Analytical calculations on Landau damping of collective modes in anisotropic Bose-Einstein condensates

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We develop an analytical approach for calculating the Landau damping of collective modes in the Bose-Einstein condensate trapped by an anisotropic harmonic potential. Based on a variational ground-state wave function obtained by solving the time-independent Gross-Pitaevskii equation beyond Thomas-Fermi approximation, we solve the Bogoliubov–de Gennes equations describing thermal excited quasiparticles and provide divergence-free analytical solutions for the Bogoliubov amplitudes and coupling matrix elements that characterize the interaction between the collective modes and the quasiparticles. With these analytical results we evaluate the Landau damping rates of several collective modes for various anisotropic parameters of the trapping potential in terms of the formulas derived from a time-dependent mean-field theory. In addition, we discuss the dependence of the damping rates on temperature, particle number, trapping frequency, and anisotropic parameter of the system, and compare our theoretical results with experimental and numerical data reported in literature.

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I. INTRODUCTION

The theory of elementary excitations, pioneered by Landau, Bogoliubov, and Feynman, is of primary importance in quantum many-body physics. The successful experimental realization of Bose-Einstein condensation (BEC) [1] in dilute atomic gases provides an excellent opportunity for the study of collective modes in trapped, weakly interacting many-body systems [2–17]. One of the challenging problems in this direction is damping of a collective mode and its temperature dependence, which have attracted much attention in recent years in both experiment [18–24] and theory [25–39].

The damping mechanism of collective modes depends on the temperature and density of the system under study. At high temperature and high density, the system is in a *collisional regime* and thus in a local thermodynamic equilibrium. The damping mechanism in this regime is of a dissipative type and the dynamics of collective modes may be described by the theory of two-fluid hydrodynamics [29]. In contrast, if the system is very dilute and at very low temperature, collisions between excitations play a minor role. The damping mechanism of a collective mode in this *collisionless regime* is not related to thermalization processes but to the coupling between the collective mode and thermally excited quasiparticles. Up to now, most experiments with trapped Bose-condensed gases have been performed in this regime [18–24]. There are two damping mechanisms in the collisionless regime, i.e., Landau damping and Beliaev damping. The former occurs by the process of a collective mode being absorbed by a quasiparticle, and then turned into another quasiparticle. The latter arises from the process of a collective mode being absorbed and then two quasiparticles being created. The Beliaev damping can be ignored for low-energy collective modes in a trapped Bose gas because of the discretization of energy levels [25–39].

Many theoretical approaches have been proposed to obtain the Landau damping and its temperature dependence in trapped BECs [25–39] in the collisionless regime. Among them time-dependent mean-field theory is widely employed since it gives an accurate description of the coupled dynamics of condensate and noncondensate components. For calculating Landau damping, various coupling matrix elements describing the interaction between the collective mode and quasiparticles must be calculated, which, however, requires solving the Gross-Pitaevskii (GP) and Bogoliubov–de Gennes (BdG) equations in order to get the ground-state wave function of the condensate and the eigenvalues and eigenfunctions of the quasiparticles, respectively. Because of the inhomogeneous character (i.e., the existence of a trapping potential) of the system, it is very difficult to obtain an analytical solution of these eigenfunctions. Up to now nearly all works on Landau damping were based on numerical simulations [25–38]. Analytical work can be done for repulsive atomic interactions with very large particle number by employing the Thomas-Fermi approximation (TFA). However, the result obtained under the TFA for the Bogoliubov amplitudes of quasiparticles and the coupling matrix elements have uncontrollable divergence [14–17,39].

In recent work, we proposed an analytical approach for calculating the Landau damping of a collective mode in harmonically trapped BECs [39]. However, the method in that work is valid only for an isotropic (i.e., spherically symmetric) harmonic trap. In the present work we generalize the result in Ref. [39] to anisotropic traps. We shall make a detailed calculation of the Landau damping of collective modes in a condensate trapped in a cylindrically symmetric trap and discuss its dependence on the parameters of the system. We stress that such generalization is necessary and also non-

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trivial for the following reasons. (i) Up to now all experimental observations on the Landau damping of collective modes in BECs have been made for anisotropic traps [18–24]. (ii) The linear eigenvalue problem for the collective modes and quasiparticles in an anisotropic BEC is harder to solve and the quasiparticle spectrum is much denser and hence displays a richer structure than in the case of an isotropic BEC. (iii) The Landau damping and its dependence on the physical parameters of the system have much richer characters than those for the isotropic trap. (iv) A general analytical theory is of interest from the theoretical point of view and this theory is useful for the investigation of frequency-shift and mode-coupling problems in BECs and for evaluating the Landau damping in superfluid Fermi gases in a BCS-BEC crossover [40].

The paper is organized as follows. In the next section, for completeness we describe briefly the time-dependent Hatree-Fock-Bogoliubov mean-field theory for Landau damping. In Sec. III we give a variational ground-state wave function of the GP equation beyond the TFA and provide divergence-free analytical solutions for the BdG equations. In Sec. IV we evaluate the Landau damping of several collective modes and discuss its dependence on the temperature, particle number, and anisotropic parameter of the system. A comparison with the experimental and numerical data reported in the literature is also made. Finally, in the last section we give a discussion and summary of our main results.

II. TIME-DEPENDENT MEAN-FIELD THEORY FOR LANDAU DAMPING

We consider a dilute Bose-condensed atomic gas trapped in an anisotropic external potential $V_{ext}(\mathbf{r})$. The grand-canonical Hamiltonian of the system in terms of the boson field operator $\psi(\mathbf{r},t)$ reads

$$H = \int d\mathbf{r}\; \psi^{\dagger}(\mathbf{r},t) H_0 \psi(\mathbf{r},t) + (g/2)\int d\mathbf{r}\; \psi^{\dagger}(\mathbf{r},t)\psi^{\dagger}(\mathbf{r},t)\psi(\mathbf{r},t)\psi(\mathbf{r},t), \tag{1}$$

where $H_0 = -\hbar^2\nabla^2/(2M) + V_{ext}(r) - \mu$ with μ the chemical potential and M the atomic mass. The parameter $g = 4\pi\hbar^2 a_{sc}/M$ describes a contact interaction, with a_{sc} being the s-wave scattering length. $\psi(\mathbf{r},t)$ satisfies the Heisenberg equation of motion

$$i\hbar\partial\psi(\mathbf{r},t)/\partial t = H_0\psi(\mathbf{r},t) + g\psi^{\dagger}(\mathbf{r},t)\psi(\mathbf{r},t)\psi(\mathbf{r},t). \tag{2}$$

To study the dynamics of the Bose-condensed gas at low temperature, one can make the Bogoliubov decomposition $\psi(\mathbf{r},t) = \Phi(\mathbf{r},t) + \tilde{\psi}(\mathbf{r},t)$, where Φ $(\equiv\langle\psi\rangle)$ and $\tilde{\psi}$ represent, respectively, condensate and noncondensate components, satisfying $\langle\tilde{\psi}\rangle = 0$. Here the symbol $\langle\cdots\rangle$ $(\langle\cdots\rangle_0)$ denotes the nonequilibrium (equilibrium) average. Then by applying a self-consistent time-dependent Hartree-Fock-Bogoliubov mean-field approximation, one obtains the equation of motion for the condensate wave function Φ,

$$i\hbar\partial\Phi/\partial t = H_0\Phi + g|\Phi|^2\Phi + 2g\Phi\tilde{n}(\mathbf{r},t) + g\Phi^*\tilde{m}(\mathbf{r},t), \tag{3}$$

where $\tilde{n}(\mathbf{r},t) \equiv \langle\tilde{\psi}^{\dagger}\tilde{\psi}\rangle$ and $\tilde{m}(\mathbf{r},t) \equiv \langle\tilde{\psi}\tilde{\psi}\rangle$ denote the normal and anomalous (thermal) particle densities, respectively.

By taking the Bogoliubov transformation $\tilde{\psi}(\mathbf{r},t) = \Sigma_j[u_j(\mathbf{r})\alpha_j(t) + v_j^*(\mathbf{r})\alpha_j^{\dagger}(t)]$ and $\tilde{\psi}^{\dagger}(\mathbf{r},t) = \Sigma_j[u_j^*(\mathbf{r})\alpha_j^{\dagger}(t) + v_j(\mathbf{r})\alpha_j(t)]$, where the quasiparticle operators $\alpha_j, \alpha_j^{\dagger}$ satisfy Bose commutation relations and the Bogoliubov amplitudes u_j and v_j satisfy the normalization condition $\int dr[u_i^*(\mathbf{r})u_j(\mathbf{r}) - v_i^*(\mathbf{r})v_j(\mathbf{r})] = \delta_{ij}$, we obtain the equations of motion describing the time evolution of the normal and anomalous densities:

$$i\hbar\partial f_{ij}(t)/\partial t = \langle[\alpha_i^{\dagger}(t)\alpha_j(t), H]\rangle, \tag{4a}$$

$$i\hbar\partial g_{ij}(t)/\partial t = \langle[\alpha_i(t)\alpha_j(t), H]\rangle, \tag{4b}$$

where $f_{ij}(t) = \langle\alpha_i^{\dagger}(t)\alpha_j(t)\rangle - f_i^0\delta_{ij}$ and $g_{ij}(t) = \langle\alpha_i(t)\alpha_j(t)\rangle$ and f_i^0 is the equilibrium density of quasiparticles, whose explicit expression will be given below.

To study the Landau damping of a collective mode, we must consider the coupling between the collective mode and the thermal excited quasiparticles. With this aim we assume $\Phi = \Phi_0(\mathbf{r}) + \delta\Phi(\mathbf{r},t)$, where $\delta\Phi(\mathbf{r},t)$ is a small fluctuation denoting the collective mode. We make the decomposition $\tilde{n}(\mathbf{r},t) = \tilde{n}_0(\mathbf{r}) + \delta\tilde{n}(\mathbf{r},t)$ and $\tilde{m}(\mathbf{r},t) = \tilde{m}_0(\mathbf{r}) + \delta\tilde{m}(\mathbf{r},t)$, where $\tilde{n}_0(\mathbf{r}) = \langle\tilde{\psi}^{\dagger}\tilde{\psi}\rangle_0$ and $\tilde{m}_0(\mathbf{r}) = \langle\tilde{\psi}\tilde{\psi}\rangle_0$ denote the static distributions of the normal and anomalous thermal particle densities, respectively. Then Eq. (3) yields

$$[H_0 + gn_0(\mathbf{r})]\Phi_0(\mathbf{r}) = 0, \tag{5}$$

$$\begin{aligned} i\hbar\partial\delta\Phi/\partial t = {} & \{H_0 + 2g[n_0(\mathbf{r}) + \tilde{n}_0(\mathbf{r})]\}\delta\Phi + g[n_0(\mathbf{r}) \\ & + \tilde{m}_0(\mathbf{r})]\delta\Phi^* + g\Phi_0\sum_{ij}\{2[u_i^*u_j + v_i^*v_j + v_i^*u_j]f_{ij}(t) \\ & + [2v_iu_j + u_iu_j]g_{ij}(t) + [2u_i^*v_j^* + v_i^*v_j^*]g_{ij}^*(t)\}, \end{aligned} \tag{6}$$

where $n_0(\mathbf{r}) = |\Phi_0(\mathbf{r})|^2$. Equation (5) is a time-independent GP equation determining the ground-state wave function $\Phi_0(\mathbf{r})$ of the condensate. Because at very low temperature the time-independent normal and anomalous thermal particle densities $\tilde{n}_0(\mathbf{r})$ and $\tilde{m}_0(\mathbf{r})$ are small in comparison with the condensate density $n_0(\mathbf{r})$ and hence negligible in Eq. (6), we take them as zero in the following calculation for simplicity. Notice that it is the time-dependent normal thermal particle density $\delta\tilde{n}(\mathbf{r},t)$ and the anomalous thermal particle density $\delta\tilde{m}(\mathbf{r},t)$ that contribute the Landau damping of the collective mode.

If u_j and v_j are chosen to satisfy the BdG equations

$$Lu_j(\mathbf{r}) + gn_0(\mathbf{r})v_j(\mathbf{r}) = \epsilon_j u_j(\mathbf{r}), \tag{7a}$$

$$Lv_j(\mathbf{r}) + gn_0(\mathbf{r})u_j(\mathbf{r}) = -\epsilon_j v_j(\mathbf{r}), \tag{7b}$$

with $L = -\hbar^2\nabla^2/(2m) + V_{ext}(\mathbf{r}) - \mu + 2gn_0(\mathbf{r})$, the Hamiltonian of the system can be expressed as $H = \text{const} + \Sigma_j\epsilon_j\alpha_j^{\dagger}\alpha_j + H'$, where ϵ_j is the eigenenergy of the quasiparticle and H' is an interacting term. Then the commutators on the right-hand side of Eqs. (4a) and (4b) can be obtained immediately; they are omitted here for saving space.

When Φ_0, i.e., the ground-state wave function of the condensate, is obtained by solving the time-independent GP Eq. (5), we can obtain the solutions of Eqs. (6), (4a), and (4b). This can be done by using a perturbation theory combined with a Fourier transform. We suppose that a collective mode of the condensate with oscillating frequency ω_0 is excited, i.e.,

$$\delta\Phi(\mathbf{r},t) = u_{osc}(\mathbf{r})\exp(-i\omega_0 t),$$

$$\delta\Phi^*(\mathbf{r},t) = v_{osc}(\mathbf{r})\exp(-i\omega_0 t). \quad (8)$$

It is easy to show that (u_{osc}, v_{osc}) obeys also the BdG Eq. (7). Then we obtain the frequency correction of the collective mode, i.e., $\omega=\omega_0+\eta-i\gamma$. The Landau damping rate of the collective mode is given by [26,39]

$$\gamma_L = \sum_{ij} \gamma_{ij}\delta(\omega_0+\omega_i-\omega_j), \quad (9)$$

with $\omega_j=\epsilon_j/\hbar$, and

$$\gamma_{ij} = (4\pi g^2/\hbar^2)|A_{ij}|^2(f_i^0 - f_j^0) \quad (10)$$

with $f_j^0=\langle\alpha_j^+\alpha_j\rangle_0=\{\exp[\epsilon_j/(k_BT)]-1\}^{-1}$. γ_{ij} is called the damping strength of the transition from state $|i\rangle$ to state $|j\rangle$ [32]. A_{ij} is the coupling matrix element describing the energy transfer between the collective mode and quasiparticles, whose expression is given by

$$A_{ij} = \int d\mathbf{r}\,\Phi_0[u_{osc}(u_iu_j^* + v_iv_j^* + v_iu_j^*) + v_{osc}(u_iu_j^* + v_iv_j^* + u_iv_j^*)]. \quad (11)$$

We see that only three-mode resonant interactions satisfying the resonance conditions $\omega_0+\omega_i-\omega_j=0$ contribute to the Landau damping.

III. DIVERGENCE-FREE ANALYTICAL SOLUTIONS OF THE BdG EQUATIONS AND COUPLING MATRIX ELEMENTS

A. Divergence-free analytical solutions of the BdG equations

In order to evaluate the Landau damping rate γ_L one must calculate the coupling matrix elements A_{ij}, which, however, requires one to solve the GP equation (5) and the BdG equations (7a) and (7b) to determine the ground-state wave function Φ_0 and the eigenvalues ϵ_j and eigenfunctions (u_j, v_j) of the quasiparticles, respectively. Exact analytical solutions for them are not easy to get because of the existence of the trapping potential. Up to now most analytical results obtained in literature are based on the TFA, i.e., neglecting the kinetic energy terms in the equations for both the ground state and the excitations. This approximation, however, is not satisfactory for the following reasons. (i) The TFA is valid only for very large (infinite large, theoretically) particle number. (ii) Under such approximation the Bogoliubov amplitudes (i.e., u_j and v_j) vary sharply at the boundary of the condensate. The kinetic energy of both the condensate and the excitations at the boundary is significant and hence cannot be simply neglected. (iii) There appears a singular point in the solutions of the Bogoliubov amplitudes at the boundary [14,15], which makes the theory uncontrollable. (iv) The existence of the singular point in the Bogoliubov amplitudes results in a divergence in the coupling matrix elements, which prevents us from getting the Landau damping rate of the collective mode in the system. Recently, this problem has been investigated in Refs. [16,17] and divergence-free solutions have been obtained beyond the TFA. In the following we give a simple description of some results related to the ground-state wave function and the eigenvalues and eigenfunctions of the BdG equations.

We consider a trapping potential of an axial symmetry with the form $V_{ext}(\mathbf{r})=m\omega_{ho}^2 r^2/2$, with $r^2=x^2+y^2+\lambda^2z^2$. Here ω_{ho} is the trapping frequency in the radial (i.e., x-y) direction, and λ is an anisotropic parameter [i.e., the ratio of the trap frequency in the radial direction to the one in the axial (i.e., z) direction]. By rescaling the variables $\bar{r}=r/R_0$, $\bar{\boldsymbol{\nabla}}=R_0\boldsymbol{\nabla}$, and introducing $\zeta=\hbar\omega_{ho}/2\mu$ (here $R_0=\sqrt{2\mu/M\omega_{ho}^2}$ is the characteristic radius of the condensate), the GP equation (5) is transformed into the dimensionless form

$$\zeta^2\sigma(\bar{\mathbf{r}}) + \bar{r}^2 - 1 + |\Phi_0(\bar{\mathbf{r}})/\Phi_0(0)|^2 = 0, \quad (12)$$

where $\sigma(\bar{\mathbf{r}})=-[\bar{\boldsymbol{\nabla}}^2\Phi_0(\bar{\mathbf{r}})]/\Phi_0$ is a quantity proportional to the kinetic energy (i.e., zero-point pressure) in the ground state of the condensate.

We solve Eq. (12) beyond the TFA by using a Fetter-like *variational* ground-state wave function [16,17]

$$\Phi_0(\bar{\mathbf{r}}) = C_0\sqrt{\lambda}(1-\bar{r}^2)^{(q+1)/2}\Theta(1-\bar{r}), \quad (13)$$

where $\bar{r}^2=\bar{s}^2+\lambda^2\bar{z}^2$ with $\bar{s}=s/R_0$ and $\bar{z}=z/R_0$, $C_0=\{\lambda N_0/[2\pi R_0^3 B(3/2,2+q)]\}^{1/2}$ is a normalized constant with $B(3/2,2+q)$ being the Beta function and $N_0=R_0^3\int d\bar{\mathbf{r}}|\Phi_0(\bar{\mathbf{r}})|^2$ being the particle number in the condensate. q is a variational parameter, determined by minimizing the ground-state energy. Once q is determined, the ground-state energy is given by $\mu=\hbar\omega_{ho}[4\lambda P/B(3/2,2+q)]^{2/5}/2$, where $P=N_0a_{sc}/a_{ho}$ is the dimensionless interatomic interaction strength.

By defining $\phi_j^{\pm}=u_j\pm v_j$ and $\bar{\omega}_j=\omega_j/\omega_{ho}$, the BdG equation (7) gains the dimensionless form

$$-\bar{\nabla}^2(1-\bar{r}^2)\phi_j^+ - (1-\bar{r}^2)\sigma\phi_j^+ + (\zeta^2/2)\times[\bar{\nabla}^4 + 3\bar{\nabla}^2\sigma + \sigma\bar{\nabla}^2 + 3\sigma^2]\phi_j^+ = 2\bar{\omega}_j^2\phi_j^+, \quad (14a)$$

$$-\bar{\nabla}^2(1-\bar{r}^2)\phi_j^- - (1-\bar{r}^2)\sigma\phi_j^- + (\zeta^2/2)\times[\bar{\nabla}^4 + \bar{\nabla}^2\sigma + 3\sigma\bar{\nabla}^2 + 3\sigma^2]\phi_j^- = 2\bar{\omega}_j^2\phi_j^-. \quad (14b)$$

Using the ground-state wave function given by Eq. (13) we can get the expression for $\sigma(\bar{\mathbf{r}})$. Notice that at low temperature only low-energy collective excitations are relevant and hence we have $\hbar\omega_{ho}\ll\hbar\omega_j\ll\mu$. Solving Eqs. (14a) and (14b) by taking ζ^2 as a small quantity, we obtain the leading-order solution of the eigenfunctions of Eq. (14),

$$\phi^{\pm}_{n_z n_s m}(\mathbf{r}) = \frac{(\zeta\bar{\omega}^{(0)}_{n_z n_s m})^{\pm 1/2}}{\sqrt{2\pi R_0^3 I_{n_z n_s m}}}(1-\bar{s}^2-\lambda^2\bar{z}^2)^{(q\mp 1)/2}\bar{s}^m P^{(2n_s)}_{n_p} \times(\bar{s},\bar{z})e^{im\varphi}, \quad (15)$$

with $\bar{\omega}^{(0)}_{n_z n_s m}$ being the leading-order eigenvalue and $I_{n_z n_s m}$ being a normalized coefficient. Here n_p $(=0,1,2,\ldots)$ is the principal quantum number, n_s $(=0,1,2,\ldots,\mathrm{int}[n_p/2])$ is the radial quantum number, $n_z=n_p-2n_s$ is the axial quantum number, and m $(=0,\pm1,\pm2,\ldots)$ is the azimuthal quantum number. The coupled axial and radial functions $P^{(2n_s)}_{n_p}$ satisfy a two-dimensional differential equation, which has a solution with the form $P^{(2n_s)}_{n_p}(\bar{s},\bar{z})=\Sigma_{k=0}^{n_p}\Sigma_{n_s=0}^{\mathrm{int}[k/2]}b_{k,n_s}\bar{z}^{k-2n_s}\bar{s}^{2n_s}$. The coefficient $b_{k,n}$ satisfies the iterate equation

$$4(n+1)(n+|m|+1)b_{k+2,n+1}+(k-2n+2)(k-2n+1)b_{k+2,n} = 4\lambda^2(n+1)(n+|m|+1)b_{k,n+1}-[X-4n(n+|m|+1+q) - \lambda^2(k-2n)(k-2n+1+2q)]b_{k,n}+(k-2n+2)(k-2n + 1)b_{k,n-1}, \quad (16)$$

with $X=2(\bar{\omega}^{(0)}_{n_z n_s m})^2-2|m|(1+q)$. For the detailed expressions of $\bar{\omega}^{(0)}_{n_z n_s m}$, $I_{n_z n_s m}$, and $b_{k,n}$, see Refs. [16,17].

B. Dimensionless Landau damping formula and coupling matrix elements

For the convenience of later calculation, we write the Landau damping rate and coupling matrix elements in dimensionless forms, which can be obtained by taking $\bar{A}_{ij}=A_{ij}/a_{ho}^3$, $\bar{\gamma}_{ij}=\gamma_{ij}/\omega_{ho}^2$, and $\bar{\gamma}=\gamma/\omega_{ho}$. Then we have

$$\bar{\gamma}_L \equiv \gamma_L/\omega_{ho} = \sum_{ij}\bar{\gamma}_{ij}\delta(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j), \quad (17)$$

where

$$\bar{\gamma}_{ij} = 4\pi(4\pi a_{sc}/a_{ho})^2|\bar{A}_{ij}|^2(f_i^0-f_j^0), \quad (18)$$

with

$$\bar{A}_{ij} = \frac{[4P/B(3/2,2+q)]^{1/10}}{8\pi\lambda^{9/10}[I_0 I_i I_j\bar{\omega}_0\bar{\omega}_i e\omega_j]^{1/2}}\left(\frac{N_0}{P}\right)^{1/2} \times\int_0^1 dx\int_0^1 dy\, W_0 W_i W_j^* F_{ij}(x,y). \quad (19)$$

In the above formulas we have defined $x=\bar{s}^2$, $y=\lambda\bar{z}/\sqrt{1-x}$, $W_j=\bar{s}^m P^{(2n_s)}_{n_p}(\bar{s},\bar{z})e^{im\varphi}$ $[j\equiv(n_z,n_s,m)]$, and $F_{ij}(x,y)=3\zeta^2\bar{\omega}_0\bar{\omega}_i\bar{a}_j(1-x)^{2q-1/2}(1-y^2)^{2q-1}+(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)(1-x)^{2q+3/2}(1-y^2)^{2q+1}$. f_j^0 is expressed as $f_j^0=[\exp(2\zeta\bar{\omega}_j/\bar{T})-1]^{-1}$, with $\bar{T}=k_BT/\mu$ being the dimensionless temperature.

For a practical calculation one must have a way to express the Dirac δ function appearing in Eq. (17). By the formula $(1/\pi)\lim_{\bar{\Delta}\to 0}(\bar{\Delta}/2)/[(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)^2+(\bar{\Delta}/2)^2]=\delta(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)$, we have $\bar{\gamma}_L=\lim_{\bar{\Delta}\to 0}\bar{\gamma}_L(\bar{\Delta})$, with

$$\bar{\gamma}_L(\bar{\Delta}) = \frac{1}{\pi}\sum_{ij}\bar{\gamma}_{ij}\frac{\bar{\Delta}/2}{[(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)^2+(\bar{\Delta}/2)^2]}. \quad (20)$$

Note that there exist some selection rules for the coupling matrix elements $\bar{A}_{ij}$ resulting from the integration for the azimuthal angle φ. From Eq. (19) with the expression of W_j, one has $\bar{A}_{ij}\propto\int_0^{2\pi}e^{i(m_0+m_i-m_j)\varphi}d\varphi$. Thus $\bar{A}_{ij}$ is nonvanishing only when

$$m_0 = m_j - m_i. \quad (21)$$

IV. RESULTS FOR LANDAU DAMPING RATE

We now make a detailed calculation of the Landau damping rate of collective modes for the BEC with an anisotropic trap in terms of the formulas presented in the last section. Note that all our results are obtained beyond the TFA, the divergence problem in the Bogoliubov amplitudes and the coupling matrix elements encountered in previous studies [14,15] disappear. In addition, unlike in the numerical approaches given in Refs. [25,27,32,35], in our present theory $\bar{A}_{ij}$ can be evaluated analytically based on the variational ground-state wave function and the explicit expressions for the eigenfunctions of the quasiparticles.

A. Damping strength for various transitions

We consider a gas of alkali-metal atoms (such as ^{87}Rb or ^{23}Na) trapped in a cylindrical symmetric harmonic potential. The particle number in the condensate at temperature T is given by $N_0(T)=N[1-(T/T_c^0)^3]$, where N is the total particle number of the system and T_c^0 is the critical temperature of BEC transition. The collective mode we are interested in is the breathing mode with the oscillating frequency $\omega_0=\omega_\pm=\omega_{ho}[2+3\lambda^2/2\pm(16-16\lambda^2+9\lambda^4)^{1/2}]^{1/2}$. The ω_+ mode, which corresponds to the $(n_z,n_s,m)=(2,0,0)$ mode for $\lambda>1$ or the $(n_z,n_s,m)=(0,1,0)$ mode for $\lambda<1$, is called the high-lying $m=0$ mode; The ω_- mode, which corresponds to $(n_z,n_s,m)=(0,1,0)$ mode for $\lambda>1$ or $(n_z,n_s,m)=(2,0,0)$ mode for $\lambda<1$, is called the low-lying $m=0$ mode [7,14,15]. Since in practice the frequency of a collective mode has a finite linewidth, the phase-matching conditions for three-mode resonant interactions, $\omega_0+\omega_i-\omega_j=0$, cannot be exactly satisfied. Thus a small mismatch for the three-mode resonant conditions should be introduced. Under this consideration, one can assume that resonances contributing to the Landau damping occur in the interval $0.82\omega_0<\omega_{ij}<1.18\omega_0$ [32], where $\omega_{ij}=\omega_j-\omega_i$. Noting that the eigenfunctions of the quasiparticles for the levels with large quantum numbers (n_z,n_s,m) have fast oscillations and their maxima are far away from the center of the condensate, the coupling matrix elements for transitions between the levels with larger (n_z,n_s,m) are thus small. In addition, the levels with larger (n_z,n_s,m) have smaller Bose occupation factors f_j^0. Therefore, the contribution to the damping strength by the energy levels of large quantum numbers (n_z,n_s,m) is not significant. Through a suitable estimation, the levels corresponding to $(0,0,m)$, $(1,0,m)$, $(0,1,m)$, $(2,0,m)$, $(1,1,m)$, $(3,0,m)$, $(0,2,m)$, $(2,1,m)$, $(4,0,m)$, $(1,2,m)$, $(3,1,m)$, $(5,0,m)$, $(0,3,m)$, $(2,2,m)$, $(4,1,m)$, $(6,0,m)$, $(1,3,m)$, $(3,2,m)$, $(5,1,m)$, and $(7,0,m)$ with $m<15$ are chosen in our calculation.

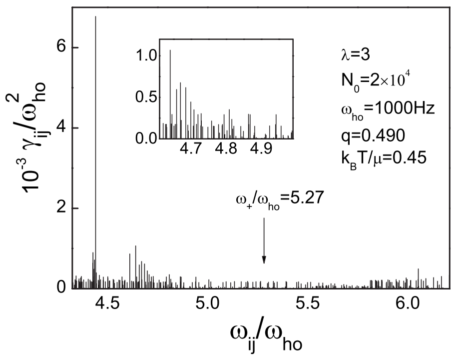

FIG. 1. Histogram of the dimensionless damping strength $\gamma_{ij}/\omega_{ho}^2$ of the ω_+ mode for ^{87}Rb atomic gas as a function of dimensionless frequency difference ω_{ij}/ω_{ho}, allowed by the resonance conditions and selection rules. The parameters of the system are $a_{sc}=5.82\times10^{-9}$ m, $\omega_{ho}=1000$ Hz, $a_{ho}=8.52\times10^{-7}$ m, $\lambda=3$, $N_0=2\times10^4$, and $\bar{T}=k_BT/\mu=0.45$. The arrow points to the value of the collective mode $\omega_+/\omega_{ho}=5.27$. The inset shows the detail of the damping strength in the interval $4.6\leqslant\omega_{ij}/\omega_{ho}\leqslant5.0$.

In Fig. 1 we have shown the histogram of the dimensionless damping strength $\bar{\gamma}_{ij}$ for the ω_+ mode of ^{87}Rb atomic gas ($a_{sc}=5.82\times10^{-9}$ m) as a function of the transition frequency difference $\bar{\omega}_{ij}$ at $k_BT/\mu=0.45$, which corresponds to $T=100$ nK. In the calculation we have taken $N_0=2\times10^4$, $\omega_{ho}=1000$ Hz, and $\lambda=3$ (i.e., a disk-shaped trap with $a_{ho}=8.52\times10^{-7}$ m). The variational parameter is given by $q=0.490$. The positions of the bars correspond to the allowed transition frequencies $\bar{\omega}_{ij}$, whereas their heights define the values of $\bar{\gamma}_{ij}$. The arrow in the figure points to the frequency of the collective mode, $\bar{\omega}_+=5.27$. The relatively large values of $\bar{\gamma}_{ij}$ correspond to the transitions between the levels $(2,0,1)\rightarrow(0,0,1)$, $(4,1,0)\rightarrow(2,0,1)$, $(5,1,0)\rightarrow(3,0,0)$, and $(5,1,1)\rightarrow(3,0,1)$.

Figure 2 shows the damping strength for the ω_+ mode for different anisotropic parameters λ, with the other parameters being the same as in Fig. 1. The results plotted in Figs. 2(a)–2(c) are for cigar-shaped ($\lambda=0.5$), sphere-shaped ($\lambda=1$), and disk-shaped ($\lambda=3$) traps, respectively. From the figure we see that the density of the vertical bars increases as λ increases. The reason is that as λ increases the spectrum density of the quasiparticles also increases, resulting in a growth of the number of the transitions allowed by the resonance conditions and selection rules.

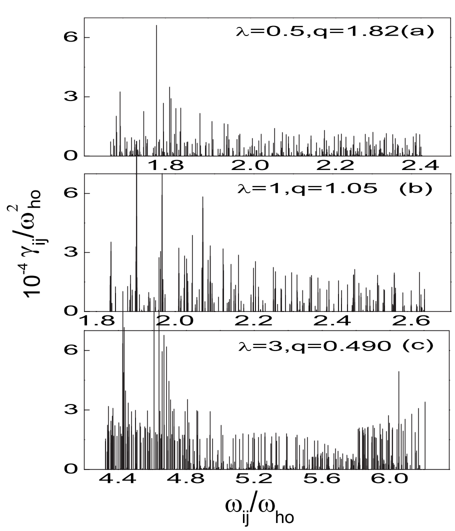

FIG. 2. Same as Fig. 1 but for different anisotropic parameters. $\lambda=$(a) 1/2; (b) 1; (c) 3.

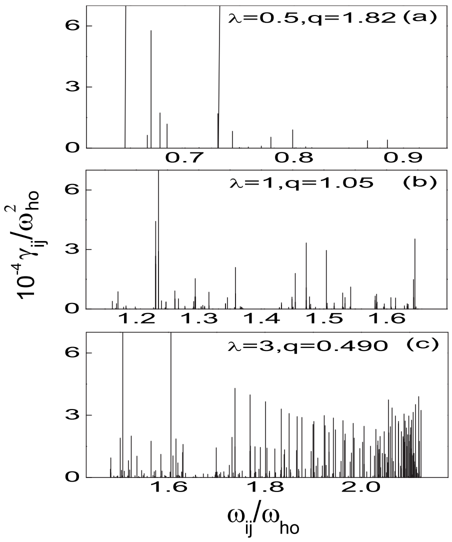

FIG. 3. Same as Fig. 2 but for the ω_- mode.

Shown in Fig. 3 is the result for the damping strength for the ω_- mode with different λ. The parameters are the same as in Fig. 1. Figures 3(a)–3(c) are for cigar-shaped ($\lambda=0.5$), sphere-shaped ($\lambda=1$), and disk-shaped ($\lambda=3$) traps, respectively. We see that for the ω_- mode the density of the vertical bars in the figure is lower than that for the ω_+ mode (see Fig. 2). This is because the number of transitions allowed by the resonance conditions and selection rules for the ω_- mode is much less than for the ω_+ mode.

B. $\bar{\gamma}_L$ vs $\bar{\Delta}$ for different dimensionless temperatures $\bar{T}$

To obtain the Landau damping rate $\bar{\gamma}_L$ one needs to evaluate the value of $\bar{\gamma}(\bar{\Delta})$ in Eq. (20). If the variation of $\bar{\gamma}_L(\bar{\Delta})$ with respect to $\bar{\Delta}$ is weak, an extrapolation back to $\bar{\Delta}\rightarrow0$ can be made and hence the value of $\bar{\gamma}_L$ can thus be obtained [32]. Shown in Fig. 4 is the result of $\bar{\gamma}_L(\bar{\Delta})$ vs $\bar{\Delta}$ for the ω_+ mode of the ^{87}Rb atomic gas. The parameters are the same as in Fig. 1 but for $\bar{T}=0.225$ (solid diamonds), 0.45 (solid squares), 0.675 (solid circles), and 0.9 (solid triangles), which correspond to $T=50$, 100, 150, and 200 nK, respectively. We see that the variation of $\bar{\gamma}_L(\bar{\Delta})$ is weak when $\bar{\Delta}$ lies between 0.04 and 0.20. In fact, $\bar{\gamma}_L(\bar{\Delta})$ has only a weak dependence on $\bar{\Delta}$ if the condition $\Delta\bar{\omega}\ll\bar{\Delta}\ll1$ is satisfied, where $\Delta\bar{\omega}$ is the average distance of the transitions, a small quantity because of the finite lifetime of the quasiparticles. By fitting the data of $\bar{\gamma}_L(\bar{\Delta})$ to an approximated straight line and extrapolating it back to $\bar{\Delta}=0$, we can obtain the Landau damping rate of the collective mode for a given $\bar{T}$. For ex-

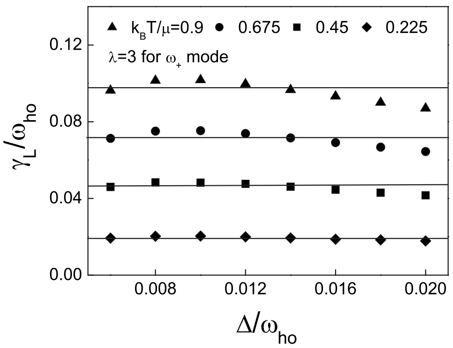

FIG. 4. Dimensionless Landau damping rate γ_L/ω_{ho} of the ω_+ mode as a function of the Lorentz width Δ/ω_{ho} with $N_0=2\times10^4$, $\lambda=3$, and $\omega_{ho}=1000$ Hz for the different dimensionless temperatures: solid diamonds, $\bar{T}=k_BT/\mu=0.225$; solid squares, $\bar{T}=0.45$; solid circles, $\bar{T}=0.675$; solid triangles, $\bar{T}=0.9$.

ample, when $\bar{T}=0.45$ (i.e., $T=100$ nK), we get $\bar{\gamma}_L=0.05$, which corresponds to the dimensional Landau damping rate $\gamma_L=50\ \mathrm{s}^{-1}$.

C. $\bar{\gamma}_L$ vs $\bar{T}$ for different N_0 and ω_{ho}

With the above results it is easy to discuss the temperature dependence of the Landau damping rates for different N_0 and ω_{ho} when the anisotropic parameter λ is given. In Fig. 5, we have shown the result of the damping rate for the ω_+ mode as a function of $\bar{T}=k_BT/\mu$ with $\lambda=3$ for different N_0 and ω_{ho}. Three different cases are plotted. The curves connected by solid triangles, circles, and squares are for $N_0=2\times10^4$ and $\omega_{ho}=1000$ Hz, $N_0=4\times10^4$ and $\omega_{ho}=1000$ Hz, $N_0=2\times10^4$ and $\omega_{ho}=2000$ Hz, respectively. From the figure we can obtain the following conclusions. (i) For given N_0 and ω_{ho}, the Landau damping rate grows with temperature. This is expected because the number of quasiparticles available in the system becomes larger when T increases. (ii) The damping rates for different atomic numbers at the same temperature display no significant difference. This can be seen, for example, by looking at the points a and b in Fig. 5, which represent the case of different N_0 but with the same temperature. (iii) The damping rates increase as the trapping frequency increases. The reason is that the atomic density increases as the trapping frequency becomes larger. This situation has been shown by the points b and c in the figure, which have the same temperature T [41] and the same particle number but different trapping frequency.

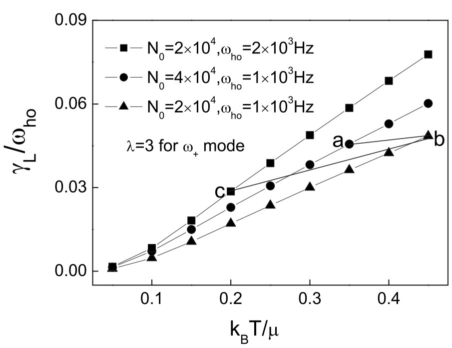

FIG. 5. Dimensionless Landau damping rate γ_L/ω_{ho} of the ω_+ mode as a function of dimensionless temperature k_BT/μ with $\lambda=3$ for different atom numbers and trapping frequencies.

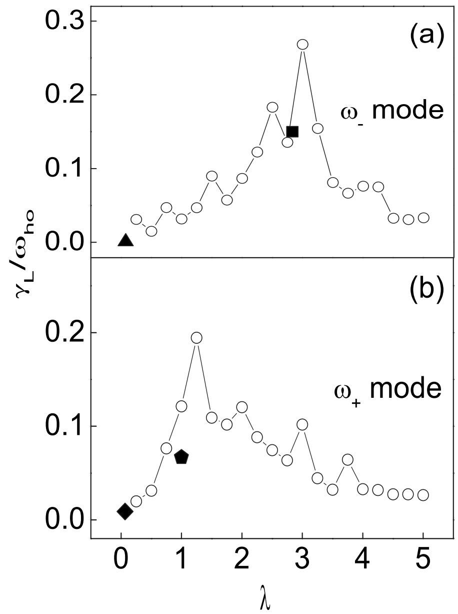

FIG. 6. Theoretical result (open circles) of the dimensionless Landau damping rates γ_L/ω_{ho} for the ω_- mode (a) and the ω_+ modes (b) as functions of the anisotropic parameter λ. The parameters are given by $N_0=2\times10^4$, $\omega_{ho}=1000$ Hz, and $T=200$ nK. The solid square, solid triangle, and solid diamond are the experimental results reported in Refs. [20,21,24], respectively. The solid star is the numerical date of Ref. [32].

D. Relation between Landau damping rate $\bar{\gamma}_L$ and anisotropic parameter λ

With the theoretical approach developed above we can also obtain the relation between the Landau damping rate and the anisotropic parameter of the trapping potential, i.e., $\bar{\gamma}_L=\bar{\gamma}_L(\lambda)$. Shown in Fig. 6 are the Landau damping rates of the ω_- mode [Fig. 6(a)] and the ω_+ mode [Fig. 6(b)] as functions of λ. The parameters are chosen as $\omega_{ho}=1000$ Hz, $N_0=2\times10^4$, and $T=200$ nK. The open circles are the calculated results based on the analytical formulas given in the last section. From the figure we see that for very small λ, the Landau damping is very small. As λ increases, the damping rates also increase. At $\lambda=\lambda_m$, the damping reaches a maximum $\bar{\gamma}_{Lm}$. Note that λ_m and $\bar{\gamma}_{Lm}$ are different for different collective modes. As λ increases further, the damping rates decrease to smaller values.

The interesting behavior of the Landau damping for different values of the anisotropic parameter shown in Fig. 6 can be understood through a detailed analysis on the Landau damping formula, given by Eq. (17), together with the expressions of the damping strength (18) and the coupling matrix element (19). By inspection of these formulas we see that the eigenfrequencies (or eigenenergies) of quasiparticles, the Bose occupation factor, and the number of transitions allowed by the resonant conditions and selection rules are functions of λ and have contributions to the Landau damping rates. As λ increases, the quasiparticle eigenfrequencies, the coupling matrix elements, and the number of allowed transitions also increase, with relatively small increasing rates. In Contrast, the Bose occupation factor decreases as λ increases. The decreasing rate of the Bose occupation factor is small for small λ, but grows fast for large λ. The competition among these factors result in the appearance of a maximum in the Landau damping rates. For given parameters the maximum occurs at $\lambda \simeq 3$ ($\lambda \simeq 1.2$) for the ω_- (ω_+) mode.

For comparison, in Fig. 6 we have also shown some experimental and numerical data reported in the literature. The solid square, solid triangle, and solid diamond in the figure are the experimental results of Refs. [20,21,24], respectively. The solid star is the numerical datum of Ref. [32]. We see that our theoretical results agree fairly well with the experimental and numerical ones [42]. Notice that, when $\lambda \approx 0$, the ω_+ mode ($\omega_+ = 2\omega_{ho}$) corresponds to the transverse breathing investigated experimentally in Ref. [24]. One can see that there is little deviation between our theory and the experiment presented in Ref. [24], shown by the solid diamond in the lower panel of Fig. 6. The reason is that for a good variational ground-state wave function the variational parameter q should be small. However, in our present theoretical scheme q is large for very small λ, resulting in the deviation. For very small λ, one should extend the present theory, e.g., to take the Thomas-Fermi radius R_0 as a new variational parameter.

V. SUMMARY

In this work, we have developed an analytical method for calculating the Landau damping of low-energy collective modes in anisotropic Bose-Einstein condensates in the frame of a time-dependent Hatree-Fock-Bogoliubov mean-field theory. Based on a variational ground-state wave function obtained by solving the time-independent Gross-Pitaevskii equation beyond the Thomas-Fermi approximation, we have solved the Bogoliubov–de Gennes equations that determine the eigenvalues and eigenfunctions of thermal excited quasiparticles. The divergence-free explicit solutions of Bogoliubov amplitudes and coupling matrix elements that characterize the interaction between the collective modes and the quasiparticles have been provided. With these analytical results we have evaluated the Landau damping rates of several collective modes for various anisotropic parameters of the trapping potential. In addition, we have discussed the dependence of the damping rates on temperature, particle number, trapping frequency, and anisotropic parameter of the system. We have also made comparison between our theoretical result and the experimental and numerical data reported in the literature and found good agreement. It must be emphasized that the analytical formulas presented in this work allow one to discuss clearly the important role played by various damping processes. The general analytical theory developed here is instructive and useful for the investigation of frequency-shift and mode-coupling problems in BECs and for evaluating the Landau damping in superfluid Fermi gases in a BCS-BEC crossover.

ACKNOWLEDGMENTS

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[41] The relation between T and $\bar{T}$ is given by $T=\bar{T}\hbar(\omega_{ho}/2k_B)\times[4\lambda P/B(3/2,2+q)]^{2/5}$.
[42] When plotting the experimental and numerical data on Fig. 6 a conversion of relevant physical quantities to our case has been made. In the conversion the difference due to different particle numbers was not considered since at the same temperature the Landau damping rate is not sensitive to the change of particle number, as shown in Sec. IV C. The difference induced by different trapping frequencies in the radial direction has been taken into account because the damping rates have been expressed in units of ω_{ho}.