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INVESTIGATION OF FOUR-WAVE MIXING OF MATTER WAVES IN BOSE–EINSTEIN CONDENSATES

CHUNLIU SUN*, CHAO HANG*, GUOXIANG HUANG*,† and BAMBI HU†,‡

*Department of Physics and Institute of Theoretical Physics, East China Normal University, Shanghai 200062, China

†Department of Physics and the Center for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China

‡Department of Physics, University of Houston, Houston, TX77204-5506, USA

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We investigate the four-wave mixing of matter wave packets created from a Bose–Einstein condensate, realized experimentally by utilizing light pulses to create two high-momentum wave packets via Bragg diffraction from a stationary condensate. Based on the Gross–Pitaevskii equation, a set of nonlinearly coupled envelope equations including self- and cross-phase modulational effects are derived systematically using a method of multiple-scales. The exact and explicit analytical solutions are provided for the coupled envelope equations and the evolution of the wave packets after turning off trapping potential is discussed and compared with experiment.

Keywords: Bose-Einstein condensation; matter waves; four-wave mixing.

The advent of the laser as an intense source of coherent light resulted in the rapid development of nonlinear optics.[1] One of the main areas of research activity in this field is resonant wave–wave mixing. In recent years, the experimental realization of Bose–Einstein condensation in weakly interacting atomic gases has opened a new direction for the study of the nonlinear properties of matter waves.[2] This enables the extension of linear atom optics to a nonlinear regime, i.e. nonlinear atom optics.[3] This is very much like how the laser led to the development of nonlinear optics in the 1960s. Different from an optical medium, where the nonlinearity is originated from the interaction between light and the medium, the nonlinearity in a Bose–Einstein condensate (BEC) comes from atom–atom collisions. Some typical nonlinear excitations, such as solitons and vortices, have been observed.[4] Following the suggestion in Ref. 5, Deng *et al.* successfully demonstrated a four-wave mixing (FWM) in a remarkable experiment by using phase-matched BEC wave packets.[6]

There are several theoretical works on the FWM for the BEC matter waves.[5,7–10] Based on Ref. 5 and the experimental result of Ref. 6, Trippenbach *et al.*[8] made a detail investigation on the FWM in a BEC. Starting from the Gross–Pitaevskii (GP) equation they derived a set of nonlinearly coupled envelope equations using a slowly–varying envelope approximation (SVEA) but only a numerical solution of these equations was given. Wu *et al.*[7] presented a different theory and provided an exact solution in the form of a triangular function on the FWM. However, the terms describing self-phase modulation effect are absent in their envelope equations.

In this work, we give a systematic derivation of the nonlinearly coupled envelope equations including the self- and cross-phase modulations for the FWM in the BEC by using a method of multiple-scales. We provide an explicit analytical (Jacob elliptic function) solution for these envelope equations and discuss the evolution of BEC matter waves after turning off trapping potential and a comparison will be made with experimental results.

It is well known that the dynamics of a weakly interacting Bose gas at zero temperature is described by the time-dependent GP equation

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t) + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)\,, \tag{1}$$

where $\Psi(\mathbf{r},t)$ is condensed state wave function (order parameter), $\int d\mathbf{r}|\Psi(\mathbf{r},t)|^2 = N$ is the atomic number in the condensate, $V(\mathbf{r},t)$ is a trapping potential, $g = 4\pi\hbar^2 a_s/m$ is the inter-atomic interaction constant with a_s the s-wave scattering length ($a_s > 0$ for a repulsive interaction). According to the basic idea of singular perturbation theory,[11] for a FWM in the BEC we can employ the asymptotic expansion

$$\Psi(\mathbf{r},t) = \epsilon\Psi^{(1)} + \epsilon^2\Psi^{(2)} + \epsilon^3\Psi^{(3)} + \cdots \tag{2}$$

where ϵ is a small parameter characterizing the relative amplitude of the wave packet, $\Psi^{(j)}$ ($j = 1, 2, 3\cdots$) are the functions of the multiple-scale variables x, y, t, $x_1 = \epsilon x$, $y_1 = \epsilon y$, $t_1 = \epsilon t$, $x_2 = \epsilon^2 x$, $y_2 = \epsilon^2 y$ and $t_2 = \epsilon^2 t$, $V(\mathbf{r},t) = \epsilon^2 V_2(x_1, y_1, t_1, x_2, y_2, t_2)$. Then Eq. (1) reads

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_r^2\right)\Psi^{(j)} = \alpha^{(j)}\,. \tag{3}$$

The explicit expressions of $\alpha^{(j)}$ are omitted here.

In the leading order ($j = 1$), Eq. (3) admits the solution

$$\Psi^{(1)} = \Phi(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\,, \tag{4}$$

where $\mathbf{k} = (k_x, k_y)$, $\mathbf{r} = (x, y)$ and Φ is an envelope function of slow variables yet to be determined. Substituting Eq. (4) into Eq. (3) in leading order, we obtain the linear dispersion relation given by $\omega(\mathbf{k}) = \hbar\mathbf{k}^2/(2m)$. Note that it is different from the Bogoliubov-type excitation spectrum since the latter is obtained for the excitations created from a stationary background.[12]

We are interested in a FWM in the BEC. For an efficient FWM a phase-matching condition is required by momentum and energy conservation[5,8]:

$$\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4\,, \tag{5}$$

$$\mathbf{k}_1^2 + \mathbf{k}_3^2 = \mathbf{k}_2^2 + \mathbf{k}_4^2\,. \tag{6}$$

Equations (5) and (6) can be fulfilled by suitably selecting the wavevectors $\mathbf{k}_j$, for details see Ref. 8. For instance, the central momenta can be chosen as $\mathbf{k}_1 = 0$, $\mathbf{k}_2 = (k_2 \cos\theta_0, k_2 \sin\theta_0, 0)$, $\mathbf{k}_3 = (k_3, 0, 0)$, and $\mathbf{k}_4 = (k_3 - k_2 \cos\theta_0, -k_2 \sin\theta_0, 0)$ with $k_3 \cos\theta_0 = k_2$, where $-\pi/2 < \theta_0 < \pi/2$ and k_2 arbitrary.[6] Initially three BEC wave packets with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ can be produced by using Bragg diffraction.[6] If these wave packets overlap spatially, the atom–atom interaction can cause the matter-wave FWM and the fourth wave with wave vector $\mathbf{k}_4$ to appear.

For describing the FWM we consider the case when the condensed state wave function is a superposition of four wave packets with different central momenta

$$\Psi^{(1)} = \sum_{l=1}^{4} \Phi_l(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) \exp(i\theta_l)\,, \tag{7}$$

where $\theta_l(\mathbf{r}, t) = \mathbf{k}_l \cdot \mathbf{r} - \omega_l t$ with $\omega_l = \hbar \mathbf{k}_l^2/(2m)$, Φ_l is the envelope of lth wave packet. $\mathbf{k}_l$ $(l = 1, 2, 3, 4)$ are chosen according to the phase-matching conditions (5) and (6).

At the second order $(j = 2)$, the solvability condition of Eq. (3) requires

$$\left(\frac{\partial}{\partial t_1} + \frac{\hbar}{m}\mathbf{k}_l \cdot \nabla_{\mathbf{r}_1}\right)\Phi_l = 0\,. \tag{8}$$

At the next order $j = 3$, a solvability condition of Eq. (3) results in the closed equations for the envelopes Φ_l:

$$\left[-i\hbar\left(\frac{\partial}{\partial t_2} + \frac{\hbar}{m}\mathbf{k}_1 \cdot \nabla_{\mathbf{r}_2}\right) - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}_1}^2 + V_2\right]\Phi_1$$
$$= -g\big[|\Phi_1|^2\Phi_1 + 2(|\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2)\Phi_1 + 2\Phi_4\Phi_2\Phi_3^* \exp(i\Delta\theta)\big], \tag{9}$$

$$\left[-i\hbar\left(\frac{\partial}{\partial t_2} + \frac{\hbar}{m}\mathbf{k}_2 \cdot \nabla_{\mathbf{r}_2}\right) - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}_1}^2 + V_2\right]\Phi_2$$
$$= -g\big[|\Phi_2|^2\Phi_2 + 2(|\Phi_1|^2 + |\Phi_3|^2 + |\Phi_4|^2)\Phi_2 + 2\Phi_4^*\Phi_3\Phi_1 \exp(-i\Delta\theta)\big], \tag{10}$$

$$\left[-i\hbar\left(\frac{\partial}{\partial t_2} + \frac{\hbar}{m}\mathbf{k}_3 \cdot \nabla_{\mathbf{r}_2}\right) - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}_1}^2 + V_2\right]\Phi_3$$
$$= -g\big[|\Phi_3|^2\Phi_3 + 2(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_4|^2)\Phi_3 + 2\Phi_4\Phi_2\Phi_1^* \exp(i\Delta\theta)\big], \tag{11}$$

$$\left[-i\hbar\left(\frac{\partial}{\partial t_2} + \frac{\hbar}{m}\mathbf{k}_4 \cdot \nabla_{\mathbf{r}_2}\right) - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}_1}^2 + V_2\right]\Phi_4$$
$$= -g\big[|\Phi_4|^2\Phi_4 + 2(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2)\Phi_4 + 2\Phi_2^*\Phi_1\Phi_3 \exp(-i\Delta\theta)\big], \tag{12}$$

with $\Delta\theta = \theta_4 + \theta_2 - \theta_1 - \theta_3 = (\mathbf{k}_4 + \mathbf{k}_2 - \mathbf{k}_1 - \mathbf{k}_3) \cdot \mathbf{r} - (\omega_4 + \omega_2 - \omega_1 - \omega_3)t$, $\Delta\omega = \omega_4 + \omega_2 - \omega_1 - \omega_3 = \epsilon^2\Delta\Omega$, representing a possible phase mismatch in experiment. The left-hand side of these equations describes the motion of the wave packets due to their kinetic and trapping potential energies, and the first fourth terms in the right-hand side describe the effect of the phase matched nonlinear interaction terms, including the contributions from self-phase modulation (denoted by $|\Phi_l|^2\Phi_l$) and cross-phase modulation (denoted by other terms such as $|\Phi_2|^2\Phi_1$, etc.). The last term on the right-hand of each equation is a source term which either creates or destroys atoms in the wave packets. This set of equations describing the FWM of matter waves have been derived using SVEA and solved numerically by Trippenbach *et al.* when the phases are completely matched, i.e. $\Delta\theta = 0$. Wu *et al.* provided an analytical (triangle function) solution by disregarding the contributions from the kinetic and trapping potential energies as well as the self-phase modulation in each equation.[7]

It is difficult to solve the Eqs. (8) and (9)–(12) analytically. However, according to the experiment of Deng *et al.*,[6] the change in space and time of the envelopes Φ_l are actually very slow so these envelopes depend only on the slow variables $\mathbf{r}_2$ and t_2. In this situation, one should use the asymptotic expansion $\Psi(\mathbf{r},t)=\epsilon\Psi^{(1)} + \epsilon^3\Psi^{(3)} + \cdots$ with $\Psi^{(j)} = \Psi^{(j)}(\mathbf{r}, t, \mathbf{r}_2, t_2)$. Then using a similar procedure as deriving Eqs. (9)–(12) we obtain

$$\frac{\partial}{\partial t_2}\Phi_1 + \frac{i}{\hbar}g\big[|\Phi_1|^2\Phi_1 + 2(|\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2)\Phi_1 + 2\Phi_4\Phi_2\Phi_3^* \exp(-i\Delta\Omega t_2)\big] = 0\,, \tag{13}$$

$$\frac{\partial}{\partial t_2}\Phi_2 + \frac{i}{\hbar}g\big[|\Phi_2|^2\Phi_2 + 2(|\Phi_1|^2 + |\Phi_3|^2 + |\Phi_4|^2)\Phi_2 + 2\Phi_4^*\Phi_3\Phi_1 \exp(i\Delta\Omega t_2)\big] = 0\,, \tag{14}$$

$$\frac{\partial}{\partial t_2}\Phi_3 + \frac{i}{\hbar}g\big[|\Phi_3|^2\Phi_3 + 2(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_4|^2)\Phi_3 + 2\Phi_4\Phi_2\Phi_1^* \exp(-i\Delta\Omega t_2)\big] = 0\,, \tag{15}$$

$$\frac{\partial}{\partial t_2}\Phi_4 + \frac{i}{\hbar}g\big[|\Phi_4|^2\Phi_4 + 2(|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2)\Phi_4 + 2\Phi_2^*\Phi_1\Phi_3 \exp(i\Delta\Omega t_2)\big] = 0\,, \tag{16}$$

where the contribution of the trapping potential has been neglected because in the experiment the FWM is observed after turning off the trapping potential.[6]

We now seek the analytic solutions of the nonlinearly coupled envelope Eqs. (13)–(16). In order to simplify the analysis, we make the transformation

$$\Phi_l = E_0 f_l(t_2)\exp(-i\varphi_l(t_2)\,)\,, \tag{17}$$

where $E_0 = \sqrt{\hbar/g}$ (a real constant) and f_l, φ_l ($l = 1,2,3,4$) being real functions. The total particle number $N = \sum_{l=1}^{4} N_l$ where $N_l = \int |\Psi_l|^2 d\mathbf{r}$ denotes the numbers of atoms in each wave packet. Separating the real and imaginary parts of Eqs. (13)–(16) and setting $F_l = \epsilon f_l$ give the equations

$$-F_1\frac{d\varphi_1}{dt} + F_1^3 + 2(F_2^2 + F_3^2 + F_4^2)F_1 + 2F_2F_3F_4\cos\vartheta = 0\,, \tag{18}$$

$$\frac{dF_1}{dt} + 2F_2F_3F_4\sin\vartheta = 0\,, \tag{19}$$

$$-F_2\frac{d\varphi_2}{dt} + F_2^3 + 2(F_1^2 + F_3^2 + F_4^2)F_2 + 2F_1F_3F_4\cos\vartheta = 0\,, \tag{20}$$

$$\frac{dF_2}{dt} - 2F_1F_3F_4\sin\vartheta = 0\,, \tag{21}$$

$$-F_3\frac{d\varphi_3}{dt} + F_3^3 + 2(F_1^2 + F_2^2 + F_4^2)F_3 + 2F_1F_2F_4\cos\vartheta = 0\,, \tag{22}$$

$$\frac{dF_3}{dt} + 2F_1F_2F_4\sin\vartheta = 0\,, \tag{23}$$

$$-F_4\frac{d\varphi_4}{dt} + F_4^3 + 2(F_1^2 + F_2^2 + F_3^2)F_4 + 2F_1F_2F_3\cos\vartheta = 0\,, \tag{24}$$

$$\frac{dF_4}{dt} - 2F_1F_2F_3\sin\vartheta = 0\,, \tag{25}$$

when returning to the original variables, here $\vartheta = \varphi_2 + \varphi_4 - \varphi_1 - \varphi_3 + \Delta\omega t$. It is easy to show that Eqs. (18)–(25) have the following conservation quantities:

$$F_1^2 + F_2^2 = F_1^2(0) + F_2^2(0) = \mu_1, \tag{26}$$

$$F_2^2 + F_3^2 = F_2^2(0) + F_3^2(0) = \mu_2, \tag{27}$$

$$F_2^2 - F_4^2 = F_2^2(0) - F_4^2(0) = \mu_3, \tag{28}$$

$$F_1^2 + F_4^2 = F_1^2(0) + F_4^2(0), \tag{29}$$

$$F_3^2 + F_4^2 = F_3^2(0) + F_4^2(0), \tag{30}$$

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 = F_1^2(0) + F_2^2(0) + F_3^2(0) + F_4^2(0), \tag{31}$$

where $F_l^2 \propto n_l = |\Psi_l|^2$, n_l denotes the number density of the lth wave packet. $F_l^2(0)$ is proportional to the initial number density of lth wave packet. μ_1, μ_2 and μ_3 are constants. Note that Eq. (31) clearly manifests the particle-number conservation required by FWM of matter waves. Combining Eqs. (18), (20), (22) and (24) produces an equation for the temporal evolution on the phase mismatch ϑ,

$$\frac{d\vartheta}{dt} = F_1^2 + F_3^2 - F_2^2 - F_4^2 + \Delta\omega + \frac{\cos\vartheta}{\sin\vartheta}\frac{d}{dt}\ln(F_1F_2F_3F_4)\,, \tag{32}$$

which by integration yields

$$F_1F_2F_3F_4\cos\vartheta = \sigma - \frac{\Delta\omega}{4}F_2^2 + \frac{1}{8}(F_1^4 + F_2^4 + F_3^4 + F_4^4)\,, \tag{33}$$

where σ is an integration constant. Using Eqs. (21) and (33), we obtain the following differential equation

$$\frac{dF_2^2}{dt} = 4\delta\left\{F_1^2F_2^2F_3^2F_4^2 - \left[\sigma - \frac{\Delta\omega}{4}F_2^2 + \frac{1}{8}(F_1^4 + F_2^4 + F_3^4 + F_4^4)\right]^2\right\}^{1/2} \tag{34}$$

with $\delta = \pm 1$. Using Eq. (31) to express F_1^2, F_3^2, F_4^2 in terms of F_2^2 leads to the result

$$4t = \pm\int_{F_2^2(0)}^{F_2^2(t)} \frac{dF_2^2}{[(F_2^2 - \alpha_1)(F_2^2 - \alpha_2)(\alpha_3 - F_2^2)(\alpha_4 - F_2^2)]^{1/2}}\,, \tag{35}$$

where α_1, α_2, α_3, α_4 are the roots of F_2^2 for $Q = 0$, with

$$Q = F_2^2(F_2^2 - \mu_1)(F_2^2 - \mu_2)(F_2^2 - \mu_3) - \left\{\sigma - \frac{\Delta\omega}{4}F_2^2 + \frac{1}{8}[(F_2^2 - \mu_1)^2 + (F_2^2 - \mu_2)^2 + (F_2^2 - \mu_3)^2 + F_2^4]\right\}^2 . \quad (36)$$

For bounded solutions, the roots of Q satisfy $0 \le \alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4$ with $\alpha_2 \le F_2^2(t) \le \alpha_3$, which are also the requirements of an elliptic function. If Eq. (36) stands, one of its root α_1 is zero. Given the solution of α_l $(l = 1, 2, 3, 4)$, Eq. (36) holds through fixing on the parameters of $\sigma, \Delta\omega$. In the general case where α_l are all distinct, the solution for F_2^2 is given by[13]

$$F_2^2(t) = \frac{\alpha_2(\alpha_3 - \alpha_1) - \alpha_1(\alpha_3 - \alpha_2)\mathrm{sn}^2(\lambda, \gamma)}{(\alpha_3 - \alpha_1) - (\alpha_3 - \alpha_2)\mathrm{sn}^2(\lambda, \gamma)}\,, \quad (37)$$

where $\lambda = [(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)]^{1/2}(t - t_0)$, $\gamma = \{[(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_1)]/[(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)]\}^{1/2}$ are the moduli of the Jacobian elliptic function, with $0 \le \gamma \le 1$ and t_0 is a constant. The solution for other $F_l(t)$ expressed by $F_2(t)$ reads

$$F_1^2(t) = F_1^2(0) + F_2^2(0) - F_2^2(t)\,, \quad (38)$$

$$F_3^2(t) = F_2^2(0) + F_3^2(0) - F_2^2(t)\,, \quad (39)$$

$$F_4^2(t) = F_4^2(0) - F_2^2(0) + F_2^2(t)\,. \quad (40)$$

The behavior of $F_l(t)$ for two sets of initial conditions is shown in Figs. 1 and 2, respectively. For convenience we have set $t_0 = 0$ so that $F_2(0) = \sqrt{\alpha_2}$. Initially $(t = 0)$ the condensate is a superposition of three wave packets (i.e. $F_j(0) \neq 0$, $j = 1, 2, 3$) and the fourth does not exists (i.e. $F_4(0) = 0$). Due to the nonlinear coupling and under the phase-matching condition, for $t > 0$ a new wave packet is produced (i.e. $F_4(t) \neq 0$). The growth of the created wave 4 is at the expense of decreasing both wave 1 and wave 3, and at the same time, increasing another wave 2, which can be seen in Eqs. (26)–(28), Figs. 1 and 2. The FWM does not occur when the nonlinear term in GP equation is absent. We see that the redistribution of four matter waves among themselves is restricted by energy, momentum and particle-number conservation laws in the FWM process.

We now discuss the conversion efficiency, N_4/N, of the FWM. For this purpose, we consider a condensate of sodium atoms under the experimental condition of the trap frequencies in the $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions being 84 Hz, 59 Hz and 42 Hz, respectively. The geometric average of the oscillator frequencies for harmonic potential is $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. Under a Thomas–Fermi approximation (TFA), the size of the condensate is given by the TF radius[8]

$$r_{\mathrm{TF}} = \sqrt{2\mu/m\bar{\omega}}\,, \quad (41)$$

where the chemical potential μ is determined by the normalization of the wave function and is given by

$$\mu = \frac{1}{2}\left(\frac{15gN}{4\pi}\right)^{2/5}(m\bar{\omega}^2)^{3/5}\,, \quad (42)$$

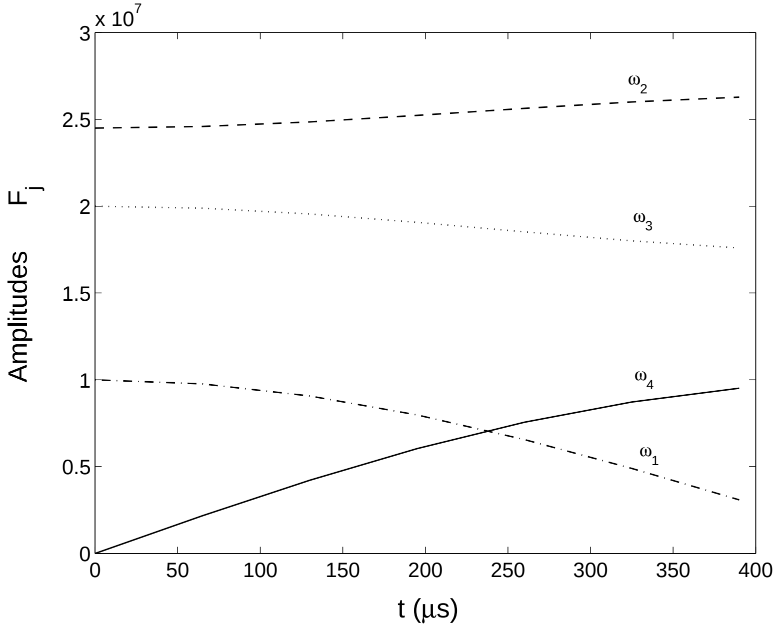

Fig. 1. Plots of the individual envelope amplitudes $F_l(t)$ for $\alpha_1 = 0$, $\alpha_2 = 6\times10^{14}$, $\alpha_3 = 7\times10^{14}$, $\alpha_4 = 9\times10^{14}$, $\gamma = 0.6547$ and $F_1(0) = 10^7$, $F_2(0) = 2.45\times10^7$, $F_3(0) = 2\times10^7$, $F_4(0) = 0$.

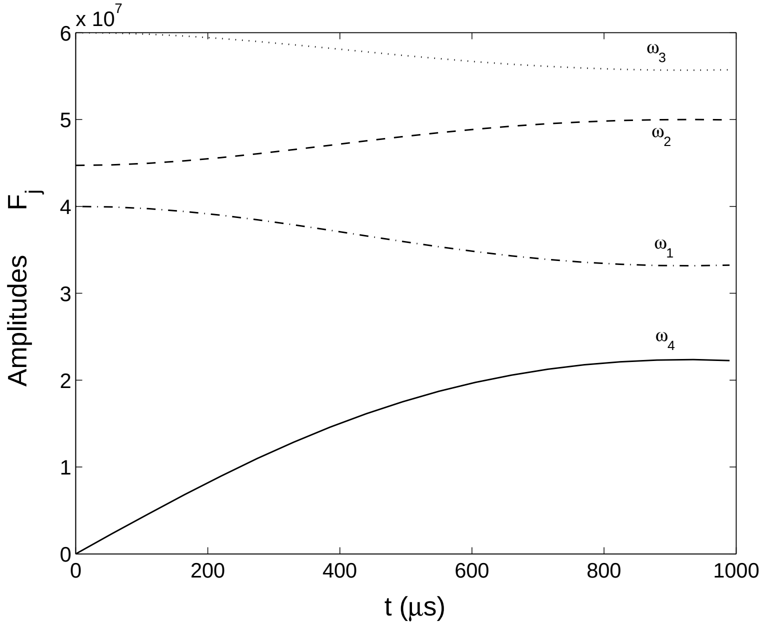

Fig. 2. Plots of the individual envelope amplitudes $F_l(t)$ for $\alpha_1 = 0$, $\alpha_2 = 2\times10^{15}$, $\alpha_3 = 2.5\times10^{15}$, $\alpha_4 = 3\times10^{15}$, $\gamma = 0.7746$ and $F_1(0) = 4\times10^7$, $F_2(0) = 4.47\times10^7$, $F_3(0) = 6\times10^7$, $F_4(0) = 0$.

hence the TF radius r_{TF} scales with N as $N^{1/5}$ and the volume of the condensate is expressed as

$$V = \frac{4\pi}{3} r_{\text{TF}^3} = \frac{4\pi}{3}\left(\frac{15g}{4\pi m}\right)^{3/5} \bar{\omega}^{3/10} N^{3/5} \,. \tag{43}$$

Therefore, V is proportional to $N^{3/5}$, with the proportion coefficient about 10^{-10}. When the initial particle density $n_l^0 = (F_l(0))^2$ $(l = 1, 2, 3)$ are given, the numbers of atom in the three initial wavepackets N_1^0, N_2^0, N_3^0 can be deduced using $N_l^0 \equiv N_l(0) = n_l^0 V$. The total particle number of the system is $N = \sum_{l=1}^{3} N_l^0 = \sum_{l=1}^{4} N_l$, where $N_l = N_l(t)$ is the atom number when the four wavepackets completely separated. For obtaining a maximum conversion efficiency, N_l is chosen by making $F_4^2(t)$ maximum, i.e. the Jacobian elliptic function sn in Eq. (37) taking a zero value. For the case of Fig. 2, we have $V = 3.25 \times 10^{-10}$, $N_1^0 = 5.2 \times 10^5$, $N_2^0 = 6.5 \times 10^5$, $N_3^0 = 11.7 \times 10^5$, $N_1 = 3.575 \times 10^5$, $N_2 = 8.125 \times 10^5$, $N_3 = 10.075 \times 10^5$ and $N_4 = 1.625 \times 10^5$. The conversion efficiency N_4/N is 6.9% with the total number $N = 2.34 \times 10^6$. In case of Fig. 1 we obtain the conversion efficiency of 9.1%. These results manifest the dependence of the conversion efficiency N_4/N on the initial distribution of atoms and the total particle number, agreeing well with the experimental measurement by Deng *et al.*[6]

In conclusion, in this work we have developed a theoretical approach for the FWM with BEC matter waves. We have derived in a systematic way a set of nonlinearly coupled envelope equations describing the FWM starting from the GP equation by using a method of multiple-scales. Comparing with the SVEA the method of multiple-scales makes the "slow variation" of the envelopes more precise and the derivation of the envelope equations more transparent physically and easy to control mathematically. In addition, we have provided an exact and explicit analytical solution of the envelope equations expressed by Jacobian elliptic function. The theoretical results obtained agree well with the experimental ones measured by Deng *et al.*[6]

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References

1. Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).
2. P. Meystre, *Atom Optics* (Springer-Verlag, New York, 2001).
3. S. L. Rolston and W. D. Phillips, *Nature* **416** (2002) 219.
4. J. Denschlag *et al.*, *Science* **287** (2000) 97; K. E. Streckerb *et al.*, *Nature* **417** (2002) 150; L. Khaykovich *et al.*, *Science* **296** (2002) 1290; M. R. Mattews *et al.*, *Phys. Rev. Lett.* **83** (1999) 2498; K. W. Madison *et al.*, *ibid.* **84** (2000) 806; J. R. Abo-Shaeer *et al.*, *Science* **292** (2001) 476.

5. M. Trippenbach *et al.*, *Opt. Express* **3** (1998) 530.
6. L. Deng *et al.*, *Nature* **398** (1999) 218.
7. Y. Wu *et al.*, *Phys. Rev.* **A61** (2000) 043604.
8. M. Trippenbach *et al.*, *Phys. Rev.* **A62** (2000) 023608.
9. J. Heurich *et al.*, *Phys. Rev.* **A63** (2000) 033605; Q. Yang *et al.*, *ibid.* **A67** (2003) 013603.
10. E. Infeld and M. Trippenbach, *Phys. Rev.* **A68** (2003) 015603.
11. A. Jeffery and T. Kawahara, *Asymptotic Methods in Nonlinear Wave Theory* (Pitman, London, 1982).
12. G. Huang *et al.*, *Phys. Rev.* **A64** (2001) 013617; *ibid.* **A65** (2002) 053605; *ibid.* **A67** (2003) 023604.
13. T. J. M. Boyd and J. G. Turner, *J. Math. Phys* **19** (1978) 1403.