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Collective Modes of a Quasi-Two-Dimensional Superfluid Fermi Gas in BCS-BEC Crossover *

ZHOU Yu(周昱)[1], MA Xiao-Dong(马晓栋)[1,2], HUANG Guo-Xiang(黄国翔)[1**]

[1] *Department of Physics, East China Normal University, Shanghai 200062*

[2] *Department of Physics, Xinjiang Normal University, Ulrumchi 830054*

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We investigate the collective modes of a quasi-two-dimensional (Q2D) superfluid Fermi gas in Bardeen–Cooper–Schrieffer Bose–Einstein condensation (BCS-BEC) crossover. For solving a generalized Gross–Pitaevskii equation by using a time-dependent variational method, we take a trial wavefunction with the form of hybrid Gaussian-parabolic type, which not only reflects the Q2D character of the system and also allows an essentially analytical approach of the problem. We present a Q2D criterion that is valid for various superfluid regimes and displays clearly the relation between the maximum condensed particle number and the parameters of trapping potential as well as atom–atom interaction. We show that due to the small particle number in the Q2D condensate, the contribution to oscillating frequencies by the quantum pressure in the strong confinement direction is significant and hence a Thomas–Fermi approximation can not be used.

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Since the first experimental realization of the quantum degenerate Fermi gas in a trap,[1] much interest has been focused on the study of ultracold fermionic atoms and fermionic superfluidity.[2−4] Since for dilute atomic systems the atom–atom interaction, characterized by s-wave scattering length a_s, can be tuned by magnetic-field-induced Feshbach resonance, one can manipulate the interaction strength over the range from $-\infty$ to $+\infty$ in a controllable way. Using this technique condensed fermionic pairs and the Bardeen–Cooper–Schrieffer Bose–Einstein condensation (BCS-BEC) crossover has been realized in a series of beautiful experiments.[2,3] At the same time, the collective excitations in various superfluid regimes have also been investigated intensively.[2,4−8] These dramatic progress raises an important question about the role of dimensionality effect in ultracold fermionic atom gases.

In this Letter, we investigate the collective excitations of a quasi-two-dimensional (Q2D) superfluid Fermi gas in BCS-BEC crossover. Based on a superfluid order-parameter equation and a polytropic approximation for equation of state, we calculate the excitation spectrum by using a time-dependent variational method. We present a unified Q2D criterion, which is valid for the various crossover regimes. We demonstrate that, due to the small particle number in the Q2D condensate, the modification to oscillating frequencies by the quantum pressure is significant and hence a Thomas-Fermi approximation (TFA)[5] cannot be used.

In the ground state of a superfluid fermionic atom gas, all particles are paired with $n/2$ being pair density. These pairs, called the condensed fermionic atom pairs, are originated from two-component fermionic atom systems (i.e. ^{6}Li or ^{40}K) with different internal states. By means of Feshbach resonance one can easily realize the transitions from BCS to BEC regimes. When $a_s < 0$ ($a_s > 0$), the system is in a BCS (BEC) regime. By defining a dimensionless quantity $\eta = 1/(k_F a_s)$, where $k_F = (3\pi^2 n)^{1/3}$ is Fermi wavenumber, one can distinguish several different superfluidity regimes,[7,8] i.e. *BCS regime* ($\eta < -1$), *BEC regime* ($\eta > 1$), and *BCS-BEC crossover regime* ($-1 < \eta < 1$). In particular, the point $\eta = 0$ is called the unitarity limit, corresponding to $a_s = \pm\infty$. Both theoretical and experimental studies demonstrate that the transition from BCS regime to BEC regime is smooth, which hints that one can study the physical property of the system in various superfluid regimes in a unified way.

In the experiments on the superfluid of ultracold fermionic atom gas, the system is confined in a finite space by an external trapping potential.[2,3] The inhomogeneous character of the system makes a microscopic approach based on a quantized model Hamiltonian difficult. However, notice that at very low T (around 10^{-8} K) low-frequency collective modes cannot decay by formation of single fermionic excitations because of the gap in their energy spectrum. Since thermal excitations play no significant role, the system can be taken as a perfect superfluid and may be well described by the generalized Gross–Pitaevskii (GGP) equation,[6−8]

$$i\hbar\frac{\partial}{\partial t}\psi = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\boldsymbol{r}) + \mu(n)\Big]\psi, \qquad (1)$$

where ψ is superfluid order parameter, $n = |\psi|^2$ is superfluid density satisfying the normalization condi-

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** To whom correspondence should be addressed. Email: gxhuang@phy.ecnu.edu.cn

tion $\int d\boldsymbol{r}n = N$ (N is the particle number in the condensate), $V_{\rm ext}$ is an external trapping potential, $\mu(n)$ is the chemical potential (also called the equation of state) obtained when $V_{\rm ext} = 0$. Different superfluid regimes are characterized by $\mu(n)$, which can be obtained by using a quantum Monte Carlo simulation[9] or some other techniques.[5] It can be shown that[6-8] $\mu(n) = \partial[n\varepsilon(n)]/\partial n$, where $\varepsilon(n)$ is the ground state energy per particle. For a dilute Fermi gas it can be written as $\varepsilon(n) = \dfrac{3}{5}\varepsilon_F(n)\sigma(\eta)$, where $\varepsilon_F(n) = \hbar^2 k_F^2/(2m)$ is the Fermi energy; $\sigma(\eta)$ is a yet unknown function. Some asymptotic expressions of $\sigma(\eta)$ have been obtained by fitting calculating data.[9,10] Interpolating these asymptotic expressions for small and large $|\eta|$ one can obtain the following general formula: $\sigma(\eta) = \alpha_1 - \alpha_2 \arctan[\alpha_3\eta(\beta_1 + |\eta|)/(\beta_2 + |\eta|)]$. The fitting parameters α_j ($j = 1,2,3$) and β_l ($l = 1,2$) have been given in Ref. [7].

A simple approach for the equation of state is to take a polytropic approximation, i.e. one assumes[5-8] $\mu(n) = \mu_0(n/n_0)^\gamma$, where μ_0 and n_0 are reference chemical potential and particle-number density of the system, introduced here for the convenience of later calculation. It is easy to show that the effective polytropic index takes the form[7,8] $\gamma(\eta) = \Big[\dfrac{2}{3}\sigma(\eta) - \dfrac{2\eta}{5}\sigma'(\eta) + \dfrac{\eta^2}{15}\sigma''(\eta)\Big]\Big/\Big[\sigma(\eta) - \dfrac{\eta}{5}\sigma'(\eta)\Big]$. There are two well-known limits for the value of the polytropic index γ. One is $\gamma = 2/3$ at $\eta = -\infty$ (BCS limit) and another one is $\gamma = 1$ at $\eta = \infty$ (BEC limit). Mathematically, the polytropic approximation, is a little rough but it has the advantage of allowing one to obtain an analytical expressions for the eigenfunctions and eigenfrequencies of collective modes[5-8] for all superfluid regimes in a unified way.

As in most experiments,[2,3] we consider a harmonic trapping potential of cylindrical symmetry, with the form $V_{\rm ext}(\boldsymbol{r}) = \dfrac{1}{2}m\omega_z^2[\lambda^2(x^2+y^2)+z^2]$. Here $\lambda = \omega_\perp/\omega_z$, with $\omega_\perp$ and ω_z being the harmonic frequencies in the radial (i.e. x and y) and axial (i.e. z) directions, respectively. For a Q2D (disc-shape) trap one has $\lambda \ll 1$.

To determine the dynamics of the Q2D condensate we employ a time-dependent variational method[11] to solve the GGP equation (1). The corresponding Lagrange density is $\mathcal{L} = i\hbar(\psi\partial\psi^*/\partial t - \psi^*\partial\psi/\partial t)/2 + \hbar^2|\nabla\psi|^2/(2m) + V_{\rm ext}(\boldsymbol{r})|\psi|^2 + \mu_0|\psi|^{2\gamma+2}/(n_0^\gamma(\gamma+1))$.

Because the system is strongly (weakly) confined in the axial (radial) direction, the condensate wavefunction in the axial (radial) direction should have a Gaussian (parabolic) form. Thus we choose the following hybrid trail variational wavefunction:[11]

$$\psi = A_n\Big(1 - \frac{x^2}{l_x^2} - \frac{y^2}{l_y^2}\Big)^{\frac{1}{2\gamma}} e^{-\frac{z^2}{2l_z^2}} e^{i(\beta_x x^2 + \beta_y y^2 + \beta_z z^2)}, \tag{2}$$

where $l_j(t)$ and $\beta_j(t)$ ($j = x,y,z$) are condensate width and phase parameters. Their time evolution determine completely the dynamics of the condensate. The normalization condition requires $A_n^2 = (1+\gamma)N/(\gamma l_x l_y l_z \pi^{3/2})$.

Using the integration $L = \int \mathcal{L} d\boldsymbol{r}$, we obtain the Lagrangian

$$\begin{aligned} L = {} & \frac{\hbar N}{2}\Big[\frac{\gamma}{2\gamma+1}(\dot\beta_x l_x^2 + \dot\beta_y l_y^2) + \dot\beta_z l_z^2\Big] \\ & + \frac{\hbar^2 N}{m}\Big[\frac{\gamma}{2\gamma+1}(\beta_x^2 l_x^2 + \beta_y^2 l_y^2) + \beta_z^2 l_z^2 + \frac{1}{4l_z^2}\Big] \\ & + \frac{mN\omega_z^2}{4}\Big[\frac{\gamma}{2\gamma+1}\lambda^2(l_x^2 + l_y^2) + l_z^2\Big] \\ & + \frac{(\gamma+1)^{\gamma-1/2}\mu_0}{\gamma^\gamma(1+2\gamma)n_0^\gamma}\frac{N^{\gamma+1}}{(l_x l_y l_z \pi^{3/2})^\gamma}, \end{aligned} \tag{3}$$

where the quantum pressure (i.e. kinetic energy) term in the radial (axial) direction has been omitted (retained) because the wavefunction varies slowly (fast) in this direction. The Euler–Lagrange equations yield the dynamic equations for the condensate widths $l_j(t)$ and phases $\beta_j(t)$. Defining the dimensionless time $\tau = \omega_z t$ and the width $d_j = l_j/a_z$ with $a_z = \sqrt{\hbar/m\omega_z}$ (i.e. the harmonic oscillator length in the axial direction), we obtain

$$\begin{aligned} \frac{d^2}{d\tau^2}d_j = {} & -\lambda_j^2 d_j + \frac{C_p}{d_j(d_x d_y d_z)^\gamma}\Big(1 - \frac{1+\gamma}{1+2\gamma}\delta_{jz}\Big) \\ & + \frac{1}{d_z^3}\delta_{jz}, \end{aligned} \tag{4}$$

where $\lambda_x = \lambda_y = \lambda$, $\lambda_z = 1$ and

$$C_p = \frac{2(\gamma+1)^{\gamma-1/2}\mu_0}{\gamma^\gamma \pi^{3\gamma/2} n_0^\gamma}\frac{mN^\gamma}{\hbar^2 a_z^{3\gamma-2}}. \tag{5}$$

To find a ground state solution we set $d_j = d_{j0}$ and by eliminating d_{x0} and d_{y0} we obtain the equation for d_{z0}

$$d_{z0}^2 = \frac{\gamma}{1+2\gamma}\Big(\frac{C_p^{1/\gamma}\lambda^2}{d_{z0}}\Big)^{\frac{\gamma}{1+\gamma}} + \frac{1}{d_{z0}^2}. \tag{6}$$

Since the first term on the right-hand side of Eq. (6) is much smaller than the second term and hence can be safely neglected. Then we obtain $d_{z0} = 1$, which means that the ground state width of the condensate wavefunction in the axial direction is just the axial harmonic oscillator length $l_{z0} = \sqrt{\hbar/m\omega_z}$. This is the minimum width the condensate shape can attain and it is also the solution for the width of a noninteracting gas. Thus the gas along z-direction has a feature of a noninteracting gas.

We now give the criterion for a Q2D superfluid Fermi gas. By a detailed calculation we obtain

$$\mu = \frac{N}{2}\hbar\omega_z\Big[1 + (C_p^{1/\gamma}\lambda^2)^{\frac{\gamma}{1+\gamma}}\Big]. \tag{7}$$

The Q2D condition of the system is $N\hbar\omega_\perp < \mu < N\hbar\omega_z$. From the result given by Eq. (7) we obtain the

Q2D criterion:

$$N < N_{\max} = \frac{1}{C_{p\mu}^{1/\gamma}\lambda^2}, \tag{8}$$

where $C_{p\mu} = C_p/N^\gamma$ is constant independent of N (see Eq. (5)). The inequality (8) shows clearly the constraint condition among the superfluid particle number N, the anisotropic parameter λ, and the polytropic index γ. Figure 1 presents the curves of maximum particle number in the condensate $N = N_{\max}(\eta,\gamma)$ for fixed λ based on the criterion (8), in which we have chosen $\omega_z = 2\pi \times 2200\,\text{Hz}$ and $\lambda = 7/2200$. The parameters n_0 and μ_0, appearing in Eq. (5), are chosen as the peak density of the condensate (i.e. $n_0 = n(0) = (\gamma+1)N/(\gamma l_{x0} l_{y0} l_{z0} \pi^{3/2})$) and the chemical potential per particle in the BCS limit (i.e. $\mu_0 = \hbar^2(3\pi^2)^{2/3} n_0^{2/3}/(2m)$), respectively. From the figure we can see that the maximum particle number in the BCS regime is less than that in the BEC regime, but both of them have the same order of magnitude (around 10^4 for given ω_z and λ). The inset shows the case for $\omega_z = 2\pi \times 700\,\text{Hz}$ and $\lambda = 1/100$. In this situation the maximum particle number is lowered to about 10^3. Thus for different λ, the maximum particle number in the superfluid can be quite different. In general, as λ decreases, $N_{\max}$ increases in a way inversely proportional to λ^2 (see Eq. (8)).

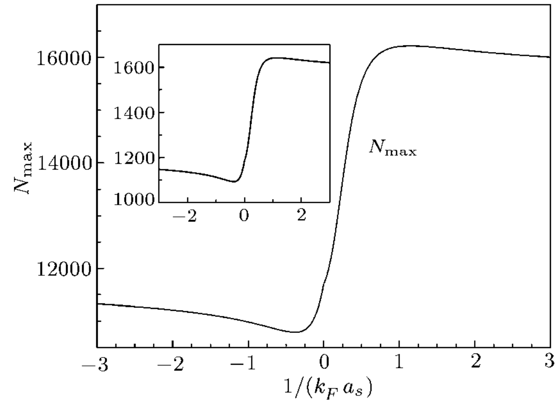

Fig. 1. The relation between the maximum particle number in Q2D Fermi superfluid and the interaction parameter $\eta = 1/(k_F a_s)$. The trapping parameters are $\omega_z = 2\pi \times 2200\,\text{Hz}$ and $\lambda = 7/2200$. The inset shows the case for $\omega_z = 700 \times 2\pi\,\text{Hz}$ and $\lambda = 1/100$.

Our next topic is to investigate the collective modes in the Q2D superfluid Fermi gas. Notice that the trail wave function Eq. (2) allows three modes to be generated. For the axially symmetric trap, the three modes are $m = 2$ mode and the $m = 0$ low- and high-lying modes, where m denotes the angular momentum quantum number. In order to reach their eigen frequencies and corresponding eigenvectors, we take $d_j = d_{j0} + \varepsilon_j(t)$ with $\varepsilon_j(t)$ being small excitations. Then we obtain

$$\begin{pmatrix} \ddot{\varepsilon}_x \\ \ddot{\varepsilon}_y \\ \ddot{\varepsilon}_z \end{pmatrix} = -\begin{pmatrix} \dfrac{(2+\gamma)C_p}{d_{x0}^{2+\gamma} d_{y0}^{\gamma} d_{z0}^{\gamma}} & \dfrac{\gamma C_p}{d_{x0}^{1+\gamma} d_{y0}^{1+\gamma} d_{z0}^{\gamma}} & \dfrac{\gamma C_p}{d_{x0}^{1+\gamma} d_{y0}^{\gamma} d_{z0}^{1+\gamma}} \\ \dfrac{\gamma C_p}{d_{x0}^{1+\gamma} d_{y0}^{1+\gamma} d_{z0}^{\gamma}} & \dfrac{(2+\gamma)C_p}{d_{x0}^{\gamma} d_{y0}^{2+\gamma} d_{z0}^{\gamma}} & \dfrac{\gamma C_p}{d_{x0}^{\gamma} d_{y0}^{1+\gamma} d_{z0}^{1+\gamma}} \\ \dfrac{\gamma^2 C_p}{(1+2\gamma) d_{x0}^{1+\gamma} d_{y0}^{\gamma} d_{z0}^{1+\gamma}} & \dfrac{\gamma^2 C_p}{(1+2\gamma) d_{x0}^{\gamma} d_{y0}^{1+\gamma} d_{z0}^{1+\gamma}} & \dfrac{(2+\gamma)\gamma C_p}{(1+2\gamma) d_{x0}^{\gamma} d_{y0}^{\gamma} d_{z0}^{2+\gamma}} + \dfrac{4}{d_{z0}^4} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}. \tag{9}$$

Letting $\varepsilon_j(t) = \varepsilon_j(0)\exp(-i\omega t) + \text{c.c.}$ and solving the above equation, we obtain the eigenvalues $\omega_{\gamma K}^2 = 2a$ and

$$\omega_{\gamma K_\pm}^2 = \frac{1}{2(1+2\gamma)}\Big[2a(1+2\gamma)(1+\gamma) + (2+\gamma) \cdot \Big(\gamma c + \frac{4(1+2\gamma)}{2+\gamma}\Big) \pm \sqrt{R}\Big], \tag{10}$$

where we have defined $a = D \equiv C_p/(d_{r0}^{2+2\gamma} d_{z0}^{\gamma})$, $c = D d_{r0}^2/d_{z0}^2$ ($d_{r0} \equiv d_{x0} = d_{y0}$), and $R = \{2a(1+2\gamma)(1+\gamma) - (2+\gamma)[\gamma c + 4(1+2\gamma)/(2+\gamma)]\}^2 + 8\gamma^3 b^2(1+2\gamma)$ with $b = D d_{r0}/d_{z0}$. The normalized eigenvectors corresponding to the eigenvalues are given respectively by $(-1,1,0)$ and $(1,1,V_{\gamma K\pm})$, where $V_{\gamma K\pm} = -\dfrac{1}{\gamma b}[2a(1+\gamma) - \omega_{\gamma K\pm}^2]$.

From these results we see that the three eigenmodes display different properties. The first eigenmode with oscillation frequency $\omega_{\gamma K}$ corresponds to $m = 2$ mode and has oscillation in x and y directions only. The other two eigenmodes with frequencies $\omega_{\gamma K_\pm}$ correspond to $m = 0$ mode and have oscillation mainly along the z direction (for the mode $\omega_{\gamma K_+}$) or mainly in the x and y directions (for the mode $\omega_{\gamma K_-}$).

In Fig. 2 we have shown the eigenfrequencies and the eigenvectors for the mode $\omega_{\gamma K_+}$ in the BCS-BEC crossover. The trapping parameters are chosen as $\omega_z = 2\pi \times 2200\,\text{Hz}$ and $\lambda = 7/2200$ (solid line) and $\omega_z = 2\pi \times 700\,\text{Hz}$ and $\lambda = 1/100$ (dotted line). Shown in Figs. 2(a) and 2(b) (Figs. 2(c) and 2(d)) are respectively the eigenfrequency and the z component of the eigenvector in the presence (absence) of quantum pressure. From the figure we see that: (i) It is an in-phase breathing mode, i.e. all three components in the eigenvector have the same sign. However, the oscillation is mainly along z direction because the z component of the eigenvector is much larger than the x and y components. (ii) The oscillating frequency in the BEC regime is larger than that in the BCS regime. (iii) The frequency ratio $\omega_{\gamma K_+}/\omega_z$ increases as λ decreases. (iv) The quantum pressure has a significant contribution to the oscillation eigenfrequency. Thus a TFA cannot be used to the Q2D problem in the strong confinement direction.

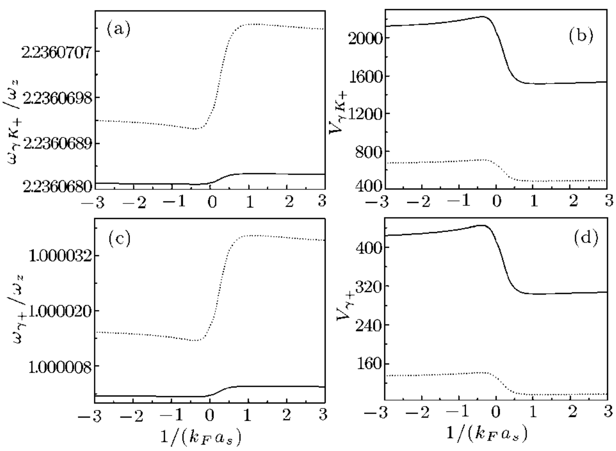

Fig. 2. The relation between the oscillating frequency and the interaction parameter $\eta = 1/(k_F a_s)$ of the in-phase breathing mode $\omega_{\gamma K_+}$. The oscillation is mainly along z direction. (a) and (b) The oscillating frequency and the z-component of the eigenvector. The trapping parameters are $\omega_z = 2\pi \times 2200$ Hz and $\lambda = 7/2200$ (solid line), $\omega_z = 2\pi \times 700$ Hz and $\lambda = 1/100$ (dotted line). (c) and (d) The same as (a) and (b) but in the absence of quantum pressure.

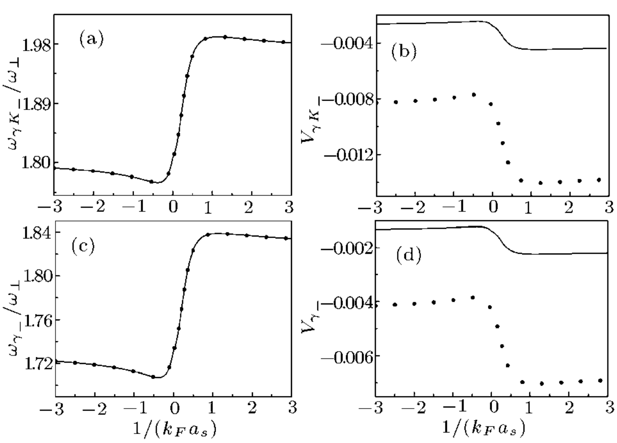

Fig. 3. The oscillating frequency of the out-of-phase breathing mode $\omega_{\gamma K_-}$ in the BCS-BEC crossover. The oscillation is mainly in the $x-y$ plane. (a) and (b) The oscillating frequency and the z-component of the eigenvector. The trapping parameters are $\omega_z = 2\pi \times 2200$ Hz and $\lambda = 7/2200$ (solid line), $\omega_z = 2\pi \times 700$ Hz and $\lambda = 1/100$ (dotted line). (c) and (d) The same as (a) and (b) but in the absence of quantum pressure.

Figure 3 shows the eigenfrequencies and the eigenvectors for the mode $\omega_{\gamma K_-}$ as a function of interaction parameter η. The trapping parameters are chosen as the same as in Fig. 2. Figures 3(a) and 3(b) (Figs. 3(c) and 3(d)) are respectively the eigenfrequency and the z component of the eigenvector in the presence (absence) of quantum pressure. Different from the in-phase breathing mode shown in Fig. 2, in this case we have a out-of-phase breathing mode, i.e. the oscillating direction in the z component has opposite sign with the oscillating directions in the xy components. However, we note that the magnitude of the z component is very small and hence the oscillation is mainly in the x and y plane. In addition, for this out-of-phase mode the frequency ratio $\omega_{\gamma K_-}/\omega_\perp$ is nearly independent of the λ, which can be seen in Figs. 3(a) and 3(c), in which the curves for two different λ are coincident.

In conclusion, we have investigated the collective modes in a Q2D superfluid Fermi gas in BCS-BEC crossover. By taking a hybrid trial wavefunction we have solved the GGP equation by means of a time-dependent variational method. We have provided a Q2D criterion for superfluid Fermi gas that is valid for various superfluid regimes and displays clearly the relation between the condensed particle number and the parameters of trapping potential as well as atom-atom interaction. We have demonstrated that, because of the small particle number in the Q2D condensate, the contribution to oscillating frequencies by the quantum pressure in the strong confinement direction is significant and hence a TFA cannot be used. To obtain a Q2D superfluid Fermi gas, one can use the method by continuously removing atoms from a highly anisotropic trap or to increase gradually the trap anisotropy from moderate to very large values whilst keeping the atom number fixed. The results presented in this work may be useful for understanding the physical properties of low-dimensional superfluid Fermi gases in BCS-BEC crossover and guide new experimental findings in future.

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