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Korteweg de Vries Description of Dark Solitons in Bose–Einstein Condensates *

HUANG Guo-Xiang(黄国翔)
Key Laboratory for Optical and Magnetic Resonance Spectroscopy and Department of Physics, East China Normal University, Shanghai 200062

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We investigate the dynamics of pulses in a cigar-shaped Bose–Einstein condensate with repulsive atom–atom interactions without using Thomas–Fermi approximation. In the linear level our results give the Bogoliubov excitation spectrum for sound propagation with speed $c = c_0/\sqrt{2}$, where c_0 is the speed for the case without a trap. We develop a Korteweg de Vries (KdV) description for dark soliton propagation in the system and show that it is the quantum pressure that contributes the dispersion necessary for the formation of the dark solitons.

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The remarkable experimental realization of the Bose–Einstein condensation of weakly interacting atomic gases[1] has strongly stimulated the exploration of nonlinear properties of matter waves. Nonlinear excitations, such as solitons and vortices, have been observed[2−4] and the four-wave mixing in a Bose–Einstein condensate (BEC) has also been realized recently.[5] These studies support the new field of nonlinear atom optics.

The properties of solitons in homogeneous systems have been intensively studied.[6] However, for a BEC, in addition to an atom–atom interaction all the atoms move in an external trap, which makes the the problem mathematically nontrivial. On the other hand, the inhomogeneity also makes the dynamic behaviour of solitons in the BEC considerably rich. For systems such as rubidium and sodium atomic vapours in the trap, in the case of no external drive force the solitary excitations in the condensate belong to the type of "dark" soliton because the interaction between atoms is repulsive. Many theoretical studies on the dynamics of dark solitons in BECs have appeared.[7−10] Recently, Tsurumi and Wadati proposed an interesting new approach for the soliton propagation in a cigar-shaped BEC with a repulsive atomic interaction.[11] In addition to a Thomas–Fermi (TF) approximation, they also assumed that an "interface" of the condensate exists, and the interface is taken as an elastic cylindrical shell, which is assumed to obey Newton's second law. Under these assumptions, they derived a Korteweg–de Vries (KdV) equation using a reductive perturbation method. Instead of a dark soliton, they obtained a bump (i.e. anti-dark) soliton relative to the background of the condensate. The velocity of the bump soliton is larger than the sound speed.[11] These unexpected results contradict most of the studies on solitons in BECs with repulsive interactions.[7−10] In this letter, we analyse the reason for this contradiction and try to answer this problem by developing another KdV description for the soliton dynamics in BECs with repulsive interactions.

At low temperatures, the dynamic behaviour of interacting Bose gases is described by the time-dependent Gross–Pitaevskii (GP) equation for the order parameter[1]

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\boldsymbol{r}) + g|\Psi|^2\right]\Psi, \qquad (1)$$

where $\int \mathrm{d}\boldsymbol{r}|\Psi|^2 = N$ is the number of atoms in the condensate, $g = 4\pi\hbar^2 a/m$ is the interacting constant with a being the s-wave scattering length ($a > 0$ for repulsive interactions). As in the experiment[4] we consider a cigar-shaped harmonic trap with the elongated axis in the z-direction. Thus we have

$$V_{\rm ext}(\boldsymbol{r}) = \frac{m}{2}[\omega_\perp^2(x^2+y^2) + \omega_z^2 z^2]$$

with

$$\omega_z \ll \omega_\perp,$$

where ω_z and $\omega_\perp$ are the frequencies of the trap in the z-direction and the transverse direction, respectively. Expressing the order parameter in terms of its modulus and phase, $\Psi = \sqrt{n}\exp(\mathrm{i}\phi)$, we obtain a set of coupled equations for n and ϕ. By introducing $(x, y, z) = a_\perp(x', y', z'), t = \omega_\perp^{-1}t', n = n_0 n'$ with $a_\perp = [\hbar/(m\omega_\perp)]^{1/2}$ and $n_0 = N/a_\perp^3$, we have the following dimensionless equations of motion after dropping the primes:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\nabla\phi) = 0, \qquad (2)$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(x^2+y^2) + \frac{1}{2}\left(\frac{\omega_z}{\omega_\perp}\right)^2 z^2 + Qn + \frac{1}{2}\left[(\nabla\phi)^2 - \frac{1}{\sqrt{n}}\nabla^2\sqrt{n}\right] = 0, \qquad (3)$$

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with

$$Q = 4\pi \frac{Na}{a_\perp}, \int \mathrm{d}\boldsymbol{r} n = 1.$$

The last term on the left-hand side of Eq. (3), i.e. $(-\nabla^2\sqrt{n})/[2\sqrt{n}]$, is called the quantum pressure. We can see that it is just the quantum pressure that provides the dispersion of the system, as will be seen below.

Because in the experiment[4] $\omega_z/\omega_\perp$ is small (≈ 0.03), as in Ref. [11] here and later we neglect the third term on the left-hand side in Eq.(3). Then we let $\sqrt{n} = F$ and assume $F = A(z,t)G(x,y)$ and $\phi = -\mu t + \varphi(z,t)$. Then $G(x,y)$ satisfies

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G + \frac{1}{2}(x^2+y^2)G = \nu G. \tag{4}$$

Equation (4) is the well-known eigenvalue problem of a two-dimensional harmonic oscillator in quantum mechanics. Its ground-state solution is $G_0(x,y) = \exp[-(x^2+y^2)/2]$ with the eigenvalue $\nu = \nu_0 = 1$. With these substitutions Eqs.(2) and (3) now are transformed into

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial z}\frac{\partial \varphi}{\partial z} + \frac{1}{2}A\frac{\partial^2\varphi}{\partial z^2} = 0, \tag{5}$$

$$-\frac{1}{2}\frac{\partial^2 A}{\partial z^2} + (-\mu+1)A + \left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right]A + \frac{1}{2}QA^3 = 0. \tag{6}$$

As in Ref.[12], to obtain Eq.(6) we have used Eq.(4) with $G = G_0$ and then multiplied G_0^* to Eq.(3) and then integrated once with respect to x and y to eliminate the dependence on x and y. This procedure is equivalent to taking the system as quasi-one-dimensional.[7,10–13]

Let $A = u_0 + a(z,t)\,(u_0 > 0)$ with $(a,\varphi) = (a_0,\varphi_0)\exp[\mathrm{i}(kz-\omega t)]$+c.c., u_0, a_0 and φ_0 being the constants, we obtain the linear dispersion relation of Eqs.(5) and (6)

$$\omega = \pm\frac{1}{2}k\left[2Qu_0^2 + k^2\right]^{1/2}, \tag{7}$$

where the positive (negative) sign corresponds to the wave propagating to the right (left). We must stress that the k^2-term in the square-root of Eq. (7) comes from the quantum pressure, denoted by the term $-(1/2)\partial^2 A/\partial z^2$ in Eq. (6). Equation (7) is a Bogoliubov-type linear excitation spectrum. We see that, to obtain the Bogoliubov excitation spectrum, the quantum pressure plays an important role. From Eq.(7) we obtain the sound speed of the system

$$c = \frac{\mathrm{d}\omega}{\mathrm{d}k}|_{k=0} = \sqrt{\frac{Q}{2}}u_0.$$

For the case of an homogeneous system (i.e. $V_{\rm ext}(\boldsymbol{r}) = 0$) the corresponding sound speed is $c_0 = \sqrt{Q}u_0$ in our notation. Thus we have $c/c_0 = 1/\sqrt{2}$. The factor $1/\sqrt{2}$ is due to the effect of the transverse confinement of the system. This result is consistent with the experiment for sound propagation.[14]

Now we consider the nonlinear excitations of the system. Using the asymptotic expansion $A = u_0 + \epsilon^2(a^{(0)} + \epsilon^2 a^{(1)} + \cdots), \varphi = \epsilon(\varphi^{(0)} + \epsilon^2\varphi^{(1)} + \cdots)$, and assuming that $a^{(j)}$ and $\varphi^{(j)} (j = 0, 1, \cdots)$ are the functions of the multiple-scale variables $\xi = \epsilon(z-ct)$ and $\tau = \epsilon^3 t$, where ϵ is a smallness parameter characterizing the relative amplitude of the excitation, and then substituting these into Eqs.(5) and (6), we obtain

$$c\frac{\partial a^{(j)}}{\partial\xi} - \frac{1}{2}u_0\frac{\partial^2\varphi^{(j)}}{\partial\xi^2} = \alpha^{(j)}, \tag{8}$$

$$Qu_0^2 a^{(j)} - cu_0\frac{\partial\varphi^{(j)}}{\partial\xi} = \beta^{(j)}, \tag{9}$$

for $j = 0, 1, \cdots$. The concrete expressions of $\alpha^{(j)}$ and $\beta^{(j)}$ are omitted here.

In the leading order ($j = 0$) we obtain $\varphi^{(0)} = (2c/u_0)\int \mathrm{d}\xi a^{(0)}$ with $a^{(0)}$ being a function to be determined. The solvability condition demands $c = \pm\sqrt{Q/2}u_0$. At the next order ($j = 1$), the solvability condition results in the closed equation for $a^{(0)}$:

$$\frac{\partial a^{(0)}}{\partial\tau} + \frac{3c}{u_0}\frac{\partial a^{(0)}}{\partial\xi} - \frac{1}{8c}\frac{\partial^3 a^{(0)}}{\partial\xi^3} = 0. \tag{10}$$

Equation (10) is the KdV equation widely studied in soliton theory.[6] We note that the dispersion term in Eq.(10) (i.e. the term with third-order derivative with respect to ξ) is also due to the contribution of the quantum pressure of the system. Letting $w = \epsilon^2 a^{(0)}$ and using the definition of ξ and τ, we obtain

$$\frac{\partial w}{\partial t} + \frac{3c}{u_0}\frac{w\partial w}{\partial Z} - \frac{1}{8c}\frac{\partial^3 w}{\partial Z^3} = 0$$

with $Z = z - ct$. The single-soliton solution is given by

$$w = -B\,\mathrm{sech}^2\left\{\left(\frac{2c^2B}{u_0}\right)^{1/2} \cdot\left[z - c\left(1-\frac{B}{u_0}\right)t - z_0\right]\right\}, \tag{11}$$

where B is a positive constant reflecting the greyness of the excitation, and z_0 is a constant denoting the initial position of the soliton. Thus, exact to the first order, the order parameter of the condensate takes the form

$$\Psi = u_0\left(1 - \tilde{B}\,\mathrm{sech}^2\{(2c^2\tilde{B})^{1/2}[z - c(1-\tilde{B})t - z_0]\}\right)\exp[\mathrm{i}(-\mu t + \phi)], \tag{12}$$

with

$$\widetilde{B} = \frac{B}{u_0}, \quad \mu = 1 + Q\frac{u_0^2}{2}$$

$$\phi = -(2\tilde{B})^{1/2} \tanh\left\{(2c^2\tilde{B})^{1/2}[z - c(1-\tilde{B})t - z_0]\right\}.$$

Now we make some remarks on the results obtained above: (i) From Eq. (12) we can see that the excitation is a dark soliton, consistent with most of the studies on soliton dynamics in BEC with a repulsive interaction.[4,7−10] (ii) The velocity of the dark soliton is $v_k = c(1-\tilde{B})$, less than the sound speed c of the system. In fact, it is just this property that makes Burger *et al.*[4] identify the dark solitons in their experiment in the BEC. To justify our approach we make a comparison between our theoretical prediction with the experimental result. In the experiment reported by Burger *et al.*,[4] the greyness $\tilde{B}$ of the dark soliton is about 0.4 and the sound speed c is 3.7mm/s. Thus by our theory [see Eq. (12)] the dark soliton velocity should be

$$v_k = c(1-\tilde{B}) = 2.2\,\mathrm{mm/s}, \tag{13}$$

which agrees well with the experimental result by Burger *et al* [see Fig. 3 of Ref. [4]]. (iii) The formation of the dark soliton given in Eq. (12) is due to the balance between the nonlinearity and the dispersion of the system. The dispersion in the KdV equation (10) comes from the quantum pressure term in Eq. (3). Thus we conclude that the quantum pressure is important for the formation of the dark soliton and hence the TF approximation (neglecting the quantum pressure) is generally invalid for soliton dynamics in BECs.

In their work, Tsurumi and Wadati[11] made the following three assumptions for the dynamics of the condensate: (i) the pulse propagation is quasi-one-dimensional; (ii) the TF approximation; and (iii) there is an “interface” for the condensate, which is taken as an elastic cylindrical shell and obeys Newton’s second law. The second assumption (the TF approximation) disregards the quantum pressure, thus the dispersion of the system is neglected. By the third assumption, a new equation for the motion of the elastic shell, not resulting from the GP equation, was introduced. The dispersion in their KdV equation results from this new equation, and has an opposite sign compared with the dispersion resulting from the quantum pressure in the system. This is the reason why they could not arrive at the Bogoliubov excitation spectrum in the linear case and why, in the weakly nonlinear case, they obtained a bump soliton with a propagating velocity larger than the sound speed of the system, thus contradicting with the experiments and most of the existing theoretical results.[4,7−10]

In conclusion, we have investigated the dynamics of nonlinear pulses in a cigar-shaped Bose–Einstein condensate with a repulsive interaction. A KdV description for dark soliton propagation is developed without using TF approximation. Our results show that the quantum pressure is important for the formation of the dark solitons in BEC. For further work we just mention the effect of inhomogeneity in the z-direction and soliton collisions.[15] A detailed study of these problems should be carried out in another work.

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