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TEMPORAL OPTICAL SOLITONS VIA MULTISTEP $\chi^{(2)}$ CASCADING*

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We consider a multistep $\chi^{(2)}$ cascading for light pulses with the dispersion of the system taken into account. Using the method of multiple scales we derive a set of coupled envelope equations governing the nonlinear evolution of the fundamental, second and third harmonic waves involved simultaneously in two nonlinear optical processes, i.e. second harmonic generation and sum frequency mixing. We show that three-wave temporal optical solitons are possible in three- and four-step cascading in the presence of a group-velocity mismatch between different pulses.

Keywords: cascading, second harmonic generation, sum frequency mixing, optical soliton
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I. INTRODUCTION

Cascading is the process by which the exchange of energy between optical pulses or beams interacting via second-order nonlinearities ($\chi^{(2)}$) leads to various effects such as nonlinear phase shifts, the generation of new pulses or beams, all-optical transistor action (i.e. gain and phase modulation), the formation of optical solitons, etc. In recent years, there has been a considerable interest in $\chi^{(2)}$ cascading because it offers a new and promising direction to explore for all-optical phenomena.[1,2]

Most studies for the $\chi^{(2)}$ cascading phenomena emphasize the amplitude and phase shift of the fundamental waves in a second harmonic interaction. The physics of such a process requires two successive second-order processes in order for the net output to be back at the input frequency (ω). This can occur via up-conversion ($\omega+\omega\to 2\omega$, i.e. second harmonic generation (SHG)) followed by down-conversion ($2\omega-\omega\to\omega$) or via down-conversion ($\omega-\omega\to 0$, i.e. optical rectification) followed by up-conversion ($\omega+0\to\omega$). In a recent paper, Koynov and Saltiel[3] introduced an interesting mechanism of three- and four-step second-order nonlinearity cascading, in which two nonlinear optical processes are involved: an SHG and a third harmonic generation (THG). For continuous waves and under the condition that two wave-mixing processes are nearly phase matched, the presence of multistep cascading leads to four times a reduction of the input intensity required for a large nonlinear phase shift. Recently, Kivshar *et al.*[4] investigated the spatial optical solitons resulting from such a multistep cascading for the light beams with diffraction but in the absence of the effect of walk-off between different beams.

In this paper we consider a multistep $\chi^{(2)}$ cascading for light pulses with the dispersion of the system being taken into account. We show that temporal optical solitons are possible in a three- and four-step cascading for light pulses displaying a temporal walk-off (i.e. group-velocity mismatch). Section II gives a detailed derivation of coupled equations governing the nonlinear evolution of the pulse envelopes corresponding to the fundamental, second and third harmonic waves, which are involved in simultaneous processes of an SHG and a THG. Section III provides some analytical coupled three-wave soliton solutions. A discussion and summary of our results are given in the last section.

II. COUPLED ENVELOPE EQUATIONS IN THE PRESENCE OF TEMPORAL WALK-OFF EFFECT

We start from the Maxwell equation for an electric field $\boldsymbol{E}(\boldsymbol{r},t)$ in a dielectric medium:

$$\nabla^2\boldsymbol{E}-\nabla(\nabla\cdot\boldsymbol{E})-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}=\frac{1}{\epsilon_0c^2}\frac{\partial^2\boldsymbol{P}}{\partial t^2}, \qquad (1)$$

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where c is the light speed in vacuum, ϵ_0 is the vacuum permittivity and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. $\boldsymbol{P}$ is the induced polarization field in the medium, which is given by

$$\begin{aligned}\frac{1}{\epsilon_0}\boldsymbol{P}(\boldsymbol{r},t) = &\int_{-\infty}^{\infty} \mathrm{d}\tau\chi^{(1)}(\tau)\cdot\boldsymbol{E}(\boldsymbol{r},t-\tau)\\ &+\int_{-\infty}^{\infty}\mathrm{d}\tau_1\mathrm{d}\tau_2\chi^{(2)}(\tau_1,\tau_2)\\ &:\boldsymbol{E}(\boldsymbol{r},t-\tau_1)\boldsymbol{E}(\boldsymbol{r},t-\tau_2)+\cdots,\end{aligned} \quad (2)$$

where $\boldsymbol{r} = (x, y, z)$, and $\chi^{(l)}\,(l = 1, 2, \cdots)$ is the jth-order susceptibility tensor of rank $l + 1$. Because the third- and higher-order susceptibilities have no contribution to a parametric process, we disregard them in the following calculation.

Assume that an electricmagnetic wave travels along the z-direction in the medium, and $\boldsymbol{E} = (E(z,t), 0, 0)$. Then Eq.(1) is simplified

$$\begin{aligned}&c^2\frac{\partial^2 E}{\partial z^2}-\frac{\partial^2 E}{\partial t^2}-\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)E(z,t-\tau)\\ =&\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\mathrm{d}\tau_2\chi^{(2)}(\tau_1,\tau_2)E(z,t-\tau_1)E(z,t-\tau_2),\end{aligned} \quad (3)$$

where $\chi^{(1)} = \chi^{(1)}_{xx}$ and $\chi^{(2)} = \chi^{(2)}_{xxx}$. It should be mentioned that the possibility of considering only one component of the wave imposes evident constraints on the symmetry of the medium.

To derive coupled envelope equations for governing the nonlinear evolution of the electric field with the dispersion of the system taken into account, we use the method of multiple scales.[5] This general technique calls, in the present problem, for the introduction of different length scales, $z_\alpha = \mu^{\alpha/2}z\,(\mu \ll 1$, a small parameter characterizing the typical amplitude of the electricmagnetic wave; $\alpha = 0, 1, 2, \cdots)$, and timescales, $t_\alpha = \mu^{\alpha/2}t$. These variables are considered to be independent. Thus the first spatial and temporal derivatives are replaced by $\partial/\partial z = \partial/\partial z_0 + \mu^{1/2}\partial/\partial z_1 + \mu\partial/\partial z_2 + \cdots$ and $\partial/\partial t = \partial/\partial t_0 + \mu^{1/2}\partial/\partial t_1 + \mu\partial/\partial t_2 + \cdots$ The electric field E is expanded into a series

$$E = \mu E^{(0)} + \mu^{3/2}E^{(1)} + \mu^2 E^{(2)} + \cdots, \quad (4)$$

with $E^{(j)}\,(j = 0, 1, \cdots)$ being functions of all z_α and t_α. Substituting the expansion (4) into Eq.(3) and equating the coefficients of the same powers of μ, we obtain a hierarchy of linear but inhomogeneous equations for $E^{(j)}$:

$$\begin{aligned}\hat{L}E^{(j)} =&c^2\frac{\partial^2 E^{(j)}}{\partial z_0^2}-\frac{\partial^2 E^{(j)}}{\partial t_0^2}-\frac{\partial^2}{\partial t_0^2}\int_{-\infty}^{\infty}\\ &\cdot\mathrm{d}\tau\chi^{(1)}(\tau)E^{(j)}(t_0-\tau) = M^{(j)},\end{aligned} \quad (5)$$

with

$$M^{(0)} = 0, \quad (6)$$

$$\begin{aligned}M^{(1)} =& -2c^2\frac{\partial^2}{\partial z_0\partial z_1}E^{(0)}+2\frac{\partial^2}{\partial t_0\partial t_1}E^{(0)}\\ &-\frac{\partial^2}{\partial t_0^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)\tau\frac{\partial}{\partial t_1}E^{(0)}(t_0-\tau)\\ &+2\frac{\partial^2}{\partial t_0\partial t_1}\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)E^{(0)}(t_0-\tau),\end{aligned} \quad (7)$$

$$\begin{aligned}M^{(3)} =& -2c^2\frac{\partial^2}{\partial z_0\partial z_1}E^{(1)}-c^2\left(2\frac{\partial^2}{\partial z_0\partial z_2}+\frac{\partial^2}{\partial z_1^2}\right)E^{(0)}\\ &+2\frac{\partial^2}{\partial t_0\partial t_1}E^{(1)}+\left(2\frac{\partial^2}{\partial t_0\partial t_2}+\frac{\partial^2}{\partial t_1^2}\right)E^{(0)}\\ &+\frac{\partial^2}{\partial t_0^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)\Big[-\tau\frac{\partial}{\partial t_1}E^{(1)}(t_0-\tau)\\ &+\left(-\tau\frac{\partial}{\partial t_2}+\frac{1}{2}\tau^2\frac{\partial^2}{\partial t_1^2}\right)E^{(0)}(t_0-\tau)\Big]\\ &+2\frac{\partial^2}{\partial t_0\partial t_1}\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)\\ &\cdot\left[E^{(1)}(t_0-\tau)-\tau\frac{\partial}{\partial t_1}E^{(0)}(t_0-\tau)\right]\\ &+\left(2\frac{\partial^2}{\partial t_0\partial t_2}+\frac{\partial^2}{\partial t_1^2}\right)\int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)E^{(0)}(t_0-\tau)\\ &+\frac{\partial^2}{\partial t_0^2}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\mathrm{d}\tau_2\chi^{(2)}(\tau_1,\tau_2)\\ &\cdot E^{(0)}(t_0-\tau_1)E^{(0)}(t_0-\tau_2),\end{aligned} \quad (8)$$

where the quantity $E^{(j)}(t_0 - \tau)$ appearing in the integrations $\int_{-\infty}^{\infty}\mathrm{d}\tau\cdots$ and $\int_{-\infty}^{\infty}\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots$ in Eqs.(5)–(8) represents $E^{(j)}(z_0, z_1, z_2, \cdots; t_0 - \tau, t_1, t_2, \cdots)$.

In the leading order $(j = 0)$ we have the solution

$$\begin{aligned}E^{(0)} =&F(z_1, z_2, \cdots; t_1, t_2, \cdots)\\ &\cdot\exp\{\mathrm{i}[k(\omega)z_0-\omega t_0]\}+\mathrm{c.c.},\end{aligned} \quad (9)$$

where c.c. represents corresponding complex conjugate term. The envelope function F depends only on the 'slow' variables z_α and t_α $(\alpha = 1, 2, \cdots)$. The relation between the frequency ω and the wave vector k (i.e. the linear dispersion relation of the system) is given by $k(\omega) = (\omega/c)[1 + \hat{\chi}^{(1)}(\omega)]^{1/2}$ with $\hat{\chi}^{(1)}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}\tau\chi^{(1)}(\tau)\exp(\mathrm{i}\omega\tau)$.

We are interested in a multistep cascading in the system. Assume that a fundamental wave with frequency ω enters into the medium. As a first step via a process of type I SHG, the wave with frequency 2ω is generated; and as a second step via a process of sum frequency generation (SFG) ($\omega + 2\omega = 3\omega$), a third harmonic wave is generated. Both processes (SHG and SFG) are supposed to be nearly phase matched. The second and third harmonic waves generated are down-converted to the fundamental wave ω via processes $(2\omega - \omega)$, $(3\omega - 2\omega)$ and $(3\omega - 2\omega,\ 2\omega - \omega)$ (see Fig.1 in Ref.[3]). Under such a consideration, the leading-order solution takes the form

$$\begin{aligned} E^{(0)} =& F_1 \exp[\mathrm{i}(k_1 z_0 - \omega_1 t_0)] \\ &+ F_2 \exp[\mathrm{i}(k_2 z_0 - \omega_2 t_0)] \\ &+ F_3 \exp[\mathrm{i}(k_3 z_0 - \omega_3 t_0)] + \mathrm{c.c.}, \end{aligned} \tag{10}$$

where $F_l\,(l = 1,2,3)$ are envelope functions of the fundamental, second harmonic and third harmonic waves, respectively. $k_l \equiv k(\omega_l)\,(l = 1,2,3)$, $\omega_2 = 2\omega_1$, $\omega_3 = \omega_1 + \omega_2 = 3\omega_1$, and

$$k_2 = 2k_1 + \Delta k_2, \tag{11}$$

$$k_3 = k_1 + k_2 + \Delta k_3, \tag{12}$$

where Δk_2 and Δk_3 are wave vector mismatches for the processes of the SHG and the SFG, respectively.

In the next order, from Eq.(5) for $j = 1$ we have

$$\begin{aligned} \hat{L}E^{(1)} =& -2\mathrm{i}\sum_{l=1}^{3} k_l \left\{ \frac{\partial F_l}{\partial z_1} + \frac{\omega_l}{c^2 k_l}\left[1 + \hat{\chi}^{(1)}(\omega_l) \right.\right. \\ &\left.\left. + \frac{\omega_l}{2}\frac{\partial \hat{\chi}^{(1)}(\omega_l)}{\partial \omega_l}\right]\frac{\partial F_l}{\partial t_1}\right\} \\ &\cdot \exp[\mathrm{i}(k_l z_0 - \omega_l t_0)] + \mathrm{c.c.} \end{aligned} \tag{13}$$

Obviously, on the right-hand side of Eq.(13) are secular terms which will produce a divergence for the solution of $E^{(1)}$. To eliminate such secular terms we must take

$$\frac{\partial F_l}{\partial z_1} + \frac{\omega_l}{c^2 k_l}\left[1 + \hat{\chi}^{(1)}(\omega_l) + \frac{\omega_l}{2}\frac{\partial \hat{\chi}^{(1)}(\omega_l)}{\partial \omega_l}\right]\frac{\partial F_l}{\partial t_1} = 0, \tag{14}$$

for $l = 1,2,3$. Note that

$$\begin{aligned} k'(\omega_l) =& \frac{\partial k(\omega_l)}{\partial \omega_l} \\ =& \frac{\omega_l}{c^2 k_l}\left[1 + \hat{\chi}^{(1)}(\omega_l) + \frac{\omega_l}{2}\frac{\partial \hat{\chi}^{(1)}(\omega_l)}{\partial \omega_l}\right], \end{aligned} \tag{15}$$

thus Eq.(14) can be written as

$$\frac{\partial F_l}{\partial z_1} + k'(\omega_l)\frac{\partial F_l}{\partial t_1} = 0, \tag{16}$$

for $l = 1,2,3$.

In the order $j = 2$, the solvability conditions yield the nonlinear evolution equations for F_l:

$$\begin{aligned} &\mathrm{i}\frac{\partial F_1}{\partial z_2} + \frac{1}{2k_1}\frac{\partial^2 F_1}{\partial z_1^2} - \frac{1}{2k_1}\left[k_1 k_1'' + (k_1')^2\right]\frac{\partial^2 F_1}{\partial t_1^2} \\ &\quad + \frac{\omega_1^2}{k_1 c^2}\left[\hat{\chi}^{(2)}(-\omega_1, \omega_2) F_1^* F_2 \mathrm{e}^{\mathrm{i}\Delta k_2 z_0} \right. \\ &\quad \left. + \hat{\chi}^{(2)}(-\omega_2, \omega_3) F_2^* F_3 \mathrm{e}^{\mathrm{i}\Delta k_3 z_0}\right] = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathrm{i}\frac{\partial F_2}{\partial z_2} + \frac{1}{2k_2}\frac{\partial^2 F_1}{\partial z_1^2} - \frac{1}{2k_2}\left[k_2 k_2'' + (k_2')^2\right]\frac{\partial^2 F_2}{\partial t_1^2} \\ &\quad + \frac{\omega_2^2}{k_2 c^2}\left[\hat{\chi}^{(2)}(\omega_1, \omega_1) F_1^2 \mathrm{e}^{-\mathrm{i}\Delta k_2 z_0} \right. \\ &\quad \left. + 2\hat{\chi}^{(2)}(-\omega_1, \omega_3) F_1^* F_3 \mathrm{e}^{\mathrm{i}\Delta k_3 z_0}\right] = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &\mathrm{i}\frac{\partial F_3}{\partial z_2} + \frac{1}{2k_3}\frac{\partial^2 F_3}{\partial z_1^2} - \frac{1}{2k_3}\left[k_3 k_3'' + (k_3')^2\right]\frac{\partial^2 F_3}{\partial t_1^2} \\ &\quad + \frac{\omega_3^2}{k_3 c^2}\hat{\chi}^{(2)}(\omega_1, \omega_2) F_1 F_2 \mathrm{e}^{-\mathrm{i}\Delta k_3 z_0} = 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} k_l'' =& \frac{\partial^2 k(\omega_l)}{\partial \omega_l^2} \\ =& \frac{1}{c^2 k(\omega_l)}\left[1 + \hat{\chi}^{(1)}(\omega_l) + 2\omega_l \frac{\partial \hat{\chi}^{(1)}(\omega_l)}{\partial \omega_l} \right. \\ &\left. + \frac{\omega_l^2}{2}\frac{\partial^2 \hat{\chi}^{(1)}(\omega_l)}{\partial \omega_l^2} - c^2\left(\frac{\partial k(\omega_l)}{\partial \omega_l}\right)^2\right], \end{aligned} \tag{20}$$

and

$$\begin{aligned} \hat{\chi}^{(2)}(\omega_{l_1}, \omega_{l_2}) = \int_{-\infty}^{\infty} &\mathrm{d}\tau_1 \mathrm{d}\tau_2 \chi^{(2)}(\tau_1, \tau_2) \\ &\cdot \exp[\mathrm{i}\omega_{l_1}\tau_1 + \mathrm{i}\omega_{l_2}\tau_2]. \end{aligned} \tag{21}$$

For simplicity, we assume $\chi^{(2)}(\tau_1, \tau_2) = \chi_{nl}^{(2)}\delta(\tau_1)\delta(\tau_2)$, i.e. there is no dispersion in the second-order susceptibility, hence we have $\hat{\chi}^{(2)}(\omega_1, \omega_2) = \chi_{nl}^{(2)}=$const. By Eq.(16) (for $l = 1,2,3$) we take $F_l = F_l(\xi, z_2)$ with $\xi = k'(\omega_1) z_1 - t_1$. Using the technique introduced by Newell and Moloney,[5] Eqs.(17)–(19) are simplified

$$\begin{aligned} &\mathrm{i}\frac{\partial F_1}{\partial z_2} - \frac{1}{2}k_1''\frac{\partial^2 F_1}{\partial \xi^2} \\ &\quad + \sigma F_1^* F_2 \mathrm{e}^{\mathrm{i}\Delta k_2 z_0} + \sigma F_2^* F_3 \mathrm{e}^{\mathrm{i}\Delta k_3 z_0} = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &\mathrm{i}\frac{\partial F_2}{\partial z_2} + \mathrm{i}\Delta_2\frac{\partial F_2}{\partial \xi} - \frac{1}{2}\left\{k_2'' + \frac{1}{k_2}\left[(k_2')^2 - (k_1')^2\right]\right\} \\ &\cdot\frac{\partial^2 F_2}{\partial \xi^2} + \sigma F_1^2 \mathrm{e}^{-\mathrm{i}\Delta k_2 z_0} + 2\sigma F_1^* F_3 \mathrm{e}^{\mathrm{i}\Delta k_3 z_0} = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &\mathrm{i}\frac{\partial F_3}{\partial z_2} + \mathrm{i}\Delta_3\frac{\partial F_3}{\partial \xi} - \frac{1}{2}\left\{k_3'' + \frac{1}{k_3}\left[(k_3')^2 - (k_1')^2\right]\right\} \\ &\cdot\frac{\partial^2 F_3}{\partial \xi^2} + 3\sigma F_1 F_2 \mathrm{e}^{\mathrm{i}\Delta k_3 z_0} = 0, \end{aligned} \tag{24}$$

where $\sigma = \omega_1^2\chi_{nl}^{(2)}/(k_1c^2)$. The parameters

$$\varDelta_2 = \mu^{-1/2}(k_1' - k_2'), \tag{25}$$

$$\varDelta_3 = \mu^{-1/2}(k_1' - k_3'), \tag{26}$$

in Eqs.(23) and (24) denote the group-velocity mismatches between the fundamental wave and the second (for $\varDelta_2$) and the third (for $\varDelta_3$) harmonic waves, respectively. They are considered to be of order unity in our present problem. In general, the group-velocity mismatches result in an effect of temporal walk-off among different light pulses. Equations (22)–(24) are coupled envelope equations governing the nonlinear evolution of the the fundamental, second harmonic and third harmonic waves involved in the simultaneous SHG and THG processes. The terms with second-order derivatives in Eqs.(22)–(24) represent the dispersion of corresponding wave modes, which are important for the formation of three-wave temporal optical solitons discussed below.

III. THREE-WAVE TEMPORAL OPTICAL SOLITON SOLUTIONS

In order to solve Eqs.(22)–(24) we perform the variable transformation $z_2 = \mu z = \mu L_{\rm D}\tau$, $\xi = \mu^{1/2}(k_1'z - t) = \mu^{1/2}T_0 s$, $F_l = \mu^{-1}F_{l0}a_l(\tau, s)$ with $F_{10} = 1/(\sqrt{6}\sigma L_{\rm D})$, $F_{20} = 1/(\sqrt{3}\sigma L_{\rm D})$ and $F_{30} = 1/(\sqrt{2}\sigma L_{\rm D})$, where T_0 denotes the pulse width and $L_{\rm D}$ the dispersion length ($= T_0^2/|k_1|''$), then Eqs.(22)–(24) transfer into the following dimensionless form:

$$\begin{aligned}
&{\rm i}\frac{\partial a_1}{\partial \tau} - \frac{\alpha_1}{2}\frac{\partial^2 a_1}{\partial s^2} + a_2^* a_3 {\rm e}^{-{\rm i}\beta_3\tau} \\
&+ \gamma_1 a_1^* a_2 {\rm e}^{-{\rm i}\beta_2\tau} = 0, &(27)\\
&{\rm i}\left(\frac{\partial a_2}{\partial \tau} - \delta_2\frac{\partial a_2}{\partial s}\right) - \frac{\alpha_2}{2}\frac{\partial^2 a_2}{\partial s^2} \\
&+ a_3 a_1^* {\rm e}^{-{\rm i}\beta_3\tau} + \gamma_2 a_1^2 {\rm e}^{{\rm i}\beta_2\tau} = 0, &(28)\\
&{\rm i}\left(\frac{\partial a_3}{\partial \tau} - \delta_3\frac{\partial a_3}{\partial s}\right) - \frac{\alpha_3}{2}\frac{\partial^2 a_3}{\partial s^2} + a_1 a_2 {\rm e}^{{\rm i}\beta_3\tau} = 0, &(29)
\end{aligned}$$

where $\alpha_1 = {\rm sgn}(k_1'')$, $\alpha_2 = \{k_2'' + [(k_2')^2 - (k_1')^2]/k_2\}/|k_1''|$, $\alpha_3 = \{k_3'' + [(k_3')^2 - (k_1')^2]/k_3\}/|k_1''|$, $\beta_2 = -(\varDelta k_2)L_{\rm D}$, $\beta_3 = -(\varDelta k_3)L_{\rm D}$, $\gamma_2 = \gamma_1/2 = \sqrt{3}/6$, $\delta_2 = L_{\rm D}(k_2' - k_1')/T_0$, and $\delta_3 = L_{\rm D}(k_3' - k_1')/T_0$; $\delta_l\,(l = 2, 3)$ are also called the temporal walk-off parameters (i.e. the group-velocity mismatches). The phase mismatches are reflected by two parameters β_2 and β_3.

We are interested in three-wave soliton solutions for Eqs.(27)–(29). Some special soliton solutions with $\gamma_l = 0 (l = 1, 2)$ (i.e. only for dispersive interaction) have been discussed recently.[6] Here we are involved in the multistep cascading thus γ_1 and γ_2 cannot be taken to be zero. We assume that the solutions have the form $a_l = U_l(\theta){\rm exp}({\rm i}\theta_l)$ with $\theta = \varOmega s - K\tau, \theta_l = K_l\tau - \varOmega_l s$ $(l = 1, 2, 3)$. Then Eqs.(27)–(29) are transformed into the following nonlinear ordinary differential equations:

$$\begin{aligned}
&-\frac{\alpha_1}{2}\varOmega^2 U_1'' + U_2^* U_3 + \gamma_1 U_1^* U_2 + {\rm i}(-K + \alpha_1\varOmega\varOmega_1)U_1' \\
&+ (-K_1 + \frac{\alpha_1}{2}\varOmega_1^2)U_1 = 0, &(30)\\
&-\frac{\alpha_2}{2}\varOmega^2 U_2'' + U_1^* U_3 + \gamma_2 U_1^2 \\
&+ {\rm i}(-K + \alpha_2\varOmega\varOmega_2 - \delta_2\varOmega)U_2' \\
&+ (-K_2 + \frac{\alpha_2}{2}\varOmega_2^2 - \delta_2\varOmega_2)U_2 = 0, &(31)\\
&-\frac{\alpha_3}{2}\varOmega^2 U_3'' + U_1 U_2 + {\rm i}(-K + \alpha_3\varOmega\varOmega_3 - \delta_3\varOmega)U_3' \\
&+ (-K_3 + \frac{\alpha_3}{2}\varOmega_3^2 - \delta_3\varOmega_3)U_3 = 0, &(32)
\end{aligned}$$

with the conditions

$$\varOmega_2 = 2\varOmega_1, \quad \varOmega_3 = \varOmega_1 + \varOmega_2, \tag{33}$$

$$K_2 = 2K_1 + \beta_2, \quad K_3 = K_1 + K_2 + \beta_3. \tag{34}$$

It is hard to gain the general solutions of Eqs.(30)–(32) by using a conventional method of integration. However, simple three-wave soliton solutions can be obtained in the following way. We assume $U_l = C_l{\rm sech}^{n_l}\theta$ with n_l being integers and $C_l\,(l = 1, 2, 3)$ constants. As a result we have $U_l'' = n_l(n_l - 1)C_l{\rm sech}^{n_l+2}\theta + n_l C_l{\rm sech}^{n_l}\theta$. Note that the formation of solitons in a nonlinear and dispersive system is due to the balance between the nonlinearity and the dispersion. In our system, dispersion is represented by the second-order derivatives in Eqs.(30)–(32). Under such considerations, to obtain soliton solutions for the system (30)–(32) one identifies that $n_1+2 = n_2+n_3 = n_1+n_2$ (from Eq.(30)), $n_2+2 = n_1+n_3 = 2n_1$ (from Eq.(31)), and $n_3+2 = n_1+n_2$ (from Eq.(32)). Thus we obtain $n_1 = n_2 = n_3 = 2$ and hence we make the assumption $U_l = C_l{\rm sech}^2\theta$ $(l = 1, 2, 3)$. Substituting this assumption into Eqs.(30)–(32), we have the following equations:

$$3\alpha_1\varOmega^2 C_1 + C_2C_3 + \gamma_1 C_1C_2 = 0, \tag{35}$$

$$-2\alpha_1\varOmega^2 C_1 + (-K_1 + \frac{\alpha_1}{2}\varOmega_1^2)C_1 = 0, \tag{36}$$

$$3\alpha_2\varOmega^2 C_2 + C_1C_3 + \gamma_2 C_1^2 = 0, \tag{37}$$

$$-2\alpha_2\varOmega^2 C_2 + (-K_2 + \frac{\alpha_2}{2}\varOmega_2^2 - \delta_2\varOmega_2)C_2 = 0, \tag{38}$$

$$3\alpha_3\varOmega^2 C_3 + C_1C_2 = 0, \tag{39}$$

$$-2\alpha_3\varOmega^2 C_3 + (-K_3 + \frac{\alpha_3}{2}\varOmega_3^2 - \delta_3\varOmega_3)C_3 = 0. \tag{40}$$

Solving Eqs.(35)–(40), we obtain (note $\gamma_2 = 2^{-1}\gamma_1$)

$$C_1 = 3s_1\Omega^2\sqrt{\alpha_2\alpha_3}\left(1 + s_2\frac{\alpha_3\gamma_1}{\sqrt{\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1}}\right)^{1/2}, \quad (41)$$

$$C_2 = \frac{3}{2}s_2\Omega_2\sqrt{\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1} \cdot \left(1 + s_2\frac{\alpha_3\gamma_1}{\sqrt{\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1}}\right), \quad (42)$$

$$C_3 = -\frac{3}{2}s_1 s_2 \mathrm{sgn}(\alpha_3)\Omega_2\sqrt{(\alpha_2/\alpha_3)(\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1)} \cdot \left(1 + s_2\frac{\alpha_3\gamma_1}{\sqrt{\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1}}\right)^{3/2}, \quad (43)$$

$$K_1 = -2\alpha_1\Omega^2 + \frac{1}{2}\alpha_1\Omega_1^2, \quad (44)$$

$$\Omega_1 = -\frac{\delta_3}{\alpha_1 - 3\alpha_3}, \quad (45)$$

$$K = \alpha_1\Omega\Omega_1, \quad (46)$$

$$\Omega^2 = \frac{\alpha_1\Omega_1^2 + \beta_2 + \delta_2\Omega_2 - \alpha_2\Omega_2^2/2}{4\alpha_1 - 2\alpha_2}, \quad (47)$$

with $s_l = \pm 1\,(l = 1, 2)$ and the conditions

$$\frac{\delta_2}{\delta_3} = \frac{\alpha_1 - 2\alpha_2}{\alpha_1 - 3\alpha_3}, \quad (48)$$

$$(16\alpha_3\alpha_1 - 9\alpha_1\alpha_2 - 5\alpha_2\alpha_3)\Omega_1^2 + [4\delta_2(3\alpha_1 - \alpha_3) - 6\delta_3(2\alpha_1 - \alpha_2)]\Omega_1 + 2\beta_2(\alpha_1 + \alpha_2 - \alpha_3) - 2\beta_2(2\alpha_1 - \alpha_2) = 0. \quad (49)$$

Consequently, we have the following three-wave soliton solution for Eqs.(27)–(29):

$$a_1 = 3s_1\rho\sqrt{\alpha_2\alpha_3}\Omega^2\mathrm{sech}^2(\Omega s - K\tau) \cdot \exp[\mathrm{i}(K_1\tau - \Omega_1 s)], \quad (50)$$

$$a_2 = \frac{3}{2}s_2\rho^2\Omega^2\sqrt{\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1} \cdot \mathrm{sech}^2(\Omega s - K\tau)\exp[\mathrm{i}(K_2\tau - \Omega_2 s)], \quad (51)$$

$$a_3 = -\frac{3}{2}s_1 s_2\rho^3\Omega^2\mathrm{sgn}(\alpha_3)\sqrt{(\alpha_2/\alpha_3)(\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1)} \cdot \mathrm{sech}^2(\Omega s - K\tau)\exp[\mathrm{i}(K_3\tau - \Omega_3 s)], \quad (52)$$

with $\rho = [1 + s_2\alpha_3\gamma_1/(\alpha_3^2\gamma_1^2 + 4\alpha_3\alpha_1)^{1/2}]^{1/2}$. We see that all three wave components are simultaneously one-hump solitons with the same central position and the same travelling velocity. The physical origin for the formation of such simultaneous solitons is the mutual self-trapping via the multistep $\chi^{(2)}$ cascading process. Note that $s_l = \pm 1\,(l = 1, 2)$, hence one has in fact a family of three-wave soliton solutions. The solution (50)–(52) is valid also for vanishing temporal walk-off. In fact, if there is no group-velocity mismatch, i.e. $\delta_l = 0\,(l = 2, 3)$, the three-wave soliton solution (50)–(52) is still a solution but requires $\alpha_1 = 2\alpha_2 = 3\alpha_3$ and $\beta_3 = 7\beta_2/9$, Ω_1 is an arbitrary parameter for this particular case.

IV. DISCUSSION AND SUMMARY

We have studied a multistep $\chi^{(2)}$ cascading for light pulses with the dispersion of the system taken into account. By use of the method of multiple scales, we have derived coupled envelope equations which describe the nonlinear evolution of the fundamental, second and third harmonic waves involved simultaneously in three- and four-step cascadings. We have demonstrated that three-wave temporal optical solitons are possible in such multistep cascading in the presence of temporal walk-off for different pulses.

The formation mechanism of the three-wave temporal solitons is the multistep cascading effect between three wave components. In this process, the fundamental and harmonics interact with one another through repeated wave–wave interactions. For instance, in the three-step cascading, the energy of the fundamental wave is first upconverted to the second harmonic waves, and further upconverted to the third harmonic waves, and then downconverted to the fundamental waves again, resulting in a self-trapping of the fundamental wave and hence the formation of the fundamental solitons. At the same time, the two- and four-step cascadings also have their contribution to the appearance of the fundamental solitons. The formation of two other solitons corresponding to the second harmonic and the third harmonic waves are also through a similar mechanism.

In order to produce the three-wave temporal optical solitons suggested here, appropriate optical materials with large $\chi^{(2)}$ response should be chosen. In addition, the nearly phase-matching conditions (11), (12) and the nearly group-velocity matching conditions (25), (26) should be taken into account. Note that the spatial solitons using two-step cascading have been observed both in one and two dimensions.[8,9] However, the creation of temporal solitons due to cascading is more difficult than the spatial solitons because the dominant role of the group-velocity mismatch, which splits the interacting pulses at different wavelengths before the interplay between gain and group-velocity dispersion (i.e. the terms of second-order derivative in Eqs.(27)–(29)) allows the trapping mechanism to set in. To solve this problem, in a recent experiment Di Trapani *et al.*[10] designed a powerful method to create the temporal solitons due to two-step cascading. In their technique tilted pulses were used. The impact of the tilt is to tune the effective group-velocity mismatch and group-velocity dispersion to proper values to form a temporal soliton in real materials (for details see Ref.[10]). Thus, to observe the solitons predicted above, one can also apply tilted pulses. Another method is to use one-dimensional photonic crystals that consist of alternating GaAs slabs.[7] A detailed investigation for the three-wave temporal optical solitons in photonic crystals, etc, will be given in a future publication.

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