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Formation and propagation of matched and coupled ultraslow optical soliton pairs in a four-level double-Λ system

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We investigate the simultaneous formation and stable propagation of ultraslow optical soliton pairs in a lifetime broadened four-state atomic system under double-Λ excitation with large one- and two-photon detunings. We show that detrimental probe field distortions due to strong dispersion effects under weak driving conditions can be well balanced by self- and cross-phase modulation effects, leading to a pair of temporal, group velocity, and amplitude matched ultraslow optical solitons of different frequencies.

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Optical solitons [1] describe a class of fascinating shape-preserving propagation phenomena of optical fields in nonlinear media. Such a remarkable propagation effect is the consequence of the interplay between nonlinear effects and dispersion properties of the medium under optical excitations. In general, the glass phase of a solid-state medium (optical fibers, for example) used in most optical soliton generation techniques implies that there is no distinctive energy level structure. Therefore, conventional optical soliton generation can be classified as far off-resonance operation. Such a "passive mode" of operation is one of the main reasons that high powered lasers are required for soliton formation. As a consequence of such a strong excitation, optical solitons produced this way generally travel with group velocities very close to the speed of light in vacuum and hence, require extensive propagation distance to generate.

In the past few years strong index enhancement techniques [2] have been vigorously pursued in the field of nonlinear and quantum optics. One of the main achievements of these techniques is the possibility of enhancing nonlinear excitations with low optical loss. There is ample evidence that these techniques can lead to many interesting physics effects [3]. Recently, we have applied these techniques to investigate [4] the formation and propagation of ultraslow [5] optical solitons in a highly resonant atomic medium. Contrary to the conventional technique where the key is the high local intensity distribution rather than the resonant structure of the medium, we study shape-preserving ultraslow propagations where resonance effects are essential. It is this structural difference and the index enhancement with low optical loss that give rise to the possibilities of ultraslow optical soliton formation and stable propagation under weak optical excitation in a short medium.

In this paper, we report the results of a generation of well-matched [6] ultraslow optical soliton pairs. Specifically, with a four-state double-Λ configuration [7–9], we show the simultaneous formation and stable propagation of a pair of temporal, group-velocity, and amplitude (TAG) matched ultraslow optical solitons in a highly resonant medium under weak excitations. We further show interactions and modulations between two paired and matched ultraslow solitons. We show that with suitable detunings and initial conditions it is possible to achieve fast dynamic switching between members of the pair, which have fundamentally different characteristics. We note that such dynamic switching features have no correspondence in conventional optical soliton generation techniques. It is remarkable that such a rich soliton-soliton interaction can be produced in a small propagation distance of less than 1 cm with driving fields of typically less than a hundred MHz.

We consider a lifetime broadened four-level double-Λ system shown in Fig. 1. In our model, two pulsed probe fields (same pulse length τ at the entrance of the medium) and two continuous wave (cw) control fields complete the respective excitations. Assuming that the probe lasers are weak so that the ground state $|0\rangle$ is not depleted, we obtain the following equations of motion for the atomic response and probe fields ($n=1, 2$);

$$\left(\frac{\partial A_n}{\partial t}\right)_z = id_n A_1 + i\Omega_{cn} A_3 + i\Omega_{pn} A_0, \tag{1a}$$

$$\left(\frac{\partial A_3}{\partial t}\right)_z = id_3 A_3 + i\Omega^*_{c1} A_1 + i\Omega^*_{c2} A_2, \tag{1b}$$

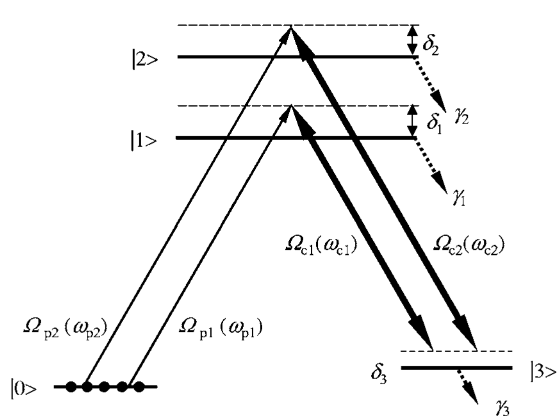

FIG. 1. Energy levels and the excitation scheme of a lifetime broadened four-state atomic system that interacts with two pulsed probe lasers (Rabi frequency $2\Omega_{p1,p2}$) and two continuous wave pump lasers (Rabi frequencies $2\Omega_{c1,c2}$).

$$|A_0|^2 + |A_1|^2 + |A_2|^2 + |A_3|^2 = 1, \tag{1c}$$

$$\left(\frac{\partial \Omega_{pn}}{\partial z}\right)_t = i\kappa_{0n} A_n A_0^*. \tag{1d}$$

Here, we have defined $d_n = \delta_n + i\gamma_n/2$ where δ_n is the detuning of the probe laser (ω_{pn}) from the $|0\rangle \to |n\rangle$ resonance, and γ_n is the decay rate of state $|n\rangle$. A_3 is the amplitude of state $|3\rangle, d_3 = \delta_3 + i\gamma_3/2$ where γ_3 is the decay rate of state $|3\rangle$ and $\delta_3 = \omega_{p1} - \omega_{c1} = \omega_{p2} - \omega_{c2}$ is the two-photon detuning between states $|0\rangle$ and $|3\rangle$. $2\Omega_{pn}$ and $2\Omega_{cn}$ are Rabi frequencies of the probe and control fields for the relevant transitions, and $\kappa_{0n} = 2\pi N \omega_{pn} |D_{0n}|^2 / (\hbar c)$ where N is the concentration and D_{0n} is the dipole moment for the $|0\rangle \to |n\rangle$ transition. For mathematical simplicity, we will assume that $\kappa_{01} = \kappa_{02}$ in the following calculations [10].

To solve Eqs. (1a)–(1d) we first assume $A_j = \Sigma_k A_j^{(k)} \epsilon^k$ where ϵ is a perturbation parameter. Substituting A_j into Eqs. (1a)–(1d) and collecting terms according to the orders of ϵ, we obtain ($n, m = 1, 2; n \neq m$)

$$\alpha_n^{(1)} = -\frac{D_m(\omega)\Lambda_{pn} - \beta_{mn}\Lambda_{pm}}{D(\omega)}, \tag{2a}$$

$$\alpha_3^{(1)} = \frac{\Omega_{c1}^*(d_2 + \omega)\Lambda_{p1} + \Omega_{c2}^*(d_1 + \omega)\Lambda_{p2}}{D(\omega)}, \tag{2b}$$

$$\frac{\partial \Lambda_{pn}}{\partial z} = i\kappa_{0n}\alpha_n^{(1)} A_0^*, \tag{2c}$$

where $D(\omega) = |\Omega_{c1}|^2(d_2 + \omega) + |\Omega_{c2}|^2(d_1 + \omega) - (d_1 + \omega)(d_2 + \omega)(d_3 + \omega)$, $D_n(\omega) = |\Omega_{cn}|^2 - (d_3 + \omega)(d_n + \omega)$, and $\beta_{mn} = \Omega_{cm}^* \Omega_{cn}$. In Eqs. (2a)–(2c), $\alpha_n^{(1)}, \alpha_3^{(1)}$, and Λ_{pn} are the time Fourier transforms of $A_n{}^{(1)}, A_3{}^{(1)}$, and Ω_{pn}, respectively (ω is the time Fourier transform variable). Equation (2) can be solved analytically, yielding [7]

$$\Lambda_{pn} = e^{-i\kappa_{01}\alpha_p z/[2D(\omega)]}\Big[W_-^{(n)} e^{-i\kappa_{01}Lz/[2D(\omega)]} + W_+^{(n)} e^{i\kappa_{01}Lz/[2D(\omega)]}\Big], \tag{3}$$

where we have introduced the following quantities:

$$\alpha_{p,q} = D_1(\omega) \pm D_2(\omega), \quad L = \sqrt{\alpha_q^2 + 4|\beta_{12}|^2},$$

$$W_\pm^{(n)} = \frac{[L \pm (-1)^n \alpha_q]\Lambda_{pn}(0,\omega) \pm 2\beta_{nm}\Lambda_{pm}(0,\omega)}{2L}.$$

We note that the only approximations made so far in obtaining these analytical solutions are an undepleted ground state and slowly varying amplitude approximations. These approximations will always be accurate if the probe fields are sufficiently weak and the laser fields have nearly transform limited bandwidths.

In general, the inverse transform of Eq. (3) requires numerical evaluation.

However, much physical insight can be gained by examining the adiabatic behavior of the process under the assumptions that $\Sigma_{n=1}^2 |\Omega_{cn}\tau|^2 / |\delta_n \tau| \gg 1, |\Omega_{pn}| \ll |\Omega_{cn}|, |\delta_n| \gg \max(|\Omega_{pn}|, \gamma_n)$, and $|d_n\tau| \gg 1$. These conditions ensure that the ground state is undepleted and the adiabatic processes remain effective. The latter requirement is the key for possible analytical solutions to Eqs. (1a)–(1d). Under these conditions the exponents ($\alpha_p \pm L$) can be approximated as linear or quadratic functions of ω. The linear dependence on ω will correctly predict the propagation velocities of the two probe fields, whereas the inclusion of the quadratic terms provides corrections to both the field amplitude and group velocity due to pulse spreading and additional pulse attenuation. We note that the quadratic approximation can be quite accurate even when $\gamma_n \tau$ is relatively large, as with the lowest $S \to P$ transitions in alkali elements. Typically, when the linear or quadratic approximation to $\alpha_p \pm L$ is accurate, it is sufficiently accurate to evaluate the coefficients in the expressions of $W_\pm^{(n)}$ at $\omega = 0$. Under these approximations and assuming that $|\Omega_{cn}|^2 \gg |d_n d_3|$, we obtain

$$W_+^{(n)} = \frac{|\Omega_{cn}|^2}{|\Omega|^2}[\Lambda_{pn}(0,\omega) + S_{mn}^* \Lambda_{pm}(0,\omega)], \tag{4a}$$

$$W_-^{(n)} = \frac{|\Omega_{cm}|^2}{|\Omega|^2}[\Lambda_{pn}(0,\omega) - S_{nm}\Lambda_{pm}(0,\omega)], \tag{4b}$$

where $|\Omega|^2 = |\Omega_{c1}|^2 + |\Omega_{c2}|^2$ and $S_{nm} = \Omega_{cn}/\Omega_{cm}$. It is clear that if we choose the condition of matched inputs, i.e., $\Omega_{p1}(0,t)\Omega_{c2} = \Omega_{p2}(0,t)\Omega_{c1}$ [11,12], then $W_-^{(n)} = 0$. Thus, we have obtained two TAG matched slow waves [6,13]

$$\Lambda_{pn} = W_+^{(n)} e^{i\kappa_{12}(L-\alpha_p)z/[2D(\omega)]}. \tag{5}$$

We note that corrections due to nonadiabatic contributions can be systematically obtained from Eq. (5). These higher order, nonadiabatic contributions play important roles when the control laser Rabi frequencies are significantly reduced, as required for achieving ultraslow group velocity in a conventional three-state electromagnetically induced transparency (EIT) operation [5]. To include these corrections we take

$$-i\frac{\kappa_{01}}{2D(\omega)}(\alpha_p - L) = iK_0 + iK_1\omega + iK_2\omega^2, \tag{6}$$

where $K_0 = \kappa_{01} d_3 / |\Omega|^2$, $K_1 = \kappa_{01}/|\Omega|^2 = 1/V_g$, and $K_2 = -K_1(d_1|\Omega_{c2}|^2 + d_2|\Omega_{c1}|^2)/|\Omega|^4$. In deriving these dispersion coefficients, we have assumed that $|\delta_{pn}| \gg |\delta_3|$. Detailed analysis shows that both K_1 and K_2 contribute to group velocity and probe-pulse spreading. In fact, K_2 introduces a z dependence to both group velocity and probe pulse width in the time domain. These dispersion effects are always observed in the propagation of ultraslow waves in highly resonant media. In the following, we show that self-phase modulation (SPM) and cross-phase modulation (CPM) can be used to balance these detrimental effects, leading to the formation and propagation of a pair of TAG matched ultraslow optical solitons.

In seeking the formation of ultraslow solitons we follow Ref. [1] by taking a trial function of the form $\Omega_{pn}(z,t) = F_{pn}(z,t)\exp(iK_0 z)$ where $F_{pn}(z,t)$ is the slowly varying envelope ($n = 1,2$). Substituting this trial function into Eqs. (1a)–(1d), keeping only resonant contributions, and defining

$\xi = z$ and $\eta = t - z/V_g$, we obtain the following two coupled nonlinear Schrödinger equations for $F_{pn}(z,t)(n,m=1,2;n \neq m)$:

$$i\frac{\partial}{\partial \xi}F_{pn} - K_2\frac{\partial^2}{\partial \eta^2}F_{pn} - G_{nm}F_{pn} = 0, \tag{7a}$$

$$G_{nm} = g_{nn}|F_{pn}|^2 + g_{mn}|F_{pm}|^2 + O(\partial/\partial t), \tag{7b}$$

$$g_{nn} = g_0 d_m B_{mm}, \quad g_{nm} = g_0(d_n B_{mm} - d_m B_{nm}). \tag{7c}$$

Here, $g_0 = 2\kappa_{01}\tau d_3|\Omega_c|^4/(D_0|D_0|^2)$, $D_0 = D(\omega=0)$, $B_{nn} = 1 + |d_n|^2/(2|\Omega_c|^2)$, and $B_{nm} = 1 - d_n d_m^*/(2|\Omega_c|^2)$. In deriving Eqs. (7a)–(7c) we have assumed $\Omega_{c1} = \Omega_{c2} = \Omega_c$ and kept only terms linear in d_3 [14]. We note that function G_{nm} contains contributions due to both SPM and CPM. The signs of these terms are dependent upon the signs of detunings, giving rich dynamics and characteristic (*bright* ⇔ *dark*) switching that are not possible with conventional soliton generation methods [15].

An important note on the nonlinear term G_{nm} is now in order. In deriving G_{nm}, we have neglected contributions due to time derivatives [i.e., $O(\partial/\partial t)$], as in the previous treatment [4]. These contributions, however, can lead to significant propagation effects at large propagation distances. For instance, the first-order time derivative will give an additional group velocity correction, whereas the second-order derivatives will contribute to the wings of the ideal soliton solutions of Eqs. (7a)–(7c), leading to the well-known soliton radiation tails [16,17].

It is readily shown that the coupled Eqs. (7a)–(7c) support solutions of bright-bright and dark-dark soliton pairs under suitable conditions. For a pair of TAG matched ultraslow fundamental bright-bright solitons, we assume $(n,m = 1,2;n \neq m)$

$$F_{pn} = F_{pn}^{(0)}\text{sech}(\eta/T_n)e^{-i\xi P_n|F_{pn}^{(0)}|^2/2}, \tag{8a}$$

$$P_n = \frac{g_{nn}g_{mm} - g_{nm}g_{mn}}{g_{mm} - g_{mn}}, \tag{8b}$$

where sech(x) is the hyperbolic secant function. In Eq. (8) T_n and $F_{pn}^{(0)}$ are arbitrary constants subject only to constrain $|F_{pn}^{(0)}T_n|^2 = 2\,\text{Re}[K_2]/\text{Re}[P_n]$. Substituting Eqs. (8a) and (8b) into Eqs. (7a)–(7c), we find that the following requirements must be simultaneously satisfied:

$$\text{Re}\left[2d_1 - d_2 + \frac{|d_1|^2}{2|\Omega_c|^2}(d_1 - 2d_2)\right] > 0(<0), \tag{9a}$$

$$\text{Re}\left[2d_2 - d_1 + \frac{|d_2|^2}{2|\Omega_c|^2}(d_2 - 2d_1)\right] > 0(<0). \tag{9b}$$

Equations (9a) and (9b) can be simultaneously satisfied because of the assumptions that $|\delta_n| \gg \gamma_n$ and $|\delta_n| \gg |\Omega_c|$. Indeed, our assumptions lead to the coupled envelope equations Eqs. (7a)–(7c) with coefficients having imaginary parts that are much smaller than the corresponding real parts. We have thus shown the formation of a pair of TAG matched and coupled ultraslow fundamental bright-bright solitons in a four-state double-Λ system.

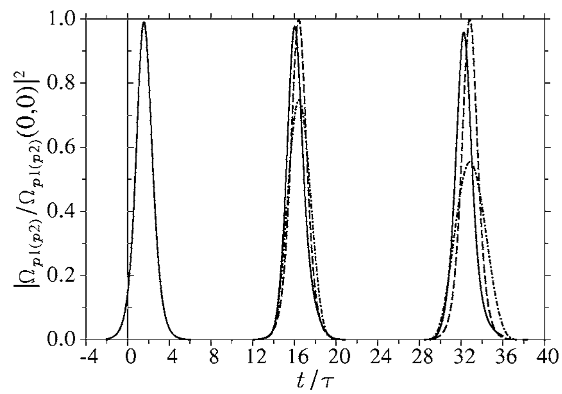

FIG. 2. Comparison of analytical solutions Eqs. (7a)–(7c) (dashed lines) and full numerical solutions of eight simultaneous differential equations without any approximation (solid lines). Parameters: $|\Omega_{p1}(0,0)\tau| \simeq |\Omega_{p2}(0,0)\tau| \simeq 7.84$, $\gamma_1\tau = \gamma_2\tau = 0.5$, $\gamma_3\tau = 10^{-3}$, $\delta_1\tau = \delta_2\tau = -80$, $\delta_3\tau = -1$, $|\Omega_{c1}\tau| = |\Omega_{c2}\tau| = 40$, $\kappa_{01}\tau = \kappa_{02}\tau = 10^5\text{cm}^{-1}$, and $\kappa_{13}\tau = \kappa_{23}\tau = 33333$ cm^{-1}. The dot-dashed lines are numerical solutions without nonlinear terms, which show significant pulse spreading. For $\tau = 10^{-7}$s, we have $V_g \simeq 1.07 \times 10^{-5}c$ and soliton width $z_0 \simeq 3.2 \times 10^{-2}$ cm.

We now use a set of realistic parameters to demonstrate the formation and stable propagation of a pair of TAG matched ultraslow bright-bright solitons. Using typical parameters for cold alkali atoms, we show in Fig. 2 the analytical solutions of Eqs. (7a)–(7c) (dashed lines) and the full numerical solutions (solid lines) obtained by directly integrating eight simultaneous differential equations for four amplitudes and four fields *without* any approximation [18,19]. Each curve contains two indistinguishable traces, representing the perfectly matched pair. We note that the dot-dashed lines are numerical solutions without nonlinear terms (i.e., without SPM and CPM terms). It exhibits severe pulse spreading as expected. Figure 2 clearly validates the significance and importance of our theory. At a longer propagation distance, the cumulative effects by contributions from time derivative terms neglected in calculating G_{nm} give noticeable corrections to both group velocity and pulse shape. Therefore, the analytical solutions and the numerical solutions will gradually separate. However, the solitons of different frequencies in each pair stay perfectly matched as they should.

We finally note that under suitable conditions it is possible to generate a stable soliton pair using only one input, i.e., $\Omega_{p2}(0,0) = 0$, with less efficiency. This generally requires that a multiple-single-channel-induced-transparency based FWM process be operative. In this case, it can be shown [20] that the exponential associated with $W_-^{(n)}$ always decays much faster than the exponential associated with $W_+^{(n)}$. This *differential* decay behavior is the key that leads to multiple destructive interferences, resulting in matched pulse propagation. When this is achieved, one equivalently has two matched inputs deep inside the medium and the soliton formation and propagation follow as described above.

In conclusion, we have shown the formation and stable propagation of a pair of TAG matched and coupled bright-bright ultraslow optical solitons in a four-state atomic system. It is remarkable that by adjusting initial conditions and detunings one can dynamically switch from bright pairs to dark pairs. This is to be contrasted with conventional optical soliton generation techniques where the transition from bright solitons to dark solitons cannot be easily accomplished without changing the working medium. Indeed, highly resonant systems provide a unique and dynamically rich regime, allowing the formation and stable propagation of ultraslow optical solitons in very limited length. This regime is not equivalent to the conventional soliton regime where strong field and extended media are required. In addition to the effect of SPM the present study also demonstrates, TAG matched and coupled ultraslow solitons via CPM. Finally, we point out that the method described here is readily applicable to other excitation schemes such as ladder, cascade and N-type schemes.

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[10] Such as two members of the same hyperfine manifold.

[11] This requires that the two input pulses have the same phase and envelope [see Ref. [12]] at the entrance of the medium. It, however, preexcludes the possibilities of bright-dark or dark-bright soliton pairs. A more generalized theory will be published elsewhere.

[12] For maximum efficiency, the two input pulses should be in-phase. This can be produced using techniques such as a combination of a laser and AOMs, EIT-based FWM [9], and inelastic two- and four-wave mixings [20].

[13] For arbitrary initial pulse shapes Eqs. (4a) and (4b) lead to two pair pulses that travel with separately matched group velocities. See Ref. [7].

[14] Higher order terms in d_3 are important when $|\delta_3\tau| \gg 1$. It leads to higher attenuation of the fields and a different set of coupled nonlinear equations that may support solitary wave solutions.

[15] In the conventional three-state single channel EIT where $\delta_j = 0$ ($j=1, 2, 3$), the SPM coefficient g_{11} is imaginary, thus stable propagation of ultraslow solitons is not supported.

[16] Further studies, using the multiple-scales theory and operator formalism, of the formation and propagation dynamics of other combinations of solitons of different gray levels using this and other schemes in various regimes and under various driving conditions, including radiation damping and diffraction effects will be published elsewhere.

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[19] It is not necessary to start with two exactly matched sech (t/τ) functions. Numerical calculations using two matched Gaussian inputs have shown that the soliton pair formation is completed at $z=0.02$ cm and the shape preservation is well maintained at even $z=1$ cm. The fit of the pulse shape to a sech (t/τ) function at $z=1.0$ cm is better than 95%.

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