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High-Dimensional Nonlinear Envelope Equations and Nonlinear Localized Excitations in Photonic Crystals*

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Abstract *We investigate the nonlinear localized structures of optical pulses propagating in a one-dimensional photonic crystal with a quadratic nonlinearity. Using a method of multiple scales we show that the nonlinear evolution of a wave packet, formed by the superposition of short-wavelength excitations, and long-wavelength mean fields, generated by the self-interaction of the wave packet, are governed by a set of coupled high-dimensional nonlinear envelope equations, which can be reduced to Davey–Stewartson equations and thus support dromionlike high-dimensional nonlinear excitations in the system.*

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1 Introduction

In recent years, considerable progress has been made on the study of electromagnetic wave propagation in photonic crystals (PCs). Since the periodic variation of refractive index of PCs can be controlled and even engineered, such systems have many important applications in the design of new optical materials and optical-device.[1] PCs have also opened a new chapter of nonlinear optics. Because of their remarkable capacities of localizing and guiding electromagnetic waves, together with local field enhancement and dispersion tunability, PCs provide a conceptually new architecture for nonlinear optical materials with enhanced nonlinearities, extended phase matching abilities, artificial anisotropy, etc.[2,3] In particular, PCs support nonlinear localized structures called optical solitons, which form due to the balance between dispersion (or diffraction) and nonlinearity, near photonic band edge and even within a band gap. However, most studies up to now have been concentrated on one-dimensional (1D) optical solitons in PCs with cubic (i.e. $\chi^{(3)}$) nonlinearity.[3] Although there appeared a lot of work for second harmonic generation and three-wave interactions in PCs with quadratic (i.e. $\chi^{(2)}$) nonlinearity,[4] to the best of our knowledge, up to now there is no report on possible high-dimensional nonlinear localized excitations in such systems.

In the present work, we investigate the nonlinear localized structures of optical pulses propagating in a 1D photonic crystal with a quadratic nonlinearity. Different from most of the previous studies, in our approach we allow a transverse variation of the envelopes of the optical pulses and hence the evolution of these envelopes is (3+1)-dimensional. Using a method of multiple-scales we obtain scalar and vector coupled envelope equations, which can be reduced to Davey–Stewartson (DS) equations and hence dromionlike (2+1)-dimensional nonlinear localized excitations are shown to be possible. The paper is arranged as follows. In Sec. 2 we derive a set of nonlinear scalar envelope equations. An extension to a vector case is given in Appendix. In Sec. 3 we discuss the reduction of these equations to the DS equations and present dromionlike solutions. The last section is a summary of our results.

2 Asymptotic Expansion and Derivation of Scalar Envelope Equations

In nonmagnetic materials, and in the absence of sources, Maxwell equations yield the vector nonlinear wave equation for the electric field $\boldsymbol{E}$ as

$$\nabla^2\boldsymbol{E}-\nabla(\nabla\cdot\boldsymbol{E})-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(\boldsymbol{E}+\frac{\boldsymbol{P}}{\epsilon_0}\Big)=0\,, \tag{1}$$

where ϵ_0 and c are the permittivity and speed of light in vacuum, respectively. The polarization of the material, $\boldsymbol{P}$, can be expressed in terms of the electric field by the expansion

$$\frac{1}{\epsilon_0}\boldsymbol{P}=\chi^{(1)}*\boldsymbol{E}+\chi^{(2)}*\boldsymbol{E}\boldsymbol{E}+\chi^{(3)}*\boldsymbol{E}\boldsymbol{E}\boldsymbol{E}+\cdots\,, \tag{2}$$

where $\chi^{(n)}=\chi^{(n)}(\boldsymbol{r},t)$ is the n-th order susceptibility of the material and asterisk denotes an n-dimensional convolution integral. For simplicity we neglect material dispersion[5] and thus the above expression reads

$$\frac{P_j}{\epsilon_0}=\chi^{(1)}_{jk}E_k+\chi^{(2)}_{jkl}E_kE_l+\chi^{(3)}_{jklm}E_kE_lE_m+\cdots\,,\quad j=1,2,3\,.$$

The system we are going to study is a 1D PC whose dielectric function is a constant in transverse (x and y) directions but a periodic function along z direction. Thus we have $\chi^{(1)}(z+d)=\chi^{(1)}(z)$, where d is lattice constant. The system has been sketched in Fig. 1. Although not independent from the nonlinear wave equation (1), it is useful to use the divergence law

$$\nabla\cdot\Big(\boldsymbol{E}+\frac{\boldsymbol{P}}{\epsilon_0}\Big)=0\,. \tag{3}$$

We note that, as a consequence of Eq. (3), $\nabla\cdot\boldsymbol{E}\neq 0$. The term $\nabla\cdot\boldsymbol{E}$ is a small perturbation and its contribution

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to the nonlinear wave equation (1) is usually neglected (the so-called transverse wave approximation). However, its presence is crucial in the following derivation as we are going to see.

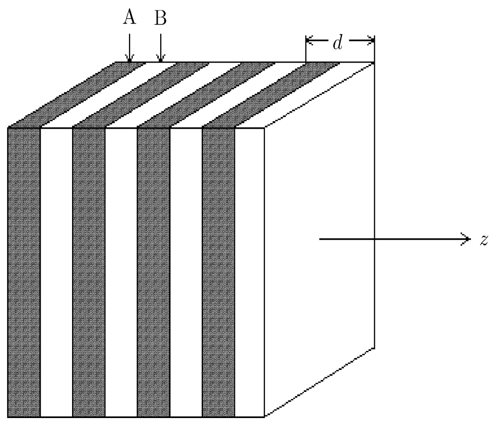

Fig. 1 The schematic illustration of a one-dimensional (1D) PC whose dielectric function is a constant in transverse (x and y) directions but a periodic function along z direction. Here, A and B denote two different kinds of material and d is lattice constant.

We are interested in the weak nonlinear excitations of the system. We introduce the asymptotic expansion $\boldsymbol{E}=\sum_{j=1}^{\infty}\varepsilon^j\boldsymbol{E}^{(j)}=\varepsilon\boldsymbol{E}^{(1)}+\varepsilon^2\boldsymbol{E}^{(2)}+\varepsilon^3\boldsymbol{E}^{(3)}+\cdots$, where ε is a small parameter characterizing the relative amplitude of the electric field. To obtain a divergence-free expansion, $\boldsymbol{E}^{(j)}$ are considered as the functions of the multi-scale variables $\boldsymbol{r}^j=\varepsilon^j\boldsymbol{r}$ and $t_j=\varepsilon^j t$ $(j=0,1,2,\ldots)$. We assume the susceptibility has a periodic variation only on fast spatial scale, i.e., $\chi^{(j)}(z)=\chi^{(j)}(z_0)$. The vector $\boldsymbol{P}$ can be also expanded in powers of ε.

Using the above expansion and collecting the terms with equal powers of ε, equation (1) can be reduced into ($j=1$ to 4)

$$\nabla_0^2\boldsymbol{E}^{(j)}-\nabla_0(\nabla_0\cdot\boldsymbol{E}^{(j)})-\frac{1}{c^2}\frac{\partial^2}{\partial t_0^2}\boldsymbol{E}^{(j)} -\frac{1}{c^2}\chi^{(1)}(z_0)\frac{\partial^2}{\partial t_0^2}\boldsymbol{E}^{(j)}=\boldsymbol{M}^{(j)}\,. \tag{4}$$

The explicit expressions of $\boldsymbol{M}^{(j)}$ can be obtained analytically but their explicit expressions are omitted here.

We consider first the propagation of an optical pulse in a uniaxial 4 mm materials (for example $BaTiO_3$). This particular choice of symmetry class guarantees that only few components of the nonlinear susceptibility tensors play an active role in the vector wave Eq. (1).[6] For convenience we take (x,y,z) axes as the crystallographic z,x,y axes. The non-zero components are then given by

$$\chi^{(1)}:\chi_{xx}^{(1)}=\chi_{yy}^{(1)}\neq\chi_{zz}^{(1)}\,,$$

$$\chi^{(2)}:\chi_{xxx}^{(2)},\chi_{xyy}^{(2)}=\chi_{xzz}^{(2)},\chi_{yxy}^{(2)}=\chi_{zxz}^{(2)},\chi_{yyx}^{(2)}=\chi_{zzx}^{(2)}\,.$$

For $\chi^{(3)}$, there are 21 nonzero elements, among which eleven are independent. We assume that input electric field of the system is polarized along one of the principal axes of crystal. The field propagating along the z axis and polarized along x axis, i.e. $\boldsymbol{E}^{(1)}=(E_x^{(1)},0,0)$ (see Fig. 1). The x component of Eq. (4) at leading order ($j=1$) reads

$$\frac{\partial^2}{\partial z_0^2}E_x^{(1)}-\frac{1}{c^2}\frac{\partial^2}{\partial t_0^2}E_x^{(1)}-\frac{1}{c^2}\chi_{xx}^{(1)}(z_0)\frac{\partial^2}{\partial t_0^2}E_x^{(1)}=0\,. \tag{5}$$

We assume that input field is a wavepacket, i.e. we take $E_x^{(1)}=A\phi_m(z_0)\,\mathrm{e}^{-\mathrm{i}\omega_m t_0}+\text{c.c.}$, where A is an envelope function of slow variables $\boldsymbol{r}^j$ and t_j $(j=1,2,\ldots)$, which is yet to be determined, and c.c. denotes complex conjugate. Substituting this solution into Eq. (5) we get

$$-c^2\frac{\partial^2}{\partial z_0^2}\phi_m(z_0)=\omega_m^2\epsilon(z_0)\phi_m(z_0)\,, \tag{6}$$

where $\epsilon(z_0)=1+\chi_{xx}^{(1)}(z_0)$. Equation (6) is an eigenvalue problem of Sturm–Liouville type. Its solutions constitute a complete set of eigenfunctions, called the Bloch states. The eigenfunctions satisfy the following orthonormalized relation:

$$\langle m'|\epsilon(z_0)|m\rangle=\int_0^L\mathrm{d}z_0\phi_{m'}^*(z_0)\epsilon(z_0)\phi_m(z_0)=\delta_{m'm}\,, \tag{7}$$

where $L=nd$ is the length over which the function ϕ_m must be periodic, with d being the period of the PC.

Since the dielectric function $\epsilon(z_0)$ is periodic, the Floquet–Bloch theorem applies. Thus one has

$$\phi_m(z_0)=\phi_{n,k}(z_0)=\Gamma_{n,k}(z_0)\,\mathrm{e}^{\mathrm{i}kz_0}\,, \tag{8}$$

where $\Gamma_{n,k}(z_0)$ is a periodic function of periodicity d, i.e. $\Gamma_{n,k}(z_0)=\Gamma_{n,k}(z_0+d)$. The subscripts n and k are the band index and crystal momentum, respectively.

At the second order ($j=2$), the x component of Eq. (4) gives

$$\begin{aligned}&\frac{\partial^2}{\partial z_0^2}E_x^{(2)}-\frac{1}{c^2}\epsilon(z_0)\frac{\partial^2}{\partial t_0^2}E_x^{(2)}\\ &\quad=-\Big[2\frac{\partial A}{\partial z_1}\frac{\partial\phi_m}{\partial z_0}+\frac{2}{c^2}\mathrm{i}\omega_m\epsilon(z_0)\frac{\partial A}{\partial t_1}\phi_m\Big]\,\mathrm{e}^{-\mathrm{i}\omega_m t_0}\\ &\qquad-\frac{4\omega_m^2}{c^2}\chi_{xxx}^{(2)}(z_0)A^2\phi_m^2\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0}+\text{c.c.}\end{aligned} \tag{9}$$

In order to solve this equation, we use the completeness of the linear eigenfunctions. We make the expansion

$$\begin{aligned}E_x^{(2)}&=E_x^{(2,0)}+\sum_l E_{x,l}^{(2,1)}\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}\omega_m t_0}\\ &\quad+\sum_l E_{x,l}^{(2,2)}\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0}+\text{c.c.}\,,\end{aligned} \tag{10}$$

where $E^{(2,0)}$, $E^{(2,1)}$, $E^{(2,2)}$ are new envelope functions of slow variables. Substituting the expansion into Eq. (9), we find

$$\sum_l E_{x,l}^{(2,1)}[\omega_l^2-\omega_m^2]\epsilon(z_0)\phi_l\,\mathrm{e}^{-\mathrm{i}\omega_m t_0}+\text{c.c.}=2\mathrm{i}\Big[c\hat{\Omega}\phi_m\frac{\partial A}{\partial z_1}+\omega_m\epsilon(z_0)\phi_m\frac{\partial A}{\partial t_1}\Big]\,\mathrm{e}^{-\mathrm{i}\omega_m t_0}+\text{c.c.}\,, \tag{11a}$$

$$\sum_l E_{x,l}^{(2,2)}[\omega_l^2 - 4\omega_m^2]\epsilon(z_0)\phi_l\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0} + \text{c.c.} = 4\omega_m^2\chi_{xxx}^{(2)}(z_0)\phi_m^2 A^2\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0} + \text{c.c.}, \tag{11b}$$

where $\hat{\Omega} = -\mathrm{i}c\partial/\partial z_0$.

We analyze Eq. (11a) with two steps. In the first step we project it onto the eigenvector ϕ_m. The left-hand side of Eq. (11a) then vanishes and the following condition for envelope function A is thus found

$$c\langle m|\hat{\Omega}|m\rangle\frac{\partial A}{\partial z_1} + \omega_m\frac{\partial A}{\partial t_1} = 0\,, \tag{12}$$

where $\langle m|\hat{\Omega}|m\rangle = \int_0^L \mathrm{d}z_0\phi_m^*\hat{\Omega}\phi_m$ and the orthonormalized relation (7) has been used. We see that the envelope A propagates with the (group) velocity

$$c_g = \frac{\mathrm{d}\omega_m}{\mathrm{d}k} = \frac{c}{\omega_m}\langle m|\hat{\Omega}|m\rangle\,.$$

In the second step we project Eq. (11a) onto the space spanned by the remaining eigenfunction vectors $\{\phi_l\}$ ($l \neq m$). Then we have

$$E_{x,l}^{(2,1)} = \frac{\partial A}{\partial z_1}\Lambda_{l,m}d\,, \tag{13}$$

where the coupling coefficient $\Lambda_{l,m}$ is defined as

$$\Lambda_{l,m} = \frac{2\mathrm{i}c}{d}\frac{\langle l|\hat{\Omega}|m\rangle}{\omega_l^2 - \omega_m^2}\,. \tag{14}$$

Similarly from Eq. (11b) we also get $E_{x,l}^{(2,2)} = A^2\Gamma_{l,m}^x$ with

$$\Gamma_{l,m}^x = \frac{4\omega_m^2\langle l|\chi_{xxx}^{(2)}(z_0)|m,m\rangle}{\omega_l^2 - 4\omega_m^2}\,,$$

$$\langle l|\chi_{xxx}^{(2)}(z_0)|m,m\rangle = \int_0^L \mathrm{d}z_0\phi_l^*\chi_{xxx}^{(2)}(z_0)\phi_m\phi_m\,.$$

Thus we obtain

$$E_x^{(2)} = \Phi_x + \sum_{l\neq m}\frac{\partial A}{\partial z_1}\Lambda_{l,m}\,\mathrm{d}\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}\omega_m t_0} + \sum_{l\neq 2m}A^2\Gamma_{l,m}^x\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0} + \text{c.c.} \tag{15}$$

Following a similar procedure, we find $E_{y,l}^{(2,1)} = 0$, $E_{y,l}^{(2,2)} = A^2\Gamma_{l,m}^y$, $E_{z,l}^{(2,1)} = V_{l,m}d\,\partial A/\partial x_1$, and $E_{z,l}^{(2,2)} = A^2\Gamma_{l,m}^z$, where

$$\Gamma_{l,m}^y = \frac{4\omega_m^2\langle l|\chi_{yxx}^{(2)}(z_0)|m,m\rangle}{\omega_l^2 - 4\omega_m^2}\,,$$

$$V_{l,m} = \frac{\mathrm{i}c\langle l|\eta(z_0)\hat{\Omega}|m\rangle}{\omega_m^2 d}\,,$$

$$\Gamma_{l,m}^z = -\langle l|\eta(z_0)\chi_{zxx}^{(2)}(z_0)|m,m\rangle$$

with $\eta(z_0) = \epsilon(z_0)/\bar{\epsilon}(z_0)$ and $\bar{\epsilon}(z_0) = 1 + \chi_{zz}^{(1)}(z_0)$. Note that for normal uniaxial crystal $\eta(z_0) \neq 1$. Thus we get

$$E_y^{(2)} = \Phi_y + \sum_{l\neq 2m}A^2\Gamma_{l,m}^y\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0} + \text{c.c.}\,, \tag{16a}$$

$$E_z^{(2)} = \Phi_z + \sum_l\frac{\partial A}{\partial x_1}V_{l,m}d\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}\omega_m t_0} + \sum_l A^2\Gamma_{l,m}^z\phi_l(z_0)\,\mathrm{e}^{-\mathrm{i}2\omega_m t_0} + \text{c.c.} \tag{16b}$$

The dc electric fields Φ_i ($i = 1, 2, 3$) in Eqs. (15) and (16) at this order are still undetermined functions. These dc fields are generated by self-excitation of the wavepacket and play an important role, as shown below.

At third-order approximation ($j = 3$), a solvability condition of Eq. (4) gives rise to

$$\mathrm{i}\kappa_1\frac{\partial A}{\partial z_2} + \mathrm{i}\kappa_2\frac{\partial A}{\partial t_2} + \kappa_3\frac{\partial^2 A}{\partial x_1^2} + \kappa_4\frac{\partial^2 A}{\partial y_1^2} + \kappa_5\frac{\partial^2 A}{\partial \xi^2} + \kappa_6\Phi_x A + \kappa_7|A|^2 A = 0\,, \tag{17}$$

where $\xi = z_1 - c_g t_1$, and

$$\kappa_1 = \frac{2}{c}\langle m|\hat{\Omega}|m\rangle\,, \tag{18a}$$

$$\kappa_2 = \frac{2\omega_m}{c^2}\,, \tag{18b}$$

$$\kappa_3 = -\frac{\mathrm{i}}{c}\sum_l V_{l,m}d\langle m|\hat{\Omega}|l\rangle\,, \tag{18c}$$

$$\kappa_4 = \langle m|m\rangle\,, \tag{18d}$$

$$\kappa_5 = \langle m|m\rangle - \frac{c_g^2}{c^2} + \frac{2\mathrm{i}}{c}\sum_{l\neq m}\Lambda_{l,m}d\langle m|\hat{\Omega}|l\rangle\,, \tag{18e}$$

$$\kappa_6 = \frac{2\omega_m^2}{c^2}\langle m|\chi_{xxx}^{(2)}(z_0)|m\rangle\,, \tag{18f}$$

$$\kappa_7 = \frac{\omega_m^2}{c^2}\Big[2\sum_{l\neq 2m}\Gamma_{l,m}^x\langle m,m|\chi_{xxx}^{(2)}(z_0)|l\rangle + 3\langle m,m|\chi_{xxxx}^{(3)}(z_0)|m,m\rangle\Big]\,. \tag{18g}$$

It is easy to show the relations $\kappa_1/\kappa_2 = c_g = \partial\omega_m/\partial k$ and $2\kappa_5/\kappa_2 = \partial^2\omega_m/\partial k^2 = \partial c_g/\partial k$.

Note that the divergence law (3) for dc terms at order $j = 3$ yields the explicit relation among Φ_x, Φ_y, Φ_z and A. Substituting this relation into the x-component of Eq. (4) for dc terms at the fourth order ($j = 4$), we obtain

$$\lambda_1\frac{\partial^2\Phi_x}{\partial x_1^2} + \frac{\partial^2\Phi_x}{\partial y_1^2} + \lambda_2\frac{\partial^2\Phi_x}{\partial\xi^2} + (\lambda_1 - 1)\frac{\partial^2\Phi_y}{\partial x_1\partial y_1} = \lambda_3\frac{\partial^2}{\partial x_1^2}|A|^2 + \lambda_4\frac{\partial^2}{\partial\xi^2}|A|^2 \tag{19}$$

with

$$\lambda_1 = \frac{1}{L}\int_0^L\eta(z_0)\,\mathrm{d}z_0\,, \tag{20a}$$

$$\lambda_2 = 1 - \frac{c_g^2}{Lc^2}\int_0^L\epsilon(z_0)\,\mathrm{d}z_0\,, \tag{20b}$$

$$\lambda_3 = -\frac{2}{L}\langle m|\frac{\chi_{xxx}^{(2)}(z_0)}{\bar{\epsilon}(z_0)}|m\rangle\,, \tag{20c}$$

$$\lambda_4 = \frac{2c_g^2}{Lc^2}\langle m|\chi_{xxx}^{(2)}(z_0)|m\rangle\,. \tag{20d}$$

For getting a close system we need an additional equation, which can be obtained from the y-component of Eq. (4) for dc terms at the fourth order. It reads

$$\frac{\partial^2\Phi_y}{\partial x_1^2} + \lambda_1\frac{\partial^2\Phi_y}{\partial y_1^2} + \lambda_2\frac{\partial^2\Phi_y}{\partial\xi^2} + (\lambda_1 - 1)\frac{\partial^2\Phi_x}{\partial x_1\partial y_1}$$

$$= \lambda_3 \frac{\partial^2}{\partial x_1 \partial y_1} |A|^2 \,. \tag{21}$$

Equations (17), (19), and (21) describe the nonlinear evolution of the envelopes A, Φ_x, and Φ_y. Similar equations have been obtained by Ablowitz *et al.* in different optical systems.[7]

The above results can be generalized to a vector case, that is, when the input electric field at the leading-order is within xy plane, i.e. $\boldsymbol{E}^{(1)} = (E_x^{(1)}, E_y^{(1)}, 0)$. A detailed derivation for vector envelope equations has been presented in Appendix.

3 Davey–Stewartson Equations and (2+1)-Dimensional Localized Solutions

We now discuss the solutions of the nonlinear envelope equations obtained in the last section. For simplicity we assume A, Φ_x, and Φ_y are only dependent on ξ, y_1, and t_2. Taking $\Phi_x = \partial A_0/\partial \xi$, from Eqs. (17) and (19) we get

$$\alpha_1 \frac{\partial^2 A_0}{\partial \xi^2} - \frac{\partial^2 A_0}{\partial y_1^2} = \alpha_2 \frac{\partial}{\partial \xi} |A|^2 \,, \tag{22a}$$

$$\mathrm{i} \frac{\partial A}{\partial t_2} + \beta_1 \frac{\partial^2 A}{\partial \xi^2} + \beta_2 \frac{\partial^2 A}{\partial y_1^2} + \beta_3 |A|^2 A - \beta_4 A \frac{\partial A_0}{\partial \xi} = 0 \tag{22b}$$

with $\alpha_1 = -\lambda_2$, $\alpha_2 = -\lambda_4$, $\beta_1 = \kappa_5/\kappa_2$, $\beta_2 = \kappa_4/\kappa_2$, $\beta_3 = \kappa_7/\kappa_2$ and $\beta_4 = -\kappa_6/\kappa_2$.

Defining $\partial A_0/\partial\xi = -[\beta_1/(\varepsilon^2 \alpha_1 \beta_4)]s$ and $A = [4\beta_1/(\varepsilon^2 \alpha_2 \beta_4)]^{1/2} u$, equations (22a) and (22b) can be rewritten as

$$\frac{\partial^2 s}{\partial z'^2} - \frac{\partial^2 s}{\partial y'^2} + 4 \frac{\partial^2}{\partial z'^2} (|u|^2) = 0 \,, \tag{23a}$$

$$\mathrm{i} \frac{\partial u}{\partial t'} + \frac{\partial^2 u}{\partial z'^2} + \frac{\alpha_1 \beta_2}{\beta_1} \frac{\partial^2 u}{\partial y'^2} + 4 \frac{\alpha_1 \beta_3}{\alpha_2 \beta_4} |u|^2 u + su = 0 \,, \tag{23b}$$

where $z' = \sqrt{1/\alpha_1}(z - c_g t)$, $y' = y$, $t' = (\beta_1/\alpha_1)t$. When α_i and β_i satisfy the conditions $\alpha_1 \beta_2/\beta_1 = 1$ and $2\alpha_1 \beta_3/(\alpha_2 \beta_4) = 1$, equation (23b) takes the form

$$\mathrm{i} \frac{\partial u}{\partial t'} + \frac{\partial^2 u}{\partial z'^2} + \frac{\partial^2 u}{\partial y'^2} + 2|u|^2 u + su = 0 \,. \tag{24}$$

Equations (23a) and (24) are standard DS-I equations, which are completely integrable and can be solved exactly by inverse scattering method.[8] One of the remarkable properties of the DS-I equations is that they allow a localized envelope solution decaying exponentially in all spatial directions.[8] The key for the fact that equations (22a) and (22b) can be simplified into the DS-I equations depends on the fact that both α_i and β_i are positive.

A single dromion solution of the DS-I Eqs. (23a) and (23b) is[9]

$$u = \frac{G}{F} \,, \quad s = 4 \frac{\partial^2}{\partial z'^2} \ln F \,, \tag{25a}$$

$$F = 1 + \exp(\eta_1 + \eta_1^*) + \exp(\eta_2 + \eta_2^*) + \gamma \exp(\eta_1 + \eta_1^* + \eta_2 + \eta_2^*) \,, \tag{25b}$$

$$G = \rho \exp(\eta_1 + \eta_2) \tag{25c}$$

with $\eta_1 = (k_r + \mathrm{i}k_i)z'' + (\Omega_r + \mathrm{i}\Omega_i)t'$, $\eta_2 = (l_r + \mathrm{i}l_i)z'' + (\omega_r + \mathrm{i}\omega_i)t'$, $\Omega_r = -2k_r k_i$, $\omega_r = -2l_r l_i$, $\Omega_i + \omega_i = k_r^2 + l_r^2 - k_i^2 - l_i^2$, $\rho = |\rho| \exp(\mathrm{i}\varphi_\rho)$, $|\rho| = 2\sqrt{2k_r l_r(\gamma - 1)}$, $z'' = (y' + z')/\sqrt{2}$, and $y'' = (y' - z')/\sqrt{2}$. The constants k_r, k_i, l_r, l_i, $|\rho|$, φ_ρ, and γ are real integration constants. If we choose $k_r l_r > 0$ we obtain $\gamma = \exp(2\varphi_\gamma)$ with $\varphi_\gamma > 0$.

By taking $k_r = \sqrt{2}\mu$, $k_i = \sqrt{2}a$, $l_r = \sqrt{2}\lambda$, $l_i = \sqrt{2}p$ ($\lambda\mu \geq 0$), $\Omega_i = 2(\mu^2 - a^2)$, $\omega_i = 2(\lambda^2 - p^2)$, $\Omega_r = -4a\mu$, and $\omega_r = -4\lambda p$ we obtain

$$u = \frac{2\mu \exp(\mathrm{i}h)}{m \cosh f_1 + n \cosh f_2} \,, \tag{26a}$$

$$s = \frac{4(m^2 + n^2)(\mu^2 + \lambda^2) - 8\mu^2}{(m \cosh f_1 + n \cosh f_2)^2} + \frac{8mn[(\mu^2 + \lambda^2)\cosh f_1 \cosh f_2 - (\mu^2 - \lambda^2)\sinh f_1 \sinh f_2]}{(m \cosh f_1 + n \cosh f_2)^2} \,, \tag{26b}$$

where $m = (\mu/[\lambda(\gamma - 1)])^{1/2}$, $n = (\mu\gamma/[\lambda(\gamma - 1)])^{1/2}$, and

$$h = \sqrt{2}az'' + \sqrt{2}py'' + 2(\mu^2 + \lambda^2 - a^2 - p^2)t' + \varphi_\rho \,, \tag{27a}$$

$$f_1 = \sqrt{2}\mu z'' - \sqrt{2}\lambda y'' - 4(a\mu - \lambda p)t' \,, \tag{27b}$$

$$f_2 = \sqrt{2}\mu z'' + \sqrt{2}\lambda y'' - 4(a\mu + \lambda p)t' + \varphi_\gamma \,. \tag{27c}$$

Obviously, the expression of u in Eq. (26a) denotes a localized envelope function decaying *exponentially* in all spatial directions, called dromion.[8,9] The mean-field component s consists of two interacting plane solitons with each plane soliton decaying in its travelling direction. These features are shown in Fig. 2.

In the case of the dromion excitation, the explicit expression in the leading-order approximation for the x component of optical pulse takes the following form:

$$E_x = 4\mu \sqrt{\frac{4\beta_1}{\alpha_2 \beta_4}} \frac{\cos \Phi}{m \cosh f_1 + n \cosh f_2} \phi_m(z) \cos(\omega_m t) \tag{28}$$

with

$$\Phi = (a + p)y + \frac{a - p}{\sqrt{\alpha_1}} z + \Big[2(\mu^2 + \lambda^2 - a^2 - p^2)\frac{\beta_1}{\alpha_1} - \frac{a - p}{\sqrt{\alpha_1}} c_g - \omega_m\Big] t + \varphi_\rho \,. \tag{29}$$

The results show that the electric field $\boldsymbol{E}$ is a high-dimensional localized function in space.

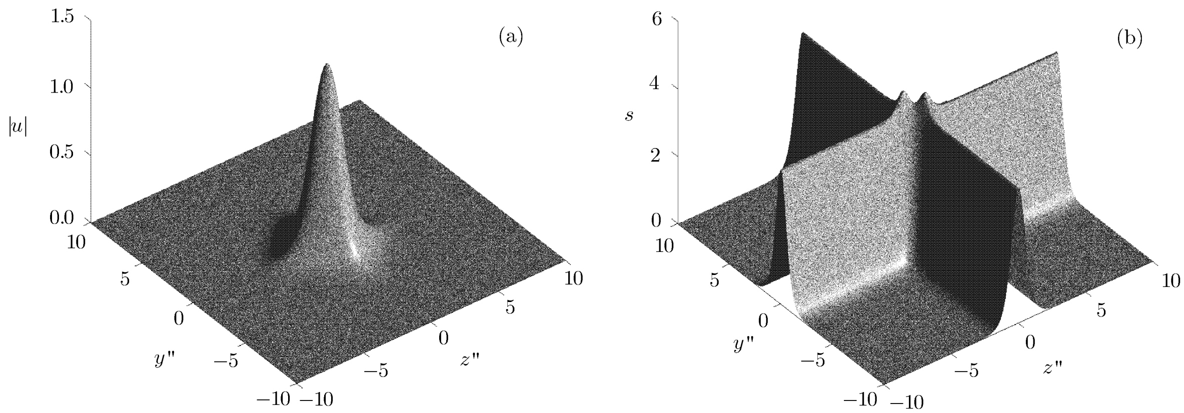

Fig. 2 (a) The modulus of short-wavelength input field component u in the case of a dromion excitation. The parameters are chosen as $\mu = 1.0$, $\lambda = 1.0$, $a = 1.0$, $p = 1.0$, $\varphi_\rho = 0$, $\varphi_\gamma = 1.0$ at $t = 0$. (b) The long wavelength mean-field component s in the case of a dromion excitation. The parameters are the same as in (a).

4 Conclusion

We have investigated the nonlinear dynamics of a multidimensional quasi-monochromatic optical pulse in a 1D photonic crystal with quadratic nonlinearity. By employing a method of multiple-scales and allowing a variation in the transverse directions for the electromagnetic field, we have derived the nonlinear envelope equations with scalar and vector forms for biaxial and uniaxial crystals, describing the interaction between the wavepacket of short waves and long wavelength mean fields generated by the self-interaction of the short waves. We have shown that these envelope equations can be reduced to well-known DS-I equations and hence dromionlike (2+1)-dimensional nonlinear localized excitations are possible. Such high-dimensional nonlinear localized excitations, although predicted in water waves,[7] plasma physics,[10] and Bose–Einstein condensates,[11] have never been reported in photonic crystals up to now. Because of their robust nature, the high-dimensional localized excitations in nonlinear photonic crystals have potential applications in optical information processing and engineering.

Appendix: Vector Envelope Equations

In this appendix we give a detailed derivation of nonlinear envelope equations for the case in which the input electric field has nonzero components along both the transverse principal axes of the material, i.e., $\boldsymbol{E}^{(1)} = (E_x^{(1)}, E_y^{(1)}, 0)$. The derivation follows closely the method developed for the scalar case presented in Sec. 2. For concreteness, we consider a (uniaxial) hexagonal $\overline{6}m2$ crystal and we take (x, y, z) axes to coincide with the crystallographic axes.

Note that the $\chi^{(1)}$-tensor of $\overline{6}m2$ symmetry class has three nonzero elements (diagonal elements), among which two are independent: $\chi_{xx}^{(1)} = \chi_{yy}^{(1)}$ and $\chi_{zz}^{(1)}$. The $\chi^{(2)}$-tensor of $\overline{6}m2$ symmetry class is

$$\chi_{yyy}^{(2)} = -\chi_{yxx}^{(2)} = -\chi_{xxy}^{(2)} = -\chi_{xyx}^{(2)}. \tag{A1}$$

The $\chi^{(3)}$-tensor of $\overline{6}m2$ symmetry class has 21 nonzero elements, among which 10 are independent

$$\chi_{xxxx}^{(3)} = \chi_{yyyy}^{(3)} = \chi_{xxyy}^{(3)} + \chi_{xyyx}^{(3)} + \chi_{xyxy}^{(3)}\,,$$
$$\chi_{xxyy}^{(3)} = \chi_{yyxx}^{(3)}, \chi_{xyyx}^{(3)} = \chi_{yxxy}^{(3)}, \chi_{xyxy}^{(3)} = \chi_{yxyx}^{(3)}\,, \tag{A2}$$

Components containing z indices are omitted here since they have no active role in the following calculation.

The electric field is expanded as in Sec. 2. Although the calculation here is more involved, the perturbation expansion and the analysis proceed almost exactly as in the scalar case. So in what follows, we are only concentrated on the differences between the two cases.

At the leading-order approximation ($j = 1$), since $\chi_{yy}^{(1)}(z_0) = \chi_{xx}^{(1)}(z_0)$, eigenvalue equations of x-component and y-component are identical. That means we can conveniently reserve one set of eigenfunctions to expand the space polarized in either direction. We still choose the eigenfunctions satisfying Eq. (6) as in Sec. 2.

At the second-order approximation ($j = 2$), we obtain ($\alpha = x, y$)

$$E_{\alpha,l}^{(2,1)} = \frac{\partial A_\alpha}{\partial z_1}\Lambda_{l,m} d\,, \tag{A3}$$

where A_α is undetermined envelope function ($A_\alpha = E_\alpha^{(1,1)}$), $\Lambda_{l,m}$ is defined by Eq. (14). The other envelope functions read

$$E_{x,l}^{(2,2)} = A_x A_y \Delta_{l,m}^x\,, \tag{A4a}$$
$$E_{y,l}^{(2,2)} = (A_y^2 - A_x^2)\Delta_{l,m}^y\,, \tag{A4b}$$
$$E_z^{(2,1)} = \Big(\frac{\partial A_x}{\partial x_1} + \frac{\partial A_y}{\partial y_1}\Big)V_{l,m} d\,, \tag{A4c}$$
$$E_z^{(2,2)} = 0\,, \tag{A4d}$$

where

$$\Delta_{l,m}^x = 8\omega_m^2 \frac{\langle l|\chi_{xxy}^{(2)}(z_0)|m, m\rangle}{\omega_l^2 - 4\omega_m^2}$$

$$= -8\omega_m^2 \frac{\langle l|\chi^{(2)}_{yyy}(z_0)|m,m\rangle}{\omega_l^2 - 4\omega_m^2}\,, \tag{A5a}$$

$$\Delta^y_{l,m} = 4\omega_m^2 \frac{\langle l|\chi^{(2)}_{yyy}(z_0)|m,m\rangle}{\omega_l^2 - 4\omega_m^2}\,. \tag{A5b}$$

At the third-order approximation ($j=3$), we find the evolution of the transverse components of the optical field. By defining $\Phi_\alpha = E_\alpha^{(2,0)}$, we obtain

$$\begin{aligned} &\mathrm{i}\kappa_1\frac{\partial A_\alpha}{\partial z_2} + \mathrm{i}\kappa_2\frac{\partial A_\alpha}{\partial t_2} + \kappa_3\Big(\frac{\partial^2 A_\alpha}{\partial\alpha_1^2} + \frac{\partial^2 A_{\bar\alpha}}{\partial\alpha_1\partial\bar\alpha_1}\Big) + \kappa_4\Big(\frac{\partial^2 A_\alpha}{\partial\bar\alpha_1^2} - \frac{\partial^2 A_{\bar\alpha}}{\partial\alpha_1\partial\bar\alpha_1}\Big) + \kappa_5\frac{\partial^2 A_\alpha}{\partial\xi^2} + \kappa_6\Theta_\alpha \\ &\quad + \kappa_7 A_\alpha|A_{\bar\alpha}|^2 + \kappa_8 A_\alpha^*(A_{\bar\alpha}^2 - A_\alpha^2) + \kappa_9[2A_\alpha|A_{\bar\alpha}|^2 + A_\alpha^* A_{\bar\alpha}^2] + \kappa_{10}A_\alpha|A_\alpha|^2 = 0\,, \end{aligned} \tag{A6}$$

with

$$\kappa_1 = \frac{2}{c}\langle m|\hat\Omega|m\rangle\,, \tag{A7a}$$

$$\kappa_2 = \frac{2\omega_m}{c^2}\,, \tag{A7b}$$

$$\kappa_3 = -\frac{\mathrm{i}}{c}\sum_l V_{l,m} d\langle m|\hat\Omega|l\rangle\,, \tag{A7c}$$

$$\kappa_4 = \langle m|m\rangle\,, \tag{A7d}$$

$$\kappa_5 = \langle m|m\rangle - \frac{c_g^2}{c^2} + \frac{2\mathrm{i}}{c}\sum_{l\neq m}\Lambda_{l,m} d\langle m|\hat\Omega|l\rangle\,, \tag{A7e}$$

$$\kappa_6 = -\frac{2\omega_m^2}{c^2}\langle m|\chi^{(2)}_{yyy}(z_0)|m\rangle\,, \tag{A7f}$$

$$\kappa_7 = -\frac{2\omega_m^2}{c^2}\sum_{l\neq 2m}\Delta^x_{l,m}\langle m,m|\chi^{(2)}_{yyy}(z_0)|l\rangle\,, \tag{A7g}$$

$$\kappa_8 = -\frac{2\omega_m^2}{c^2}\sum_{l\neq 2m}\Delta^y_{l,m}\langle m,m|\chi^{(2)}_{yyy}(z_0)|l\rangle\,, \tag{A7h}$$

$$\begin{aligned}\kappa_9 = \frac{\omega_m^2}{c^2}\langle m,m|(\chi^{(3)}_{xxyy}(z_0) + \chi^{(3)}_{xyyx}(z_0) \\ + \chi^{(3)}_{xyxy}(z_0))|m,m\rangle\,, \end{aligned}\tag{A7i}$$

$$\kappa_{10} = 3\frac{\omega_m^2}{c^2}\langle m,m|\chi^{(3)}_{xxxx}|m,m\rangle\,, \tag{A7j}$$

$$\Theta_x = A_x\Phi_y + A_y\Phi_x\,, \tag{A7k}$$

$$\Theta_y = A_x\Phi_x - A_y\Phi_y\,, \tag{A7l}$$

where α denotes either x or y, and $\bar\alpha$ is the other transverse coordinate. As in the scalar case, the evolution of the fields Φ_x and Φ_y is captured at $O(\varepsilon^4)$, which reads

$$\begin{aligned} &\lambda_1\frac{\partial^2\Phi_\alpha}{\partial\alpha_1^2} + \frac{\partial^2\Phi_\alpha}{\partial\bar\alpha_1^2} + \lambda_2\frac{\partial^2\Phi_\alpha}{\partial\xi^2} + (\lambda_1 - 1)\frac{\partial^2\Phi_{\bar\alpha}}{\partial\alpha_1\partial\bar\alpha_1} \\ &= \lambda_3\Big[\frac{\partial^2}{\partial\alpha_1^2}M_\alpha + \frac{\partial^2}{\partial\alpha_1\partial\bar\alpha_1}M_{\bar\alpha}\Big] + \lambda_4\frac{\partial^2}{\partial\xi^2}M_\alpha\,, \end{aligned} \tag{A8}$$

with

$$\lambda_1 = \frac{1}{L}\int_0^L \eta(z_0)\,\mathrm{d}z_0\,, \tag{A9a}$$

$$\lambda_2 = 1 - \frac{c_g^2}{Lc^2}\int_0^L \epsilon(z_0)\,\mathrm{d}z_0\,, \tag{A9b}$$

$$\lambda_3 = \frac{2}{L}\langle m|\frac{\chi^{(2)}_{yyy}(z_0)}{\bar\epsilon(z_0)}|m\rangle\,, \tag{A9c}$$

$$\lambda_4 = -\frac{2c_g^2}{Lc^2}\langle m|\chi^{(2)}_{yyy}(z_0)|m\rangle\,, \tag{A9d}$$

$$M_x = A_x A_y^* + A_x^* A_y\,, \tag{A9e}$$

$$M_y = |A_x|^2 - |A_y|^2\,. \tag{A9f}$$

We observe that, the vector equations contain more coupling terms, especially the direct coupling between the two optical fields and the coupling of either optical fields to the dc field in the other transverse coordinate. Again, we note that, the vector envelope (A6) and (A8) are only valid for materials belonging to $\bar{6}m2$ symmetry class. The materials with other symmetry classes will result in similar vector equations.

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