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Physics Letters A 354 (2006) 406–409

PHYSICS LETTERS A

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Solitons in a $(N+1)$-level medium

Guoxiang Huang *, Chao Hang

Department of Physics, East China Normal University, Shanghai 200062, China

Received 28 June 2005; received in revised form 13 January 2006; accepted 17 January 2006

Available online 8 February 2006

Communicated by A.R. Bishop

Abstract

The Maxwell–Schrödinger equations for a generalized V-type system with $N+1$ atomic levels are solved exactly. Three types of coupled soliton solutions are given explicitly. These optical solitons can have a propagating velocity much less than the light speed in vacuum.

PACS: 42.50.Md; 42.50.Gy

Solitons have been observed in many states of matter ranging from solid, such as optical fiber (optical soliton [1]), to Bose–Einstein condensed atomic vapor (matter wave solitons [2–4]). In recent years, much work has been done on the study of soliton excitations in resonant optical media, including the self-induced transparency (SIT) [5] in two-level atoms, optical simultons in three- [6–9], four- [10], and five-level [7] media, lasing without inversion [11], phaseonium [12], electromagnetically induced transparency (EIT) [13], and ultraslow optical solitons [14,15]. A constant focus of interest is to obtain analytical soliton solutions of Maxwell–Schrödinger (MS) equations that control the evolution of optical pulses and atomic-state probability amplitudes.

Recently, optical pulse propagation in multi-level media has received considerable attention [16]. Due to the existence of multiple dark states, such systems can be used to realize many interesting quantum interference effects, including coherent population transfer, and multiple EIT. However, many theoretical approaches for solving the MS equations of multi-level systems involve linear, steady-state or adiabatic approximations and hence exclude the possibility of getting an analytical soliton solution. In this work, we go beyond these approximations and show that the propagation of shape-preserving optical pulses in the form of coupled optical solitons is possible in a generalized V-type system with $N+1$ levels.

We consider the propagation dynamics of a laser field containing N optical pulses that propagate in z-direction and interact resonantly with a $(N+1)$-level atomic medium. The level configuration of the system takes a form of generalized V-type, in which the laser pulses couple N upper levels $|j\rangle$ ($j=1,2,\ldots,N$) to a single lower level ($|0\rangle$), as shown Fig. 1. Obviously, such model is a direct generalization of a usual three-level V-type system that has been investigated intensively in recent years [17,18]. The electric-field vector for the N opti-

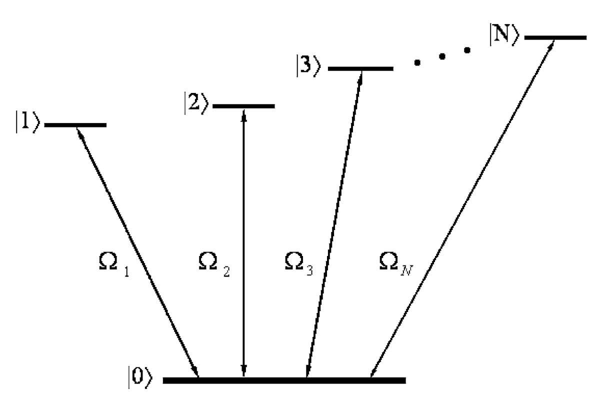

Fig. 1. Schematic diagram of the $(N+1)$-level system considered. A lower level is coupled to N upper-levels by N resonant laser fields.

* Corresponding author.
E-mail address: gxhuang@phy.ecnu.edu.cn (G. Huang).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2006.01.081

cal pulses can be written as

$$\mathbf{E} = \sum_{j=1}^{N} \mathbf{e}_j \mathcal{E}_j(z,t) \exp\big[i(k_j z - \omega_j t)\big] + \text{c.c.}, \tag{1}$$

where $\mathbf{e}_j$ is the unit vector representing the polarization of jth pulse, ω_j and $\mathcal{E}_j$ are the center frequency and amplitude of the electric field of the jth optical pulse, respectively. The Hamiltonian of the system is $\hat{H} = \hat{H}_0 + \hat{H}'$, where $\hat{H}_0$ describes an free atom and $\hat{H}'$ describes the interaction between the atom and the optical field. In Schrödinger picture, the state vector of the system is expressed by $|\Psi(t)\rangle = \sum_{j=0}^{N} c_j(z,t)|j\rangle$, where $|j\rangle$ is the eigenstate of $\hat{H}_0$. In a rotating-wave approximation, the Hamiltonian takes the form

$$\hat{H} = \sum_{j=0}^{N} \epsilon_j |j\rangle\langle j| - \left[\hbar \sum_{j=1}^{N} \Omega_j(z,t) \exp\big[i(k_j z - \omega_j t)\big] |j\rangle\langle 0| + \text{H.c.}\right], \tag{2}$$

where ϵ_j is the energy of state $|j\rangle$, $\Omega_j = \mathbf{e}_j \cdot \mathbf{p}_{j0}\mathcal{E}_j/\hbar$ is the half Rabi frequency corresponding to jth optical pulse ($\mathbf{p}_{j0}$ is the electric dipole matrix element associated with the transition from $|0\rangle$ to $|j\rangle$), and H.c. represents Hermitian conjugate.

To investigate the time evolution of the system it is more convenient to employ an interaction picture, which is obtained by making the transformation $c_j(z,t) = a_j(z,t)\exp[i(k_j z - \epsilon_j t/\hbar)]$, with $k_0 = 0$. The Hamiltonian in the interaction picture reads

$$\hat{H}_{\text{int}} = -\hbar \left[\sum_{j=1}^{N} \Omega_j(z,t)|j\rangle\langle 0| + \text{H.c.}\right]. \tag{3}$$

The equations of motion for the atomic probability amplitude a_j is given by

$$i\frac{\partial}{\partial t} a_0 + \sum_{l=1}^{N} \Omega_l^* a_l = 0, \tag{4a}$$

$$i\frac{\partial}{\partial t} a_j + \Omega_j a_0 = 0, \tag{4b}$$

($j = 1, 2, \ldots, N$) with $\sum_{l=0}^{N} |a_l|^2 = 1$.

Under a slowly-varying envelope approximation the Maxwell equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}$$

with

$$\mathbf{P} = \mathcal{N}_a \sum_{j=1}^{N} \big[\mathbf{p}_{0j} a_j a_0^* \exp\big[i(k_j z - \omega_j t)\big] + \text{c.c.}\big]$$

yields the equations of motion for the Rabi frequencies ($j = 1, 2, \ldots, N$)

$$i\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\Omega_j + \kappa_0 a_0^* a_j = 0. \tag{5}$$

For simplicity the propagation coefficient $\kappa_{0j} = \mathcal{N}_a \omega_j |\mathbf{p}_{0j}|^2/(2\epsilon\hbar c)$ (with $\mathcal{N}_a$ being the atomic density and c being the light speed in vacuum) has been assumed to be equal [8,9] for all transitions, i.e., $\kappa_{0j} = \kappa_0$. Experimentally, this can be achieved using an atomic element with ground (excited) state having large angular momenta J or F.

Note that a damping term $i\gamma_j a_j$ representing the dissipation contributed by spontaneous emission and dephasing should be added into Eq. (4b).[1] However, such term can be neglected for pulse propagation in a *coherent transient regime* that we are interested in here. This can be explained easily as follows. Taking $t = t'\tau_0$, where t' is dimensionless time and τ_0 is the characteristic temporal width of the optical pulses, Eq. (4b) is transferred into the dimensionless form $i(\partial/\partial t' + \gamma_j \tau_0)a_j + \Omega_j \tau_0 a_0 = 0$. Consider, for example, a cold atom system having decay rate γ_j less than 10 MHz. If τ_0 is around one nanosecond, then one has $\gamma_j \tau_0 < 10^{-2}$ and hence the damping term in the equation plays no significant role. Thus in this coherent transient regime pulses can propagate for a long distance with negligible energy loss [19]. In the following, we shall disregard such damping term in our calculation.

Note that Eqs. (4a) and (4b) admit $N(N-1)/2$ conservation laws

$$a_j \Omega_l - a_l \Omega_j = A_{jl}^{\text{trap}} \tag{6}$$

($j = 1, 2, \ldots, N$ and $l = j+1, j+2, \ldots, N$) if all Ω_j can be taken as constants. The quantities A_{jl}^{trap} are called trapping amplitudes [8], which means that whatever population is initially in A_{jl} will remain there.

Eqs. (4) and (5) are coupled partial differential equations with dispersion and nonlinearity. A general exact solution for such equations is not available yet. Here we try to get some coupled soliton solutions by employing some special ansatz [20]. For this aim we assume $a_j = a_j(\zeta)$ and $\Omega_j = \Omega_j(\zeta)$ with $\zeta = Kz - \tau/\tau_0$ and $\tau = t - z/c$. Three types of coupled soliton solutions are obtained, which are given as follows.

(I) By assuming $a_0 = A_0 \tanh\zeta$, $a_j = A_j \operatorname{sech}\zeta$, $\Omega_j = B_j \operatorname{sech}\zeta$ ($j = 2, 3, \ldots, N$), where A_j and B_j are constants yet to be determined. Substitution of these expressions into the Eqs. (4) and (5) yields a series of algebraic equations of A_j and B_j. Solving this equations we get $A_0 = i$, $A_j = -\tau_0 B_j$, $K = \kappa_0 \tau_0$, with $\tau_0 = (\sum_{l=1}^{N} B_l^2)^{-1/2}$. Thus we obtain the exact coupled soliton solution ($j = 1, 2, \ldots, N$)

$$a_0 = i\tanh(Kz - \tau/\tau_0), \tag{7a}$$

$$a_j = -\tau_0 B_j \operatorname{sech}(Kz - \tau/\tau_0), \tag{7b}$$

$$\Omega_j = B_j \operatorname{sech}(Kz - \tau/\tau_0), \tag{7c}$$

where B_j (pulse amplitudes) remain as arbitrary, real constants. The physical process described by the solution (7) can be understood as follows. Initially ($\zeta = -\infty$), light fields Ω_j ($j = 1, 2, 3, \ldots, N$) are not established and population is in the atomic ground state $|0\rangle$. At $\zeta = 0$, all light fields are in

[1] Because the ground state $|0\rangle$ has no decay thus it is not necessary to add a damping term to Eq. (4a) (i.e., $\gamma_0 = 0$).

their maximum intensity and population is transferred from the ground state to the excited states. At $\zeta = \infty$, particles return back into the ground state from the excited states. As a result, the atomic medium is transparent for all optical pulses. Such transparency for N optical pulses in the $(N+1)$-level system is obviously a direct generalization of the SIT for one optical pulse in a two-level system proposed first by MacCall and Hahn [5]. Note that during propagation N optical pulses are automatically matched, i.e., they have the same waveform (i.e., sech-shaped envelope) with common propagating velocity V, which satisfies

$$\frac{1}{V} = \frac{1}{c} + \frac{\kappa_0}{\sum_{l=1}^{N} B_l^2}. \tag{8}$$

Note that the value of V can be manipulated by changing the atomic density of the medium and intensity of optical pulses. It can be much less c for higher atomic density $\mathcal{N}_a$ (thus larger κ_0) and smaller pulse amplitudes B_j. For example, for a typical alkali system one take $N = 5$, $\kappa_0 = 1.0 \times 10^9\ \mathrm{cm}^{-1}\,\mathrm{s}^{-1}$, and $B_j \approx 1.0 \times 10^8\ \mathrm{s}^{-1}$. Using these parameters we get $V = 4.0 \times 10^{-2} c$. Thus *slow-light solitons* can be realized in such multi-level system. The formation of the slow-light solitons of multiple components (i.e., coupled solitons) is due to the exact balance between nonlinear and dispersive effects of the system.

Remarkably, the coupled soliton solution given by Eq. (7) satisfies the following relations:

$$A_{jl}^{\mathrm{trap}} = a_j \Omega_l - a_l \Omega_j = 0$$
$$(j = 1, 2, 3, \ldots, N,\ l = j+1, \ldots, N), \tag{9}$$

which means that trapping amplitudes A_{jl}^{trap} can be constant in more general conditions than that for which they were defined, i.e., constant trapping field states can be obtained even if the laser fields are time-dependent. These soliton-related trapping states were first noticed by Eberly for a three-level system [8]. They have a deep relation to the multiple dark states of the system [16].

(II) By assuming $a_1 = A_1 \tanh\zeta$, $a_j = A_j \operatorname{sech}\zeta$ $(j = 0, 2, 3, \ldots, N)$, $\Omega_1 = B_1 \operatorname{sech}\zeta$ and $\Omega_l = B_l \operatorname{sech}\zeta$ $(l = 2, 3, \ldots, N)$, and substituting these expressions into MS equation (4) yield a set of algebraic equations on the undetermined coefficients A_j and B_j. Solving these equations one gets $A_0 = \tau_0 B_1$, $A_1 = 1$ and $a_j = -B_j/B_1$ $(j = 2, 3, \ldots, N)$. Thus we have another type of coupled soliton solution:

$$a_0 = \tau_0 B_1 \operatorname{sech}(Kz - \tau/\tau_0), \tag{10a}$$

$$a_1 = \tanh(Kz - \tau/\tau_0), \tag{10b}$$

$$a_j = -\frac{B_j}{B_1} \operatorname{sech}(Kz - \tau/\tau_0), \tag{10c}$$

$$\Omega_1 = B_1 \operatorname{sech}(Kz - \tau/\tau_0), \tag{10d}$$

$$\Omega_j = B_j \tanh(Kz - \tau/\tau_0), \tag{10e}$$

where $K = -\kappa_0 \tau_0$ with $\tau_0 = (1 - \sum_{l=2}^{N} B_l^2/B_1^2)^{1/2}/B_1$. Here the pump field amplitudes B_j $(j = 1, 2, \ldots, N)$ are arbitrary real constants except for the constraint $\sum_{l=2}^{N} B_l^2 \leqslant B_1^2$. The propagating velocity V of the coupled soliton is given by

$$\frac{1}{V} = \frac{1}{c} - \frac{\kappa_0}{B_1^2}. \tag{11}$$

Hence one can get superluminal solitons by increasing $\mathcal{N}_0$ and decreasing B_1.

The physical process described by the coupled soliton solution (10) for pulse propagation and population distribution of the system are the following. Initially $(\zeta = -\infty)$, fields Ω_j $(j = 2, 3, \ldots, N)$ are established and population is in the excited state $|1\rangle$. At $\zeta = 0$ all particles are pumped into the ground state $|0\rangle$ and the excited states $|j\rangle$ $(j = 2, 3, \ldots, N)$. The field Ω_1 is established and the fields Ω_j $(j = 2, 3, \ldots, N)$ vanish. At $\zeta = \infty$, the fields Ω_j $(j = 2, 3, \ldots, N)$ recover to their initial values, Ω_1 vanishes and all particles redistribute in the excited state $|1\rangle$. Thus the system is transparent for the N optical pulses. The solution expressed by Eq. (10) can be taken as a multi-level generalization of a three-level EIT in a V-type system [18]. Ω_1 corresponds to a probe field and Ω_j $(j = 2, 3, \ldots, N)$ correspond to control fields. Note the counter-intuitive character of the pulses, i.e., the probe field Ω_1 does not turn on until after the control fields Ω_j $(j = 2, 3, \ldots, N)$ are fully established.

It is easy to show that above coupled soliton solution (10) satisfies the following relations:

$$A_{1l}^{\mathrm{trap}} = a_1 \Omega_l - a_l \Omega_1 = B_l \quad (l = 2, 3, \ldots, N), \tag{12}$$

$$a_j \Omega_l - a_l \Omega_j = 0 \quad (j = 2, 3, \ldots, N,\ l = j+1, \ldots, N). \tag{13}$$

(III) In addition to the coupled soliton solutions given above, we have also found the following solution

$$a_0 = i \operatorname{sech}(Kz - \tau/\tau_0), \tag{14a}$$

$$a_j = \tau_0 B_j \tanh(Kz - \tau/\tau_0), \tag{14b}$$

$$\Omega_j = B_j \operatorname{sech}(Kz - \tau/\tau_0), \tag{14c}$$

$j = 1, 2, \ldots, N$, where $K = -\tau_0 \kappa_0$ with $\tau_0 = (\sum_{l=1}^{N} B_l^2)^{-1/2}$. B_j $(j = 1, 2, \ldots, N)$ remain arbitrary real constants. The propagating velocity V of the coupled soliton in this case is still given by Eq. (8) and thus a new type of coupled slow-light soliton is possible. The solution (14) describes the following physical process: all particles are initially populated in all upper levels; then all light fields are established and the particles are transferred into the ground state $|0\rangle$; at last all particles return into upper levels and hence the medium is transparent for N optical pulses. It can be easily shown that the trapping amplitudes A_{jl}^{trap} are also constants.

Now we study, by a numerical simulation, the stability of the soliton solutions obtained above. In our simulation the space and time derivatives in Eq. (4) are performed by using a pseudo-spectral method (as used in Ref. [21]) and a fourth-order Runge–Kutta method for superior conservation of energy and other invariants, respectively. In the simulation the analytical soliton solutions obtained are naturally taken as initial conditions of Eqs. (4) and (5). Shown in Fig. 2 is the pulse propagation when taking the coupled soliton solution (7) as an initial condition. Here, for illustration we select $N = 5$ (i.e., we simulate a 6-level V-type system) and the parameters of the system

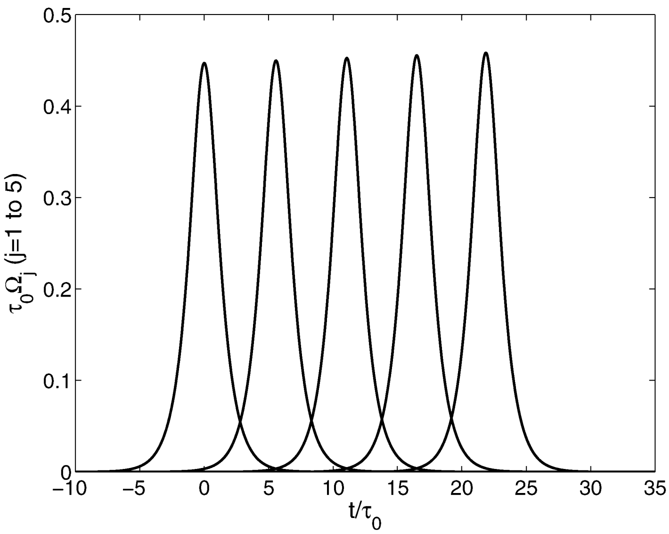

Fig. 2. Propagation of the coupled soliton (7). From left to right, the curves correspond to $z = 1.0$, 2.0, 3.0, 4.0, 5.0 cm. The soliton exhibits great robustness during the propagation.

are chosen as $\kappa_0 = 1.0 \times 10^9$ cm^{-1} s^{-1}, and $B_j \approx 1.0 \times 10^8$ s^{-1} ($j = 1$–5). Thus, one has $\tau_0 \approx 4.47 \times 10^{-9}$ s. From left to right, the curves in Fig. 2 correspond to $z = 1.0$, 2.0, 3.0, 4.0, and 5.0 cm, respectively. We see that the soliton exhibits great robustness during propagation even for a long distance.

In conclusion, we have solved the Maxwell–Schrödinger equations for a $(N + 1)$-level system exactly by using some special ansatz. We have provided several types of coupled soliton solutions explicitly and showed that the N optical pulses in the system can be automatically matched and have a very slow propagating velocity. The stability of the coupled solitons has also been studied by using numerical simulations. These interesting properties of the coupled optical solitons may have promising application in optical information processing and engineering.

Acknowledgements

This work was supported by the NSF-China under Grant Nos. 90403008 and 10434060, and State Key Development Program for Basic Research of China under Grant No. 2005CB724508.

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