand, equation (21) holds in the canonical ensemble except near and above T_c, while our analysis also holds for the temperature near T_c^0. In figure 2 the dotted curve shows the numerical result of Wilkens *et al* [9].

In figure 2 the arrow marks the temperature T_m which corresponds to the maximum fluctuations $\langle\delta^2 N_0\rangle_{\max}$. Below the temperature T_m, from (19), one obtains the analytical result for the fluctuations of the condensate.

$$\langle\delta^2 N_0\rangle = A\left(\frac{m_{k_B T}}{\hbar^2}\right)^2 V^{4/3}. \tag{22}$$

It is interested to find that the coefficient differs by a factor two, compared with (21). The thin full line shows (22) in figure 2.

We should note that our results are reliable, although the disputable saddle-point method is used to investigate the fluctuations of the condensate. It is well known that the applicability of the saddle-point approximation for the condensed Bose gases has been the subject of a long debate [7, 24]. Recently, the analysis in [12] showed that the fluctuations are overestimated, and do not appear to vanish properly with temperature using the usual saddle-point method. Our discussions for fluctuations are reasonable for two reasons.

(a) As proved in [13], the free energy (7) of the non-interacting Bose gases is still correct, even when dealing carefully with the failure of the standard saddle-point method below the critical temperature.
(b) In the usual statistical method $\langle N_0\rangle$ and $\langle\delta^2 N_0\rangle$ are obtained through the first and second partial derivatives of the partition function, respectively.

When the saddle-point approximation is used to calculate the partition function of the system, the error will be overestimated in the second partial derivative of the partition function so that we cannot obtain correct fluctuations of the condensate with the usual method. However, in this paper we have used the reliable result (7). The distribution function of the ground state occupation number is obtained directly from (7), without resorting to the second partial derivative of the partition function. $\langle N_0 \rangle$ and $\langle \delta^2 N_0 \rangle$ are obtained from the distribution function in this paper.

3. Fluctuations of the condensate based on the lowest-order perturbation theory

In the case of interacting Bose gases, the role of interactions on the fluctuations of the condensate is still an open and unsolved problem. Giorgini *et al* [14] predicted the anomalous behaviour of the fluctuations in a weakly interacting Bose gas confined in a box, while Idziaszek *et al* [15] considered that the fluctuations are normal. Research has shown that different approximation models for the interacting Bose gases will lead to different predictions concerning the fluctuations of the condensate. The method developed to obtain the fluctuations of the ideal Bose gas in this paper can be used straightforwardly to discuss the fluctuations of the interacting Bose gases when different models of approximation are adopted.

Let us first discuss the fluctuations of the condensate in the case of the lowest-order perturbation theory, which is also discussed in [15]. In terms of the lowest-order perturbation theory, the partition function of the system within the canonical ensemble is given by

$$Z_{\mathrm{int}}[N] = \sum_{\sum N_{\boldsymbol{n}}=N} \exp\left[-\beta\left(\sum N_{\boldsymbol{n}}\varepsilon_{\boldsymbol{n}} + E_{\mathrm{int}}\right)\right] \tag{23}$$

where the interaction energy of the system takes the form [25, 26]

$$E_{\mathrm{int}} = \frac{4\pi a\hbar^2}{mV}\left(N^2 - \tfrac{1}{2}N_0^2\right). \tag{24}$$

In (24) a is the scattering length. Separating out the ground state $\boldsymbol{n} = 0$ from the state $\boldsymbol{n} \neq 0$, one obtains the following form for the partition function:

$$Z_{\mathrm{int}}[N] = \sum_{N_0=0}^{N} \{\exp\left[-\beta N_0\varepsilon_0 - \beta E_{\mathrm{int}}\right] Z_0\left(N - N_0\right)\} \tag{25}$$

where $Z_0\left(N - N_0\right)$ denotes the partition function of a fictitious $N - N_0$ non-interacting bosons. Using the free energy $A_0\left(N - N_0\right)$ of the fictitious system, the partition function is thus

$$Z_{\mathrm{int}}[N] = \sum_{N_0=0}^{N} \exp\left[q\left(N, N_0\right)\right] \tag{26}$$

where $q\left(N, N_0\right)$ takes the form

$$q\left(N, N_0\right) = -\beta N_0\varepsilon_0 - \beta E_{\mathrm{int}} - \beta A_0\left(N - N_0\right). \tag{27}$$

Analogously to the case of ideal Bose gases, let us first investigate the largest term in the sum of $Z_{\mathrm{int}}[N]$. The largest term is determined by the requirement

$$\frac{\partial}{\partial N_0} q\left(N, N_0\right)|_{N_0=N_0^p} = 0.$$

Therefore, one obtains the most probable value N_0^p of the interacting bosons.

$$N_0^p = N - \sum_{n\neq 0} \frac{1}{\exp\left[\beta\varepsilon_n\right]\left(z_0^p\right)^{-1} - 1} \tag{28}$$

where z_0^p is determined by

$$\ln z_0^p = \beta\varepsilon_0 + \beta\frac{\partial}{\partial N_0^p} E_{\rm int} = \beta\varepsilon_0 - \frac{2a\lambda^2 N_0^p}{V}. \tag{29}$$

From (28) and (29) one obtains

$$N_0^p \simeq N - \frac{V}{\lambda^3}\left[\zeta\left(\tfrac{3}{2}\right) - 2\sqrt{\pi}\left(\frac{2a\lambda^2 N_0^p}{V}\right)^{1/2}\right]. \tag{30}$$

Other terms in (26) will contribute to the fluctuations of the system. Assuming $\frac{\partial}{\partial N_0} q\left(N, N_0\right) = \alpha\left(N, N_0\right)$, one obtains the result for N_0

$$N_0 = N - \sum_{n\neq 0} \frac{1}{\exp\left[\beta\varepsilon_n\right]\left(z_0\right)^{-1} - 1} \tag{31}$$

where z_0 is determined by

$$\ln z_0 = \beta\varepsilon_0 - \frac{2a\lambda^2 N_0}{V} + \alpha\left(N, N_0\right). \tag{32}$$

From (28), (29) and (31), (32) it is straightforward to obtain the distribution function $G_{\rm int}\left(N, N_0\right)$ of the interacting Bose gases.

$$G_{\rm int}\left(N, N_0\right) = G_{\rm ideal}\left(N, N_0\right)\ R_{\rm int}\left(N, N_0\right) \tag{33}$$

where $G_{\rm ideal}\left(N, N_0\right)$ is the distribution function (17) of the ideal Bose gases, while $R_{\rm int}\left(N, N_0\right)$ takes the form

$$R_{\rm int}\left(N, N_0\right) = R_1\left(N, N_0\right)\ R_2\left(N, N_0\right)\ R_3\left(N, N_0\right). \tag{34}$$

In (34),

$$R_1\left(N, N_0\right) = \exp\left[-\frac{\left(\zeta\left(\frac{3}{2}\right)\right)^{3/2}}{\sqrt{2\pi}}\left(\frac{a}{\lambda_0}\frac{N_0^p}{N}\right)^{1/2}\frac{\left(N_0 - N_0^p\right)^2}{Nt^2}\theta\left(N_0 - N_0^p\right)\right] \tag{35}$$

$$R_2\left(N, N_0\right) = \exp\left[\frac{\zeta\left(\frac{3}{2}\right)a}{\lambda_0}\frac{N_0^2 - \left(N_0^p\right)^2}{Nt}\right] \tag{36}$$

$$R_3\left(N, N_0\right) = \exp\left[-\left|\frac{2\zeta\left(\frac{3}{2}\right)a}{\lambda_0}\frac{N_0^p\left(N_0 - N_0^p\right)}{Nt}\right|\right]. \tag{37}$$

We should note that $G_{\rm int}\left(N, N_0\right)$ is not a Gaussian distribution function because of the non-Gaussian factors $R_1\left(N, N_0\right)$ and $R_2\left(N, N_0\right)$, while Idziaszek *et al* [15] utilized the Gaussian distribution as an assumption to investigate the fluctuations of the interacting system. In $R_{\rm int}\left(N, N_0\right)$, $R_1\left(N, N_0\right)$ comprises the factor $\left(a/\lambda_0\right)^{1/2}$ and represents the leading correction to the distribution function due to interatomic interaction, while $R_2\left(N, N_0\right)$ and $R_3\left(N, N_0\right)$ are high-order corrections to the distribution function. We should note that the leading contribution $R_1\left(N, N_0\right)$ is not a Gaussian function.

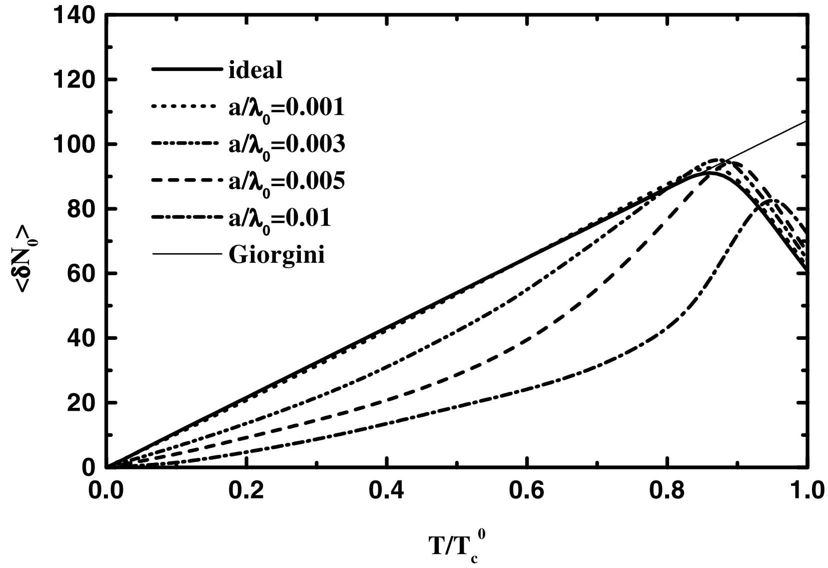

Figure 3. Temperature dependence of δN_0 for interacting Bose gases based on the lowest-order perturbation theory. The thick full curve displays the numerical result for the ideal Bose gas. We give the numerical result for the repulsive interactions with $a/\lambda_0 = 1 \times 10^{-3}, 2 \times 10^{-3}, 5 \times 10^{-3}, 1 \times 10^{-2}$. The crossover from interacting to non-interacting Bose gases is clearly demonstrated. The thin full line is obtained from (8) in [14].

From the distribution function $G_{\text{int}}(N, N_0)$ the mean occupation number and fluctuations of the condensate are determined by

$$\langle N_0 \rangle = \frac{\sum_{N_0=0}^{N} N_0 G_{\text{int}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{int}}(N, N_0)} \tag{38}$$

$$\left\langle \delta^2 N_0 \right\rangle = \frac{\sum_{N_0=0}^{N} N_0^2 G_{\text{int}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{int}}(N, N_0)} - \left(\frac{\sum_{N_0=0}^{N} N_0 G_{\text{int}}(N, N_0)}{\sum_{N_0=0}^{N} G_{\text{int}}(N, N_0)} \right)^2 . \tag{39}$$

From (30), (33) and (38), (39) we can obtain the fluctuations of the interacting boson gases. In figure 3 we give the numerical result for the repulsive interactions with $a/\lambda_0 = 1 \times 10^{-3}$, $2 \times 10^{-3}, 5 \times 10^{-3}, 1 \times 10^{-2}$. The crossover from interacting to ideal Bose gases (thick full curve) is clearly demonstrated in figure 3, while in [14] the fluctuations (thin full line) of the interacting Bose gases are irrelevant to the scattering length. When $a \to 0$ it is easy to recover the fluctuations of the ideal Bose gases. According to (39) the leading contributions to the fluctuations are anomalous, i.e. proportional to $N^{4/3}$, while there are also normal contributions proportional to N due to interatomic interaction. This conclusion contradicts that of [15] which predicts normal behaviour, where the lowest-order perturbation theory is also used to discuss the fluctuations of the condensate.

4. Fluctuations of the condensate based on Bogoliubov theory

Let us investigate the fluctuations of the condensate within the framework of the Bogoliubov theory of a uniform weakly interacting Bose gas confined in a box. According to Bogoliubov theory [27, 28], the total number of particles out of the condensate is given by

$$N_T = \sum_{n\neq 0} N_n = \sum_{n\neq 0} \left(u_n^2 + v_n^2\right) f_n \tag{40}$$

where

$$u_n^2 + v_n^2 = \frac{\left(\left(\varepsilon_n^{\mathrm{B}}\right)^2 + g^2 n_0^2\right)^{1/2}}{2\varepsilon_n^{\mathrm{B}}} \tag{41}$$

$$u_n v_n = -\frac{g n_0}{2\varepsilon_n^{\mathrm{B}}} \tag{42}$$

and f_n is the number of quasi-particles present in the system at thermal equilibrium

$$f_n = \frac{1}{\exp\left[\varepsilon_n^{\mathrm{B}}/k_{\mathrm{B}}T\right] - 1}. \tag{43}$$

In addition, the energy of the quasi-particles entering (41) and (42) is given by the well known Bogoliubov spectrum

$$\varepsilon_n^{\mathrm{B}} = \left((\varepsilon_n + g n_0)^2 - g^2 n_0^2\right)^{1/2} \tag{44}$$

where $g = 4\pi\hbar^2 a/m$ is the coupling constant, and $n_0 = N_0/V$ is the condensate density. At low $|\boldsymbol{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2}$, one obtains $u_n^2 \simeq v_n^2 \propto 1/|\boldsymbol{n}|$ and $f_n \propto 1/|\boldsymbol{n}|$. This results in $1/|\boldsymbol{n}|^2$ divergence in (43) at low $|\boldsymbol{n}|$. Although this sort of divergence will not lead to a large contribution to the number of low-energy quasi-particles, it gives the leading contribution to the fluctuations of the condensate, as pointed out in [14]. We will investigate the fluctuations due to low-energy quasi-particles in the following.

In (40) N_n can be regarded as the effective occupation number of the thermal atoms, while

$$N_n^{\mathrm{B}} = \frac{N_n}{u_n^2 + v_n^2} = f_n \tag{45}$$

is the occupation number of the quasi-particles. From the form of f_n, we can construct the partition function of the quasi-particles in the frame of the canonical ensemble

$$Z_{\mathrm{B}} = \sum_{\{n\}} \exp\left[-\beta \sum_n N_n^{\mathrm{B}} \varepsilon_n^{\mathrm{B}}\right]. \tag{46}$$

From (45) Z_{B} becomes

$$Z_{\mathrm{B}} = \sum_{\{\sum N_n = N\}} \exp\left[-\beta \sum_n N_n \varepsilon_n^{\mathrm{eff}}\right] \tag{47}$$

where $\varepsilon_n^{\mathrm{eff}} = \varepsilon_n^{\mathrm{B}}/\left(u_n^2 + v_n^2\right)$ can be regarded as the effective energy level of the thermal atoms. In this case Z_{B} is the partition function of a fictitious boson system comprising N non-interacting bosons whose energy level is determined by $\varepsilon_n^{\mathrm{eff}}$. From (47) the most probable value N_0^p is given by

$$N_0^p = N - \sum_{n\neq 0} \frac{1}{\exp\left[\left(\varepsilon_n^{\mathrm{eff}} - \varepsilon_0^{\mathrm{eff}}\right)/k_{\mathrm{B}}T\right] - 1}. \tag{48}$$

Obviously the occupation number of low $|\boldsymbol{n}|$ in (48) coincides with that of (40). Analogously, the other N_0 is thus

$$N_0 = N - \sum_{n\neq 0} \frac{1}{\exp\left[\left(\varepsilon_{\boldsymbol{n}}^{\mathrm{eff}} - \varepsilon_0^{\mathrm{eff}}\right)/k_{\mathrm{B}}T\right]\exp\left[-\alpha\left(N, N_0\right)\right] - 1}. \tag{49}$$

From (48) and (49) one obtains

$$\alpha\left(N, N_0\right) \approx -\frac{N_0 - N_0^p}{\sum_{n\neq 0}\left(u_{\boldsymbol{n}}^2 + v_{\boldsymbol{n}}^2\right)^2 f_{\boldsymbol{n}}^2} \tag{50}$$

where we have used the approximation $f_{\boldsymbol{n}} \approx k_{\mathrm{B}}T/\varepsilon_{\boldsymbol{n}}^{\mathrm{B}}$ for low-energy quasi-particles. Therefore, the Gaussian distribution function of the system is given by

$$G_{\mathrm{B}}\left(N, N_0\right) = \exp\left[-\frac{\left(N_0 - N_0^p\right)^2}{2\sum_{n\neq 0}\left(u_{\boldsymbol{n}}^2 + v_{\boldsymbol{n}}^2\right)^2 f_{\boldsymbol{n}}^2}\right] \approx \exp\left[-\frac{\left(\zeta\left(\frac{3}{2}\right)\right)^{4/3}\left(N_0 - N_0^p\right)^2}{(2\pi)^2 A N^{4/3} t^2}\right]. \tag{51}$$

Obviously the mean occupation number $\langle N_0\rangle$ and fluctuations $\left\langle \delta^2 N_0\right\rangle$ are given by

$$\langle N_0\rangle = \frac{\sum_{N_0=0}^{N} N_0 G_{\mathrm{B}}\left(N, N_0\right)}{\sum_{N_0=0}^{N} G_{\mathrm{B}}\left(N, N_0\right)} \tag{52}$$

$$\left\langle \delta^2 N_0\right\rangle = \frac{\sum_{N_0=0}^{N} N_0^2 G_{\mathrm{B}}\left(N, N_0\right)}{\sum_{N_0=0}^{N} G_{\mathrm{B}}\left(N, N_0\right)} - \left(\frac{\sum_{N_0=0}^{N} N_0 G_{\mathrm{B}}\left(N, N_0\right)}{\sum_{N_0=0}^{N} G_{\mathrm{B}}\left(N, N_0\right)}\right)^2. \tag{53}$$

From (51)–(53) one obtains the fluctuations of the condensate based on Bogoliubov theory. At the critical temperature, $G_{\mathrm{B}}\left(T = T_{\mathrm{c}}\right) = \exp\left[-N_0^2/\theta\right]$, where $\theta =$

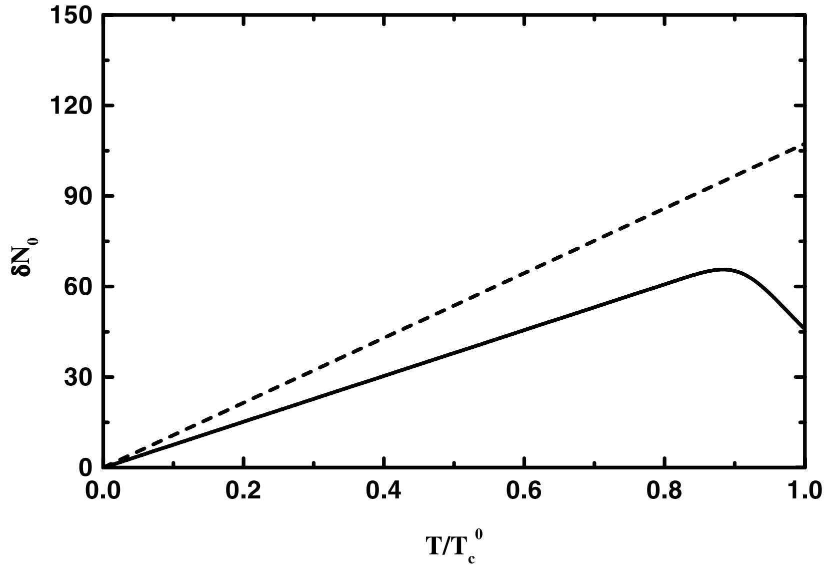

Figure 4. Temperature dependence of δN_0 for interacting Bose gases based on Bogoliubov theory. The full curve is obtained from the numerical result of (53), while the broken line displays (8) in [14].

$2\sum_{n\neq 0}\left(u_n^2+v_n^2\right)^2 f_n^2 = (2\pi)^2 A N^{4/3}/\left(\zeta\left(\frac{3}{2}\right)\right)^{4/3}$. In this case, we obtain the analytical result of the condensate fluctuations,

$$\left\langle \delta^2 N_0\right\rangle_{T=T_c} = \left(\frac{1}{2}-\frac{1}{\pi}\right)\theta = \left(\frac{1}{2}-\frac{1}{\pi}\right)\frac{(2\pi)^2 A}{\left(\zeta\left(\frac{3}{2}\right)\right)^{4/3}} N^{4/3}. \tag{54}$$

Equation (54) clearly shows that the anomalous behaviour of the condensate fluctuations originates from the low-energy quasi-particles, which gives the anomalous factor $N^{4/3}$ through θ.

In figure 4 the full curve displays our results based on the Bogoliubov theory, while the broken line shows the result of [14].

5. Discussion and conclusion

In this paper we investigate the fluctuations of the condensate in a weakly interacting Bose gas confined in a box. A canonical ensemble is developed to calculate the fluctuations of the condensate when different models of interacting Bose gases are used. We found that both the lowest-order perturbation theory and Bogoliubov theory give anomalous behaviour of the fluctuations for the interacting Bose gases confined in a box.

Different from the usual method, the distribution function $P\left(N_0|N\right)/P\left(N_0^p|N\right)$ (i.e. the ratio of the probability between N_0 and the most probable value N_0^p) of the ground state occupation number is obtained directly to calculate the fluctuations of the condensate. In some senses, we give a simple method to recover the applicability of the saddle-point approximation to discuss the condensate fluctuations, through the avoidance of the second derivative in the usual method.

For the present experiments of BEC, the harmonically trapped atoms are in a situation of almost complete isolation from the outer environment surrounding the trap, therefore a canonical (or microcanonical) ensemble should be used to calculate the fluctuations of the condensate. On the other hand, one obtains a more accurate mean ground state occupation number within the canonical ensemble, compared with the grand canonical ensemble. This paper may serve as another method to investigate the thermodynamic properties of the harmonically trapped interacting Bose gases such as the critical temperature, condensate fraction and fluctuations of the condensate.

The remaining challenge is to extend the idea of this paper to the case of a microcanonical ensemble where the energy of the system is also invariant. In addition, the role of interactions on the fluctuations of the condensate is expected to be much more dramatic in the case of attractive forces. We will investigate these problems in a subsequent work.

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