Anomalous fluctuations of two-dimensional Bose-Einstein condensates

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We investigate the particle-number fluctuations due to the collective excitations created in a two-dimensional (2D) and a quasi-2D Bose-Einstein condensates (BECs) at low temperature. We find that the fluctuations display an anomalous behavior, i.e., for the 2D BEC they are proportional to N^2, where N is the total number of particles. For the quasi-2D BEC, the particle-number fluctuations are proportional not only to N^2 but also to the square root of the trapping frequency in the strongly confined direction.

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The remarkable experimental realization of Bose-Einstein condensates (BECs) has stimulated intensive theoretical and experimental studies on weakly interacting Bose gases [1]. Much attention has been paid to the investigations of the thermodynamic properties especially the particle-number fluctuations of three-dimensional (3D) interacting Bose gases [2–7]. Recently, quasi-1D and quasi-2D BECs have been realized [8,9], which with no doubt provides many new opportunities to explore the fascinating quantum statistical property of macroscopic quantum systems in low dimensions. For a 2D Bose gas, the particle-number fluctuations due to the exchange of atoms between condensate and thermal atoms have been addressed recently in Refs. [10,11].

It is well known that for the temperature far below the critical temperature, the collective excitations play a dominant role for the fluctuations of BEC. The fluctuations of condensate due to collective excitations in 3D Bose gases have been discussed in Refs. [2,5,12]. The scaling behavior of a 2D interacting condensate confined in a box was investigated in Ref. [5]. In the present paper, we shall discuss the fluctuations originated from collective excitations for 2D and quasi-2D interacting condensates confined in a magnetic trap.

For the Bose-condensed gas confined in a magnetic trap, the total number of particles N in the system is conserved and hence a canonical (or microcanonical) ensemble should be used. Within the canonical ensemble, a fairly general method has been developed recently [12,13] for studying the thermodynamic properties of the interacting Bose-condensed gases based on the calculation of probability distribution functions. The purpose of this work is to investigate the particle-number fluctuations of a 2D BEC due to the collective excitations within a canonical ensemble.

For this purpose we first extend our previous theory developed in Refs. [12,13] on particle-number fluctuations in BECs by including an effect of the quantum depletion. For a Bose gas confined in a trap, based on Bogoliubov theory [14] one can obtain that at low temperature, the total number of particles out of the condensate due to the collective excitations is given by

$$\langle N_T\rangle = N-\langle N_0\rangle=\sum_{nl\neq 0}\langle N_{nl}\rangle$$
$$=\sum_{nl\neq 0}\left[\left(\int u_{nl}^2dV+\int v_{nl}^2dV\right)f_{nl}+\int v_{nl}^2dV\right], \quad (1)$$

where $\langle N_T\rangle$ and $\langle N_0\rangle$ are, respectively, the number of excited atoms and atoms in the condensate. $f_{nl}=1/[\exp(\beta\varepsilon_{nl})-1]$ is the average number of collective excitation nl present in the system at thermal equilibrium [$\beta=1/(k_BT)$, k_B is Boltzmann constant and T is temperature], while $\varepsilon_{nl}=\hbar\omega_{nl}$ is the energy of the collective mode characterized by the quantum numbers n and l. In the above equation, the term $\int v_{nl}^2dV$ represents the effect of the quantum depletion which does not vanish even at $T=0$. The quantities u_{nl} and v_{nl} are determined by the following coupled equations:

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+2gn(\mathbf{r})\right)u_{nl}+gn_0(\mathbf{r})v_{nl}=\varepsilon_{nl}u_{nl}, \quad (2)$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+2gn(\mathbf{r})\right)v_{nl}+gn_0(\mathbf{r})u_{nl}$$
$$=-\varepsilon_{nl}v_{nl}, \quad (3)$$

where $V_{ext}(\mathbf{r})$ is the trapping potential confining the Bose gas, and μ and g are, respectively, the chemical potential of the system and interatomic interaction constant. $n(\mathbf{r})$ and $n_0(\mathbf{r})$ are the density distributions of the Bose gas and the condensate, respectively. From Eq. (1), we see that $\langle N_{nl}\rangle$ can be taken as the average number of the excited atoms related to the collective mode nl, while $\langle N_{nl}^B\rangle=f_{nl}$ is the average number of the collective excitation nl. From the form of Eq. (1), we see that the ratio between $\langle N_{nl}\rangle$ and $\langle N_{nl}^B\rangle$ is given by

$$\eta_{nl}=\frac{\langle N_{nl}\rangle}{\langle N_{nl}^B\rangle}=\frac{\left(\int u_{nl}^2dV+\int v_{nl}^2dV\right)f_{nl}+\int v_{nl}^2dV}{f_{nl}}. \quad (4)$$

Within the canonical ensemble, the partition function of the system with N atoms takes the form

$$Z[N]=\sum{}' \exp\left[-\beta\left(N_{\mathbf{0}}\varepsilon_0+\sum_{nl\neq 0}N_{nl}^B\varepsilon_{nl}\right)\right]. \quad (5)$$

In the above equation, we have omitted the interaction between collective excitations. The prime in the summation represents the condition that the total number of atoms in the system should be conserved within the canonical ensemble, i.e., $\Sigma_{nl\neq 0}N_{nl}=\Sigma_{nl\neq 0}\eta_{nl}N_{nl}^{B}=N-N_{\mathbf{0}}$. One should note that in Eq. (5), N_{nl}^{B} is the occupation number of the collective excitation nl and should be an integer. For the convenience of calculations, by separating out the ground state $nl=0$ from the state $nl\neq 0$, we have

$$Z[N]=\sum_{N_{\mathbf{0}}=0}^{N}\{\exp[-\beta N_{\mathbf{0}}\varepsilon_0]Z_0(N_T)\}, \tag{6}$$

where $Z_0(N_T)$ stands for the partition function of a fictitious system comprising $N_T=N-N_{\mathbf{0}}$ excited atoms which takes the form

$$Z_0(N_T)=\sum_{\Sigma_{nl\neq 0}\eta_{nl}N_{nl}^{B}=N_T}\exp[-\beta N_{nl}^{B}\varepsilon_{nl}]. \tag{7}$$

The free energy of the fictitious system is $A_0(N_T)=-k_BT\ln Z_0(N_T)$.

By using the developed saddle-point method proposed in Refs. [12,13], we introduce a generating function $G_0(T,z)$ that is given by

$$G_0(T,z)=\sum_{N_T=0}^{\infty}z^{N_T}Z_0(N_T). \tag{8}$$

Due to the fact that there is a confinement condition $\Sigma_{nl\neq 0}\eta_{nl}N_{nl}^{B}=N_T$ for N_T in the above equation, we have

$$G_0(T,z)=\Pi_{nl\neq 0}\left\{\sum_{N_{nl}^{B}=0}^{\infty}z^{\eta_{nl}N_{nl}^{B}}\exp[-\beta N_{nl}^{B}\varepsilon_{nl}]\right\}$$

$$=\Pi_{nl\neq 0}\frac{1}{1-z^{\eta_{nl}}\exp[-\beta\varepsilon_{nl}]}. \tag{9}$$

$Z_0(N_T)$ can be obtained by noting that it is the coefficient of z^{N_T} in the expansion of $G_0(T,z)$. Thus, we have

$$Z_0(N_T)=\frac{1}{2\pi i}\oint dz\frac{G_0(T,z)}{z^{N_T+1}}. \tag{10}$$

Similar to the case of the developed saddle-point method for the ideal Bose gas [12,13], it is easy to get the following useful relations:

$$N_T=\sum_{nl\neq 0}\frac{\eta_{nl}}{\exp[\beta\varepsilon_{nl}]z_0^{-\eta_{nl}}-1} \tag{11}$$

and

$$-\beta\frac{\partial}{\partial N_{\mathbf{0}}}A_0(N_T)=\ln z_0, \tag{12}$$

where z_0 is the well-known saddle point.

By using the above relations, the probability distribution of the system with N_0 atoms in the condensate can be obtained based on the method developed in Refs. [12,13]. The normalized probability distribution function is given by [12,13]

$$G_n(N,N_0)=A_n\exp\left[\int_{N_0^p}^{N_0}\alpha(N,N_0)dN_0\right], \tag{13}$$

where A_n is a normalized constant and N_0^p is the most probable value of the atomic number in the condensate. If taking the ground-state energy of the system as the zero point of energy, $\alpha(N,N_0)$ is determined by

$$N_0^p-N_0=\sum_{nl\neq 0}\left[\frac{\eta_{nl}}{\exp[\beta\varepsilon_{nl}]\exp[-\alpha(N,N_0)]-1}\right.$$

$$\left.-\frac{\eta_{nl}}{\exp[\beta\varepsilon_{nl}]-1}\right]. \tag{14}$$

For temperature below the critical temperature of BEC and for $\beta\hbar\omega_\perp\ll 1$ (this condition is satisfied in the present-day experiments of BECs), the above equation can be approximated to be

$$N_0^p-N_0=\sum_{nl\neq 0}\left[\frac{1}{\exp[\beta\varepsilon_{nl}/\eta_{nl}]\exp[-\alpha(N,N_0)/\eta_{nl}]-1}\right.$$

$$\left.-\frac{1}{\exp[\beta\varepsilon_{nl}/\eta_{nl}]-1}\right]. \tag{15}$$

In fact, one can get the result given by Eq. (15) through a simple physical picture. Taking the ground-state energy as the zero point of energy, the total energy of the system is then

$$E=\sum_{nl}N_{nl}^{B}\varepsilon_{nl}=\sum_{nl}N_{nl}\varepsilon_{nl}^{eff}, \tag{16}$$

where ε_{nl}^{eff} can be regarded as the effective energy level of a single particle characterized by the quantum numbers n and l. From Eqs. (4) and (16), ε_{nl}^{eff} takes the form

$$\varepsilon_{nl}^{eff}=\frac{\varepsilon_{nl}}{\int u_{nl}^2dV+\int v_{nl}^2dV+\int v_{nl}^2dV/f_{nl}}. \tag{17}$$

Thus at low temperature, the system can be regarded as a fictitious noninteracting Bose system with the effective energy level given by ε_{nl}^{eff}. The canonical partition function of the system reads as

$$Z_{cp}=\sum{}'\exp\left[-\beta\sum_{nl}N_{nl}\varepsilon_{nl}^{eff}\right], \tag{18}$$

where the prime in the summation represents the condition $\Sigma_{nl\neq 0}N_{nl}=N_T$. It is easy to confirm that one can get the result given by Eq. (15) from this canonical partition function.

From Eq. (15), after a straightforward calculation, one obtains

$$G_n(N,N_0)=A_n \exp\left[-\frac{(N_0-N_0^p)^2}{2\Xi}\right], \tag{19}$$

where

$$\Xi=\sum_{nl\neq 0}\left[\left(\int u_{nl}^2 dV+\int v_{nl}^2 dV\right)^2\left(\frac{k_BT}{\varepsilon_{nl}}\right)^2 + 2\left(\int u_{nl}^2 dV+\int v_{nl}^2 dV\right)\int v_{nl}^2 dV\left(\frac{k_BT}{\varepsilon_{nl}}\right) + \left(\int v_{nl}^2 dV\right)^2\right]. \tag{20}$$

Below the critical temperature, $N_0^p \gg 1$, the fluctuations of the condensate contributed from the collective excitations are given by

$$\langle \delta^2 N_0\rangle=\langle N_0^2\rangle-\langle N_0\rangle^2=\Xi. \tag{21}$$

Generally speaking, Eqs. (20) and (21) can be used to investigate the fluctuations originated from the collective excitations in any dimension. Now we specify the case of 2D. For Eqs. (2) and (3), u_{nl} and v_{nl} are given by [15]

$$u_{nl}\approx -v_{nl}\approx i\sqrt{\frac{gn_0(\mathbf{r})}{2\varepsilon_{nl}}}\chi_{nl}. \tag{22}$$

For a Bose gas confined in a harmonic potential, $V_{ext}(\mathbf{r})=m\omega_\perp^2(x^2+y^2)/2$, χ_{nl} and $\varepsilon_{nl}(=\hbar\omega_{nl})$ are determined by the eigenequation

$$-\frac{\omega_\perp^2}{2}\nabla\cdot[(R_\perp^2-r_\perp^2)\nabla\chi_{nl}]=\omega_{nl}^2\chi_{nl}, \tag{23}$$

where $R_\perp$ is the radius of the condensate. After a straightforward calculation, one obtains the excitation frequency $\omega_{nl}=\omega_\perp\sqrt{2n^2+2n|l|+2n+|l|}$ and

$$\chi_{nl}=\frac{A_{nl}}{R_\perp}e^{-il\phi}H_{nl}\left(\frac{r_\perp}{R_\perp}\right), \tag{24}$$

where A_{nl} is a normalized constant determined by $\int|\chi_{nl}|^2dV=1$. In the above equation, $H_{nl}(x)$ takes the form

$$H_{nl}(x)=x^{|l|}\sum_{j=0}^{n} b_j x^{2j}, \tag{25}$$

where $b_0=1$ and the coefficients b_j satisfy the recurrence relation $b_{j+1}/b_j=(4j^2+4j+4j|l|-4n^2-4n-4n|l|)/(4j^2+4j|l|+8j+4|l|+4)$.

Substituting the above results into Eqs. (20) and (21), we obtain the fluctuations of particle number in the condensate for the 2D Bose gas:

$$\langle\delta^2N_0\rangle=\frac{mg}{\hbar^2}\left[\frac{N^2t^2(1-t^2)}{\pi\zeta(2)}\gamma_1+\frac{N^{3/2}t(1-t^2)}{\pi[\zeta(2)]^{1/2}}\gamma_2+\frac{N(1-t^2)}{4\pi}\gamma_3\right], \tag{26}$$

where $t=T/T_c^0$ with $T_c^0=[N/\zeta(2)]^{1/2}\hbar\omega_\perp/k_B$ the critical temperature corresponding to an ideal Bose gas in 2D. The coefficients γ_1, γ_2, and γ_3 are given by

$$\gamma_1=\sum_{nl\neq 0}\frac{\beta_{nl}^2}{(2n^2+2n|l|+2n+|l|)^2},$$

$$\gamma_2=\sum_{nl\neq 0}\frac{\beta_{nl}^2}{(2n^2+2n|l|+2n+|l|)^{3/2}},$$

$$\gamma_3=\sum_{nl\neq 0}\frac{\beta_{nl}^2}{(2n^2+2n|l|+2n+|l|)}, \tag{27}$$

with

$$\beta_{nl}=\frac{\int_0^1(1-x^2)x(H_{nl})^2dx}{\int_0^1 x(H_{nl})^2dx}. \tag{28}$$

By a numerical calculation, we obtain $\gamma_1=0.87$, $\gamma_2=1.43$, and $\gamma_3=4.37$.

From Eq. (26), we have the following two conclusions:

(i) $\langle\delta^2N_0\rangle$ is proportional to the interatomic interaction constant g. Therefore, for an ideal Bose gas, there is no contribution to the condensate fluctuations due to the collective excitations. This is physically reasonable because the collective excitations, which are dominant at low temperature, originate from the interaction between atoms.

(ii) The leading term for the fluctuations of the particle number in the condensate is proportional to N^2. This anomalous behavior comes from the low-dimensional property of the system. Noting that in the case of 3D Bose-condensed gas, the fluctuations of the condensate due to the collective excitations are proportional to $N^{4/3}$ [2,12]. Thus the lower the dimension of the system is, the larger the condensate fluctuations are. Thus at low temperature, a 2D Bose gas confined in a harmonic trapping potential, which has been realized recently by Görlitz *et al.* [8], is an ideal system for the observation of anomalous behavior of the fluctuations of particle number in condensates.

In real experiments [8,9], the Bose gases are confined in a quasi-2D harmonic trap. For a quasi-2D Bose gas, the coupling constant is given by $g\approx 2\sqrt{2\pi}\hbar^2a_s/(ml_z)$ [16], which is fixed by a s-wave scattering length a_s and the oscillator length $l_z=(\hbar/m\omega_z)^{1/2}$ in the z direction, where ω_z is the trap frequency in the z direction. In this case, we have

$$\langle \delta^2 N_0 \rangle = \frac{2\sqrt{2\pi}a_s}{l_z}\left[\frac{N^2 t^2(1-t^2)}{\pi\zeta(2)}\gamma_1 + \frac{N^{3/2}t(1-t^2)}{\pi[\zeta(2)]^{1/2}}\gamma_2 + \frac{N(1-t^2)}{4\pi}\gamma_3\right]. \tag{29}$$

From the above equation, we see that on one hand, $\langle \delta^2 N_0 \rangle$ is proportional to N^2 (for large N), and on the other hand, it is also proportional to $\omega_z^{1/2}$ because the factor $1/l_z$ appearing in Eq. (29). This shows that the trapping frequency in the z direction plays an important role for the particle number fluctuations due to the collective excitations. Thus, one can control the particle-number fluctuations by adjusting the trapping frequency in the z direction. Compared with the contribution due to the thermal atoms [see Eq. (39) in Ref. [10]], the particle-number fluctuations due to the collective excitations are strongly dependent on the trapping frequency in the z direction. From the relation between $\langle \delta^2 N_0 \rangle$ and ω_z, we see that the confinement of the Bose-condensed gas has the effect of increasing the fluctuations due to the collective excitations. This result is consistent with the role of the dimensionality in the fluctuations of the condensate.

In conclusion, we have studied the particle-number fluctuations of a condensed 2D Bose gas confined in a harmonic trapping potential by using the probability distribution obtained through a modified saddle-point method. We have found that the condensate fluctuations are proportional to the interatomic interaction constant g and the square of total particle number of the system. This anomalous behavior of the fluctuations makes it very promising to experimentally observe the effect of the particle-number fluctuations. The theoretical method provided here is quite general and can be applied to investigate the particle-number fluctuations of quasi-1D Bose-condensed gases.

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[1] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999); J.R. Anglin and W. Ketterle, Nature (London) **416**, 211 (2002); S.L. Rolston and W.D. Phillips, *ibid.* **416**, 219 (2002).
[2] S. Giorgini *et al.*, Phys. Rev. Lett. **80**, 5040 (1998).
[3] Z. Idziaszek *et al.*, Phys. Rev. Lett. **82**, 4376 (1999).
[4] F. Illuminati *et al.*, J. Phys. B **32**, L461 (1999).
[5] F. Meier and W. Zwerger, Phys. Rev. A **60**, 5133 (1999).
[6] V.V. Kocharovsky *et al.*, Phys. Rev. Lett. **84**, 2306 (2000).
[7] D. Jaksch *et al.*, Phys. Rev. A **58**, 1450 (1998).
[8] A. Görlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001).
[9] M. Greiner *et al.*, Phys. Rev. Lett. **87**, 160405 (2001).
[10] H.W. Xiong *et al.*, J. Phys. B **35**, 2105 (2002).
[11] R.K. Bhaduri *et al.*, J. Phys. B **35**, 2817 (2002).
[12] H.W. Xiong *et al.*, Phys. Rev. A **65**, 033609 (2002).
[13] H.W. Xiong *et al.*, J. Phys. B **34**, 4203 (2001).
[14] N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947); S. Giorgini *et al.*, J. Low Temp. Phys. **109**, 309 (1997).
[15] W.C. Wu and A. Griffin, Phys. Rev. A **54**, 4204 (1996).
[16] D.S. Petrov *et al.*, Phys. Rev. Lett. **84**, 2551 (2000).