Dynamics of dark solitons in quasi-one-dimensional Bose-Einstein condensates

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We develop a systematic analytical approach to consider the dynamics of linear and nonlinear excitations in trapped quasi-one-dimensional Bose-Einstein condensates with repulsive atom-atom interactions. We show that, for a condensate strongly confined in two transverse directions, the ground state of the system involves the high-order eigenmodes of the transverse confining potential in the transverse directions and effective high-order Thomas-Fermi wave functions in the axial direction. The linear excitations of the system have a Bogoliubov-type spectrum with the excitation frequency varying slowly along the axial direction. We find that, in a weak nonlinear approximation, the amplitude of a nonlinear excitation is governed by a variable coefficient Korteweg–de Vries equation with additional terms contributed from the transverse structure and the inhomogeneity in the axial direction of the condensate, which results in varying amplitude, width, and velocity for dark solitons. Because of the inhomogeneity the dark solitons undergo deformation and emit radiations when traveling along the axial direction. We finally demonstrate that a dark soliton will disintegrate into several ones plus a residual wave train when passing over a steplike potential.

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I. INTRODUCTION

The successful experimental demonstration of Bose-Einstein condensation in weakly interacting atomic gases [1–5] has opened the possibility to investigate the nonlinear properties of matter waves. Several macroscopically excited Bose-condensed states, e.g., dark solitons and vortices, have been observed [6–9] and the four-wave mixing has also been realized in Bose-Einstein condensates (BECs) [10]. These studies have stimulated a large amount of research activities on nonlinear atom optics [11].

Up to now there have been several theoretical approaches for the soliton dynamics in trapped one-component BECs. One of them is based on the assumption that the particle number in the condensate is very large and hence the kinetic energy of particle can be neglected so that a Thomas-Fermi (TF) approximation can be used for getting the ground-state wave function of the condensate. The condensate in this case is three-dimensional (3D) and the dark solitons are the excitations from the TF ground-state [12]. However, the dark soliton obtained in this way is dynamically unstable for a long-wavelength transverse perturbation [13] and it will decay into vortices [12]. This phenomenon has been observed in recent experiments [14]. Another theoretical approach is taking the condensate as one dimensional (1D), which implies that the confinement of atoms in two transverse (radial) directions are very strong and hence the transverse part of the order parameter are taken as being "frozen" to the ground-state wave function of the transverse confining potential, which has a Gaussian-type form for a 2D harmonic oscillator potential [15–22]. But this treatment is less rigorous because the contribution of the higher-order eigenmodes of the transverse confining potential in the transverse directions has been completely disregarded. We also mention the study on solitons and breathers in periodic traps and in array of BECs [23,24], where the transverse structure of the condensate has also been neglected.

Recently, the trapped low-dimensional BECs have been realized experimentally in optical and magnetic traps [25], in which the energy level spacing in one or two dimensions exceeds the interaction energy between atoms. Some authors refer to such an energy restriction as quasi-low-dimensional [26]. As mentioned in Ref. [25], the trapped quasi-low-dimensional condensates will offer many possibilities for investigating the nonlinear excitations such as solitons and vortices, which are more stable than in 3D, where the solitons suffer from the transverse instability and the vortices can bend [9,12–14]. Thus both theory and experiment call for a detailed study on the soliton excitations in quasi-low-dimensional BECs.

The aim of this paper is to investigate analytically the soliton excitations in trapped quasi-1D BECs in a consistent and systematic way. Note that although in recent decades the soliton excitations have been widely studied in many fields [27], the theoretical approach on soliton dynamics in inhomogeneous systems has not yet developed well. In the case of trapped quasi-1D BECs, a soliton moves in a trapping potential along the axial direction. The situation is similar to the motion of a surface wave soliton in a water channel with deformed walls or an uneven bottom [29]. Thus we expect that, in addition to some transverse structure contributed by not only the ground-state but also the higher-order eigenmodes of strong transverse confining potential, the soliton will undergo a deformation, i.e., it will have varying amplitude, width, and velocity due to the inhomogeneity arising from the weak trapping potential in the axial direction. We also anticipate that the soliton will radiate phonons because of the inhomogeneity and disintegrate into several different ones plus a residual wave train when passing over a steplike potential.

To demonstrate the conjecture stated above, in this work we develop a systematic analytical approach to investigate the motion of dark solitons in a trapped quasi-1D BEC based on a generalized method of multiple scales. There are several advantages in this approach: (i) it contains explicit dimensionless small parameters denoting the relative magnitudes of the confining potential and the excitation under study, and hence is controllable in asymptotic expansion, (ii) it reduces original 3D nonlinear order-parameter equation to a 1D amplitude equation that can be handled easily; (iii) it provides a clear-cut physical picture for some physical processes and for the formation for some coherent structures such as dark solitons. The paper is organized as follows. In Sec. II we give the dimensionless form of the order parameter equation and present its ground-state solution, which involves the higher-order eigenmodes of 2D transverse confining potential in the transverse directions and effective high-order TF wave functions in the axial direction. Section III discusses the linear excitations that arose from the ground-state. A Bogoliubov-type excitation spectrum with the excitation frequency and the sound speed depending on an effective axial trapping potential are obtained. In Sec. IV we derive the amplitude equation for a weak nonlinear excitation, i.e., a varying coefficient Korteweg–de Vries (KdV) equation with additional terms originating from the inhomogeneity and the transverse confinement. Section V presents dark soliton solutions of the amplitude equation and studies their radiation. The dark soliton disintegration for a steplike potential in the axial direction is investigated in Sec. VI. Finally, the last section contains the discussion and the summary of our results.

II. GROUND-STATE WAVE FUNCTION

The dynamic behavior of a weakly interacting Bose gas at low temperature is described by the time-dependent Gross-Pitaevskii (GP) equation [4,5]

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})+g|\Psi|^2\right]\Psi, \tag{1}$$

where Ψ is order parameter (also called condensed-state wave function), $\int d\mathbf{r}|\Psi|^2=N$ is the number of atom in the condensate, and $g=4\pi\hbar^2 a_s/m$ is the interaction constant with m the atomic mass and a_s the s-wave scattering length ($a_s>0$ for a repulsive interaction). We consider an anisotropic cigar-shaped harmonic trap of the form

$$V_{ext}(\mathbf{r})=\frac{m}{2}[\omega_x^2x^2+\omega_\perp^2(y^2+z^2)],\quad \omega_x\ll\omega_\perp, \tag{2}$$

where ω_x and $\omega_\perp$ are the frequencies of the trap in the axial (x) and the transverse (y and z) directions, respectively. A generalization to a more general potential with the form $V_{ext}(\mathbf{r})=V_\parallel(x)+V_\perp(y,z)$ is straightforward, where $V_\parallel(V_\perp)$ are the weak (strong) confining parts of the potential in the axial (transverse) directions, respectively.

Expressing the order parameter in terms of its modulus and phase, i.e., $\Psi=\sqrt{n}\exp(i\phi)$, we obtain a set of coupled equations for n and ϕ. In order to obtain a consistent perturbation analysis for the solution of Eq. (1), we introduce the dimensionless variables $x'=l_0^{-1}x,(y',z')=a_\perp^{-1}(y,z),t'=\omega_\perp t,n'=n_0^{-1}n$ with $l_0=(4\pi n_0 a_s)^{-1/2}$ (healing length), and $a_\perp=[\hbar/(m\omega_\perp)]^{1/2}$ (harmonic oscillator length in the transverse directions) and $n_0=N/(l_0a_\perp^3)$, we obtain the following dimensionless equations of motion after dropping the primes:

$$\frac{\partial n}{\partial t}+\nabla_\perp\cdot(n\nabla_\perp\phi)+\left(\frac{a_\perp}{l_0}\right)^2\frac{\partial}{\partial x}\left(n\frac{\partial\phi}{\partial x}\right)=0, \tag{3}$$

$$\left[\frac{\partial\phi}{\partial t}-\frac{1}{2}\nabla_\perp^2+\frac{1}{2}(y^2+z^2)+\frac{1}{2}(\nabla_\perp\phi)^2\right]\sqrt{n}+\left(\frac{a_\perp}{l_0}\right)^2\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{1}{2}\left(\frac{\omega_x}{\omega_\perp}\right)^2\left(\frac{a_\perp}{l_0}\right)^{-4}x^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+n\right]\sqrt{n}=0, \tag{4}$$

where $\nabla_\perp=(\partial/\partial y,\partial/\partial z)$ is the gradient operator in the transverse directions. We see that the system is characterized by two dimensionless parameters $\alpha\equiv a_\perp/l_0=[n_0g/(\hbar\omega_\perp)]^{1/2}$ and $\beta\equiv\omega_x/\omega_\perp$. The former denotes the ratio between the atomic interaction and the strength of the transverse confinement and the later describes the anisotropy of the trapping potential. The normalization condition of Ψ now reads $\int d\mathbf{r}\,n=1$.

Although an exact solution of Eqs. (3) and (4) is not available, we can simplify the problem by considering the relative importance of the physical quantities appearing in the system. Then we can obtain an approximated analytical solution of the problem based on a perturbation expansion. To this end we consider a trapped quasi-1D condensate that has the property

$$a_\perp\ll l_0, \tag{5}$$

$$\hbar\omega_x\ll n_0g\ll\hbar\omega_\perp. \tag{6}$$

Thus we have $\varepsilon\equiv\alpha^2=n_0g/(\hbar\omega_\perp)\ll1$, which can be taken as a small expansion parameter in our perturbation analysis given below. This condition also excludes the instability of dark solitons due to a long-wavelength transverse perturbation [13]. By Eq. (6) we get $\beta\ll\varepsilon\ll1$. In order to obtain a consistent asymptotic expansion, a relation between β and ε should be determined. We assume that $\beta=\Omega_x^2\varepsilon^{5/2}$ with Ω_x a number of order unity. Then Eqs. (3) and (4) become

$$\frac{\partial F}{\partial t}+\nabla_\perp F\cdot\nabla_\perp\phi+\frac{1}{2}F\nabla_\perp^2\phi+\varepsilon\left(\frac{\partial F}{\partial x}\frac{\partial\phi}{\partial x}+\frac{1}{2}F\frac{\partial^2\phi}{\partial x^2}\right)=0, \tag{7}$$

$$\left[\frac{\partial\phi}{\partial t}-\frac{1}{2}\nabla_\perp^2+\frac{1}{2}(y^2+z^2)+\frac{1}{2}(\nabla_\perp\phi)^2\right]F+\varepsilon\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+V_\parallel(X)+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+F^2\right]F=0, \tag{8}$$

where $F=\sqrt{n}, V_{\parallel}(X)=(1/2)\Omega_x^2 X^2$ with $X=\varepsilon^{3/2}x$. Note that the results given in the following do not rely on the concrete form of $V_{\parallel}(X)$. Thus hereafter we assume that $V_{\parallel}(X)$ is an arbitrary function. Equations (7) and (8), in which only *one* dimensionless small parameter appears, are our basic equations for studying the ground-state, linear, and nonlinear excitations of the system.

The ground-state of the system corresponds to set $\partial\phi/\partial t=-\tilde{\mu}$ (dimensionless chemical potential) and $\mathbf{v}=\nabla\phi=0$ (i.e., no flow in the system). From Eq. (7) we see that $F=F_{GS}$ is time-independent in the ground-state. Then Eq. (8) is reduced to

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}(y^2+z^2)-\tilde{\mu}\right]F_{GS}+\varepsilon\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+V_{\parallel}(X)+F_{GS}^2\right]F_{GS}=0. \tag{9}$$

To solve the ground-state equation (9) we make the perturbation expansion

$$F_{GS}=F_{GS}^{(0)}+\varepsilon F_{GS}^{(1)}+\varepsilon^2 F_{GS}^{(2)}+\cdots, \tag{10}$$

$$\tilde{\mu}=\mu^{(0)}+\varepsilon\mu^{(1)}, \tag{11}$$

where $F_{GS}^{(j)}=F_{GS}^{(j)}(y,z,X)$. Obviously, y and z play a role of "fast" variables while X is a "slow" variable of the system. Thus one has $\partial/\partial x=\varepsilon^{3/2}\partial/\partial X$. The expansion on $\tilde{\mu}$ can include higher-order terms $\mu^{(j)}(j=2,3,\dots)$ but we find that they are not necessary and hence are taken as zero. Substituting Eqs. (10) and (11) into Eq. (9) we obtain

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}(y^2+z^2)-\mu^{(0)}\right]F_{GS}^{(j)}=M^{(j)}, \tag{12}$$

$j=0,1,2,3,\dots$ with

$$M^{(0)}=0, \tag{13}$$

$$M^{(1)}=[\mu^{(1)}-V_{\parallel}(X)]F_{GS}^{(0)}-(F_{GS}^{(0)})^3, \tag{14}$$

$$M^{(2)}=[\mu^{(1)}-V_{\parallel}(X)]F_{GS}^{(1)}-3(F_{GS}^{(0)})^2F_{GS}^{(1)}, \tag{15}$$

$$M^{(3)}=[\mu^{(1)}-V_{\parallel}(X)]F_{GS}^{(2)}-3(F_{GS}^{(0)})^2F_{GS}^{(2)}-3F_{GS}^{(0)}(F_{GS}^{(1)})^2, \tag{16}$$

$$\cdots. \tag{17}$$

In the leading order ($j=0$) one has an eigenvalue problem of a 2D harmonic oscillator. Its eigensolution has the form $F_{GS}^{(0)}=A_{GS,n_yn_z}^{(0)}(X)\psi_{n_y}(y)\psi_{n_z}(z)$ with the eigenvalue $\mu^{(0)}=\omega_{n_yn_z}=(n_y+n_z+1)$, where n_y and n_z are nonnegative integers. $\psi_n(y)=N_n\exp(-y^2/2)H_n(y)$ with $N_n=[1/(\sqrt{\pi}2^n n!)]^{1/2}$. $H_n(y)$ is a Hermitain polynomial of order n. $\psi_n(y)$ satisfies the orthogonality condition $\int_{-\infty}^{\infty}dy\,\psi_{n_1}(y)\psi_{n_2}(y)=\delta_{n_1n_2}$. Note that the transverse level spacing, being equal to $\hbar\omega_{\perp}$ in physical unit, is one that is larger than the interaction energy between particles according to the assumption (6). Since we consider ground-state thus we take $n_y=n_z=0$, i.e.,

$$F_{GS}^{(0)}=A(X)\psi_0(y)\psi_0(z) \tag{18}$$

with $\omega_{00}=1$, where $A(X)=A_{GS,00}^{(0)}(X)$ is a function to be determined yet.

In the next order ($j=1$) we have the equation

$$\hat{L}F_{GS}^{(1)}=[\mu^{(1)}-V_{\parallel}(X)]A(X)\psi_0(y)\psi_0(z)-A^3(X)\psi_0^3(y)\psi_0^3(z), \tag{19}$$

where

$$\hat{L}=-\frac{1}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}(y^2+z^2)-1. \tag{20}$$

Note that the eigenfunctions of the operator $\hat{L}$ constitutes a complete set. Thus $F_{GS}^{(1)}$ can be expressed as

$$F_{GS}^{(1)}=\sum_{n_y,n_z}A_{GS,n_yn_z}^{(1)}(X)\psi_{n_y}(y)\psi_{n_z}(z). \tag{21}$$

Substituting Eq. (21) into Eq. (19) and using the orthogonality of $\psi_{n_y}(y)\psi_{n_z}(z)$, one obtains

$$A_{GS,n_yn_z}^{(1)}(X)(\omega_{n_yn_z}-1)=[\mu^{(1)}-V_{\parallel}(X)]A(X)\delta_{n_y0}\delta_{n_z0}-A^3(X)I_{n_yn_z}, \tag{22}$$

where $I_{n_yn_z}=\int_{-\infty}^{\infty}dy\,dz\,\psi_0^3(y)\psi_0^3(z)\psi_{n_y}(y)\psi_{n_z}(z)$. From (22), for $n_y=n_z=0$ we have

$$A(X)=I_0^{-1/2}[\mu^{(1)}-V_{\parallel}(X)]^{1/2} \tag{23}$$

with $I_0=I_{00}=1/\sqrt{2\pi}$. Thus we obtain an *effective* TF wave function for the axial part of $F_{GS}^{(0)}$. Equation (23) is indeed a result of a solvability condition of Eq. (19). The correction of the dimensionless chemical potential $\mu^{(1)}$ plays a role of an effective chemical potential in the TF wave function. If one of n_y and n_z is not zero, Eq. (22) gives

$$A_{GS,n_yn_z}^{(1)}(X)=-A^3(X)\frac{I_{n_yn_z}}{\omega_{n_yn_z}-1}. \tag{24}$$

Hence we have

$$F_{GS}^{(1)}=A_{GS,00}^{(1)}(X)\psi_0(y)\psi_0(z)-A^3(X){\sum_{n_y,n_z}}'\frac{I_{n_yn_z}}{\omega_{n_yn_z}-1}\psi_{n_y}(y)\psi_{n_z}(z), \tag{25}$$

where $A_{GS,00}^{(1)}$ is a function yet to be determined. The prime in the second term of Eq. (25) means that n_y and n_z are not taken to vanish simultaneously. In the same way, one can solve Eq. (12) for $j=2$. We obtain in this order $A_{GS,00}=[3I_1/(2I_0)]A^3$ with $I_1=\Sigma'_{n_y,n_z}I^2_{n_yn_z}/(\omega_{n_yn_z}-1)$, and

$$F^{(2)}_{GS}=A^{(2)}_{GS,00}(X)\,\psi_0(y)\psi_0(z)+A^5(X)\sum_{n_y,n_z}{}' \; a^{(2)}_{GS,n_yn_z}\psi_{n_y}(y)\psi_{n_z}(z), \tag{26}$$

where $A^{(2)}_{GS,00}$ is a undetermined function and

$$a^{(2)}_{GS,n_yn_z}=\frac{1}{\omega_{n_yn_z}-1}\left[-\frac{9I_1I_{n_yn_z}}{2I_0}-\frac{I_0I_{n_yn_z}}{\omega_{n_yn_z}-1}+3\sum_{n'_y,n'_z}{}' \; \frac{I_{n'_yn'_z}J_{n_yn'_yn_zn'_z}}{\omega_{n'_yn'_z}-1}\right], \tag{27}$$

with

$$J_{n_yn'_yn_zn'_z}=\int_{-\infty}^{\infty}dy\,dz\;\psi_0^2(y)\psi_0^2(z)\psi_{n_y}(y)\psi_{n'_y}(y)\psi_{n_z}(z)\psi_{n'_z}(z). \tag{28}$$

In the order $j=3$ we obtain $A^{(2)}_{GS,00}=A^5\,a^{(2)}_{GS,00}$ with

$$a^{(2)}_{GS,00}=\frac{3}{2I_0}\left[\frac{3I_1^2}{4I_0}-\sum_{n_y,n_z}{}' \; I_{n_yn_z}a^{(2)}_{GS,n_yn_z}-\sum_{n_y,n_z}{}' \sum_{n'_y,n'_z}{}' \; \frac{I_{n_yn_z}I_{n'_yn'_z}J_{n_yn'_yn_zn'_z}}{(\omega_{n_yn_z}-1)(\omega_{n'_yn'_z}-1)}\right]. \tag{29}$$

Therefore, up to the second-order approximation we obtain the ground-state solution expressed as

$$F_{GS}=A(X)h_0(y,z)+\varepsilon A^3(X)h_1(y,z)+\varepsilon^2A^5(X)h_2(y,z)+O(\varepsilon^3), \tag{30}$$

where

$$h_0(y,z)=\psi_0(y)\psi_0(z), \tag{31}$$

$$h_1(y,z)=\frac{3I_1}{2I_0}\psi_0(y)\psi_0(z)-\sum_{n_y,n_z}{}' \; \frac{I_{n_yn_z}}{\omega_{n_yn_z}-1}\psi_{n_y}(y)\psi_{n_z}(z), \tag{32}$$

$$h_2(y,z)=a^{(2)}_{GS,00}\psi_0(y)\psi_0(z)+\sum_{n_y,n_z}{}' \; a^{(2)}_{GS,n_yn_z}\psi_{n_y}(y)\psi_{n_z}(z). \tag{33}$$

From Eq. (30) we see that the ground-state of the trapped 1D condensate displays some structure. In addition to the Gaussian-type ground state, $\psi_0(y)\psi_0(z)$, it involves also the higher-order eigenmodes of the transverse confining potential. Furthermore, by the expression of $A(X)$ given in Eq. (23), we know that in the axial direction the ground-state of the condensate is composed of effective higher-order TF wave functions. Figure 1(a) shows the norm ($\sqrt{n}$) of the ground-state wave function for the case of the harmonic trapping potential when only the leading term, i.e., $U=A(X)h_0(y,z)$, in Eq. (30) is considered (taking $z=0$ for illustration), in which the transverse part is clearly a Gaussian function. In order to visualize the contribution coming from the high-order transverse modes, in Fig. 1(b) we have plotted the first-order correction term of Eq. (30) $[V=\varepsilon A^3(X)h_1(y,z)]$ for the case of $\varepsilon=0.08$ and $\Omega_x=9$. Because the integral $I_{n_yn_z}$ decreases rapidly as n_y and n_z increase, n_y and n_z are only taken to be 6 in our calculation. We see that, in the transverse (y) direction, the ground-state wave function indeed displays some structures. There is a dip along the y direction due to mainly the contribution of the Hermite function $H_2(y)=2y^2-2$. The correction to the Gaussian distribution in the transverse directions, however, is negligible when ε becomes small.

The correction of the dimensionless chemical potential $\mu^{(1)}$ can be obtained in the following way. From the normalized condition $\int d\mathbf{r}\,n=1$ one has

$$\int_{-\infty}^{\infty}dX\,A^2(X)\approx\alpha^3. \tag{34}$$

For the harmonic potential $V_\parallel(X)=\Omega_x^2X^2/2$, we have

$$\int_{-R_x}^{R_x}dX\left[\mu^{(1)}-\frac{1}{2}\Omega_x^2X^2\right]=I_0\alpha^3, \tag{35}$$

where $R_x=(2\mu^{(1)}/\Omega_x^2)^{1/2}$. Note that $\Omega_x=\beta\alpha^{-5}$, by Eq. (35) we get

$$\mu^{(1)}=\left(\frac{3I_0\beta}{4\sqrt{2}\alpha^2}\right)^{2/3}. \tag{36}$$

The chemical potential of the system with physical unit is given by $\mu=\hbar\omega_\perp\tilde{\mu}=\hbar\omega_\perp+\alpha^2\hbar\omega_\perp\mu^{(1)}$. Using Eq. (36) and the definitions of α and β, we obtain

$$\mu=\frac{\hbar^2}{2m}\left[\frac{2}{a_\perp^2}+\left(\frac{6\pi I_0a_sN}{a_\perp^2a_x^2}\right)^{2/3}\right], \tag{37}$$

where $a_x=[\hbar/(m\omega_x)]^{1/2}$ is the harmonic oscillator length in the axial direction. The correction of the chemical potential [the second term of Eq. (37)] is due to the contribution of atom-atom interaction.

III. LINEAR EXCITATIONS

We now consider the linear excitations from the ground-state given in the preceding section. Because the system is strongly confined in the transverse directions and the trapping potential in the axial direction, $V_\parallel$, is a function of the slowly variable X, the system can be considered as a waveguide. The excitations can be sound waves propagating in the axial direction with a smaller wavelength comparing with the

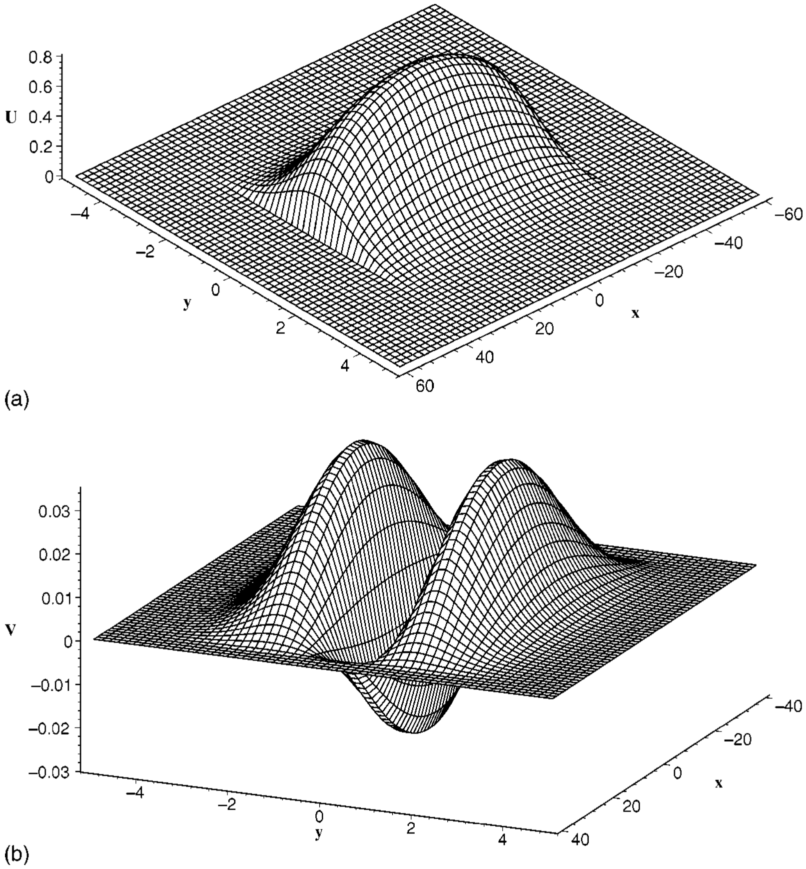

FIG. 1. (a) The norm of the ground-state wave function ($\sqrt{n}$) for the case of a harmonic trapping potential with $\Omega_x=9$ when only the leading term, i.e., $U\equiv A(X)h_0(y,z)$, in Eq. (30) is considered (taking $z=0$ for illustration). The transverse part is clearly a Gaussian function. (b) The contribution due to first-order correction in Eq. (30) $[V\equiv\varepsilon A^3(X)h_1(y,z)]$ for the case $\varepsilon=0.08$ and $\Omega_x=9$. n_y and n_z are taken up to 6 in the calculation. A dip along the y direction appears due to mainly the contribution of the Hermite function $H_2(y)=2y^2-2$.

axial size of the condensate. Here we are interested in sound wavelike excitations of the system and hence we assume

$$F=F(x,y,z,\tau), \tag{38}$$

$$\phi=-\tilde{\mu}t+\tilde{\phi}(x,\tau), \tag{39}$$

where $\tau=\varepsilon t$. Then Eqs. (7) and (8) become

$$\frac{\partial F}{\partial\tau}+\frac{\partial F}{\partial x}\frac{\partial\tilde{\phi}}{\partial x}+\frac{1}{2}F\frac{\partial^2\tilde{\phi}}{\partial x^2}=0, \tag{40}$$

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}(y^2+z^2)-\tilde{\mu}\right]F +\varepsilon\left[\frac{\partial\tilde{\phi}}{\partial\tau}-\frac{1}{2}\frac{\partial^2}{\partial x^2}+V_{\parallel}(X)+\frac{1}{2}\left(\frac{\partial\tilde{\phi}}{\partial x}\right)^2+F^2\right]F=0. \tag{41}$$

Making the perturbation expansion

$$F-F_{GS}=\varepsilon f^{(1)}+\varepsilon^2 f^{(2)}+\cdots, \tag{42}$$

$$\tilde{\phi}=\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}+\cdots, \tag{43}$$

together with $\tilde{\mu}=\mu^{(0)}+\varepsilon\mu^{(1)}$ and assuming $f^{(j)}=f^{(j)}(x,y,z,X,\tau)$ and $\phi^{(j)}=\phi^{(j)}(x,X,\tau)$, by Eq. (41) we obtain

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}(y^2+z^2)-\mu^{(0)}\right]f^{(j)}=N^{(j)}, \tag{44}$$

$j=1,2,\ldots$ with $N^{(1)}=0$, $N^{(2)}=I_0A^2(X)f^{(1)}-F_{GS}^{(0)}\partial\phi^{(1)}/\partial\tau+(1/2)\partial^2 f^{(1)}/\partial x^2-3(F_{GS}^{(0)})^2f^{(1)}$. When getting Eq. (44) we have used the ground-state equation (12) and thus $\mu^{(0)}=1$, $F_{GS}^{(0)}=A(X)\psi_0(y)\psi_0(z)$, and $\mu^{(1)}$ is given by Eq. (36). Equation (44) for $j=1$ gives rise to the solution

$$f^{(1)}=a_{LE,00}^{(1)}(x,X,\tau)\psi_0(y)\psi_0(z), \tag{45}$$

where $a_{LE,00}^{(1)}$ is a undetermined function. Here for simplicity we have assumed that the system has excited only one mode related to the Gaussian-type wave function $\psi_0(y)\psi_0(z)$. Multimode excitations related to $\psi_{n_y}(y)\psi_{n_z}(z)$ can also be considered in a similar way.

A solvability condition of Eq. (44) for $j=2$ results in

$$\frac{1}{2}\frac{\partial^2 a_{LE,00}^{(1)}}{\partial x^2}-A(X)\frac{\partial\phi^{(1)}}{\partial\tau}-2I_0A^2(X)a_{LE,00}^{(1)}=0, \tag{46}$$

and

$$f^{(2)}=a_{LE,00}^{(2)}(x,X,\tau)\psi_0(y)\psi_0(z) -3A^2(X)a_{LE,00}^{(1)}(x,X,\tau)\sum_{n_y,n_z}{}' \frac{I_{n_yn_z}}{\omega_{n_yn_z}-1}\psi_{n_y}(y)\psi_{n_z}(z), \tag{47}$$

where $a_{LE,00}^{(2)}$ can be obtained in the next order but it is not needed here.

The expansion of Eq. (40) up to $O(\varepsilon)$-order yields the equation

$$\frac{\partial f^{(1)}}{\partial \tau}+\frac{1}{2}F_{GS}^{(0)}\frac{\partial^2\phi^{(1)}}{\partial x^2}=0. \tag{48}$$

By using the solution (45) it is reduced to

$$\frac{\partial a_{LE,00}^{(1)}}{\partial \tau}+\frac{1}{2}A(X)\frac{\partial^2\phi^{(1)}}{\partial x^2}=0. \tag{49}$$

Equations (46) and (49) are basic equations for linear excitations. To get the linear dispersion law of the excitations we take the plane-wave solution

$$(a_{LE,00}^{(1)},\phi^{(1)})=(a_0^{(1)},\phi_0^{(1)})\exp[i(qx-\omega\tau)]+\text{c. c.}, \tag{50}$$

where $a_0^{(1)}$ and $\phi_0^{(1)}$ are independent on x and τ but may be the functions of X. Substituting Eq. (50) into Eqs. (46) and (49) we obtain

$$\omega=\omega(q,X)=\pm\frac{1}{2}q[4I_0A^2(X)+q^2]^{1/2}. \tag{51}$$

It is a Bogoliubov-type excitation spectrum but with the excitation frequency ω depending on the slow variable X. The sound speed is found to be local and is given by

$$c_{sound}=\pm I_0^{1/2}A(X)=\pm[\mu^{(1)}-V_{\parallel}(X)]^{1/2}. \tag{52}$$

Thus the amplitude, frequency, and thus the sound speed of the excitations are the functions of the slow variable X due to the inhomogeneity in the axial direction. Such local property for the sound speed comes from the inhomogeneous background, i.e., the space-dependent ground-state of the condensate. The phenomenon is similar to the sound propagation in a slowly varying nonuniform medium [28]. The positive-negative sign of Eq. (52) means that the sound wave can propagate in two opposite directions in the elongated axis of the condensate.

IV. AMPLITUDE EQUATION FOR NONLINEAR EXCITATIONS

As mentioned above, an excitation in the trapped BEC has local amplitude and sound speed. We expect that this is also the case for a nonlinear excitation of the system. Note that for weak nonlinear excitations Eqs. (40) and (41) are still valid. In order to study the dynamics of a weak nonlinear excitation in such nonuniform system we introduce the multiple-scale variable

$$\xi=\varepsilon^{1/2}\left(\int c^{-1}(\varepsilon^{3/2}x)\,dx-\tau\right), \tag{53}$$

and make the asymptotic expansions $F-F_{GS}=\varepsilon f^{(1)}+\varepsilon^2 f^{(2)}+\varepsilon^3 f^{(3)}+\cdots,\tilde{\phi}=\varepsilon^{1/2}[\phi^{(1)}+\varepsilon\phi^{(2)}+\varepsilon^2\phi^{(3)}+\cdots]$, with $\mu=\mu^{(0)}+\mu^{(1)}, f^{(j)}=f^{(j)}(y,z,\xi,X)$ and $\phi^{(j)}=\phi^{(j)}(\xi,X)$. Thus we have the derivative expansion

$$\frac{\partial}{\partial x}=\varepsilon^{1/2}c^{-1}(X)\frac{\partial}{\partial \xi}+\varepsilon^{3/2}\frac{\partial}{\partial X}, \tag{54}$$

$$\frac{\partial}{\partial \tau}=-\varepsilon^{1/2}\frac{\partial}{\partial \xi}. \tag{55}$$

Then Eqs. (40) and (41) are transferred into

$$\hat{L}f^{(j)}=P^{(j)}, \tag{56}$$

$$\frac{\partial f^{(j)}}{\partial \xi}-\frac{1}{2}c^{-2}(X)F_{GS}^{(0)}\frac{\partial^2\phi^{(j)}}{\partial \xi^2}=Q^{(j)}, \tag{57}$$

$j=1,2,\ldots,$ where $\hat{L}$ is defined by Eq. (20) and the explicit expressions of $P^{(j)}$ and $Q^{(j)}$ are given in the the Appendix. Solving Eq. (56) order by order we obtain

$$f^{(1)}=a_{NL,00}^{(1)}(\xi,X)\psi_0(y)\psi_0(z), \tag{58}$$

$$f^{(2)}=a_{NL,00}^{(2)}(\xi,X)\psi_0(y)\psi_0(z) -3A^2(X)a_{NL,00}^{(1)}(\xi,X)\sum_{n_y,n_z}{}' \frac{I_{n_yn_z}}{\omega_{n_yn_z}-1}\psi_{n_y}(y)\psi_{n_z}(z) \tag{59}$$

with the solvability condition

$$A(X)\frac{\partial\phi^{(j)}}{\partial \xi}-2I_0A^2(X)a_{NL,00}^{(j)}=R^{(j)}, \tag{60}$$

$j=1,2,\ldots,$ where $a_{NL,00}^{(j)}(j=1,2)$ are undetermined functions. Note that Eq. (60) with $j=2$ is obtained by the solvability condition of the equation $\hat{L}f^{(3)}=P^{(3)}$.

Using (58) and (59), Eq. (57) is simplified as

$$\frac{\partial a_{NL,00}^{(j)}}{\partial \xi}-\frac{1}{2}c^{-2}(X)A(X)\frac{\partial^2\phi^{(j)}}{\partial \xi^2}=S^{(j)}, \tag{61}$$

$j=1,2,\ldots,$ where the definitions of $R^{(j)}$ and $S^{(j)}$ on the right-hand side of Eqs. (60) and (61) are also presented in the Appendix.

Since we are interested in the weak nonlinear excitation in the system, we need an equation controlling the leading-order approximation of the excitation, i.e., the equation for $a_{NL,00}^{(1)}$, appeared in Eq. (58). For this aim we solve Eqs. (60) and (61). In the order $j=1$ we obtain

$$[1-I_0A^2(X)c^{-1}(X)]\frac{\partial a_{NL,00}^{(1)}}{\partial\xi}=0, \tag{62}$$

$$\phi^{(1)}=2I_0A(X)\int d\xi\, a_{NL,00}^{(1)}(\xi,X). \tag{63}$$

From Eq. (62) we see that to get a nontrivial solution for $a_{NL,00}^{(1)}$ we must set

$$c(X)=\pm I_0^{1/2}A(X)=\pm[\mu^{(1)}-V_{\parallel}(X)]^{1/2}. \tag{64}$$

It is just the sound speed given in the last section [see Eq. (52)]. Equation (63) presents a relation between $\phi^{(1)}$ and $a_{NL,00}^{(1)}$. Once $a_{NL,00}^{(1)}$ is obtained one can get the leading-order approximation for the phase $\phi^{(1)}$ by directly integrating $a_{NL,00}^{(1)}$.

In the order $j=2$, Eqs. (60) and (61) give rise to the closed equation for $a_{NL,00}^{(1)}(\equiv U)$,

$$\frac{\partial^3U}{\partial\xi^3}+m_1(X)U\frac{\partial U}{\partial\xi}+m_2(X)\frac{\partial U}{\partial X}+m_3(X)U+m_4(X)\frac{\partial U}{\partial\xi}=0 \tag{65}$$

with

$$m_1(X)=-24I_0^2A^3(X), \tag{66}$$

$$m_2(X)=-8\,\delta_1I_0^{5/2}A^5(X), \tag{67}$$

$$m_3(X)=-12\,\delta_1I_0^{5/2}A^4(X)\frac{\partial A}{\partial X}, \tag{68}$$

$$m_4(X)=12I_0I_1A^6(X), \tag{69}$$

where $\delta_1=\pm1$, representing the two possible propagating directions of the excitation. Equations (65) is a variable coefficient KdV equation with additional terms contributed from the inhomogeneity in the axial direction (denoted by $\partial A/\partial X$) and the strong transverse confinement of the condensate (denoted by I_1). Such nonlinear amplitude equation was also obtained for water waves propagating in a channel with deformed walls or an uneven bottom [29]. It is obvious that the derivation and the results given in Secs. II–IV can be easily generalized to any trapping potential with a strong transverse confinement.

V. DEFORMATION AND RADIATION OF A DARK SOLITON

In this section, we discuss the soliton solutions of the variable coefficient KdV equation with additional terms, Eq. (65). Using the variable transformation $U=-\rho(\sigma)u(\sigma,\zeta)$ with $\rho=-6m_1^{-1}m_4^{-2}$, $\zeta=m_4^{-1}\xi-\int dX\,m_2^{-1}$, and $\sigma=\int dX\,m_2^{-1}m_4^{-3}$, Eq. (65) becomes

$$\frac{\partial u}{\partial\sigma}+6u\frac{\partial u}{\partial\zeta}+\frac{\partial^3u}{\partial\zeta^3}=-\gamma(\sigma)\,u, \tag{70}$$

with

$$\gamma(\sigma)=-\frac{27}{2}\frac{d}{d\sigma}(\ln A). \tag{71}$$

If the term on the right-hand side of Eq. (70) does not exist, i.e., $\gamma(\sigma)=0$, we have a standard KdV equation that is a completely integrable system and can be solved exactly by the inverse scattering transform [27]. Its single-soliton solution has the form

$$u=u_s(w)=2a^2\mathrm{sech}^2w,\qquad w=a(\zeta-b), \tag{72}$$

with $b=4a^2\sigma+\zeta_0$, where a and ζ_0 are two arbitrary constants characterizing the amplitude and initial position of the soliton. Note that if $\gamma(\sigma)$ vanishes the system is reduced to a uniform waveguide in which $V_{\parallel}=V_0=$const (maybe taking as zero) and hence $m_j(j=1,2,4)$ take constant values and $m_3=0$. Hence $f^{(1)}$ in Eq. (58) reads

$$f^{(1)}=\frac{12a^2}{m_1m_4^2}\mathrm{sech}^2[a(\zeta-4a^2\sigma-\zeta_0)]\psi_0(y)\psi_0(z), \tag{73}$$

with $\zeta=m_4^{-1}\xi-m_2^{-1}X,\sigma=m_2^{-1}m_4^{-3}X$, and $\xi=\varepsilon^{1/2}(c^{-1}x-\tau)$, where $c=\sqrt{\mu^{(1)}}$ is a constant. Because $m_1<0$, the excitation digs a ''hole'' on the background condensate and hence is a dark soliton. We see that in this case although the transverse confinement of the waveguide modifies the amplitude and velocity, but these quantities are still constants and thus the soliton can propagate in the axial direction without deformation. However, in the presence of the axial nonuniform, i.e., when $V_{\parallel}\neq$ constant, the situation will be quite different, as will be seen below.

To consider the effect due to the axial nonuniform, here we assume that A is a slowly varying function of X, i.e., we take $\gamma(\sigma)$ as a small quantity and thus being a perturbation of the KdV equation. We apply the perturbation theory of soliton [32] to get the soliton solutions of Eq. (70). Anticipating that the axial nonuniform will result in the modulation of the soliton parameters a and ζ_0 appeared in u_s [see (72)] and some additional radiation (i.e., phonons), one has

$$u=u_s(w)+\delta u, \tag{74}$$

where $u_s(w)$ has the form of Eq. (72) but a and b now are controlled by the equations [32]

$$\frac{\partial a}{\partial\sigma}=-\frac{1}{4a}\int_{-\infty}^{\infty}dw\,\mathrm{sech}^2w\;\gamma(\sigma)u_s(w), \tag{75}$$

$$\frac{\partial b}{\partial\sigma}=4a^2-\frac{1}{4a^3}\int_{-\infty}^{\infty}dw\,(\tanh w+w\,\mathrm{sech}^2w)\,\gamma(\sigma)u_s(w). \tag{76}$$

The radiation part δu, contributed by the continuous spectrum of a scattering problem when solving the KdV equation using the inverse scattering transform (correspondingly, the soliton part u_s is relevant to the discrete spectrum of the scattering problem, see [27]), is given by

$$\delta u = -P\int_{-\infty}^{\infty} dk\,\frac{g^{(1)}(k)}{ik(k^2+4)a^3}[1-e^{ik(k^2+4)a^3\sigma}]\Phi_0(w,k), \tag{77}$$

$$g^{(1)}(k) = -\int_{-\infty}^{\infty} dw\,\gamma(\sigma)u_s(w)\Psi_0(w,k), \tag{78}$$

where P represents the principal value of the integral, Φ_0 and Ψ_0 are defined by [32]

$$\Phi_0(w,k) = \frac{1}{\sqrt{2\pi}\,k(k^2+4)}[k(k^2+4)+4i(k^2+2)\tanh w - 8k\tanh^2 w - 8i\tanh^3 w]e^{ikw}, \tag{79}$$

$$\Psi_0(w,k) = \frac{1}{\sqrt{2\pi}\,(k^2+4)}[k^2-4ik\tanh w - 4\tanh^2 w]e^{-ikw}. \tag{80}$$

Substituting Eqs. (71) and (72) into Eqs. (75) and (76), completing the integrations and then using the definition of $m_j(X)$ given in Eqs. (66)–(69), we obtain

$$a = a_0 A^9(X), \tag{81}$$

$$b = -\frac{4\delta_1 a_0^2}{8\times 12^3 I_0^{11/2} I_1^3}\int \frac{dX}{A^5(X)}, \tag{82}$$

where a_0 is an integral constant. The soliton part of the solution (74) can be obtained by the expression (72), but with a and b being replaced by Eqs. (81) and (82), respectively. Then we have

$$U_s = -\rho\, u_s = -\tilde{a}_0^2 A^3(X)\,\mathrm{sech}^2\Theta, \tag{83}$$

where $\tilde{a}_0 = (\sqrt{2}\cdot 12 I_0^2 I_1)^{-1} a_0$, and Θ, the phase of the soliton, is defined by

$$\Theta = \sqrt{2} I_0^{1/2}\tilde{a}_0 A^3\left[\delta_1\varepsilon^{-1}\int\frac{dX}{A} + \delta_1\frac{3I_1}{2I_0}\left(1+\frac{2I_0}{3I_1}\tilde{a}_0^2\right)A^6\int\frac{dX}{A} - \varepsilon^{3/2} I_0^{1/2} t\right]. \tag{84}$$

From Eqs. (83) and (84) we see that, due to the axial inhomogeneity, the amplitude, width, and velocity of the dark soliton are not constants but varying slowly along the axial direction. This means that the dark soliton undergoes a deformation, i.e., its shape will change when propagating along the elongated direction of the condensate. This result agrees with the recent experimental observation reported by Burger *et al.* [7] and Denschlag *et al.* [8]. Needless to say that in experiment the dissipation originating from the interaction between the soliton and thermal cloud also results in the change of the soliton parameters [7]. It is obvious that the result presented above is generic and does not depend on the concrete form of $V_{\parallel}(X)$.

Furthermore, we show that, if taking the harmonic potential $V_{\parallel}(X) = \Omega_x^2 X^2/2$, the dark soliton will display an oscillating motion in the trap. This can be seen by considering the phase of the soliton, given by Eq. (84), and calculating the soliton position as a function of time. It is easy to show that the position of the soliton satisfies the equation of motion

$$\frac{dX}{dt} = \delta_1\varepsilon^{5/2} I_0^{1/2}\frac{A(X)}{1+\varepsilon\frac{3I_1}{2I_0}\left(1+\frac{2I_0}{3I_1}\tilde{a}_0^2\right)A^6(X)}. \tag{85}$$

Equation (85) can be reduced to the one provided by Busch and Anglin [17] if on the right-hand side the second term in the denominator is neglected. Solving Eq. (85) we get

$$X = R_x \sin\left[\frac{1}{\sqrt{2}}\,\frac{1}{(1+B_0)}\,\frac{\omega_x}{\omega_\perp} t\right], \tag{86}$$

where for the definitions of R_x and $\mu^{(1)}$, see Eq. (36). The constant B_0 reads

$$B_0 = \varepsilon^4\frac{27 I_1}{2I_0^2}\,\frac{1}{\sqrt{\Omega_x}}\left(1+\frac{2I_0}{3I_1}\tilde{a}_0^2\right). \tag{87}$$

Thus the oscillating frequency of the dark soliton position is different from the trap frequency $\omega_x/\omega_\perp$ (dimensionless from). It is decreased by a factor $1/[\sqrt{2}(1+B_0)]$. The factor $1/\sqrt{2}$ is due to the nonlinear effect of the lowest-order transverse confining mode. B_0 is contributed by the higher-order transverse confining modes of the transverse trap potential, which is absent if these higher-order modes are not taken into account, as done in Ref. [17]. This type of oscillating behavior has not been observed in experiment. The reason is that at finite temperature, the dark soliton is thermodynamical unstable. The interaction of the soliton with thermal cloud causes dissipation that accelerates the soliton. In most cases, the soliton has disappeared before reaching the boundary of the condensate [7].

Now we consider the radiation part of the solution. Using Eq. (80), from Eq. (78) we obtain

$$g^{(1)}(k) = \frac{\sqrt{2\pi}}{3}\,\frac{a^2 k}{\sinh(\pi k/2)}\,\gamma(\sigma). \tag{88}$$

When obtaining Eq. (88), some useful integration formulas provided in Ref. [32] have been used. Then by Eq. (77), through a detailed calculation we get

$$\delta u = \frac{\gamma(\sigma)}{3a}\left[-\int_{-\infty}^{(\zeta-\zeta_0)/(3\sigma)^{1/3}} d\kappa\,\mathrm{Ai}(\kappa) + \theta(\zeta-\zeta_0-4a^2\sigma)\right], \tag{89}$$

where $\mathrm{Ai}(\kappa)$ is the Airy function, defined by

$$\mathrm{Ai}(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\,\exp[i(s\kappa+s^3/3)], \tag{90}$$

$\theta(x)$ is the step function with $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$.

From Eq. (89) we see that δu represents a continuous wave radiated by the soliton. The continuous wave can be taken as a superposition of many phonons. One can take such process as soliton radiation, in which initially there is a dark soliton in the homogeneous region where $V_{\parallel}$=constant, propagating stably to an inhomogeneous region where $V_{\parallel}=V_{\parallel}(X)$. When arriving at this region the soliton begins to deform and emit phonons. If the inhomogeneity is significant, the soliton will emit a large amount of phonons and hence possibly disappears.

VI. DISINTEGRATION OF A DARK SOLITON

According to the results presented in the last section, a dark soliton in a trapped quasi-1D condensate will deform and emit phonons when propagating in an inhomogeneous region. For this phenomenon to happen, the width of the inhomogeneous region should be larger than the soliton width, which is the order of magnitude of the healing length. But there may exist such situation in which the inhomogeneous region is local and its width is small (less or equal to the soliton width). In this case the dark soliton can pass the region adiabatically, and then undergo a fission. The fission phenomenon of soliton has been widely studied and observed for water-wave solitons traveling onto a slowly varying beach [29–31]. In this section we show that this type of fission is also possible in BECs.

For convenience we take $u=-v$ for Eq. (70), then it becomes

$$\frac{\partial v}{\partial \sigma}-6v\,\frac{\partial v}{\partial \zeta}+\frac{\partial^3 v}{\partial \zeta^3}=-\,\gamma(\sigma)\,v. \tag{91}$$

Note that Eq. (91) admits the following two integrals of motion (conservative quantities)

$$A^{-27/2}\int_{-\infty}^{\infty} d\zeta\, v=C_1, \tag{92}$$

$$A^{-27}\int_{-\infty}^{\infty} d\zeta\, v^2=C_2, \tag{93}$$

where C_1 and C_2 are constants. Assume that $V_{\parallel}(X)$ has a steplike shape (as shown in Fig. 1). For instance, one has $V_{\parallel}(X)=V_0(1-\tanh X)/2$. But the results presented in the following are generic and hence are not limited to such particular potential.

The potential shown in Fig. 1 can be divided into three regions. Region I (upstream) is $X\leqslant X_1$, where $V_{\parallel}(X)\approx V_0$ (positive constant). Region II ($X_1<X<X_2$) is a transition region where $V_{\parallel}(X)$ has an obvious, steplike change. Region III (downstream) is $X\geqslant X_2$ where $V_{\parallel}(X)$ is another constant (taking as zero).

Now we study the disintegration of a dark soliton. Let us assume that initially one has excited a soliton at the region I. Because in this region $V_{\parallel}=0$ thus $\gamma(\sigma)=0$. The excitation is hence a standard KdV soliton of Eq. (91) [with $\gamma(\sigma)=0$], which has fixed amplitude, width, and velocity,

$$v_1=-2a_1^2\mathrm{sech}^2[a_1(\zeta-4a_1^2\sigma-\zeta_{01})], \tag{94}$$

where a_1 and ζ_{10} are constants. Equation (94) can be written as

$$v_1=-D_1\mathrm{sech}^2\left[\sqrt{\frac{D_1}{2}}(\zeta-2D_1\sigma-\zeta_{01})\right], \tag{95}$$

where $D_1=2a_1^2$ is the soliton amplitude. The soliton travels to the right with the velocity $2D_1$. When passing over the region II and arriving at the starting point $X=X_2$ of the region III, its parameters undergo a transformation, i.e., the soliton becomes

$$v_2=-D_2\mathrm{sech}^2[D_3(\zeta-D_4\sigma-\zeta_{02})]. \tag{96}$$

The parameters $D_j(j=2,3)$ can be determined by using the integrals of motion (92) and (93). It is easy to get

$$D_2=D_1\left[\frac{A(X_2)}{A(X_1)}\right]^{27/2},\quad D_3=\sqrt{\frac{D_1}{2}}. \tag{97}$$

Thus we have

$$v_2=-D_2\mathrm{sech}^2\left[\sqrt{\frac{D_1}{2}}(\zeta-D_4\sigma-\zeta_{02})\right]. \tag{98}$$

The parameter D_4 is still undetermined but it is not needed in our following analysis.

Note that the wave packet (98) is no longer a soliton because it does not satisfy the KdV equation (91) although in the region III $\gamma(\sigma)$ also vanishes. The question is about the evolution of the wave packet (98).

In order to answer this question we take Eq. (98) as an initial condition of Eq. (91). Because $\gamma(\sigma)=0$ in the region III, we have the following initial value problem:

$$\frac{\partial v}{\partial \sigma}-6v\,\frac{\partial v}{\partial \zeta}+\frac{\partial^3 v}{\partial \zeta^3}=0,\quad X>X_2, \tag{99}$$

$$v(\zeta,\sigma=0)=-D_2\mathrm{sech}^2\left[\sqrt{\frac{D_1}{2}}(\zeta-\zeta_{02})\right], \tag{100}$$

where D_2 is given in Eq. (97). According to the inverse-scattering theory of the KdV equation [27], related eigenvalue problem of Eqs. (99) and (100) is

$$\frac{d^2\psi}{d\zeta^2}+\left\{D_2\mathrm{sech}^2\left[\sqrt{\frac{D_1}{2}}(\zeta-\zeta_{02})\right]+\lambda\right\}\psi=0, \tag{101}$$

where λ and ψ are the eigenvalue and eigenfunction to be sought. In general, depending on D_1 and D_2 the eigenvalue λ consists of two parts. One part is a discrete spectrum, λ_n, which is relevant to the soliton solution of Eq. (99). The number of soliton, N, equals the number of discrete spectrum λ_n. If $N>1$ we have a multiple-soliton solution that corresponds to the disintegration of the incident soliton (95). Now we discuss how to calculate λ_n and N.

Equation (101) can be written as

$$\frac{d^2\psi}{d\zeta'^2}+[s(s+1)\mathrm{sech}^2\zeta'+\lambda']\psi=0, \tag{102}$$

where $\zeta'=\sqrt{D_1/2}(\zeta-\zeta_{02})$, $\lambda'=2\lambda/D_1$ and

$$s(s+1)=\frac{2D_2}{D_1}=2\left[\frac{A(X_2)}{A(X_1)}\right]^{27/2}. \tag{103}$$

There are two cases for the solutions of the eigenequation (102). The first one is that s in Eq. (103) is a positive integer. In this circumstance all eigenvalues are discrete [33]

$$\lambda'=\lambda_n'=2\lambda_n/D_1=-(s-n)^2 \tag{104}$$

with $n=0,1,2,\ldots,s-1$. The eigenfunctions corresponding to these discrete eigenvalues are bound states. By soliton theory [27,33], the KdV equation (99) for this case has N-soliton solution with $N=s$. Thus the incident soliton (95) in the upstream will disintegrate into N solitons in the downstream.

The second case is that s is a positive number, e.g., $s=[s]+c$, where $[s]$ is a positive integer not larger than s and c is a positive number less than one. In this case the eigenvalue of Eq. (102) possesses not only discrete but also continuous spectrum. The discrete spectrum is still given by Eq. (104) but with $n=0,1,2,\ldots,[s]$. Thus in this case Eq. (99) admits the multisoliton solution with the number of soliton being $N=[s]+1$. In addition, corresponding to the continuous spectrum the eigenfunctions are extended states. Corresponding to these extended states Eq. (99) has continuous wave (or wave train) solution. Since a continuous wave can be taken as the superposition of many phonons, in the second case due to the steplike trap the incident soliton will disintegrate into $[s]+1$ solitons plus a residual wave train.

Combined with the two cases discussed above, it is easy to show that the number of disintegrated solitons, N, satisfies the following inequality

$$N(N-1)<2\left[\frac{A(X_2)}{A(X_1)}\right]^{27/2}\leqslant N(N+1), \tag{105}$$

where the equality in Eq. (105) is valid only if s is a positive integer. By the inverse scattering theory of the KdV equation [27,33], the asymptotic amplitude of nth soliton is $2\lambda_n$. Through the relation (104) we have $2\lambda_n=-(s-n)^2D_1$. Thus the asymptotic expression for the disintegrated N solitons reads

$$v=-\sum_{n=0}^{N}2k_n^2\,\mathrm{sech}^2[k_n(\zeta-4k_n^2\sigma-\zeta_{0n})],\quad X\gg X_2, \tag{106}$$

where $k_n=(s-n)\sqrt{D_1}$. Therefore, for $f^{(1)}$ in Eq. (58), we obtain

$$f^{(1)}=\frac{12a_1^2}{m_1m_4^2}\mathrm{sech}^2[a_1(\zeta-4a_1^2\sigma-\zeta_{01})]\psi_0(y)\psi_0(z),$$
$$X\leqslant X_1, \tag{107}$$

i.e., only one soliton in the upstream, where $a_1=\sqrt{D_1/2}$ [see Eq. (95)], and

$$f^{(1)}=\frac{12k_n^2}{m_1m_4^2}\sum_{n=0}^{N}\mathrm{sech}^2[k_n(\zeta-4k_n^2\sigma-\zeta_{0n})]\psi_0(y)\psi_0(z),$$
$$X\gg X_2, \tag{108}$$

i.e., there is a train of soliton in the downstream. The phonon part contributed by the continuous spectrum in the downstream is not given explicitly here. Note that $m_1<0$, thus the solitons are dark ones (relative to the condensate background). The disintegrated solitons propagate to the right and the soliton with the larger amplitude has greater velocity.

By Eq. (105) we can predict the number of the disintegrated dark solitons. Obviously, when $A(X_2)/A(X_1)\leqslant 1$ there is no disintegration but when passing over the transition region (the region II) the incident soliton will radiate phonons. Note that $A(X)=I_0^{-1/2}[\mu^{(1)}-V_\parallel(X)]^{1/2}$. $A(X_2)/A(X_1)\leqslant 1$ means that in this situation one must have $V_\parallel(X_1)\leqslant V_\parallel(X_2)$.

Soliton disintegration occurs when $A(X_2)/A(X_1)>1$, i.e., $V_\parallel(X_1)>V_\parallel(X_2)$. As mentioned before, without loss of generality we can assume $V_\parallel(X_1)=V_0=r\mu^{(1)}$ and $V_\parallel(X_2)=0$. In this case, from Eq. (105) we obtain the soliton that will disintegrate into two ones plus phonons if $0<r\leqslant 0.1502$ (the phonons disappear when $r=0.1502$). If $0.1502<r\leqslant 0.2331$ we have three disintegrated solitons plus phonons (again when $r=0.2331$, the phonon part vanishes). If $0.2331<r<0.2890$ one gets four disintegrated solitons plus phonons (the phonons disappear when $r=0.2890$), and so on. Thus by adjusting the depth of the step potential, i.e., V_0, one can control the number of the disintegrated solitons.

From the results given above we see that when a dark soliton in the region where $V_\parallel$ is larger (thus $|\Psi|^2$ is smaller) passes over a transition region and goes into the region where $V_\parallel$ is smaller (thus $|\Psi|^2$ is larger), it undergoes a fission. But in the reverse situation, i.e., when travelling from a region of smaller $V_\parallel$ (thus larger $|\Psi|^2$) to a region of larger $V_\parallel$ (thus smaller $|\Psi|^2$), it does not show disintegration except for radiating phonons. A schematic representation of a dark soliton disintegration has been shown in Fig. 2.

VII. DISCUSSION AND SUMMARY

We have studied, in a systematic and consistent way, the ground state, linear, and nonlinear excitations in trapped one-dimensional Bose-Einstein condensates with a repulsive atom-atom interaction. We have shown analytically that for a condensate with a strong transverse confinement, the ground-state of the system involves the high-order eigenmodes of the transverse trapping potential in the transverse directions and effective high-order Thomas-Fermi wave functions in the

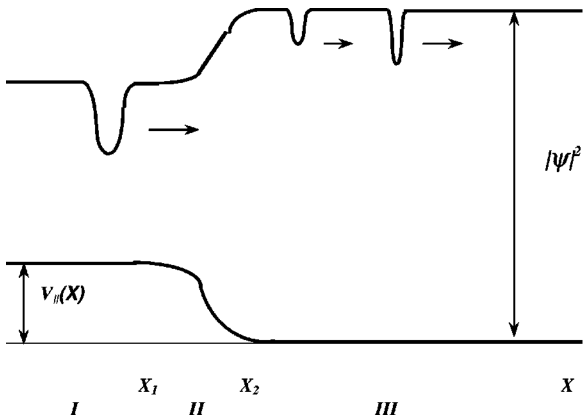

FIG. 2. A schematic representation of a dark soliton disintegration. When a dark soliton in the region I ($X \leqslant X_1$), where $V_\parallel$ is larger and hence $|\Psi|^2$ is smaller, passes over the transition region II ($X_1 \leqslant X \leqslant X_2$) and goes into the region III ($X \geqslant X_3$), where $V_\parallel$ is smaller and hence $|\Psi|^2$ is larger, it undergoes a fission. Only two disintegrated dark solitons are shown that correspond to take $V_0 = r\mu^{(1)}$ with $r = 0.1502$. But inversely if traveling from the region III to the region I, the dark soliton does not disintegrate except for radiating some phonons.

elongated axial direction. For a linear excitation with wavelength much less than the axial length of the condensate, its dispersion law is Bogoliubov-type with excitation frequency changing slowly along the axial direction. We have found that, for a weak nonlinear excitation, its amplitude is controlled by a variable coefficient Korteweg–de Vries equation with additional terms coming from the transverse structure and the axial nonuniform in the condensate, which result in slowly changing amplitude, width, and velocity for dark solitons. We have also shown that due to the inhomogeneity the dark solitons may emit radiation when propagating along the elongated direction. Finally, using the inverse scattering theory for the Korteweg–de Vries equation, we have demonstrated that, when a dark soliton passes over a local, steplike potential, it will disintegrate into multiple dark solitons plus a residual wave train. Note that not like many approaches for soliton excitations in the Bose-Einstein condensate, where an assumption of small condensate has been used [15,34,35], in our approach the condensate can be large because the ground-state F_{GS} is assumed to be of order unity. On the other hand, the solitons we obtained here are the excitations excited from the ground-state of the system.

The method of multiple scales has been widely used in fluid physics and nonlinear optics [31,36]. The theoretical approach presented above based on a generalized method of multiple-scales are not limited to a harmonic trapping potential. It can be easily generalized to any potential with a strong transverse confinement and to a trapped two-dimensional condensate. Our theory can at least partially explain the experimental observations reported by Burger *et al.* [7] and Dutton *et al.* [9], where the condensates can be taken as approximately one-dimensional ones and the dark solitons observed display slowly changing amplitude, width, and velocity. In addition, some phononlike radiations are also obvious in their experiments.

Recently, trapped low-dimensional Bose-Einstein condensates have been realized in a more rigorous sense [25], in which the energy-level spacing in the transverse directions is larger than the atom-atom interaction energy and hence the conditions given by Eqs. (5) and (6) can be easily satisfied. The steplike potential in the axial direction can also be easily realized using present-day optical methods. This paves the way to the study of radiation and disintegration of the soliton in such systems and tests our theoretical predictions provided in this paper.

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APPENDIX

The definition of $P^{(j)}$ and $Q^{(j)}$ ($j=1,2,\dots$) in Eqs. (56) and (57) are given by

$$P^{(1)}=0, \tag{A1}$$

$$P^{(2)}=I_0A^2(X)f^{(1)}+F_{GS}^{(0)}\frac{\partial\phi^{(1)}}{\partial\xi}-3(F_{GS}^{(0)})^2f^{(1)}, \tag{A2}$$

$$\begin{aligned}P^{(3)}=&I_0A^2(X)f^{(2)}-3(F_{GS}^{(0)})^2f^{(2)}-3(F_{GS}^{(0)})[2F_{GS}^{(1)}f^{(1)}\\&+(f^{(1)})^2]+(F_{GS}^{(1)}+f^{(1)})\frac{\partial\phi^{(1)}}{\partial\xi}+F_{GS}^{(0)}\frac{\partial\phi^{(2)}}{\partial\xi}\\&-\frac{1}{2}F_{GS}^{(0)}c^{-2}(X)\left(\frac{\partial\phi^{(1)}}{\partial\xi}\right)^2+\frac{1}{2}c^{-2}(X)\frac{\partial^2f^{(1)}}{\partial\xi^2},\end{aligned} \tag{A3}$$

$$Q^{(1)}=0, \tag{A4}$$

$$\begin{aligned}Q^{(2)}=&c^{-2}(X)\frac{\partial f^{(1)}}{\partial\xi}\frac{\partial\phi^{(1)}}{\partial\xi}+c^{-1}(X)\frac{\partial F_{GS}^{(0)}}{\partial X}\frac{\partial\phi^{(1)}}{\partial\xi}\\&+\frac{1}{2}F_{GS}^{(0)}\left[c^{-1}(X)\frac{\partial^2\phi^{(1)}}{\partial\xi\partial X}+\frac{\partial}{\partial X}\left(c^{-1}(X)\frac{\partial\phi^{(1)}}{\partial\xi}\right)\right]\\&+\frac{1}{2}c^{-2}(X)[F_{GS}^{(1)}+f^{(1)}]\frac{\partial^2\phi^{(1)}}{\partial\xi^2},+\cdots.\end{aligned} \tag{A5}$$

The explicit expressions of $R^{(j)}$ and $S^{(j)}$ appearing in Eqs. (60) and (61) read

$$R^{(1)}=0, \tag{A6}$$

$$R^{(2)}=-6I_1A^4(X)a_{NL,00}^{(1)}-\frac{3}{2}\frac{I_1}{I_0}A^3(X)\frac{\partial\phi^{(1)}}{\partial\xi}-\frac{1}{2}c^{-2}(X)\frac{\partial^2 a_{NL,00}^{(1)}}{\partial\xi^2}-a_{NL,00}^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}A(X)c^{-2}(X)\left(\frac{\partial\phi^{(1)}}{\partial\xi}\right)^2+3I_0A(X)(a_{NL,00}^{(1)})^2, \tag{A7}$$

$$S^{(1)}=0, \tag{A8}$$

$$S^{(2)}=\frac{3I_1}{4I_0}A^3(X)c^{-2}(X)\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+c^{-1}(X)\frac{\partial A}{\partial X}\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}A(X)\left[c^{-1}(X)\frac{\partial^2\phi^{(1)}}{\partial\xi\partial X}+\frac{\partial}{\partial X}\left(c^{-1}(X)\frac{\partial\phi^{(1)}}{\partial\xi}\right)\right]+c^{-2}(X)\frac{\partial a_{NL,00}^{(1)}}{\partial\xi}\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}c^{-2}(X)a_{NL,00}^{(1)}\frac{\partial^2\phi^{(1)}}{\partial\xi^2}. \tag{A9}$$

The other higher-order $P^{(j)}, Q^{(j)}, R^{(j)}$, and $S^{(j)}$ are not needed in our discussion.

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