Chinese Physics Letters

PAPER

Quantum Squeezing of Matter-Wave Solitons in Bose-Einstein Condensates

Quantum Squeezing of Matter-Wave Solitons in Bose–Einstein Condensates

Jinzhong Zhu(朱锦忠)[1] and Guoxiang Huang(黄国翔)[1,2,3*]

[1]*State Key Laboratory of Precision Spectroscopy, East China Normal University, Shanghai 200241, China*
[2]*NYU-ECNU Joint Institute of Physics, New York University Shanghai, Shanghai 200062, China*
[3]*Collaborative Innovation Center of Extreme Optics, Shanxi University, Taiyuan 030006, China*

(Received 10 July 2023; accepted manuscript online 13 September 2023)

We investigate the quantum squeezing of matter-wave solitons in atomic Bose–Einstein condensates. By calculating quantum fluctuations of the solitons via solving the Bogoliubov–de Gennes equations, we show that significant quantum squeezing can be realized for both bright and dark solitons. We also show that the squeezing efficiency of the solitons can be enhanced and manipulated by atom–atom interaction and soliton blackness. The results reported here are beneficial not only for understanding quantum property of matter-wave solitons, but also for promising applications of Bose-condensed quantum gases.

DOI: 10.1088/0256-307X/40/10/100504

Solitons, fascinating nonlinear wave packets, can form in extended media through the balance between nonlinearity and dispersion (and/or diffraction).[1] Among solitons found in various physical systems, matter-wave solitons have attracted tremendous attention and studied extensively since the remarkable experimental realization of Bose–Einstein condensates (BECs) of cold atomic gases.[2–23]

Most researches on matter-wave solitons carried out up to date[2–23] are based on c-number Gross–Pitaevskii (GP) equation, which is obtained by using the mean-field approximation (MFA).[24] With such an approach, effects of quantum fluctuations around solitons are disregarded. However, in many cases quantum fluctuations can induce significant quantum diffusion of matter-wave solitons,[25–28] and hence the MFA is invalid.

In this Letter, we develop a quantum theory of matter-wave solitons in a quasi one-dimensional (1D) BEC beyond the MFA. In our scheme, the quantum fluctuations around the solitons display significant effects and hence cannot be neglected. Based on such a scheme, we consider the possibility of quantum squeezing of bight and dark solitons. By analytically calculating quantum fluctuations of the solitons via exactly solving the non-Hermitian eigenvalue problem of the Bogoliubov–de Gennes (BdG) equations describing the quantum fluctuations, we find that significant quantum squeezing for both the bright and dark solitons can occur in the system.

Moreover, we demonstrate that for the bright soliton (which can be obtained in the BEC with attractive atom–atom interaction) the squeezing efficiency can be enhanced by the atom–atom interaction, while for the dark soliton (which can be obtained in the BEC with repulsive atom–atom interaction) the squeezing efficiency can not only be enhanced by the atom–atom interaction but also be controlled by the soliton blackness. The results given here are useful not only for a deep understanding of the quantum property of matter-wave solitons, but also for practical applications of BECs with significant quantum fluctuations.

Model. We start to consider a cigar-shaped ultracold quantum gas with a local (two-body) atom–atom interaction, trapped by an external simple harmonic oscillator potential $V_{\rm ext}(\boldsymbol{r}) = (m/2)[\omega_\perp^2(x^2+y^2)+\omega_z^2 z^2]$, with m the atomic mass, $\omega_\perp$ (ω_z) the transverse (axial) trapping frequency ($\omega_\perp \gg \omega_z$). The Hamiltonian of the system reads

$$\hat{H} = \int d\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r},t)\Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\boldsymbol{r})\Big]\hat{\Psi}(\boldsymbol{r},t) + \frac{G}{2}\int d\boldsymbol{r}\hat{\Psi}^\dagger(\boldsymbol{r},t)\hat{\Psi}^\dagger(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t),$$

where $d\boldsymbol{r} = dxdydz$, $\hat{\Psi}$ is atomic annihilation operator obeying the commutation relation $[\hat{\Psi}(\boldsymbol{r},t),\hat{\Psi}^\dagger(\boldsymbol{r}',t)] = \delta(\boldsymbol{r}-\boldsymbol{r}')$, parameter $G = 4\pi\hbar^2 a_{\rm s}/m$ characterizes the strength of the atom–atom interaction, with $a_{\rm s}$ being the s-wave scattering length.[24] The symbol of $a_{\rm s}$ can be adjusted by using the technique of Feshbach resonance.[24]

Based on the above Hamiltonian, one can obtain the Heisenberg equation of motion as follows:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\boldsymbol{r},t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\boldsymbol{r})\Big]\hat{\Psi}(\boldsymbol{r},t) + G\hat{\Psi}^\dagger(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t). \tag{1}$$

Since the trapping of the gas in the axial direction is much smaller than that in the transverse directions, one can take $\hat{\Psi}(\boldsymbol{r},t) = w(x,y)\hat{\psi}(z,t)$, where $w(x,y)$ satisfies

$$-\frac{\hbar^2}{2m}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)w(x,y) + \frac{m}{2}\omega_\perp^2(x^2+y^2)w(x,y) = \nu w(x,y),$$

i.e., the eigen equation of 2D harmonic oscillator. The normalized ground-state solution to this eigen equation is given by $w_0(x,y) = [m\omega_\perp/(\pi\hbar)]^{1/2}e^{-m\omega_\perp(x^2+y^2)/(2\hbar)}$, with the eigenvalue $\nu = \hbar\omega_\perp$. If the energy of the atom–atom interaction is much smaller than the atomic kinetic energy in the transverse directions, one can take

*Corresponding author. Email: gxhuang@phy.ecnu.edu.cn

$w(x,y) = w_0(x,y)$. Multiplying $w_0(x,y)$ on the left side of Eq. (1) and then carrying out the integration on the variables x and y, Eq. (1) becomes

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(z,t) = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m}{2}\omega_z^2 z^2 + \hbar\omega_\perp\Big)\hat{\psi}(z,t) + g_{1\mathrm{D}}\hat{\psi}^\dagger(z,t)\hat{\psi}(z,t)\hat{\psi}(z,t), \tag{2}$$

with $g_{1\mathrm{D}} = 2a_s\hbar\omega_\perp$. In this way, the system is effectively reduced to a quasi-1D one.[23]

For convenience of later calculations, it is better convert Eq. (2) to a dimensionless form. This can be realized by taking the transformation $\hat{\psi} = \sqrt{N\rho_0}e^{-i\mu\tau}\hat{\phi}(\zeta,\tau)$, $\zeta = z/l_\perp$, $\tau = \omega_\perp t$, and $\varOmega = \omega_z/\omega_\perp$, with N being the total atomic number, ρ_0 the 1D atomic density, and $l_\perp = \sqrt{\hbar/(m\omega_\perp)}$ the transverse harmonic-oscillator length. Then Eq. (2) becomes

$$i\frac{\partial\hat{\phi}}{\partial\tau} = \Big[-\frac{1}{2}\frac{\partial^2}{\partial\zeta^2} + \frac{\varOmega^2}{2}\zeta^2 - \mu + 1\Big]\hat{\phi} + g\hat{\phi}^\dagger\hat{\phi}\hat{\phi}.$$

Here $g = 2Na_s\rho_0$ and $\mu = \mu'/(\hbar\omega_\perp)$; $\mu' = \int d\boldsymbol{r}\langle\Psi_G|\hat{\psi}^\dagger\hat{\psi}|\Psi_G\rangle$ is chemical potential, with $|\Psi_G\rangle$ representing the initial state of the system. Notice that quasi 1D BECs have been widely used to study the dynamics of matter-wave solitons.[2–16,19–22] For example, in the experiment[12] a cigar-shaped ^{87}Rb BEC was used, with $(\omega_\perp,\omega_z) = 2\pi\times(133, 5.9)$ Hz. Since $\omega_z \ll \omega_\perp$, one has $\varOmega \ll 1$, and the trapping potential $U_{\mathrm{ext}}(\zeta) = (\varOmega^2/2)\zeta^2$ for finite ζ is practically a high-order small quantity.[29] For simplicity, we shall take $\varOmega = 0$ in the following discussion.

As a result, we obtain the dimensionless quantum nonlinear Schrödinger (QNLS) equation

$$i\frac{\partial}{\partial\tau}\hat{\phi}(\zeta,\tau) + \Big(\frac{1}{2}\frac{\partial^2}{\partial\zeta^2} + \mu - 1\Big)\hat{\phi}(\zeta,\tau) - g\hat{\phi}^\dagger(\zeta,\tau)\hat{\phi}(\zeta,\tau)\hat{\phi}(\zeta,\tau) = 0. \tag{3}$$

The effective Hamiltonian for the system described by the QNLS Eq. (3) is given by

$$\hat{H}_{\mathrm{eff}} = \int d\zeta\,\hat{\phi}^\dagger\Big(-\frac{1}{2}\frac{\partial^2}{\partial\zeta^2} - \mu + 1 + \frac{g}{2}\hat{\phi}^\dagger\hat{\phi}\Big)\hat{\phi}, \tag{4}$$

with the commutation relation $[\hat{\phi}(\zeta,\tau),\hat{\phi}^\dagger(\zeta',\tau)] = \delta(\zeta-\zeta')$.

BdG Equations for Quantum Fluctuations. We are interested in the quantum fluctuations around a soliton-like BEC in the system. To this end, we assume that the atomic number in the condensate is large, so that the quantum fluctuations, though they can display significant effect, are still weaker compared with the condensate. Thereby one can make the Bogoliubov decomposition[30]

$$\hat{\phi}(\zeta,\tau) = \phi_0(\zeta,\tau) + \delta\hat{\phi}(\zeta,\tau), \tag{5}$$

where ϕ_0 is wavefunction (complex c-number) describing the condensate, $\delta\hat{\phi}$ is an operator representing the quantum fluctuations around the condensate background.[24]

Substituting the Bogoliubov decomposition (5) into the QNLS Eq. (3) and keeping only linear terms of $\delta\hat{\phi}$, we obtain the following equations:

$$i\frac{\partial}{\partial\tau}\phi_0 + \Big(\frac{1}{2}\frac{\partial^2}{\partial\zeta^2} + \mu - 1\Big)\phi_0 - g|\phi_0|^2\phi_0 = 0, \tag{6a}$$

$$i\frac{\partial}{\partial\tau}\hat{\varPhi}(\zeta,\tau) + \hat{\mathcal{T}}\hat{\varPhi}(\zeta,\tau) = 0. \tag{6b}$$

Here $\hat{\varPhi}(\zeta,\tau) = (\delta\hat{\phi},\delta\hat{\phi}^\dagger)^{\mathrm{T}}$ (the superscript T means transpose), and $\hat{\mathcal{T}}$ is a matrix operator, defined by

$$\hat{\mathcal{T}} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B} & -\mathcal{A} \end{pmatrix}, \tag{7}$$

with $\mathcal{A} = (1/2)\partial^2/\partial\zeta^2 - 2g|\phi_0|^2 + \mu - 1$ and $\mathcal{B} = g|\phi_0|^2$.

To understand the property of the quantum fluctuations from the condensate described by ϕ_0, we must first find ϕ_0 (e.g., soliton solution) via solving the GP Eq. (6a), and then seek the eigenmodes of the matrix operator $\hat{\mathcal{T}}$ through solving the operator Eq. (6b). In order to be able to express (expand) all possible quantum fluctuations, these eigenmodes must be complete. Note that operator $\hat{\mathcal{T}}$ is not Hermitian, but pseudo-Hermitian, i.e., $\hat{\mathcal{T}}^\dagger = \sigma_3\hat{\mathcal{T}}\sigma_3$, with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Pauli matrix). In recent years, tremendous efforts have been paid to the research of non-Hermitian physics, and it has been proved that a pseudo-Hermitian operator can possess all-real spectrum.[31,32] Thus, if one can find all the eigenmodes of $\hat{\mathcal{T}}$ (and $\hat{\mathcal{T}}^\dagger$), the set of complete and bi-orthonormal eigenmodes can be constructed, by which the effective Hamiltonian (4) can be diagonalized.

For this aim, we adopt the method developed recently in Refs. [33–37] to acquire the complete and bi-orthogonal eigenmodes of the BdG Eq. (6b) through making the Bogoliubov transformation

$$\delta\hat{\phi}(\zeta,\tau) = \sum_n [u_n(\zeta)\hat{a}_n(\tau) + v_n^*(\zeta)\hat{a}_n^\dagger(\tau)] + \int dk\big[u_k(\zeta)\hat{a}_k(\tau) + v_k^*(\zeta)\hat{a}_k^\dagger(\tau)\big]. \tag{8}$$

Here the indices n and k are quantum numbers characterizing respectively the discrete and continuous modes; $\hat{a}_n(\tau)$ and $\hat{a}_k(\tau)$ are respectively atomic annihilation operators for the discrete and continuous modes, satisfying respectively the commutation relations $[\hat{a}_n(\tau),\hat{a}_m^\dagger(\tau)] = \delta_{mn}$ and $[\hat{a}_k(\tau),\hat{a}_{k'}^\dagger(\tau)] = \delta(k-k')$; $u_n(\zeta)$, $v_n(\zeta)$, $u_k(\zeta)$, and $v_k(\zeta)$ are eigenmode functions for the discrete and continuous spectra, respectively. Taking $\hat{\varPhi}(\zeta,\tau) = \hat{\varPhi}(\zeta)\exp(i\lambda\tau)$ and substituting it into Eq. (6b), we obtain the eigenvalue equations (i.e., BdG equations)

$$\hat{\mathcal{T}}\hat{\varPhi}(\zeta) = \lambda\hat{\varPhi}(\zeta). \tag{9}$$

Next, we shall consider two particular cases in which ϕ_0 takes forms of bright and dark solitons, respectively.

Quantum Fluctuations of Bright Solitons. For the case of attractive atom–atom interaction (i.e., $g < 0$[6,7]), the GP Eq. (6a) admits bright soliton solution

$$\phi_0^{\mathrm{BS}} = \eta_0\sqrt{g}\,\mathrm{sech}(Z)\,e^{i[v_0(\zeta-\zeta_0)-v_0^2\tau/2+\theta_0]}. \tag{10}$$

Here $Z = \eta_0 g(\zeta - v_0\tau - \zeta_0)$, $\mu = -\eta_0^2 g^2/2 + 1$, BS means "bright soliton"; η_0, θ_0, v_0, and ζ_0 are free real parameters, related to the soliton amplitude, initial phase, velocity, and initial position, respectively. Substituting the solution (10) into the expression of $\hat{\mathcal{T}}$ and solving the BdG Eq. (9), we can obtain all the eigenmodes of $\hat{\mathcal{T}}$, including the continuum modes (with eigenvalues $\lambda_k = -k^2 - 1$)

and discrete modes (with two degenerate eigenvalues $\lambda_1 = \lambda_2 = 0$, called zero modes). In addition, using the relation $\hat{\mathcal{T}}^\dagger = \sigma_3 \hat{\mathcal{T}} \sigma_3$, we can also acquire all the eigenmodes of $\hat{\mathcal{T}}^\dagger$. It can be shown that these eigenmodes constitute a complete set, and they are bi-orthogonal in the dual spaces of $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}^\dagger$. For detailed expressions and their completeness and bi-orthogonality of these eigenmodes, see Refs. [35,37].

Based on the above results and the Bogoliubov transformation (8), the effective Hamiltonian (4) can be diagonalized into the form

$$\hat{H}_{\rm eff}^{\rm BS} = \frac{2\eta_0^3 g^2}{3} + \frac{\eta_0^2 g^2}{2}\Big[\hat{P}_2^2 - \hat{Q}_1^2 + \int_{-\infty}^{+\infty} dk \lambda_k \hat{a}_k^\dagger(\tau)\hat{a}_k(\tau)\Big]. \quad (11)$$

Here $\hat{Q}_n = (\hat{a}_n + \hat{a}_n^\dagger)/\sqrt{2}$ and $\hat{P}_n = (\hat{a}_n - \hat{a}_n^\dagger)/(\sqrt{2}i)$ are respectively the position and momentum operators related to the zero modes, satisfying commutation relations $[\hat{Q}_n, \hat{P}_{n'}] = i\delta_{nn'}$ $(n, n' = 1, 2)$.

Based on the diagonalized Hamiltonian Eq. (11) and the Heisenberg equations of motion for $\hat{Q}_n$, $\hat{P}_n$, and $\hat{a}_k$, we obtain the following solutions:

$$\hat{Q}_1(\tau) = \hat{Q}_1(0), \quad (12a)$$
$$\hat{P}_1(\tau) = \eta_0^2 g^2 \tau \hat{Q}_1(0) + \hat{P}_1(0), \quad (12b)$$
$$\hat{P}_2(\tau) = \hat{P}_2(0), \quad (12c)$$
$$\hat{Q}_2(\tau) = A_0^2 g^2 \tau \hat{P}_2(0) + \hat{Q}_2(0), \quad (12d)$$
$$\hat{a}_k(\tau) = \hat{a}_k(0) e^{i\eta_0^2 g^2 \lambda_k \tau/2}, \quad (12e)$$

where $\hat{Q}_j(0)$, $\hat{P}_j(0)$, and $\hat{a}_k(0)$ are values of $\hat{Q}_j(\tau)$, $\hat{P}_j(\tau)$, and $\hat{a}_k(\tau)$ at $\tau = 0$, respectively. From this result we can see that the quantum fluctuations of the bright soliton are contributed mainly by two zero modes, which induce quantum phase diffusion and atomic number fluctuations as well as position and momentum fluctuations.

Quantum Fluctuations of Dark Solitons. For the case of repulsive atom–atom interaction (i.e., $g > 0$ [2,4]), the GP Eq. (6a) admits dark soliton solution

$$\phi_0^{\rm DS}(\zeta,\tau) = \eta_0\sqrt{g}(\cos\vartheta \tanh\varsigma + i\sin\vartheta)e^{i\theta_0}. \quad (13)$$

Here $\varsigma = \eta_0 g\cos\vartheta(\zeta - \zeta_0 - \eta_0 g\tau\sin\vartheta)$, $\mu = \eta_0^2 g^2 + 1$, DS means "dark soliton"; η_0 and θ_0 are constants characterizing the amplitude and the overall phase of the soliton; ϑ $(0 \le \vartheta \le \pi/2)$ is a constant characterizing the blackness of the dark soliton, defined by $\eta^2 g\cos^2\vartheta$ (i.e., the difference between the minimum of the soliton intensity and the background intensity $\eta^2 g$); $\eta g\sin\vartheta$ and ζ_0 are the velocity and initial position of the soliton, respectively. When $\vartheta = 0$, Eq. (13) reduces to a black soliton.

Similar to bright soliton, we can obtain all the eigenmodes of the matrix operator $\hat{\mathcal{T}}$ by solving the BdG Eq. (9), including continuum modes with eigenvalues $\lambda_k = |k|[-2\tan\vartheta + \sqrt{k^2 + 4(1+\tan^2\vartheta)}]$ and unique zero mode with eigenvalue $\lambda_1 = 0$. For detailed expressions of these eigenmodes and their completeness and bi-orthogonality, see Ref. [36]. With these results the Hamiltonian (4) can be diagonalized as

$$\hat{H}_{\rm eff}^{\rm DS} = \frac{\eta_0^2 g^2}{2}\cos^2\vartheta\Big[\hat{P}_1^2 + \int_{-\infty}^{+\infty} dk\lambda_k \hat{a}_k^\dagger(\tau)\hat{a}_k(\tau)\Big]. \quad (14)$$

Based on this diagonalized Hamiltonian, by using Heisenberg equations we can obtain the solution

$$\hat{P}_1(\tau) = \hat{P}_1(0), \quad (15a)$$
$$\hat{Q}_1(\tau) = \eta_0^2 g^2 \tau\cos^2\vartheta \hat{P}_1(0) + \hat{Q}_1(0), \quad (15b)$$
$$\hat{a}_k(\tau) = \hat{a}_k(0) e^{i\eta_0^2 g^2(\lambda_k\tau/2)\cos^2\vartheta}. \quad (15c)$$

This result shows that the quantum fluctuations of the dark soliton originate mainly from the zero mode, which generates the diffusion of the position and momentum of the dark soliton. It should be pointed out that no fluctuations of phase and atom-number occur here. This is due to the reason that, in the present approach, the dark soliton has a non-zero boundary at infinity, which means that the system contains infinite-many atoms. Practically the BEC has a finite size, and the dark soliton is excited on a finite background. In such a case, it is possible to obtain the second zero mode of quantum fluctuations, and hence a phase diffusion of the dark soliton may occur, an interesting topic deserving to be explored further.

Quantum Squeezing of Bright and Dark Solitons. In recent years, quantum squeezing has attracted much attention and found a wealth of important applications, especially in quantum precision measurements.[38–40] Based on the results given above, we can explore the possibility of quantum squeezing of the matter-wave solitons. Because the quantum fluctuations from continuous spectra are much smaller than those from the zero modes,[41] in the following calculation we shall neglect the contribution from the continuous spectra.

To investigate the quantum squeezing, we introduce quadrature operators related to $\hat{a}_n$ at the angle Θ,[42]

$$\hat{X}_{n,\Theta}(\tau) = \frac{1}{\sqrt{2}}[\hat{a}_n(\tau)\,e^{-i\Theta} + \hat{a}_n^\dagger(\tau)\,e^{i\Theta}] = \hat{Q}_n(\tau)\cos\Theta + \hat{P}_n(\tau)\sin\Theta, \quad (16)$$

satisfying the commutation relation $[\hat{X}_{n,\Theta}, \hat{X}_{n',\Theta+\frac{\pi}{2}}] = i\delta_{n,n'}$, where Θ is the detection angle. The quantum fluctuations can be expressed by $\hat{X}_{n,\Theta}(\tau)$ in the form

$$\delta\hat{\phi}(\zeta,\tau) = \sum_n \Big\{U_n(\zeta)\Big[\cos\Theta\hat{X}_{n,\Theta}(\tau) - \sin\Theta\hat{X}_{n,\Theta+\frac{\pi}{2}}(\tau)\Big] + iV_n(\zeta)\Big[\sin\Theta\hat{X}_{n,\Theta}(\tau) + \cos\Theta\hat{X}_{n,\Theta+\frac{\pi}{2}}(\tau)\Big]\Big\}, \quad (17)$$

where $U_n(\zeta) = [u_n(\zeta) + v_n(\zeta)]/\sqrt{2}$ and $V_n(\zeta) = [u_n(\zeta) - v_n(\zeta)]/\sqrt{2}$. In order to make difference between the cases of the bright and dark solitons, we write the above formula as the form of $\hat{X}_{n,\Theta}^\alpha(\tau) = \frac{1}{\sqrt{2}}[\hat{a}_n^\alpha(\tau)\,e^{-i\Theta} + (\hat{a}_n^\alpha)^\dagger(\tau)\,e^{i\Theta}] = \hat{Q}_n^\alpha(\tau)\cos\Theta + \hat{P}_n^\alpha(\tau)\sin\Theta$, with $\alpha = $ BS ($\alpha = $ DS) for bright (dark) soliton. The quantum squeezing of the solitons can be characterized by using quadrature variance $\langle(\hat{X}_{n,\Theta}^\alpha)^2\rangle \equiv \langle\Psi_G|(\hat{X}_{n,\Theta}^\alpha)^2|\Psi_G\rangle$ (i.e., the average under the initial state $|\Psi_G\rangle$, which contains the soliton with no quantum fluctuation[35–37]).

Figure 1 shows the result of $\langle(\hat{X}_{n,\Theta}^\alpha)^2\rangle$ as functions of τ and $\Theta/(2\pi)$, by taking $\eta_0 = g = 1$ and $\vartheta = \pi/4$. Plots in Figs. 1(a) and 1(b) are the quadrature variances $\langle(\hat{X}_{1,\theta}^{\rm BS})^2\rangle$ and $\langle(\hat{X}_{2,\theta}^{\rm BS})^2\rangle$ of the bright soliton, while Fig. 1(c) is the quadrature variance $\langle(\hat{X}_{1,\theta}^{\rm DS})^2\rangle$ of the dark soliton. We can see that three quadrature variances are sensitive to the selection of Θ. When $\tau = 0$, all the variances take the vacuum value 1/2. However, when Θ and τ locate in the blue domains, the quadrature variances are much smaller than their vacuum values, which means that the matter-wave solitons can be significantly squeezed by the quantum fluctuations contributed by the zero modes. These solitons can also be made to be anti-squeezed, indicated by the domains with red color.

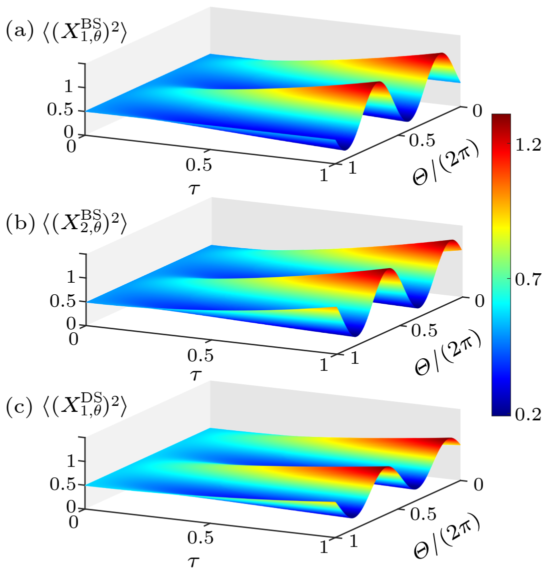

Fig. 1. Quantum squeezing of matter-wave solitons. [(a), (b)] Quadrature variances $\langle(\hat{X}^{\rm BS}_{1,\theta})^2\rangle$, $\langle(\hat{X}^{\rm BS}_{2,\theta})^2\rangle$ of the bright soliton as functions of τ and $\Theta/(2\pi)$, for $\eta_0 = g = 1$ and $\vartheta = \pi/4$. (c) Quadrature variance $\langle(\hat{X}^{\rm DS}_{1,\theta})^2\rangle$ of the dark soliton, also for $\eta_0 = g = 1$ and $\vartheta = \pi/4$.

The degree of the quantum squeezing can be characterized by the squeezing ratio, i.e., the ratio of the quadrature variance between the value at time τ and that at $\tau = 0$,[41]

$$R_n^\alpha = \frac{\langle[\hat{X}^\alpha_{n,\Theta}(\tau)]^2\rangle}{\langle[\hat{X}^\alpha_{n,\Theta}(0)]^2\rangle}. \tag{18}$$

By minimizing the quadrature variance $\langle(\hat{X}^\alpha_{n,\Theta})^2\rangle$ with respect to Θ, we can obtain the optimum squeezing by selecting $\Theta = \Theta_{\rm opt}$. Then one can take the optimum detection angle $\Theta_{\rm opt}$ to acquire the minimum and maximum values of the quadrature for angle $\Theta = \Theta_{\rm opt}$ and $\Theta = \Theta_{\rm opt} + \pi/2$, respectively.

Shown in Fig. 2(a) are minimum squeezing ratios $(R_n^\alpha)_{\rm min}$ (α = BS, DS are for the bright, dark solitons, respectively) as functions of τ for $\eta_0 = g = 1$. In the figure, the dashed blue line is for the bright soliton; the lines with red squares, yellow crosses, and purple dots are for the dark soliton with blackness parameter $\vartheta = 0$, $\pi/6$, and $\pi/3$, respectively. We see that for both the bright and dark solitons $(R_n^\alpha)_{\rm min}$ increases as τ increases. Plotted in Fig. 2(b) is the minimum squeezing ratio $(R_1^{\rm BS})_{\rm min}$ of the bright soliton with $\eta_0 = 1$, as a function of τ for $g = 0.8$, 1, and 1.2, respectively. One can see that the value of the squeezing ratio has a strong dependence on the strength of atom–atom interaction (characterized by the parameter g); the larger the atom–atom interaction, the stronger the quantum squeezing.

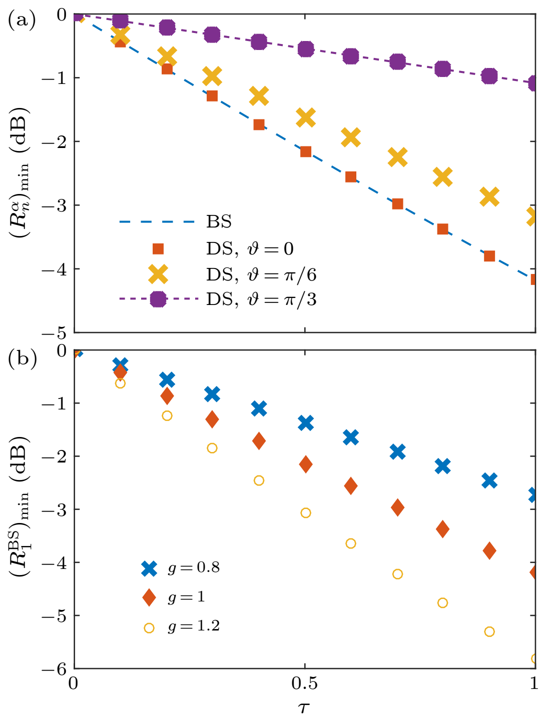

Fig. 2. Squeezing ratio for soliton squeezing. (a) Minimum squeezing ratio $(R_n^\alpha)_{\rm min}$ ($\alpha = {\rm BS, DS}$) as a function of τ for $\eta_0 = g = 1$. BS means the result for the bright soliton; DS means the result for dark soliton, with blackness parameter $\vartheta = 0$, $\pi/6$, and $\pi/3$, respectively. (b) Minimum squeezing ratio $(R_1^{\rm BS})_{\rm min}$ of the bright soliton with $\eta_0 = 1$, as a function of τ for $g = 0.8$, 1, and 1.2, respectively.

In summary, we have studied the quantum squeezing of matter-wave solitons in atomic BECs with both the attractive and repulsive interactions. Through the calculation of the quantum fluctuations of the solitons by solving the related BdG equations, we have shown that significant quantum squeezing can reach both bright and dark solitons. We have also shown that the squeezing efficiency of the solitons can be enhanced and adjusted by atom–atom interactions and soliton blackness. The theoretical approach developed here can be extended to the BEC with multiple spin components; the results obtained are helpful not only for a deep understanding on the quantum property of matter-wave solitons, but also for promising applications for quantum precision measurements.

Acknowledgement. This work was supported by the National Natural Science Foundation of China (Grant No. 11975098).

References

[1] Ablowitz M J and Clarkson P A 2003 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (New York: Cambridge University Press)

[2] Burger S, Bongs K, Dettmer S, Ertmer W, Sengstock K, Sanpera A, Shlyapnikov G V, and Lewenstein M 1999 *Phys. Rev. Lett.* **83** 5198

[3] Fedichev P O, Muryshev A E, and Shlyapnikov G V 1999 *Phys. Rev. A* **60** 3220

[4] Denschlag J, Simsarian J E, Feder D L, Clark C W, Collins L A, Cubizolles J, Deng L, Hagley E W, Helmerson K,

Reinhardt W P, Rolston S L, Schneider B I, and Phillips W D 2000 *Science* **287** 97
[5] Huang G X, Velarde M G, and Makarov V A 2001 *Phys. Rev. A* **64** 013617
[6] Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y, and Salomon C 2002 *Science* **296** 1290
[7] Strecker K E, Partridge G B, Truscott A G, and Hulet R G 2002 *Nature* **417** 150
[8] Muryshev A, Shlyapnikov G V, Ertmer W, Sengstock K, and Lewenstein M 2002 *Phys. Rev. Lett.* **89** 110401
[9] Huang G X, Szeftel J, and Zhu S H 2002 *Phys. Rev. A* **65** 053605
[10] Sinha S, Cherny A Y, Kovrizhin D, and Brand J 2006 *Phys. Rev. Lett.* **96** 030406
[11] Jackson B, Proukakis N P, Barenghi C F 2007 *Phys. Rev. A* **75** 051601
[12] Stellmer S, Becker C, Soltan-Panahi P, Richter E M, Dörscher S, Baumert M, Kronjäger J, Bongs K, and Sengstock K 2008 *Phys. Rev. Lett.* **101** 120406
[13] Cockburn S P, Nistazakis H E, Horikis T P, Kevrekidis P G, Proukakis N P, and Frantzeskakis D J 2010 *Phys. Rev. Lett.* **104** 174101
[14] Marchant A L, Billam T P, Wiles T P, Yu M M H, Gardiner S A, and Cornish S L 2013 *Nat. Commun.* **4** 1865
[15] Xu Y, Zhang Y, and Wu B 2013 *Phys. Rev. A* **87** 013614
[16] Nguyen J H V, Dyke P, Luo D, Malomed B A, and Hulet R G 2014 *Nat. Phys.* **10** 918
[17] Zhang Y C, Zhou Z M, Malomed B A, and Pu H 2015 *Phys. Rev. Lett.* **115** 253902
[18] Jiang X D, Fan Z W, Chen Z P, Pang W, Li Y Y, and Malomed B A 2016 *Phys. Rev. A* **93** 023633
[19] Nguyen J H V, Luo D, and Hulet R G 2017 *Science* **356** 422
[20] Cheiney P, Cabrera C R, Sanz J, Naylor B, Tanzi L, and Tarruell L 2018 *Phys. Rev. Lett.* **120** 135301
[21] Hang C, Gabadadze G, and Huang G 2019 *Phys. Lett. B* **793** 390
[22] Mateo A M and Yu X 2022 *Phys. Rev. A* **105** L021301
[23] Shi Z Y and Huang G X 2023 *Phys. Rev. E* **107** 024214
[24] Pethick C J and Smith H 2008 *Bose–Einstein Condensation in Dilute Gases* 2nd edn (Cambridge: Cambridge University Press)
[25] Lewenstein M and You L 1996 *Phys. Rev. Lett.* **77** 3489
[26] Yi S, MüstecaplıO Ö E, and You L 2003 *Phys. Rev. Lett.* **90** 140404
[27] Dziarmaga J 2004 *Phys. Rev. A* **70** 063616
[28] Villari L D M, Faccio D, Biancalana F, and Conti C 2018 *Phys. Rev. A* **98** 043859
[29] Significant effect may occur when soliton moves to the boundary of the cigar-shaped condensate even if is small. For simplicity, we do not consider such a situation here.
[30] Bogoliubov N 1947 *J. Phys. (Moscow)* **11** 23
[31] Konotop V V, Yang J, and Zezyulin D A 2016 *Rev. Mod. Phys.* **88** 035002
[32] Ashida Y, Gong Z, and Ueda M 2020 *Adv. Phys.* **69** 249
[33] Huang G X, Deng L, Yan J R, and Hu B 2006 *Phys. Lett. A* **357** 150
[34] Huang G X and Compagnon N 2008 *Phys. Lett. A* **372** 321
[35] Zhu J Z, Zhang Q, and Huang G X 2021 *Phys. Rev. A* **103** 063512
[36] Zhu J Z and Huang G X 2022 *Phys. Rev. A* **105** 033515
[37] Zhu J Z, Mu Y, and Huang G X 2023 *Phys. Rev. A* **107** 033517
[38] Ma J, Wang X G, Sun C P, and Nori F 2011 *Phys. Rep.* **509** 89
[39] Schnabel R 2017 *Phys. Rep.* **684** 1
[40] Sinatra A 2022 *Appl. Phys. Lett.* **120** 120501
[41] Haus H A and Lai Y 1990 *J. Opt. Soc. Am. B* **7** 386
[42] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)