Nonlinear modulation of multidimensional lattice waves

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The equations governing weakly nonlinear modulations of *N*-dimensional lattices are considered using a quasidiscrete multiple-scale approach. It is found that the evolution of a short wave packet for a lattice system with cubic and quartic interatomic potentials is governed by the generalized Davey-Stewartson (GDS) equations, which include mean motion induced by the oscillatory wave packet through cubic interatomic interaction. The GDS equations derived here are more general than those known in the theory of water waves because of the anisotropy inherent in lattices. The generalized Kadomtsev-Petviashvili equations describing the evolution of long-wavelength acoustic modes in two- and three-dimensional lattices are also presented. Then the modulational instability of an *N*-dimensional Stokes lattice wave is discussed based on the *N*-dimensional GDS equations obtained. Finally, the one- and two-soliton solutions of two-dimensional GDS equations are provided by means of Hirota's bilinear transformation method.

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I. INTRODUCTION

Since the pioneering work of Fermi, Pasta, and Ulam [1] on the nonlinear dynamics in lattices, the understanding of the dynamical localization in ordered, spatially extended discrete systems has experienced considerable progress. In particular, one-dimensional (1D) lattice solitons, which are localized nonlinear excitations due to the balance between nonlinearity and dispersion, are shown to exist [2]. Similar to the cases in fluid physics and nonlinear optics, most of the analytical approaches on lattice solitons are based on weakly nonlinear theory. The basic idea of the weakly nonlinear theory is that linearized lattice equations are assumed to provide a satisfactory first approximation for those finite-amplitude disturbances which are, in some sense, sufficiently small. Successive approximations may then be developed by an asymptotic expansion in ascending powers of a characteristic wave amplitude. The weakly nonlinear theory has been shown to be very successful in revealing many important physical processes, e.g., resonant wave-wave interactions, modulational instability, the formation of solitons, etc., in a clear-cut way. A very useful method for the asymptotic expansion is the method of multiple scales, which in the case of lattices reduces the system to a set of partial differential equations for the slowly changing envelope (or amplitude) while the original system is a set of differential-difference equations, and usually cannot be solved exactly. There are two basic advantages of the multiple-scale expansion: (i) it contains a unique explicit small parameter, and hence is controllable, and (ii) it allows us to obtain solutions in an explicit form. It is well known that, for a 1D lattice wave with a large spatial extension, the envelope of the lattice wave is governed by the nonlinear Schrödinger (NLS) equation for a short-wavelength packet [3] and the Korteweg–de Vries (KdV) equation for a long-wavelength acoustic mode [4].

In recent years, much attention has been paid to coherent structures in multidimensional lattices (see, e.g., Ref. [5]). In particular, we mention a generalization of the KdV equation in a 2D lattice with only a cubic interatomic potential, i.e., the Kadomtsev-Petviashivili (KP) equation, derived for a lattice wave traveling in a given direction [6] and coupled 2D NLS equations describing quadratic solitons due to the second-harmonic generation in a 2D lattice of the two-component dipoles [7]. However, to the best of our knowledge, up to now 2D and 3D generalization of the NLS equation with a mean motion induced by oscillatory wave packets in lattice systems (i.e., due to long-wavelength acoustic mode) has not been developed. Meanwhile, such motion introduces dramatic changes in the lattice dynamics.

In the present paper, using a quasidiscrete multiple-scale approach [3,8–10], we derive generalized Davey-Stewartson (GDS) equations in multidimensional lattices with cubic and quartic interatomic potentials. Because of the anisotropy inherent in lattice systems (i.e., without continuous translation and rotation symmetries), in the case of two dimensions the GDS equations presented here are more general than those obtained in water waves [11], which are physically isotropic. We also derive a generalized KP equation governing the evolution of a long-wavelength acoustic excitation traveling in any direction.

The organization of the paper is as follows. In Sec. II, we formulate the model and deduce the equations for slowly varying amplitudes in an *N*D lattice. In Sec. III, we concentrate on excitations in a 2D lattice. The dynamic equations in the long-wavelength limit are presented in Sec. IV. In Sec. V, we discuss modulational instability of an *N*-dimensional Stokes lattice wave on the basis of the GDS equations. Section VI provides some one- and two-soliton solutions for the

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2D GDS equations based on Hirota's bilinear transformation method and demonstrates the effect of anisotropy on the soliton formation. The outcomes are summarized in the final section.

II. MODEL AND ASYMPTOTIC EXPANSION

The system we consider is a monatomic scalar lattice with nearest-neighbor interatomic interactions. The equations of motion describing the system are given by

$$\frac{d^2}{dt^2}u(\mathbf{n})=\sum_{j=1}^{d} J_{2j}[u(\mathbf{n}+\mathbf{a}_j)+u(\mathbf{n}-\mathbf{a}_j)-2u(\mathbf{n})]+\sum_{j=1}^{d} J_{3j}\{[u(\mathbf{n}+\mathbf{a}_j)-u(\mathbf{n})]^2-[u(\mathbf{n}-\mathbf{a}_j)-u(\mathbf{n})]^2\}+\sum_{j=1}^{d} J_{4j}\{[u(\mathbf{n}+\mathbf{a}_j)-u(\mathbf{n})]^3+[u(\mathbf{n}-\mathbf{a}_j)-u(\mathbf{n})]^3\}. \tag{1}$$

Here $u(\mathbf{n})$ is the displacement from its equilibrium position of the particle having the mass M and located at the site $\mathbf{n}=\Sigma_{j=1}^{d}n_j\mathbf{a}_j$, n_j being integers, $\mathbf{a}_j$ being the lattice vectors, and d being the dimension of the lattice, $J_{\alpha j}=K_{\alpha,j}/M(\alpha=2,3,4)$, $K_{2,j}, K_{3,j}$, and $K_{4,j}$ are harmonic, cubic, and quartic nearest-neighbor force constants, respectively. Notice that the anisotropy of the lattice is included in the consideration (i.e., in a generic case $K_{\alpha,j}\neq K_{\alpha,i}$ for $i\neq j$). We include the cubic potential here since most of the realistic interatomic potentials (such as the potentials of Born-Mayer-Coulomb, Lennard-Jones, Morse, Toda, etc.) display strong cubic nonlinearity (i.e., $J_{\alpha,3}\neq 0$) [8,9]. In the most direct physical applications (namely, to atomic crystals), the dimension d can be either 2 or 3, although more formal lattices with d being bigger than 3 are available. In the present section, we deal with the last, more general, case.

In order to investigate weakly nonlinear modulation of a lattice wave packet, we use the quasidiscrete multiple-scale method [3,8–10] to derive the envelope equations describing the development of the modulation of the packet along the line of Davey and Stewartson for water waves [11]. Namely, we set

$$u(\mathbf{n})=\sum_{\nu=1} \epsilon^{\nu}u_{\nu}(\mathbf{r},\tau;\phi(\mathbf{n},t)) \tag{2}$$

with

$$\mathbf{r}=\epsilon(\mathbf{n}-\mathbf{v}t),\quad \tau=\epsilon^2 t,\quad \phi(\mathbf{n},t)=\mathbf{q}\cdot\mathbf{n}-\omega t, \tag{3}$$

where ϵ is a formal small parameter representing the relative amplitude of the excitation, $\mathbf{q}$ is the wave vector: $\mathbf{q}=\Sigma_{j=1}^{d}q_j\mathbf{b}_j$, $\mathbf{b}_j$ being the vectors of the reciprocal lattice: $\mathbf{b}_i\cdot\mathbf{a}_j=\delta_{ji}$, and ω is the frequency of the respective harmonic. The constant vector $\mathbf{v}$ as well as the link between ω and $\mathbf{q}$, i.e., the dispersion relation, are to be determined by solvability conditions.

There are some comments to be made here. In a generic case, the entries of the expansion (2) depend on the whole hierarchy of "slow" variables, i.e., one should consider the set of variables $\{\mathbf{r}_\nu,t_\nu\}$ $(\nu=1,2,\ldots)$, where $\mathbf{r}_\nu=\epsilon^{\nu}(\mathbf{n}-\mathbf{v}_\nu t)$ and $t_\nu=\epsilon^{\nu}t$, which are regarded as independent. In the case of the effect of quadratic and cubic nonlinearity, only the scales up to $\mathbf{r}_2$ and t_2 turn out to be relevant. In the present paper we restrict our consideration to the solutions independent of $\mathbf{r}_2$. For this reason, we introduce only the "lowest-order" slow variables $\mathbf{r}=\mathbf{r}_1$ and $\tau=t_2$.

Substituting Eqs. (2) and (3) into Eq. (1) and equating the coefficients of the same powers of ϵ, we obtain the hierarchy of equations as follows:

$$Lu_{\nu}\equiv\omega^2\frac{\partial^2u_\nu}{\partial\phi^2}-\sum_j J_{2j}(u_\nu^{(j)}+u_\nu^{(-j)})=M_\nu,\quad \nu=1,2,\ldots. \tag{4}$$

Here $u_\nu^{(\pm j)}\equiv u_\nu(\mathbf{r},\tau;\phi(\mathbf{n},t)\pm q_j)-u_\nu(\mathbf{r},\tau;\phi(\mathbf{n},t))$,

$$M_1=0, \tag{5a}$$

$$M_2=-2\omega(\mathbf{v}\cdot\nabla)\frac{\partial u_1}{\partial\phi}+\sum_j J_{2j}a_j\frac{\partial}{\partial x_j}(u_1^{(j)}-u_1^{(-j)})+\sum_j J_{3j}[(u_1^{(j)})^2-(u_1^{(-j)})^2], \tag{5b}$$

$$M_3=-(\mathbf{v}\cdot\nabla)^2u_1-2\omega(\mathbf{v}\cdot\nabla)\frac{\partial u_2}{\partial\phi}+2\omega\frac{\partial^2u_1}{\partial\phi\partial\tau}+\sum_j J_{2j}a_j\frac{\partial}{\partial x_j}(u_2^{(j)}-u_2^{(-j)})+\sum_j\frac{J_{2j}}{2}\left(a_j\frac{\partial}{\partial x_j}\right)^2\times(u_1^{(j)}+u_1^{(-j)}+2u_1)+2\sum_j J_{3j}\left((u_2^{(j)}u_1^{(j)}-u_2^{(-j)}u_1^{(-j)})+u_1^{(j)}a_j\frac{\partial}{\partial x_j}(u_1^{(j)}+u_1)-u_1^{(-j)}a_j\frac{\partial}{\partial x_j}(u_1^{(-j)}+u_1)\right)+\sum_j J_{4j}[(u_1^{(j)})^3+(u_1^{(-j)})^3], \tag{5c}$$

$\nabla\equiv\partial/\partial\mathbf{r}$, $a_j=|\mathbf{a}_j|$, and x_m is the mth coordinate of the vector $\mathbf{r}$, $\mathbf{r}=\Sigma_m x_m\mathbf{a}_\mathbf{m}/a_m$.

For further consideration we have to specify the effect we are looking for and this will determine the form of the lowest-order $(j=1)$ solution of Eq. (4). Namely, we will be interested in the weakly nonlinear modulation of a lattice wave originated by the interaction between a long-wavelength acoustic mode and a high-frequency mode. Thus we choose

$$u_1=A_0(\mathbf{r},\tau)+\{A_1(\mathbf{r},\tau)\exp[i\phi(\mathbf{n},t)]+\text{c.c.}\}, \tag{6}$$

where the real function A_0 stands for a mean motion induced by the oscillatory wave packet, which has the complex envelope function A_1, and c.c. denotes the corresponding complex conjugate term. Then

$$u_1^{(\pm j)}=[\exp(\pm iq_j)-1]A_1e^{i\phi(\mathbf{n},t)}+\text{c.c.},$$

and in the first order [see Eqs. (4) and (5a)] we immediately arrive at the dispersion relation of the underline linear lattice

$$\omega^2\equiv[\omega(\mathbf{q})]^2=2\sum_j J_{2j}(1-\cos q_j). \tag{7}$$

Next we take into account that

$$\mathbf{v}_g\equiv\frac{d\omega}{d\mathbf{q}}=\frac{1}{\omega}\sum_j J_{2j}\sin(q_j)\mathbf{a}_j/a_j, \tag{8}$$

which is the group velocity of the linear wave. Then, subject to assumption (6) the second-order equation of system (4) takes the form

$$Lu_2=2i\omega\{[(\mathbf{v}_g-\mathbf{v})\cdot\nabla](A_1e^{i\phi}-\overline{A}_1e^{-i\phi})+\chi^{(2)}(A_1^2e^{2i\phi}-\overline{A}_1^2e^{-2i\phi})\} \tag{9}$$

where

$$\chi^{(2)}=\sum_m\frac{J_{3m}}{\omega}(\cos q_m-1)\sin q_m \tag{10}$$

is the effective quadratic nonlinearity.

The solvability condition for the system (9) (in other words the conditions of the absence of secular terms in u_2) means the orthogonality of the right-hand side of Eq. (9) to the kernel of the operator L, i.e., to Eq. (6). Hence the right hand side of Eq. (9) must not contain the terms proportional to $\exp(\pm i\phi)$ and we conclude that $\mathbf{v}=\mathbf{v}_g$, i.e., $\mathbf{v}$ introduced in Eq. (3), is merely the group velocity of the carrier wave. Next we can look for the solution u_2 (it must be orthogonal to the first-order approximation, i.e., to the kernel of the operator L) in a form of the expansion over the eigenfunctions of the operator L. Having done this, one ensures that the only nonzero term of such an expansion is given by

$$u_2=i\alpha A_1^2\exp(2i\phi)+\text{c.c.},$$

$$\alpha=-\frac{2\omega\chi^{(2)}}{4[\omega(\mathbf{q})]^2-[\omega(2\mathbf{q})]^2}. \tag{11}$$

Formula (11) is valid unless the condition $\omega(2\mathbf{q})=2\omega(\mathbf{q})$ is satisfied. As is evident, this is the condition of the resonant second-harmonic generation [10,14]. It can be satisfied in a lattice with a complex cell, but it is not difficult to ensure that $\omega(2\mathbf{q})\neq 2\omega(\mathbf{q})$ for all $\mathbf{q}$ in a monatomic lattice with the nearest-neighbor interactions.

Passing to the third order of the multiple scale expansion, we introduce the (symmetric) group velocity dispersion tensor (GVDT) by the formula ($v_j=\partial\omega/\partial q_j$)

$$\omega_{ij}\equiv\frac{1}{\omega}[J_{2j}\cos(q_j)a_ia_j\delta_{ij}-v_iv_j], \tag{12}$$

and the (symmetric) effective GVDT Ω_{ij},

$$\Omega_{ij}\equiv\frac{1}{\omega}[J_{2j}a_ia_j\delta_{ij}-v_iv_j]. \tag{13}$$

The solvability condition for the third-order terms gives rise to the closed system of equations for A_0 and A_1:

$$\sum_{l,m}\Omega_{lm}\frac{\partial^2}{\partial x_l\partial x_m}A_0=-2\sum_m\delta_m\frac{\partial}{\partial x_m}|A_1|^2, \tag{14}$$

$$i\frac{\partial A_1}{\partial\tau}+\frac{1}{2}\sum_{l,m}\omega_{lm}\frac{\partial^2}{\partial x_l\partial x_m}A_1=\chi|A_1|^2A_1+A_1\sum_m\delta_m\frac{\partial}{\partial x_m}A_0, \tag{15}$$

where

$$\delta_m=\frac{2a_m}{\omega}J_{3m}(1-\cos q_m), \tag{16}$$

$$\chi=\frac{2}{\omega}\sum_m[2\alpha J_{3m}(1-\cos q_m)\sin q_m+3J_{4m}(1-\cos q_m)^2]. \tag{17}$$

We call Eqs. (14) and (15) the ND GDS equations.

III. GENERALIZED DAVEY-STEWARTSON EQUATIONS

Let us now focus our attention on a special case of a 2D lattice [i.e., $\mathbf{r}=(x_1,x_2)$]. For the sake of simplicity, the lattice will be considered symmetric, $J_{\alpha,j}=J_\alpha$ ($\alpha=2,3,4$ and $j=1,2$), and orthogonal: $\mathbf{a}_1\cdot\mathbf{a}_2=0$, with the lattice constant equal to unity, $|a_j|=1$. In order to diagonalize the effective GVDT Ω_{lm} in a general case, we rotate the original Cartesian system [with the coordinate basis (1, 0) and (0, 1)] to a new one with the coordinate basis $\mathbf{e}_1=(\lambda_1,\lambda_2)$ and $\mathbf{e}_2=(-\lambda_2,\lambda_1)$, where

$$\lambda_j=\frac{v_j}{v_g}=\frac{\sin q_j}{\sqrt{\sin^2q_1+\sin^2q_2}} \tag{18}$$

and v_j is the jth component of the group velocity defined in Eq. (8) (as is evident, $\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}$). In this way one of the directions of the new basis, namely $\mathbf{e}_1$, coincides with the direction of the group velocity of the carrier wave, i.e., $\mathbf{v}_g=v_g\mathbf{e}_1$. The other direction is orthogonal to it. As a result, x_1 and x_2 in Eqs. (14) and (15) take the form $x_1=\epsilon(n_1-\lambda_1v_gt)$ and $x_2=\epsilon(n_2-\lambda_2v_gt)$, and the envelope equations (14) and (15) are reduced to

$$\alpha_{11}\frac{\partial^2A_0}{\partial\xi^2}+\alpha_{22}\frac{\partial^2A_0}{\partial\eta^2}=-2\left(\beta_1\frac{\partial}{\partial\xi}+\beta_2\frac{\partial}{\partial\eta}\right)|A_1|^2, \tag{19}$$

$$i\frac{\partial A_1}{\partial\tau}+\mathcal{L}A_1=A_1\left(\beta_1\frac{\partial}{\partial\xi}+\beta_2\frac{\partial}{\partial\eta}\right)A_0+\chi|A_1|^2A_1, \tag{20}$$

where

$$\mathcal{L}=\gamma_{11}\frac{\partial^2}{\partial\xi^2}+\gamma_{22}\frac{\partial^2}{\partial\eta^2}+\gamma_{12}\frac{\partial^2}{\partial\xi\partial\eta}, \tag{21}$$

$$\xi=\mathbf{r}\cdot\mathbf{e}_1=\lambda_1x_1+\lambda_2x_2=\epsilon(\lambda_1n_1+\lambda_2n_2-v_gt), \tag{22}$$

$$\eta=\mathbf{r}\cdot\mathbf{e}_2=-\lambda_2x_1+\lambda_1x_2=\epsilon(-\lambda_2n_1+\lambda_1n_2), \tag{23}$$

$$\alpha_{11}=\frac{1}{\omega}(J_2-v_g^2),\quad \alpha_{22}=\frac{J_2}{\omega}, \tag{24}$$

$$\beta_1=\frac{2J_3}{\omega}[\lambda_1(1-\cos q_1)+\lambda_2(1-\cos q_2)],$$

$$\beta_2=\frac{2J_3}{\omega}[\lambda_1(1-\cos q_2)-\lambda_2(1-\cos q_1)], \tag{25}$$

$$\gamma_{11}=\frac{1}{2\omega}[-v_g^2+J_2(\lambda_1^2\cos q_1+\lambda_2^2\cos q_2)],$$

$$\gamma_{22}=\frac{J_2}{2\omega}(\lambda_2^2\cos q_1+\lambda_1^2\cos q_2), \tag{26}$$

$$\gamma_{12}=\frac{J_2}{\omega}\lambda_1\lambda_2(\cos q_2-\cos q_1), \tag{27}$$

$$\chi=\frac{2}{\omega}\{2J_3\alpha[\sin q_1(1-\cos q_1)+\sin q_2(1-\cos q_2)] + 3J_4[(1-\cos q_1)^2+(1-\cos q_2)^2]\}, \tag{28}$$

$$\alpha=\frac{4J_3[\sin q_1(1-\cos q_1)+\sin q_2(1-\cos q_2)]}{4[\omega(\mathbf{q})]^2-[\omega(2\mathbf{q})]^2}. \tag{29}$$

Equations (19) and (20) represent a generalized form of the conventional DS equations. They include the dispersion, diffraction, and nonlinearity of the system. One of their important features is that there exists a coupling between the mean field (denoted by A_0) and the envelope of the carrier wave (denoted by A_1). The mean field A_0 generates a strain field in the system. If $J_3=0$, a case for a symmetric interatomic potential, we have $A_0=0$, thus the mean motion and hence the strain field vanish. Another important feature for Eqs. (19) and (20) is their property of anisotropy. For different wave vector $\mathbf{q}=(q_1,q_2)$, the coefficients of the equations take different values and some of these coefficients may become vanishing for some particular directions of $\mathbf{q}$.

The conventional DS equations were derived first in surface water waves [11] and now are a well-known 2D soliton model in soliton theory [12]. Note that for water waves, the system is isotropic (i.e., it possesses a continuous rotation symmetry). The envelope equations are the same for all propagating directions of the waves and hence the coefficients appearing in the equations are independent of q_1 and q_2, and correspondingly β_2 and γ_{12} vanish [11] (see also Ref. [13]). However, *for the lattice system the modulating equations take a more general form because the lattice is anisotropic (without the continuous rotation symmetry)*. We mention that although the coefficients α_{ij} $(i=1,2)$ are both positive, signs of the coefficients γ_{ij} may change depending on the choice of the wave vector in the first Brillouin zone.

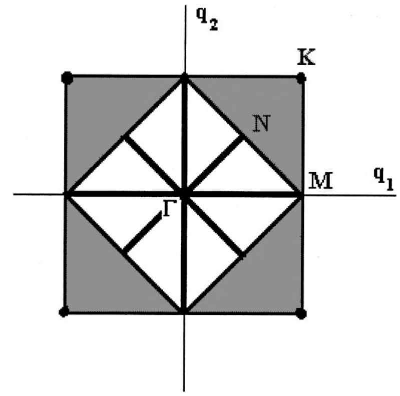

FIG. 1. The first Brillouin zone for the 2D quadratic lattice. The filled-in and empty polygons correspond to the operator $\mathcal{L}$ [it is defined by Eq. (21)] of the elliptic and hyperbolic types, respectively. Along the intervals shown by the bold lines (i.e., in the directions [100], [010], [110], and $[1\bar{1}0]$), the system (19) and (20) is reduced to the conventional DSII equation.

We now discuss several particular cases for the 2D GDS equations derived above. In the following circumstances (i.e., in some special points and lines of the Brillouin zone, see Fig. 1), the 2D GDS equations reduce to the conventional DS equations:

(i) $q_1q_2=0$ (then $\lambda_1\lambda_2=0$ and $\beta_2=\gamma_{12}=0$),

(ii) $q_1=q_2=q$ (then $\lambda_1=\lambda_2=2^{-1/2}$ and $\beta_2=\gamma_{12}=0$),

(iii) $q_1=-q_2=q$ (then $\lambda_1=-\lambda_2=2^{-1/2}$ and $\beta_1=\gamma_{12}=0$).

More precisely, since $\alpha_{11}>0$ and $\alpha_{22}>0$ for any $\mathbf{q}$, at $\gamma_{11}\gamma_{22}<0$, Eqs. (19) and (20) can be classified as DSII equations, while for $\gamma_{11}\gamma_{22}>0$ they form a dynamic system that can be identified neither with DSI nor DSII equations appearing in the theory of water waves (see, e.g., [12]).

In the case of a pure quadratic potential, $J_3=0$, we have that the evolution equations for A_1 and A_0 are decoupled. Then the GDS equations reduce to a generalized 2D NLS equation [i.e., the NLS equation plus a cross-derivative term $\partial^2A_1/(\partial\xi\partial\eta)$]. Finally, if $q_2=0$ and $\partial/\partial\eta=0$, the 2D GDS equations (19) and (20) recover the envelope equations derived in Refs. [8,10], which gives rise to standard 1D lattice solitons [8,10].

In the 3D case, Eqs. (19) and (20) are replaced by the *3D GDS equations*:

$$\alpha_{11}'\frac{\partial^2A_0}{\partial\xi^2}+\alpha_{22}'\frac{\partial^2A_0}{\partial\eta^2}+\alpha_{33}'\frac{\partial^2A_0}{\partial\zeta^2} =-2\left(\beta_1'\frac{\partial}{\partial\xi}+\beta_2'\frac{\partial}{\partial\eta}+\beta_3'\frac{\partial}{\partial\zeta}\right)|A_1|^2, \tag{30}$$

$$i\frac{\partial A_1}{\partial \tau}+\gamma'_{11}\frac{\partial^2 A_1}{\partial \xi^2}+\gamma'_{22}\frac{\partial^2 A_1}{\partial \eta^2}+\gamma'_{33}\frac{\partial^2 A_1}{\partial \zeta^2}+\left(\gamma'_{12}\frac{\partial^2}{\partial\xi\partial\eta}+\gamma'_{23}\frac{\partial^2}{\partial\eta\partial\zeta}+\gamma'_{31}\frac{\partial^2}{\partial\zeta\partial\xi}\right)A_1 = A_1\left(\beta'_1\frac{\partial}{\partial\xi}+\beta'_2\frac{\partial}{\partial\eta}+\beta'_3\frac{\partial}{\partial\zeta}\right)A_0+\chi'|A_1|^2A_1, \tag{31}$$

where α'_{jj}, β'_j, γ'_{ij} $(j=1,2,3)$, and χ' are constants dependent on $\mathbf{q}=(q_1,q_2,q_3)$ and the parameters of the system, which are not needed here and not written down explicitly. The definitions of ξ,η, and ζ are given by

$$\xi=\epsilon(\lambda_1 n_1+\lambda_2 n_2+\lambda_3 n_3-v_g t), \tag{32}$$

$$\eta=\epsilon(-\lambda_2 n_1+\lambda_1 n_2), \tag{33}$$

$$\zeta=\epsilon(-\lambda_3 n_1+\lambda_1 n_3), \tag{34}$$

where ω and λ_j are defined by Eqs. (7) and (18).

IV. LONG-WAVELENGTH LIMIT

Note that the envelope equations (14) and (15) are invalid for $\mathbf{q}=0$ since in this case there is a divergence in their coefficients. From the physical point of view, this happens because vanishing $\mathbf{q}$ corresponds to a long-wavelength acoustic mode in the lattice. In this case a different asymptotic expansion must be used to obtain divergence-free envelope equations. For simplicity, we consider the case of a symmetric 2D square lattice. In this situation, the asymptotic expansion (2) must be replaced by

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots, \tag{35}$$

with

$$u_\nu=u_\nu(\xi,\eta,\tau)\qquad \nu=0,1,2,\ldots, \tag{36}$$

$$\xi=\epsilon(\lambda_1 n_1+\lambda_2 n_2-ct), \tag{37}$$

$$\eta=\epsilon^2(-\lambda_2 n_1+\lambda_1 n_2), \tag{38}$$

$$\tau=\epsilon^3 t, \tag{39}$$

where $c=\sqrt{J_2}$ is the speed of sound and $\lambda_l (l=1,2)$ are determined by the solvability conditions required at $O(\epsilon^2)$ order. A solvability condition in the fourth order of the expansion yields the *generalized KP equation*,

$$\frac{\partial}{\partial\xi}\left[\frac{\partial v}{\partial\tau}+\frac{c}{24}(\lambda_1^4+\lambda_2^4)\frac{\partial^3 v}{\partial\xi^3}+\frac{\tilde{J}_3}{c}(\lambda_1^3+\lambda_2^3)v\frac{\partial v}{\partial\xi}+\frac{3J_4}{2c}(\lambda_1^4+\lambda_2^4)v^2\frac{\partial v}{\partial\xi}\right]+\frac{c}{2}\frac{\partial^2 v}{\partial\eta^2}=0, \tag{40}$$

where $v=\partial u_0/\partial\xi$. In deriving Eq. (40), we have assumed that $J_3=\epsilon\tilde{J}_3$ with $\tilde{J}_3$ of order unity. The parameters λ_l $(l=1,2)$ are direction-dependent and we find that their values can be obtained by using Eq. (18) but taking the limit $\mathbf{q}\to 0$. Thus the values of the coefficients in Eq. (40) are dependent on the ways of $\mathbf{q}$ approaching zero. For instance,

(i) $\lambda_1=1$, $\lambda_2=0$ if $q_2=0$, $q_1\to 0$;

(ii) $\lambda_1=\lambda_2=1/\sqrt{2}$ if $q_1=q_2=q\to 0$.

The reason for the different values of the coefficients corresponding to different directions is also due to the anisotropy of the system. It is easy to see that the KP equation obtained in Ref. [6] is our particular case with quartic nonlinearity being absent (i.e., $J_4=0$). Equation (40) admits solitary-wave solutions [12].

It is relevant to mention here that the coefficient of the term $\partial^2 v/\partial\eta^2$ is positive, which means that the line (i.e., η-independent) solitons of Eq. (40) are stable while this equation does not admit any kind of lump (i.e., decaying when $\xi^2+\eta^2\to 0$) solution.

In the same way, in 3D case Eq. (40) is generalized to

$$\frac{\partial}{\partial\xi}\left[\frac{\partial v}{\partial\tau}+a_1\frac{\partial^3 v}{\partial\xi^3}+a_2 v\frac{\partial v}{\partial\xi}+a_3 v^2\frac{\partial v}{\partial\xi}\right]+a_4\frac{\partial^2 v}{\partial\eta^2}+a_5\frac{\partial^2 v}{\partial\zeta^2}=0, \tag{41}$$

where ξ,η, and ζ are the same as Eq. (32)–(34). $a_l (l=1,2,3,4,5)$ are real constants dependent on λ_j $(j=1,2,3)$ [given by Eq. (18)] with $\mathbf{q}\to 0$.

V. MODULATIONAL INSTABILITY OF A PLANE LATTICE WAVE WITH A MEAN MOTION

In recent years, the use of nonlinear envelope (or amplitude) equations for studying the stability of patterns and waves in systems in and outside of equilibrium is widely employed [15,16]. The modulational stability of a plane water wave (e.g., a uniform Stokes wave) was analyzed by Davey and Stewartson based on the DS equations they derived [11]. In the same way, the ND GDS equations (14) and (15) obtained here can be used to study the modulational stability of a uniform Stokes lattice wave in N dimensions. A Stokes lattice wave here means a linear plane lattice wave with the wave vector $\mathbf{q}$.

Note that the uniform vibrating solution of Eqs. (14) and (15) reads

$$A_0=0,\qquad A_1=U_0\exp(-i\Omega\tau), \tag{42}$$

which, when incorporating the carrier wave [see Eq. (6)], corresponds to a plane lattice wave with the wave vector $\mathbf{q}$ and the modified frequency $\omega(\mathbf{q})+\Omega$, where U_0 is a constant and $\Omega=\chi U_0^2$. Assume that a perturbation is added into the uniform vibrating solution (42), i.e.,

$$A_0(x_1,x_2,\ldots,\tau)=\hat{\kappa}_+\exp\left(i\sum_m Q_m x_m\right)+\hat{\kappa}_-\exp\left(-i\sum_m Q_m x_m\right), \tag{43}$$

$$A_1(x_1,x_2,\ldots,\tau)=U_0\exp(-i\Omega\tau)\left[1+\hat{\varepsilon}_+\exp\left(i\sum_m Q_m x_m\right)+\hat{\varepsilon}_-\exp\left(-i\sum_m Q_m x_m\right)\right], \tag{44}$$

with $\hat{\kappa}_\pm(\tau)=\kappa_\pm(0)\exp[(\sigma_R\pm i\sigma_I)\tau]$ and $\hat{\varepsilon}_\pm(\tau)=\varepsilon_\pm(0)\exp[(\sigma_R\pm i\sigma_I)\tau]$, where $\mathbf{Q}=(Q_1,Q_2,\ldots,Q_N)$ and $\sigma_I=\sigma_I(\mathbf{Q})$ are, respectively, the wave vector and frequency of the perturbation, $\sigma_R=\sigma_R(\mathbf{Q})$ denotes the growth rate of the perturbation, and $\kappa_\pm(0)$ and $\varepsilon_\pm(0)$ are small constants with the condition $\hat{\kappa}^*_-(0)=\kappa_+(0)$ because of the reality of A_0. Substituting Eqs. (43) and (44) into Eqs.(14) and (15), we obtain a set of linear equations on $\kappa_\pm(0)$ and $\varepsilon_\pm(0)$:

$$-(\alpha_{11}Q_1^2+\alpha_{22}Q_2^2)\kappa_+(0)+2iU_0^2(\beta_1Q_1+\beta_2Q_2)\times[\varepsilon_+(0)+\varepsilon^*_-(0)]=0, \tag{45}$$

$$(\alpha_{11}Q_1^2+\alpha_{22}Q_2^2)\kappa_-(0)+2iU_0^2(\beta_1Q_1+\beta_2Q_2)\times[\varepsilon_-(0)+\varepsilon^*_+(0)]=0, \tag{46}$$

$$(\Omega+i\sigma-\gamma_{11}Q_1^2-\gamma_{22}Q_2^2-\gamma_{12}Q_1Q_2-2\chi U_0^2)\varepsilon_+(0)-\chi U_0^2\varepsilon^*_-(0)-i(\beta_1Q_1+\beta_2Q_2)\kappa_+(0)=0, \tag{47}$$

$$(\Omega+i\sigma^*-\gamma_{11}Q_1^2-\gamma_{22}Q_2^2-\gamma_{12}Q_1Q_2-2\chi U_0^2)\varepsilon_-(0)-\chi U_0^2\varepsilon^*_+(0)+i(\beta_1Q_1+\beta_2Q_2)\kappa_-(0)=0, \tag{48}$$

where $\sigma=\sigma_R+i\sigma_I$. A solvability condition of Eqs. (45)–(48) results in

$$(\sigma_R+i\sigma_I)^2=\left(\sum_{l,m}\omega_{lm}Q_lQ_m\right)\left\{U_0^2\left[-\chi+\frac{2\left(\sum_m\delta_mQ_m\right)^2}{\sum_{l,m}\Omega_{lm}Q_lQ_m}\right]-\frac{1}{4}\sum_{l,m}\omega_{lm}Q_lQ_m\right\}. \tag{49}$$

Note that the right side of Eq. (49) is real. Thus when

$$\left(\sum_{l,m}\omega_{lm}Q_lQ_m\right)\left\{U_0^2\left[-\chi+\frac{2\left(\sum_m\delta_mQ_m\right)^2}{\sum_{l,m}\Omega_{lm}Q_lQ_m}\right]-\frac{1}{4}\sum_{l,m}\omega_{lm}Q_lQ_m\right\}>0, \tag{50}$$

one has $\sigma_I=0$. As a result, if the condition (50) is satisfied, we have the growth rate

$$\sigma_R=\pm\left\{\left(\sum_{l,m}\omega_{lm}Q_lQ_m\right)\left\{U_0^2\left[-\chi+\frac{2\left(\sum_m\delta_mQ_m\right)^2}{\sum_{l,m}\Omega_{lm}Q_lQ_m}\right]-\frac{1}{4}\sum_{l,m}\omega_{lm}Q_lQ_m\right\}\right\}^{1/2}. \tag{51}$$

Thus one always has a positive σ_R branch if the condition (50) is satisfied. In this case, the perturbation grows exponentially and hence the uniform vibrating solution (42) is modulationally unstable.

For the 2D GDS equations (19) and (20), the condition of the modulational instability (50) reads

$$(\gamma_{11}Q_1^2+\gamma_{22}Q_2^2+\gamma_{12}Q_1Q_2)\times\left\{U_0^2\left[-\chi+\frac{2(\beta_1Q_1+\beta_2Q_2)^2}{\alpha_{11}Q_1^2+\alpha_{22}Q_2^2}\right]-\frac{1}{2}(\gamma_{11}Q_1^2+\gamma_{22}Q_2^2+\gamma_{12}Q_1Q_2)\right\}>0. \tag{52}$$

Thus due to the anisotropy of the lattice (i.e., $\beta_2\gamma_{12}\neq 0$), the criterion (52) gives much richer behavior for the stability of the Stokes wave than that in isotropic systems (e.g., water waves). In particular, for a given Stokes lattice wave there exist two (or maybe four, depending on the Stokes lattice wave) wave vectors **Q** for which the instability evolves with the biggest increment. This phenomenon recalls the so-called strengthening of inhomogeneities, known in the theory of beam propagation in the Kerr medium [19]. There is, however, an essential difference originated by the anisotropy: the biggest exponent is characterized by the amplitude of the value of the wave vector and also by the lattice direction. The position of the points providing the largest increment depends on the choice of the wave vector of the Stokes lattice wave.

The outcome of this type of instability may result in the formation of solitons [2] or the appearance of homoclinic structures (see Sec. 3.3 of Ref. [12]).

VI. SOLITON SOLUTIONS

We now consider the soliton solutions of the nonlinear evolution equations derived above. Taking 2D GDS equations (19) and (20) as an example, to obtain the soliton solutions we employ Hirota's bilinear transformation method, an ingenious technique of finding exact multisoliton solitons for nonlinear evolution equations [17,18]. Introducing the dependent variable transformation

$$A_0=-4\left(\beta_1\frac{\partial}{\partial\xi}+\beta_2\frac{\partial}{\partial\eta}\right)\ln F,\qquad A_1=G/F \tag{53}$$

with F (real) and G (complex) being the functions of τ, ξ, and η, Eqs. (19) and (20) are transformed into the following bilinear form:

$$(\alpha_{11}D_\xi^2+\alpha_{22}D_\eta^2)FF=|G|^2, \tag{54}$$

$$(iD_\tau+\gamma_{11}D_\xi^2+\gamma_{22}D_\eta^2+\gamma_{12}D_\xi D_\eta)GF=0, \tag{55}$$

$$[(\gamma_{11}-2\beta_1^2)D_\xi^2+(\gamma_{22}-2\beta_2^2)D_\eta^2+(\gamma_{12}-4\beta_1\beta_2)D_\xi D_\eta]FF+\chi|G|^2=0, \tag{56}$$

where D_τ, D_ξ, and D_η are Hirota's bilinear operators defined by [17,18]

$$D_\xi^m D_\eta^n D_\tau^p GF\equiv\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi'}\right)^m\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\eta'}\right)^n\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\tau'}\right)^p\times G(\xi,\eta,\tau)F(\xi',\eta',\tau')|_{\xi'=\xi,\eta'=\eta,\tau'=\tau}. \tag{57}$$

In order to get a one-soliton solution, we assume

$$F=1+L\exp(\Phi+\Phi^*),\qquad G=\exp(\Phi) \tag{58}$$

with

$$\Phi=(p_R+ip_I)\xi+(q_R+iq_I)\eta+(s_R+is_I)\tau+\Phi_{0R}+i\Phi_{0I}, \tag{59}$$

where $L,p_R,p_I,q_R,q_I,s_R,s_I,\Phi_{0R}$, and Φ_{0I} are real, yet to be determined constants. Substituting Eq. (58) into Eqs. (54)–(56), we obtain the set of algebraic equations

$$8L(\alpha_{11}p_R^2+\alpha_{22}q_R^2)-1=0, \tag{60}$$

$$\gamma_{11}(p_R^2-p_I^2)+\gamma_{12}(p_Rq_R-p_Iq_I)+\gamma_{22}(q_R^2-q_I^2)-s_I=0, \tag{61}$$

$$2(\gamma_{11}p_Rp_I+\gamma_{22}q_Rq_I)+\gamma_{12}(p_Rq_I+p_Iq_R)+s_R=0, \tag{62}$$

$$\chi+8L[(\gamma_{11}-2\beta_1^2)p_R^2+(\gamma_{22}-2\beta_2^2)q_R^2+(\gamma_{12}-4\beta_1\beta_2)p_Rq_R]=0. \tag{63}$$

From Eq. (60) we get

$$L=\frac{1}{8(\alpha_{11}p_R^2+\alpha_{22}q_R^2)}. \tag{64}$$

Equations (61) and (62) give rise to the ''dispersion relations''

$$s_I=\gamma_{11}(p_R^2-p_I^2)+\gamma_{12}(p_Rq_R-p_Iq_I)+\gamma_{22}(q_R^2-q_I^2), \tag{65}$$

$$s_R=-2(\gamma_{11}p_Rp_I+\gamma_{22}q_Rq_I)-\gamma_{12}(p_Rq_I+p_Iq_R), \tag{66}$$

with p_R,p_I,q_R, and q_I being arbitrary constants. Equations (63) gives a condition for the one-soliton solution.

From Eq. (53) and the results given above, we have

$$A_0=-4(\beta_1p_R+\beta_2q_R)[1+\tanh(\theta-\delta_0)], \tag{67}$$

$$A_1=[2(\alpha_{11}p_R^2+\alpha_{22}q_R^2)]^{1/2}\mathrm{sech}(\theta-\delta_0)\exp(i\varphi), \tag{68}$$

with $\theta=p_R\xi+q_R\eta+s_R\tau+\Phi_{0R}$, $\varphi=p_I\xi+q_I\eta+s_I\tau+\Phi_{0I}$, and $\delta_0=(1/2)\ln[8(\alpha_{11}p_R^2+\alpha_{22}q_R^2)]$ (Φ_{0R} and Φ_{0I} are arbitrary constants). Thus the single-soliton solution obtained is a line soliton, which consists of two parts, a vibrating wave packet (A_1, an envelope soliton) and a mean displacement field (A_0, a kink).

The two-soliton solutions of Eqs. (19) and (20) can be obtained by choosing

$$F=1+L_1\exp(\Phi_1+\Phi_1^*)+L_2\exp(\Phi_2+\Phi_2^*)+(L_3+iL_4)\times\exp(\Phi_1+\Phi_2^*)+(L_3-iL_4)\exp(\Phi_1^*+\Phi_2)+L_5\exp(\Phi_1+\Phi_2+\Phi_1^*+\Phi_2^*), \tag{69}$$

$$G=\exp(\Phi_1)+\exp(\Phi_2)+(M_1+iM_2)\exp(\Phi_1+\Phi_2+\Phi_1^*)+(M_3+iM_4)\exp(\Phi_1+\Phi_2+\Phi_2^*), \tag{70}$$

with $\Phi_j=(p_{jR}+ip_{jI})\xi+(q_{jR}+iq_{jI})\eta+(s_{jR}+is_{jI})\tau+\Phi_{jR}^0+i\Phi_{jI}^0(j=1,2)$, where p_{jR}, p_{jI}, q_{jR}, q_{jI}, s_{jR}, s_{jI}, Φ_{jR}^0, and Φ_{jI}^0 are real constants. When Eqs. (69) and (70) are substituted into the bilinear equations (54)–(56), we obtain a set of nonlinear algebraic equations for the real coefficients $L_j(j=1,2,\ldots,5)$ and $M_j(j=1,2,3,4)$ appearing in Eqs. (69) and (70). Solving these equations one can get the expressions of L_j and M_j, as well as the ''dispersion relations'' $s_{jR,I}=s_{jR,I}(p_{jR},p_{jI},q_{jR},q_{jI})(j=1,2)$, which have been given in Appendix A. To guarantee Eqs. (69) and (70) are two-soliton solutions, the following conditions must be imposed:

$$\gamma_{12}=4\beta_1\beta_2, \tag{71}$$

$$\frac{\alpha_{11}}{\gamma_{11}-2\beta_1^2}=\frac{\alpha_{22}}{\gamma_{22}-2\beta_2^2}=-\frac{1}{\chi}. \tag{72}$$

In addition, for p_{2R} and q_{2R}, there is a constraint

$$\alpha_{22}(\alpha_{11}p_{2R}^2+\alpha_{22}q_{2R}^2)\chi=(\alpha_{11}\beta_2^2+\alpha_{22}\beta_1^2)p_{2R}^2+2\alpha_{22}\beta_2^2q_{2R}^2. \tag{73}$$

It is easy to show that the integrable conditions of the standard DS equations (i.e., the ones amenable to being solved by the inverse-scattering transform) derived in the water wave problem are the particular case of the conditions (71) and (72) (see Appendix B). This fact implies that the GDS equations (19) and (20) may be integrable under the conditions (71) and (72).

We note that different equalities in these conditions, however, reflect different physical properties. In particular, Eq. (71) and the first equality in Eq. (72) result in an equation for the wave vector only [i.e., having the form $f(q_1,q_2)=0$, where $f(q_1,q_2)$ does not depend on the lattice parameters, i.e., on J_2] for which the existence of solitons is possible. Then the second equality in Eq. (72) allows one to find the particular values of the nonlinear coefficients. In other words, the above conditions specify the set of points in the first Brilloun zone and necessary values of the nonlinear forces. What is important for the next consideration is that such points in the Brillouin zone do exist. Indeed, as an example we mention that the above conditions are satisfied for all points $\mathbf{q}=(q_1,0)$ and $\mathbf{q}=(0,q_2)$.

Equations (69) and (70) describe two obliquely interacting solitons in the (ξ,η) space. The interaction results in a phase shift (i.e., position shift) for each soliton.

It is possible to get N-soliton solutions of the 2D GDS equations (19) and (20) using their bilinear representation, Eqs. (54)–(56), under the integrable conditions (71) and (72). We note that due to the anisotropy inherent in the lattice system (i.e., $\beta_2\gamma_{12}\neq 0$), the existence of the two-soliton solution requires the condition $\gamma_{12}=4\beta_1\beta_2$ [Eq. (71)], which is absent for isotropic systems (e.g., water waves).

VII. CONCLUSION

Using a quasidiscrete multiple-scale method, we have derived the envelope equations of weakly nonlinear modulations of N-dimensional lattice waves. The equations are obtained for the case of interaction of a high-frequency mode with a long-wavelength acoustic one (also called mean field) and can be classified as generalized Davey-Stewartson equations. In the case at hand, due to the anisotropy of the lattice system, the GDS equations in two dimensions are reduced either to the DS equations or to a form that does not appear in the theory of water waves [11]. The mean field coupled to the oscillatory short wave packet results from the cubic interatomic potential in the lattice. Additionally, generalized Kadomtsev-Petviashvili equations describing the evolution of a long-wavelength acoustic mode in the lattice are also presented. We have also studied the modulation instability of Stokes waves and provided some exact soliton solutions for the two-dimensional GDS equations based on Hirota's bilinear transformation method.

The results reported here recover the known ones in one-dimensional systems, which give rise to standard lattice solitons. On the other hand, the method can also be used to study the weakly nonlinear modulations of the wave packets in vector lattices or in lattices with a complex cell. The derivation procedure involves more cumbersome calculation, but the envelope equations obtained still take a form similar to Eq. (14) and (15) for high-frequency wave packets and Eqs. (40) and (41) for long-wavelength acoustic modes.

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APPENDIX A

The expressions of L_j and M_j for two-soliton solutions appearing in Eqs. (69) and (70) are given by

$$L_1=\frac{1}{8(\alpha_{11}p_{1R}^2+\alpha_{22}q_{1R}^2)},$$

$$L_2=\frac{1}{8(\alpha_{11}p_{2R}^2+\alpha_{22}q_{2R}^2)},$$

$$L_3=-\frac{1}{2}\frac{\alpha_{11}\Gamma_{-+}^{-}+\alpha_{22}\Sigma_{-+}^{-}}{[\alpha_{11}\Gamma_{-+}^{-}+\alpha_{22}\Sigma_{-+}^{-}]^2+4[\alpha_{11}\Delta_{-+}^{p}+\alpha_{22}\Delta_{-+}^{q}]^2},$$

$$L_4=-\frac{\alpha_{11}\Delta_{-+}^{p}+\alpha_{22}\Delta_{-+}^{q}}{[\alpha_{11}\Gamma_{-+}^{-}+\alpha_{22}\Sigma_{-+}^{-}]^2+4[\alpha_{11}\Delta_{-+}^{p}+\alpha_{22}\Delta_{-+}^{q}]^2},$$

$$L_5=\frac{1}{64}\frac{A_{5n}}{A_{5d}},$$

$$M_1=\frac{1}{8}\frac{M_{1n}}{M_{1d}},\qquad M_2=-\frac{1}{2}\frac{M_{2n}}{M_{2d}},\qquad M_3=\frac{1}{8}\frac{M_{3n}}{M_{3d}},$$

$$M_4=-\frac{1}{2}\frac{M_{4n}}{M_{4d}},$$

$$L_{5n}=\alpha_{11}^2(\Gamma_{--}^{+})^2+\alpha_{22}(\Sigma_{--}^{+})^2+2\alpha_{11}\alpha_{22}(\Gamma_{--}^{-}\Sigma_{--}^{-}+4\Delta_{--}^{p}\Delta_{--}^{q}),$$

$$L_{5d}=(\alpha_{11}p_{1R}^2+\alpha_{22}q_{1R}^2)(\alpha_{11}p_{2R}^2+\alpha_{22}q_{2R}^2)[\alpha_{11}^2(\Gamma_{-+}^{+})^2+\alpha_{22}^2(\Sigma_{-+}^{+})^2+2\alpha_{11}\alpha_{22}(\Sigma_{-+}^{-}\Gamma_{-+}^{-}-4\Delta_{-+}^{p}\Delta_{-+}^{q})],$$

$$\begin{aligned}M_{1n}=&\alpha_{11}^2\{(p_{1R}^2-p_{2R}^2)^2+(p_{1I}-p_{2I})^2[(p_{1I}-p_{2I})^2-2(3p_{1R}^2-p_{2R}^2)]\}+\alpha_{22}^2\{(q_{1R}^2-q_{2R}^2)^2+(q_{1I}-q_{2I})^2[(q_{1I}-q_{2I})^2-2(3q_{1R}^2-q_{2R}^2)]\}+2\alpha_{11}\alpha_{22}\{[(p_{1I}-p_{2I})^2\\&-(p_{1R}^2+p_{2R}^2)][(q_{1I}-q_{2I})^2-(q_{1R}^2+q_{2R}^2)]\\&-4(p_{1I}-p_{2I})(q_{1I}-q_{2I})(p_{1R}q_{1R}-p_{2R}q_{2R})\\&-4q_{1R}q_{2R}p_{1R}p_{2R}\},\end{aligned}$$

$$M_{1d}=M_{2d}=(\alpha_{11}p_{1R}^2+\alpha_{22}q_{1R}^2)[\alpha_{11}^2(\Gamma_{-+}^{+})^2+\alpha_{22}(\Sigma_{-+}^{+})^2+2\alpha_{11}\alpha_{22}(\Gamma_{-+}^{-}\Sigma_{-+}^{-}+4\Delta_{-+}^{p}\Delta_{-+}^{q})],$$

$$M_{2n}=\alpha_{11}^2p_{1R}(p_{1I}-p_{2I})[(p_{1I}-p_{2I})^2-p_{1R}^2+p_{2R}^2]$$
$$+\alpha_{22}q_{1R}(q_{1I}-q_{2I})[(q_{1I}-q_{2I})^2-q_{1R}^2+q_{2R}^2]$$
$$+\alpha_{11}\alpha_{22}\{(q_{1I}-q_{2I})[q_{1R}\Gamma_{-+}^-+2p_{1R}p_{2R}(q_{1I}+q_{2R})]$$
$$+(p_{1I}-p_{2I})[p_{1R}\Sigma_{-+}^-+2q_{1R}q_{2R}p_{2R}]\},$$

$$M_{3n}=\alpha_{11}^2\{(p_{1R}^2-p_{2R}^2)^2+(p_{1I}-p_{2I})^2[(p_{1I}-p_{2I})^2-2(3p_{2R}^2$$
$$-p_{1R}^2)]\}+\alpha_{22}^2\{(q_{1R}^2-q_{2R}^2)^2+(q_{1I}-q_{2I})^2[(q_{1I}$$
$$-q_{2I})^2-2(3q_{2R}^2-q_{1R}^2)]\}+2\alpha_{11}\alpha_{22}\{[(p_{1I}-p_{2I})^2$$
$$-(p_{1R}^2+p_{2R}^2)][(q_{1I}-q_{2I})^2-(q_{1R}^2+q_{2R}^2)]-4(p_{1I}$$
$$-p_{2I})(q_{1I}-q_{2I})(p_{1R}q_{1R}-p_{2R}q_{2R})$$
$$-4q_{1R}q_{2R}p_{1R}p_{2R}\},$$

$$M_{3d}=M_{4d}=(\alpha_{11}p_{2R}^2+\alpha_{22}q_{2R}^2)[\alpha_{11}^2(\Gamma_{-+}^+)^2+\alpha_{22}(\Sigma_{-+}^+)^2$$
$$+2\alpha_{11}\alpha_{22}(\Gamma_{-+}^-\Sigma_{-+}^-+4\Delta_{-+}^p\Delta_{-+}^q)],$$

$$M_{4n}=-\alpha_{11}^2(p_{1I}-p_{2I})p_{2R}[(p_{1I}-p_{2I})^2+p_{1R}^2-p_{2R}^2]$$
$$-\alpha_{22}^2(q_{1I}-q_{2I})q_{2R}[(q_{1I}-q_{2I})^2+q_{1R}^2-q_{2R}^2]$$
$$-\alpha_{11}\alpha_{22}\{(q_{1I}-q_{2I})[q_{2R}\Gamma_{-+}^-+2p_{1R}p_{2R}(q_{1I}+q_{2R})]$$
$$+(p_{1I}-p_{2I})[p_{2R}\Sigma_{-+}^-+2q_{1R}q_{2R}p_{1R}]\},$$

where

$$\Gamma_{\sigma_1\sigma_2}^{\pm}=(p_{1I}+\sigma_1p_{2I})^2\pm(p_{1R}+\sigma_2p_{2R})^2,$$

$$\Sigma_{\sigma_1\sigma_2}^{\pm}=(q_{1I}+\sigma_1q_{2I})^2\pm(q_{1R}+\sigma_2q_{2R})^2,$$

$$\Delta_{\sigma_1\sigma_2}^{p}=(p_{1I}+\sigma_1p_{2I})(p_{1R}+\sigma_2p_{2R}),$$

$$\Delta_{\sigma_1\sigma_2}^{q}=(q_{1I}+\sigma_1q_{2I})(q_{1R}+\sigma_2q_{2R}),$$

with $\sigma_j=\pm1(j=1,2)$.

The "dispersion relations" are given by

$$s_{1R}=-4\beta_1\beta_2(p_{1R}q_{1I}+p_{1I}q_{1R})+2\beta_2^2\left(\frac{\alpha_{11}p_{1I}p_{1R}}{\alpha_{22}}\right.$$
$$\left.-q_{1I}q_{1R}\right)+2\beta_1^2\left(\frac{\alpha_{22}q_{1I}q_{1R}}{\alpha_{11}}-p_{1I}p_{1R}\right),$$

$$s_{1I}=\frac{1}{\alpha_{11}\alpha_{22}}\{[-\alpha_{11}^2\beta_2^2(p_{1R}^2-p_{1I}^2)-\alpha_{22}^2\beta_1^2(q_{1R}^2-q_{1I}^2)$$
$$+\alpha_{11}\alpha_{22}[\beta_1^2(p_{1R}^2-p_{1I}^2)+4\beta_1\beta_2(p_{1R}q_{1R}-p_{1I}q_{1I})$$
$$+\beta_2^2(q_{1R}^2-q_{1I}^2)]\},$$

$$s_{2R}=-4\beta_1\beta_2(p_{2R}q_{2I}+p_{2I}q_{2R})+2\beta_2^2\left(\frac{\alpha_{11}p_{2I}p_{2R}}{\alpha_{22}}\right.$$
$$\left.-q_{2I}q_{2R}\right)+2\beta_1^2\left(\frac{\alpha_{22}q_{2I}q_{2R}}{\alpha_{11}}-p_{2I}p_{2R}\right),$$

$$s_{2I}=\frac{1}{\alpha_{11}\alpha_{22}}\{[-\alpha_{11}^2\beta_2^2(p_{2R}^2-p_{2I}^2)-\alpha_{22}^2\beta_1^2(q_{2R}^2-q_{2I}^2)$$
$$+\alpha_{11}\alpha_{22}[\beta_1^2(p_{2R}^2-p_{2I}^2)+4\beta_1\beta_2(p_{2R}q_{2R}-p_{2I}q_{2I})$$
$$+\beta_2^2(q_{2R}^2-q_{2I}^2)]\},$$

where p_{jR}, p_{jI}, q_{jR}, and $q_{jI}(j=1,2)$ are arbitrary constants.

APPENDIX B

One type of the standard DS equations which can be solved by the inverse scattering transform is (see p. 240 in Ref. [12] for the case of $r=-q^*$)

$$\frac{\partial^2\phi}{\partial x^2}-\sigma^2\frac{\partial^2\phi}{\partial y^2}=-2\frac{\partial^2}{\partial x^2}(|q^2|),$$

$$i\frac{\partial q}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 q}{\partial x^2}+\frac{1}{2}\frac{\partial^2 q}{\partial y^2}=q\phi+|q|^2q$$

with $\sigma^2=\pm1$. Taking the transformation $x\to\xi$, $y\to\eta$, $t\to(2/\sigma^2)\tau$, $q\to(1/\sqrt{2})A_1$, and $\phi\to-(\sigma^2/2)(\partial A_0/\partial\xi)$, the above equations become

$$\sigma^2\frac{\partial^2A_0}{\partial\xi^2}-\frac{\partial^2A_0}{\partial\eta^2}=2\frac{\partial}{\partial\xi}(|A_1|^2),$$

$$i\frac{\partial A_1}{\partial\tau}+\frac{\partial^2A_1}{\partial\xi^2}+\sigma^{-2}\frac{\partial^2A_1}{\partial\eta^2}=\sigma^{-2}|A_1|^2A_1-A_1\frac{\partial A_0}{\partial\xi}.$$

Comparing with Eqs. (19) and (20), for the last two equations we have

$$\alpha_{11}=\sigma^2,\quad \alpha_{22}=-1,\quad \beta_1=-1,\quad \beta_2=0,$$

$$\gamma_{11}=1,\quad \gamma_{12}=0,\quad \gamma_{22}=\sigma^{-2},$$

$$\chi=\sigma^{-2},$$

which satisfy the integrable conditions (71) and (72).

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