Commun. Theor. Phys. (Beijing, China) **41** (2004) pp. 699–706
© International Academic Publishers
Vol. 41, No. 5, May 15, 2004

Three-Wave Resonant Interaction of Bogoliubov Excitations in a Bose–Einstein Condensate with Diffraction*

SUN Chun-Liu and HUANG Guo-Xiang

Department of Physics and Key Laboratory for Optical and Magnetic Resonance Spectroscopy, East China Normal University, Shanghai 200062, China

(Received September 15, 2003)

Abstract *We investigate the three-wave resonant interaction (TWRI) of Bogoliubov excitations in a disk-shaped Bose–Einstein condensate with the diffraction of the excitations taken into account. We show that the phase-matching condition for the TWRI can be satisfied by a suitable selection of the wavevectors and the frequencies of the three exciting modes involved in the TWRI. Using a method of multiple-scales we derive a set of nonlinearly coupled envelope equations describing the TWRI process and give some explicit solitary-wave solutions.*

PACS numbers: 03.75.Kk, 42.65.Ky
Key words: soliton, three-wave resonant, diffraction

1 Introduction

The remarkable experimental realization of Bose–Einstein condensation in weakly interacting atomic gases has opened a new direction for the study on the nonlinear properties of matter waves. The most spectacular experimental progress achieved recently concern the demonstration of atomic four-wave mixing,[1] the discovery of superradiance,[2] the development of matter-wave amplification,[3,4] and the observation of dark and bright solitons as well as vortices in Bose–Einstein condensates (BECs).[5,6] At the same time, an intensive theoretical study in this area has appeared,[7−16] and new phenomena such as atom holography through BEC,[17] coherent matter-wave amplification, and superradiance in degenerate Fermi gases,[18] etc., have been predicted. These researches have enabled the extension of linear atom optics to a nonlinear regime, i.e., nonlinear atom optics,[19] very much like the laser, which leads to the development of nonlinear optics in the 1960s.

The study of collective excitations (or called quasi-particles) is one of the main areas of interest for the research activity in Bose-condensed gases. Much progress has been made on study of the linear collective excitations in Bose–Einstein condensates (BECs).[20] The research on the nonlinear collective excitations in BECs has also attracted much attention. Up to now, the investigation on the nonlinear collective excitations can be classified into two types. One of them is the low-energy excitations with the size the same as that of condensate. The eigen-frequencies of such excitations are discrete, i.e., they are standing wave modes. The nonlinear frequency shift and mode coupling have been explored both theoretically and experimentally.[21−27] One of the interesting works in this aspect is the experimental observation by Hodby *et al.* on the Beliaev coupling of three discrete standing modes of BEC in an anisotropic harmonic trap.[28] The other type of excitations explored are those with the size much smaller than that of the condensate. In this case the excitations have higher energy and their eigen-frequencies are continuous (or quasi-continuous), characterizing the intrinsic bulk property of the condensate.[29] Such collective excitations can propagate a fairly long time before reaching the boundary of the condensate. The most typical nonlinear excitations of such kind explored in BECs are solitary excitations, including dark and bright solitons.[5] Recently, Ozeri *et al.* investigated the mixing of three propagating wave-modes with energy down-conversion in a homogeneous, single-component BEC with a repulsive interatomic interaction and observed the oscillations of excitation numbers between different plane-wave modes involved in the mixing.[30]

In a recent work, we have studied the three-wave resonant interaction (TWRI) in a disk-shaped BEC with a repulsive interatomic interaction and found that three-wave solitons are possible nonlinear excitations of the system.[31] However, the TWRI envelope equations derived and hence the soliton solutions obtained in that work are not valid for the excitations with a narrower width because the effect of diffraction of the excitations has been neglected. In order to consider the effect of diffraction one must include the second spatial derivatives into the TWRI envelope equations. Thus it is possible to find new types of three-wave solitons in the BEC when the diffraction of the excitations is taken into account. It is this topic that will be addressed here. Since the TWRI under consideration is a process of energy up-conversion, at zero temperature such a process can be well described by an order parameter equation, i.e. the Gross–Pitaevskii

*The project supported by National Natural Science Foundation of China under Grant No. 10274021

(GP) equation. We shall show that the three-wave resonance condition for the TWRI can be fulfilled by suitably choosing the wavevectors and frequencies of three exciting modes. Using a method of multiple-scales we derive the nonlinearly coupled envelope equations including diffraction effect and give their explicit soliton solutions.

2 Model and Phase-Matching Condition for TWRI

The dynamics of a weakly interacting Bose gas at zero temperature is described by the time-dependent GP equation,[20]

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}=\Big[-\frac{\hbar^2}{2m}\nabla^2+V_{\rm ext}(\boldsymbol{r})+g|\Psi|^2\Big]\Psi\,,\qquad(1)$$

where Ψ is an order parameter, $\int \mathrm{d}\boldsymbol{r}|\Psi|^2=N$ is the atomic number in the condensate, $g=4\pi\hbar^2a_s/m$ is the interaction constant with a_s the s-wave scattering length ($a_s>0$ for a repulsive interaction). We consider a disk-shaped harmonic trap of the form $V_{\rm ext}(\boldsymbol{r})=m[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2]/2$ with $\omega_\perp\ll\omega_z$, where $\omega_\perp$ and ω_z are the frequencies of the trap in the transverse (x and y) and axial (z) directions, respectively. Expressing the order parameter in terms of its modulus and phase, $\Psi=\sqrt{n}\exp(\mathrm{i}\phi)$, we get a set of coupled equations for n and ϕ. By introducing $(x,y,z)=a_z(x',y',z')$, $t=\omega_z^{-1}t'$, $n=n_0n'$ with $a_z=[\hbar/(m\omega_z)]^{1/2}$ and $n_0=N/a_z^3$, we obtain the following dimensionless equations of motion after dropping the primes

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\nabla\phi)=0\,,\qquad(2)$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}z^2+V_{\|}(x,y)+Qn+\frac{1}{2}\Big[(\nabla\phi)^2-\frac{1}{\sqrt{n}}\nabla^2\sqrt{n}\Big]=0\qquad(3)$$

with $Q=4\pi Na_s/a_z$ and $\int\mathrm{d}\boldsymbol{r}n=1$.

$$V_{\|}(x,y)=(\omega_\perp/\omega_z)^2(x^2+y^2)/2$$

is the dimensionless trapping potential in the x and y directions. Since $\omega_\perp/\omega_z$ is very small, we can neglect $V_{\|}$ in Eq. (3), i.e., the excitations can propagate only within the disk plane due to the strong confinement in the z direction. In this situation we can make the quasi-2D approximation,[14] $\sqrt{n}=P(x,y,t)G_0(z)$, $\phi=-\mu t+\varphi(x,y,t)$, where $G_0(z)=\exp(-z^2/2)$ is the ground-state wave function of the 1D harmonic oscillator with the harmonic oscillator with the harmonic potential $z^2/2$ in the z-direction, μ is the chemical potential of the condensate and φ is a phase function contributed from the excitation, which is assumed to be a function of x and y because as mentioned above the generated excitation can propagate only in the x and y directions. Then equations (2) and (3) are reduced to

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial P}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{P}{2}\Big(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\Big)=0\,,\qquad(4)$$

$$-\frac{1}{2}\Big(\frac{\partial^2P}{\partial x^2}+\frac{\partial^2P}{\partial y^2}\Big)-\Big(\mu-\frac{1}{2}\Big)P+\Big[\frac{\partial\varphi}{\partial t}+\frac{1}{2}\Big(\frac{\partial\varphi}{\partial x}\Big)^2+\frac{1}{2}\Big(\frac{\partial\varphi}{\partial y}\Big)^2\Big]P+Q'P^3=0\,,\qquad(5)$$

where $Q'=I_0Q$ is an effective interaction constant with

$$I_0=\int_{-\infty}^{\infty}\mathrm{d}zG_0^4(z)/\int_{-\infty}^{\infty}\mathrm{d}zG_0^2(z)=1/\sqrt{2}\,.$$

In principle, one can take into account the contribution of the higher-order eigen-modes of the harmonic oscillator in the z-direction, as done in Ref. [15] for a cigar-shaped trap. However, as here we have assumed $n_0g\ll\hbar\omega_z$, the contribution from these higher-order eigen-modes is small and can thus be safely neglected. On the other hand for the thin disk-shaped trap ($\omega_\perp/\omega_z\ll1$) the trapping potential in the (x,y) plane is a slowly-varying function of x and y and hence the size of the condensate in the radial direction is much larger than the size of the excitations (with the order of the healing length) considered below. In the propagation of the excitations for short times, the boundary of the condensate does not come into play and we can therefore take the condensate as uniform in the (x,y) plane (i.e. neglecting the affect from $V_{\|}(x,y)$). The effect of the condensate boundary will be considered elsewhere.

The linear dispersion relation of an excitation can be obtained by assuming in Eqs. (4) and (5)

$$P=u_0+a(x,y,t)\quad(u_0>0)\,.$$

Here $(a,\varphi)=(a_0,\varphi_0)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]+\text{c.c.}$ with $\boldsymbol{k}=(k_x,k_y)$, and $\boldsymbol{r}=(x,y)$ (u_0,a_0 and φ_0 are constants). The result reads

$$\omega(\boldsymbol{k})=\frac{1}{2}k(4c^2+k^2)^{1/2}\,,\qquad(6)$$

where $k^2=k_x^2+k_y^2$, and $c=\sqrt{Q'}u_0$ is the sound speed of the system. Equation (6) is a Bogoliubov-type excitation spectrum in two dimensions. A precise measurement for such an excitation spectrum in BEC has been done by Steinhauer *et al.* recently.[29]

We are interested in a TWRI among three collective modes $(\boldsymbol{k}_j,\omega_j)$ $(j=1,2,3)$ excited in the background of the condensate. One of necessary conditions for the TWRI is the phase-matching condition,

$$\omega_1+\omega_2=\omega_3\,,\qquad(7)$$

$$\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3\,,\qquad(8)$$

where $\omega_j\equiv\omega(\boldsymbol{k}_j)$. From Eq. (6) it is easy to show that these conditions can be satisfied if we select

$$\boldsymbol{k}_1=(k_1\cos\vartheta,k_1\sin\vartheta)\,,\qquad(9)$$

$$\boldsymbol{k}_3 = (k_3, 0)\,, \tag{10}$$

$$\boldsymbol{k}_2 = \boldsymbol{k}_3 - \boldsymbol{k}_1 = (k_3 - k_1\cos\vartheta, -k_1\sin\vartheta)\,, \tag{11}$$

where the angle ϑ satisfies the following relation,

$$\cos\vartheta = \frac{1}{2k_3k_1}\Big\{k_3^2 + k_1^2 + 2c^2 - 2\Big[c^4 + \Big(k_3\sqrt{c^2+\frac{1}{4}k_3^2} - k_1\sqrt{c^2+\frac{1}{4}k_1^2}\Big)^2\Big]^{1/2}\Big\}\,. \tag{12}$$

Equation (12) has solution for any non-vanishing k_1 and k_3. Another necessary condition for the TWRI is that the nonlinearity describing the interaction between the collective modes must have a quadratic component, similar to a $\chi^{(2)}$ nonlinearity in a nonlinear optical medium. From Eqs. (4) and (5) we see that the equations describing the excitations $(P-u_0, \varphi)$ are of not only quadratic but also cubic nonlinearities. Consequently, in the disk-shaped condensate a TWRI of collective excitation is indeed possible if the angle ϑ is chosen according to Eq. (12).

3 Envelope Equations for TWRI with Diffraction

Let us now derive the nonlinear envelope equations for the TWRI with the diffraction of the excitations taken into account. We employ the asymptotic expansion

$$P = u_0 + \epsilon(a^{(1)} + \epsilon^{1/2}a^{(2)} + \epsilon a^{(3)} + \cdots)\,, \tag{13}$$

$$\varphi = \epsilon(\varphi^{(1)} + \epsilon^{1/2}\varphi^{(2)} + \epsilon\varphi^{(3)} + \cdots)\,, \tag{14}$$

where ϵ is a small parameter characterizing the relative amplitude of the excitation, and $a^{(j)}$, $\varphi^{(j)}$ $(j=1,2,3,\ldots)$ are the functions of the multiple-scale variables x, y, t, $\epsilon^{1/2}x$, $\epsilon^{1/2}y$, ϵx, and ϵy. Substituting the above expansion to Eqs. (4) and (5) we obtain

$$\frac{\partial a^{(j)}}{\partial t} + \frac{1}{2}u_0\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\varphi^{(j)} = \alpha^{(j)}\,, \tag{15}$$

$$-\frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)a^{(j)} + 2c^2a^{(j)} + u_0\frac{\partial}{\partial t}\varphi^{(j)} = \beta^{(j)}\,, \tag{16}$$

$j=1,2,3,\ldots$ The explicit expressions of $\alpha^{(j)}$ and $\beta^{(j)}$ are omitted here.

At leading order $(j=1)$, the solution reads

$$\varphi^{(1)} = A_0 + (A\exp(\mathrm{i}\theta) + \mathrm{c.c.})\,, \tag{17}$$

$$a^{(1)} = \frac{\mathrm{i}}{2}\frac{u_0k^2}{\omega}A\exp(\mathrm{i}\theta) + \mathrm{c.c.}\,, \tag{18}$$

where A_0 (real) and A (complex) are yet to be determined envelope functions of the slow variables $\epsilon^{1/2}x$, $\epsilon^{1/2}y$, ϵx, and ϵy introduced necessarily to eliminate the secular terms appearing in the higher-order approximations. $\theta = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t$ with $\omega(\boldsymbol{k})$ being just the linear dispersion relation given by Eq. (6), and c.c. represents a corresponding complex conjugate term.

In the process of the TWRI, three wave modes are involved and hence the leading-order solution should be taken as

$$\varphi^{(1)} = A_0 + \sum_{j=1}^{3}(A_j\exp(\mathrm{i}\theta_j) + \mathrm{c.c.})\,, \tag{19}$$

$$a^{(1)} = \sum_{j=1}^{3}B_j\exp(i\theta_j) + \mathrm{c.c.}\,, \tag{20}$$

where $B_j = (\mathrm{i}u_0k_j^2/2\omega_j)A_j$, $\theta_j = \boldsymbol{k}_j\cdot\boldsymbol{r} - \omega_j t$, and B_j is the envelope of the j-th wave mode. $\boldsymbol{k}_j$ and ω_j $(j=1,2,3)$ are chosen according to the TWRI phase-matching conditions (7) and (8). At the second order $(j=2)$ the solvability conditions of Eqs. (15) and (16) result in

$$v_{1x}\frac{\partial B_1}{\partial x_1} + v_{1y}\frac{\partial B_1}{\partial y_1} = 0\,, \tag{21}$$

$$v_{2x}\frac{\partial B_2}{\partial x_1} + v_{2y}\frac{\partial B_2}{\partial y_1} = 0\,, \tag{22}$$

$$v_{3x}\frac{\partial B_3}{\partial x_1} + v_{3y}\frac{\partial B_3}{\partial y_1} = 0\,, \tag{23}$$

where v_{jx} and v_{jy} $(j=1,2,3)$ are the components of the group velocity of the j-th wave modes, $v_j = \nabla_{k_j}\omega_j$.

At the third order $(j=3)$ the solvability conditions of Eqs. (15) and (16) give rise to the closed equations governing the evolution of the envelopes B_j. After making the transformation $b_j = \epsilon B_j$ $(j=1,2,3)$ and using Eqs. (21) $\sim$ (23), these equations take the forms

$$\mathrm{i}\Big(\frac{\partial b_1}{\partial x} + \frac{v_{1y}}{v_{1x}}\frac{\partial b_1}{\partial y}\Big) - g_1\frac{\partial^2 b_1}{\partial y^2} + \sigma_1 b_2^* b_3\,\mathrm{e}^{\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{x}} = 0\,, \tag{24}$$

$$\mathrm{i}\Big(\frac{\partial b_2}{\partial x} + \frac{v_{2y}}{v_{2x}}\frac{\partial b_2}{\partial y}\Big) - g_2\frac{\partial^2 b_2}{\partial y^2} + \sigma_2 b_1^* b_3\,\mathrm{e}^{\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{x}} = 0\,, \tag{25}$$

$$\mathrm{i}\Big(\frac{\partial b_3}{\partial x} + \frac{v_{3y}}{v_{3x}}\frac{\partial b_3}{\partial y}\Big) - g_3\frac{\partial^2 b_3}{\partial y^2} + \sigma_3 b_1 b_2\,\mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{x}} = 0\,, \tag{26}$$

when returning to the original variables. Equations (24) $\sim$ (26) describe the evolution of the wave envelopes for the TWRI. Different from the result in Ref. [31], here the diffraction effect of each wave mode has been included, reflected by the second-order derivative terms appearing in the envelope equations. The coefficients of the diffractive terms are given by $g_j = -(1 + v_{jy}^2/v_{jx}^2)/(2k_{jx})$. The explicit expressions for the nonlinear coefficients σ_j are omitted here. A possible phase mismatch for the TWRI has also been considered, which is denoted by the phase factor $\Delta\boldsymbol{k}\cdot\boldsymbol{x}$ with $\Delta\boldsymbol{k} = \boldsymbol{k}_3 - \boldsymbol{k}_1 - \boldsymbol{k}_2$.

4 Exact Soliton Solutions of TWRI Equations

We now present some exact soliton solutions of Eqs. (24) $\sim$ (26). For simplicity, we assume the phase-mismatch $\Delta\boldsymbol{k}$ only has x component Δk_x, i.e. the y component $\Delta k_y = 0$. We set

$$b_1 = \sqrt{\sigma_1\omega_1/(\sigma_3\omega_3)}F_1\,,$$
$$b_2 = \sqrt{\sigma_2\omega_3/(\sigma_3\omega_2)}F_2\,,$$

and $b_3 = F_3$. Then equations (24) $\sim$ (26) are transformed into the following forms

$$\mathrm{i}\Big(\frac{\partial F_1}{\partial x} + \frac{v_{1y}}{v_{1x}}\frac{\partial F_1}{\partial y}\Big) - g_1\frac{\partial^2 F_1}{\partial y^2} + \sigma F_2^* F_3\,\mathrm{e}^{\mathrm{i}\Delta k_x x} = 0\,, \quad (27)$$

$$\mathrm{i}\Big(\frac{\partial F_2}{\partial x} + \frac{v_{2y}}{v_{2x}}\frac{\partial F_2}{\partial y}\Big) - g_2\frac{\partial^2 F_2}{\partial y^2} + \sigma\frac{\omega_2}{\omega_1}F_1^* F_3\,\mathrm{e}^{\mathrm{i}\Delta k_x x} = 0\,, \quad (28)$$

$$\mathrm{i}\Big(\frac{\partial F_3}{\partial x} + \frac{v_{3y}}{v_{3x}}\frac{\partial F_3}{\partial y}\Big) - g_3\frac{\partial^2 F_3}{\partial y^2} + \sigma\frac{\omega_3}{\omega_1}F_1 F_2\,\mathrm{e}^{-\mathrm{i}\Delta k_x x} = 0\,. \quad (29)$$

To solve the above equations we assume[32]

$$F_j = F_{j0}U_j(\zeta)\exp(\mathrm{i}\zeta_j)$$

with

$$\zeta = \Omega s - K\xi\,,$$
$$\zeta_j = K_j\xi - \Omega_j s \quad (j = 1,2,3)\,,$$
$$s = T_0^{-1}(y - v_{1y}/v_{1x}x)\,,$$
$$\xi = x|g_1|/T_0^2\,,$$

where T_0 denotes the soliton width. Substituting this ansatz into Eqs. (27) $\sim$ (29) we obtain a set of equations for U_j with the relations

$$K_3 = K_1 + K_2 + \beta\,, \quad (30)$$
$$\Omega_3 = \Omega_1 + \Omega_2\,, \quad (31)$$
$$K/\Omega = \alpha_1\Omega_1 = \alpha_2\Omega_2 - \gamma_2 = \alpha_3\Omega_3 - \gamma_3\,, \quad (32)$$

where

$$\alpha_1 = 2\,\mathrm{sgn}(g_1)\,, \qquad \alpha_2 = 2g_2/|g_1|\,,$$
$$\alpha_3 = 2g_3/|g_1|\,, \qquad L_D = T_0^2/|g_1|\,,$$
$$\beta = -(\Delta k_x)L_D\,,$$
$$\gamma_2 = -(L_D/T_0)(v_{2y}/v_{2x} - v_{1y}/v_{1x})\,,$$
$$\gamma_3 = -(L_D/T_0)(v_{3y}/v_{3x} - v_{1y}/v_{1x})\,.$$

Then we get the following three types of soliton solutions.

Type 1

If α_1, α_2, and α_3 have the same sign, the solutions of Eqs. (27) $\sim$ (29) take the form

$$F_1 = 3\frac{\omega_1|g_1|}{\sigma T_0^2\sqrt{\omega_2\omega_3}}\delta_1\sqrt{\alpha_2\alpha_3}\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)]\,, \quad (33)$$

$$F_2 = 3\frac{\sqrt{\omega_1/\omega_3}|g_1|}{\sigma T_0^2}\delta_2\sqrt{\alpha_1\alpha_3}\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)]\,, \quad (34)$$

$$F_3 = -3\frac{\sqrt{\omega_1/\omega_2}|g_1|}{\sigma T_0^2}\delta_1\delta_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)]\,, \quad (35)$$

where

$$\delta_j = \pm 1 \quad (j = 1,2)\,, \qquad K = \alpha_1\Omega\Omega_1\,, \qquad K_1 = \alpha_1\Omega_1^2/2 - 2\alpha_1\Omega^2\,, \qquad K_2 = \alpha_2\Omega_2^2/2 - \gamma_2\Omega_2 - 2\alpha_2\Omega^2\,,$$
$$\Omega^2 = [\alpha_3\Omega_3^2 - \alpha_1\Omega_1^2 - \alpha_2\Omega_2^2 + 2(\gamma_2\Omega_2 - \gamma_3\Omega_3 - \beta)]/4(\alpha_3 - \alpha_1 - \alpha_2)\,.$$

The parameters Ω_j $(j = 1,2,3)$ are determined by Eqs. (31) and (32). From Eqs. (33) $\sim$ (35) we see that the three wave modes are simultaneously one-hump spatial solitons with the same central position. The physical reason for the formation of such simultaneous solitons is due to the mutual self-trapping through a cascading process between different excitation modes.

Shown in Fig. 1 is the modulus of the order parameter in the $z = 0$ plane, i.e.

$$|\Psi| = u_0 + \epsilon a^{(1)} = u_0 + P_1 + P_2 + P_3\,,$$

where u_0 is the condensate background and P_j $(j = 1,2,3)$ is the amplitude of the j-th wave mode with

$$P_j = b_j\exp(\mathrm{i}\theta_j) + \mathrm{c.c}\,.$$

The parameters are chosen as $u_0 = 1.0$, $c = 1.0$, $k_1 = 0.2$, $k_3 = 0.4$, $\Delta k_x = 0.01$, $T_0 = 1.0$, and $s_j = 1.0$ $(j = 1,2,3)$ at time $t = 1.0$.

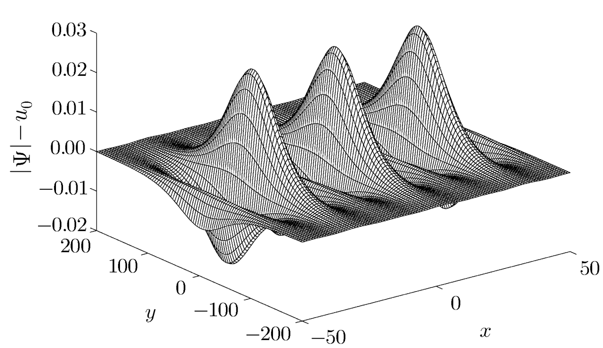

Fig. 1 The quality $|\Psi| - u_0$ in the case of the three-wave soliton excitation with diffraction, given by the solutions (33) $\sim$ (35). The parameters are chosen as $u_0 = 1.0$, $c = 1.0$, $k_1 = 0.2$, $k_3 = 0.4$, $\Delta k_x = 0.01$, $T_0 = 1.0$, $s_j = 1.0$ $(j = 1,2,3)$ at time $t = 1.0$.

For illustrating clearly, the cross section of each wave mode for different positions x is shown in Fig. 2.

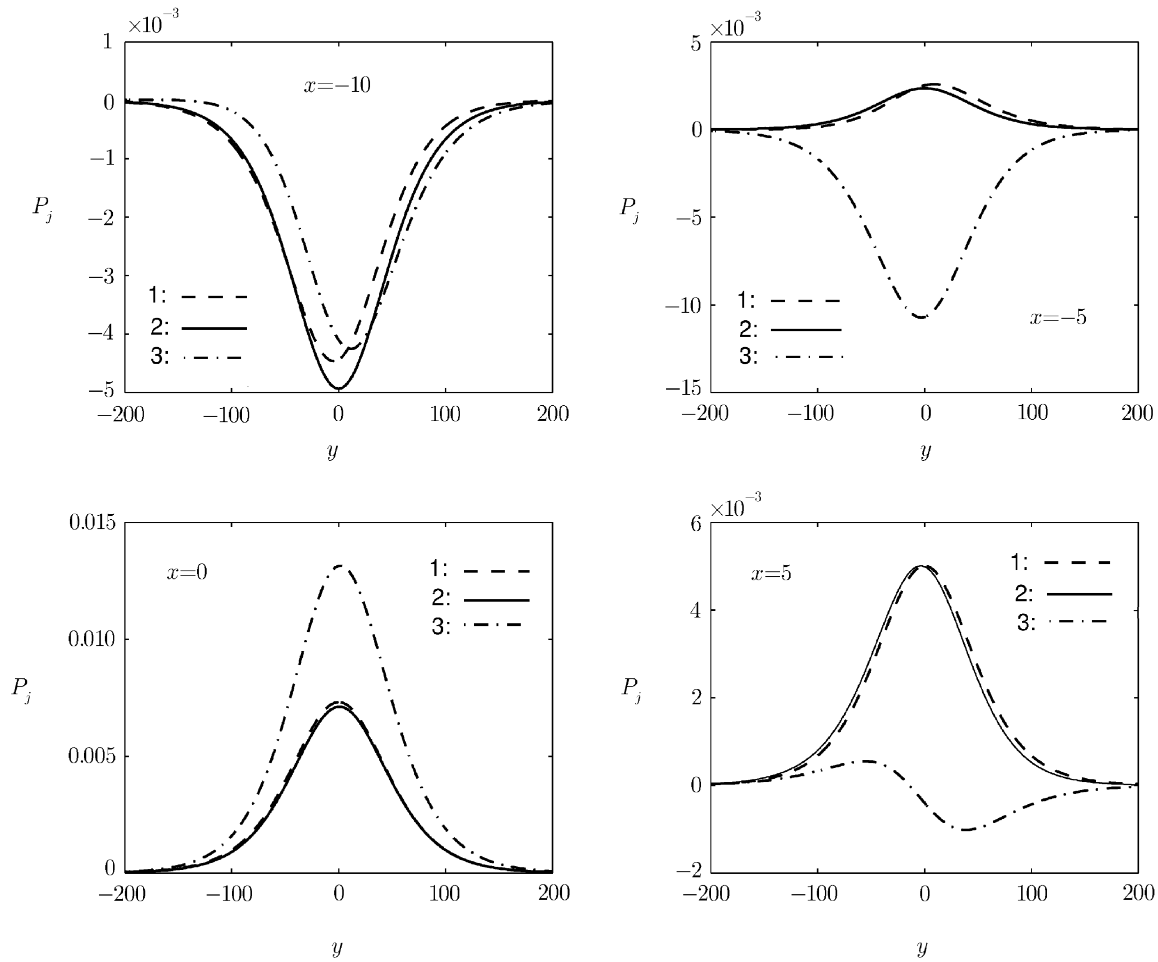

Fig. 2 The amplitudes of the three modes P_j $(j = 1, 2, 3)$ for $x = -10.0$, $x = -5.0$, $x = 0.0$, and $x = 5.0$, respectively. The parameters are the same as used in Fig. 1.

Type 2

The case $\mathrm{sign}(\alpha_1) = \mathrm{sign}(\alpha_2) = \mathrm{sign}(\alpha_3)$ also gives rise to the solutions

$$F_1 = -3\frac{\omega_1|g_1|}{\sigma T_0^2\sqrt{\omega_2\omega_3}}\delta_1\sqrt{\alpha_2\alpha_3}\Omega^2\Big[\frac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)\Big]\exp[\mathrm{i}(K_1\xi - \Omega_1 s)]\,, \tag{36}$$

$$F_2 = -3\frac{\sqrt{\omega_1\omega_3}|g_1|}{\sigma T_0^2}\delta_2\sqrt{\alpha_1\alpha_3}\Omega^2\Big[\frac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)\Big]\exp[\mathrm{i}(K_2\xi - \Omega_2 s)]\,, \tag{37}$$

$$F_3 = 3\frac{\sqrt{\omega_1/\omega_2}|g_1|}{\sigma T_0^2}\delta_1\delta_2\sqrt{\alpha_1\alpha_2}\mathrm{sgn}(\alpha_3)\Omega^2\Big[\frac{2}{3} - \mathrm{sech}^2(\Omega s - K\xi)\Big]\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \tag{38}$$

with

$$K = \alpha_1\Omega\Omega_1\,, \qquad K_1 = \frac{\alpha_1}{2}\Omega_1^2 + 2\alpha_1\Omega^2\,, \qquad K_2 = \frac{\alpha_2}{2}\Omega_2^2 - \gamma_2\Omega_2 + 2\alpha_2\Omega^2\,,$$
$$\Omega^2 = \frac{-\alpha_3\Omega_3^2 + \alpha_1\Omega_1^2 + \alpha_2\Omega_2^2 + 2(-\gamma_2\Omega_2 + \gamma_3\Omega_3 + \beta)}{4(\alpha_3 - \alpha_1 - \alpha_2)}\,.$$

The solutions (36) $\sim$ (38) represent three dark solitons. Figure 3 shows the cross sections in different positions y for the amplitudes of the three wave modes at time $t = 1.0$. We see that the intensity of each of the wave modes has two minima.

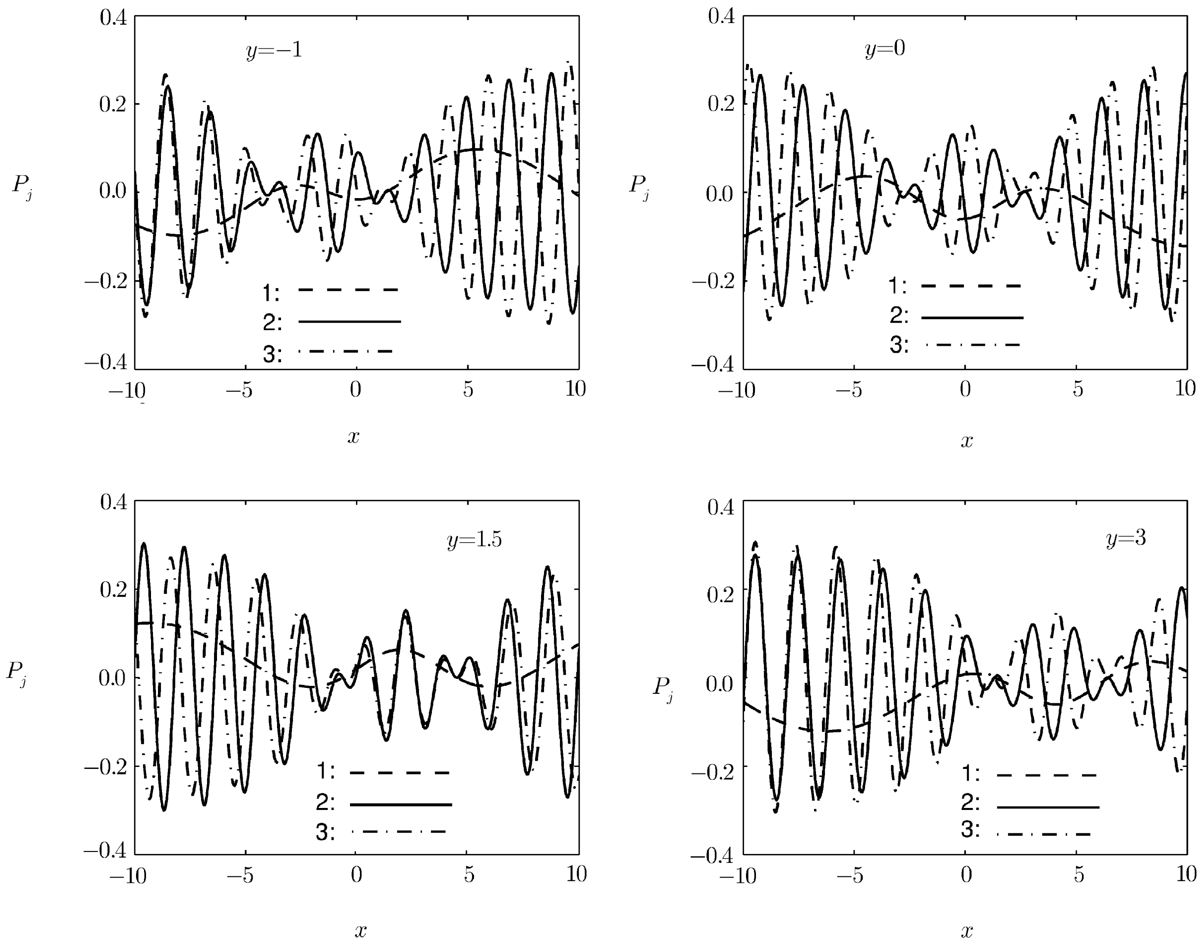

Fig. 3 The amplitudes of the three modes P_j $(j = 1, 2, 3)$ at $y = -1.0$, $y = 0.0$, $y = 1.5$, $y = 3.0$, respectively, given by the solutions (36) $\sim$ (38). The parameters are chosen as $u_0 = 1.0$, $c = 1.0$ $k_1 = 2.0$, $k_3 = 5.0$, $\Delta k_x = 0.1$, $T_0 = 1.0$, $s_j = 1.0$ $(j = 1, 2, 3)$ at time $t = 1.0$.

Type 3

If $\alpha_1\alpha_2 < 0$ and $\alpha_2\alpha_3 > 0$, one has the solutions

$$F_1 = -3\frac{\omega_1|g_1|}{\sigma T_0^2\sqrt{\omega_2\omega_3}}\delta_1\delta_2\sqrt{\alpha_2\alpha_3}\mathrm{sgn}(\alpha_3)\Omega^2\mathrm{sech}^2(\Omega s - K\xi)\exp[\mathrm{i}(K_1\xi - \Omega_1 s)]\,, \tag{39}$$

$$F_2 = 3\frac{\sqrt{\omega_1/\omega_3}|g_1|}{\sigma T_0^2}\mathrm{i}\delta_1\sqrt{-\alpha_1\alpha_3}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_2\xi - \Omega_2 s)]\,, \tag{40}$$

$$F_3 = 3\frac{\sqrt{\omega_1/\omega_2}|g_1|}{\sigma T_0^2}\mathrm{i}\delta_2\sqrt{-\alpha_1\alpha_2}\Omega^2\mathrm{sech}(\Omega s - K\xi)\tanh(\Omega s - K\xi)\exp[\mathrm{i}(K_3\xi - \Omega_3 s)] \tag{41}$$

with

$$K = \alpha_1\Omega\Omega_1\,, \qquad K_1 = \frac{\alpha_1}{2}\Omega_1^2 + \alpha_1\Omega^2\,, \qquad K_2 = \frac{\alpha_2}{2}(\Omega_2^2 - \Omega^2) - \gamma_2\Omega_2\,,$$

$$\Omega^2 = \frac{\alpha_3\Omega_3^2 - \alpha_1\Omega_1^2 - \alpha_2\Omega_2^2 + 2(\gamma_2\Omega_2 - \gamma_3\Omega_3 - \beta)}{2\alpha_1 - \alpha_2 + \alpha_3}\,.$$

From Eqs. (39) $\sim$ (41) we see that one fundamental wave mode is a two-hump soliton, but another fundamental wave mode and the harmonic wave mode are two-hump solitons, different from the soliton solutions in Types 1 and 2, see

Fig. 4. Obviously, other types of three-wave soliton solutions are also possible for Eqs. (27) $\sim$ (29), which are not presented here.

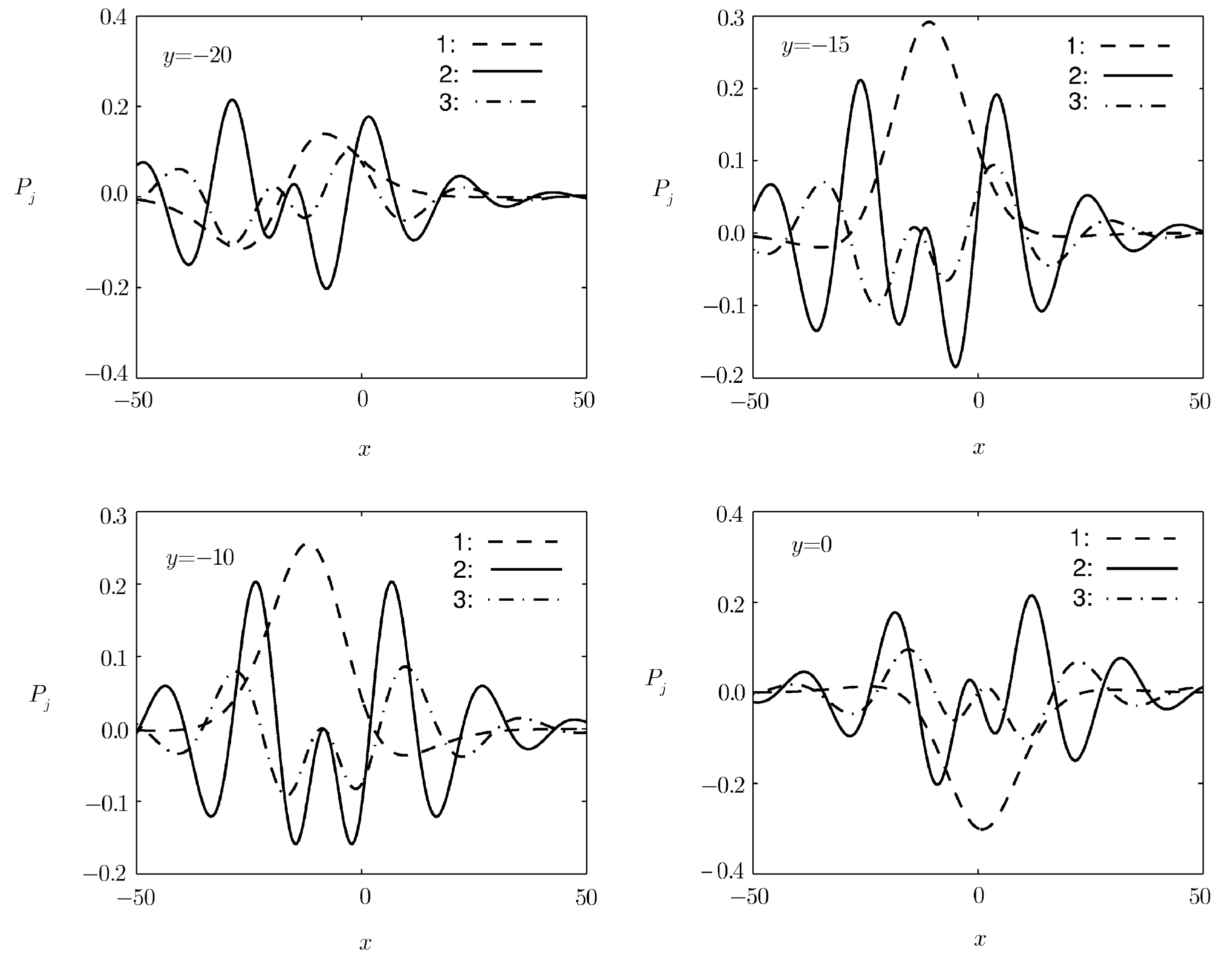

Fig. 4 The amplitudes of the three wave modes P_j ($j = 1, 2, 3$) at different positions $y = -20.0$, $y = -15.0$, $y = -10.0$, and $y = 0.0$, respectively, given by the solutions (39) $\sim$ (41). The parameters are chosen as $u_0 = 1.0$, $c = 1.0$, $k_1 = 0.2$, $k_3 = -0.8$, $\Delta k_x = 0.01$, $T_0 = 1.0$, $s_j = 1.0$ ($j = 1, 2, 3$) at time $t = 1.0$.

5 Discussion and Summary

We have investigated the TWRI of Bogoliubov excitations in a disk-shaped Bose–Einstein condensate. Different from previous approach here the effect of diffraction of the excitations is taken into account. We have shown that the phase-matching condition for the TWRI can be fulfilled by suitably choosing the wavevectors and the frequencies of the three exciting modes involved in the TWRI. Using a method of multiple-scales we have derived a set of nonlinearly coupled envelope equations describing the TWRI process, in which the second-order derivatives representing the diffraction appear. We have also provided some explicit three-wave soliton solutions for these envelope equations.

Note that the three-wave soliton solutions obtained here are different from those found in Ref. [31]. Due to the diffraction effect the three wave modes involved in the TWRI can be three simultaneous solitons. The formation of such simultaneous solitons is due to the cascading process contributed from the nonlinear coupling between the three wave modes. In this process, the fundamental and the harmonic waves interact with themselves through repeated wave-wave interactions. For instance the energy of one fundamental wave is first upconverted to another fundamental wave and the harmonic wave and then downconverted again, resulting in a mutual self-trapping of each wave and thus the formation of three simultaneous solitary waves. The results presented in this work can be useful to understanding the nonlinear property of larger-amplitude excitations and as a guide for new experiment findings in the study of Bose–Einstein condensates.

Disk-shaped traps have been used to realize the first BEC in 1995[33] and observed linear excitations lately.[34] To experimentally demonstrate the TWRI phenomenon, one can use a disk-shaped BEC and employ a two-photon Bragg transition technique. This technique has been used recently by Katz *et al.* for studying the Beliaev damping of Bogoliubov excitations (a down-converted TWRI) in a BEC.[35] If the amplitude of the excitations is made large and the wavevectors and frequencies of two fundamental waves are chosen according to the phase-matching condition of the TWRI, it is possible to observe the TWRI and related three-wave solitons predicted in this work. Theoretically, one can discuss further the second-harmonic generation and four-wave mixing of the collective excitations in the system.

Acknowledgments

The authors are indebted to CUI Wei-Na, HANG Chao, and Nicolas Compagnion for their useful discussions.

References

[1] L. Deng, *et al.*, Nature (London) **398** (1999) 218.
[2] S. Inouye, *et al.*, Science **285** (1999) 571.
[3] S. Inouye, *et al.*, Nature (London) **402** (1999) 641.
[4] M. Kozuma, *et al.*, Science **286** (1999) 2309.
[5] S. Burger, *et al.*, Phys. Rev. Lett. **83** (1999) 5198; J. Denschlag, *et al.*, Science **287** (2000) 97; Z. Dutton, *et al.*, Science **293** (2001) 663; K.E. Strecker, *et al.*, Nature **417** (2002) 150; L. Khaykovich, *et al.*, Science **296** (2002) 1290.
[6] M.R. Mattews, *et al.*, Phys. Rev. Lett. **83** (1999) 2498; K.W. Madison, *et al.*, Phys. Rev. Lett. **84** (2000) 806; J.R. Abo-Shaeer, *et al.*, Science **292** (2001) 476; B.P. Anderson, *et al.*, Phys. Rev. Lett. **86** (2001) 2926.
[7] V.M. Perez-Garcia, *et al.*, Phys. Rev. **A57** (1998) 3837; A.E. Muryshev, *et al.*, Phys. Rev. **A60** (1999) R2665; Th. Busch and J.R. Anglin, Phys. Rev. Lett. **84** (2000) 2298; E.V. Goldstein and P. Meystre, Phys. Rev. **A59** (1999) 1509; M. Trippenbach, *et al.*, Phys. Rev. **A62** (2000) 023608.
[8] Y. Wu, *et al.*, Phys. Rev. **A61** (2000) 043604.
[9] W.M. Liu, *et al.*, Phys. Rev. Lett. **84** (2000) 2294.
[10] B. Wu, *et al.* Phys. Rev. Lett. **88** (2002) 034101.
[11] W. Hai, *et al.*, Phys. Rev. **E66** (2002) 026202.
[12] Y.L. Ma and S.T. Chui, Phys. Rev. **A65** (2002) 053610.
[13] Q. Yang, *et al.*, Phys. Rev. **A67** (2003) 013603.
[14] G. Huang, *et al.*, Phys. Rev. **A64** (2001) 013617.
[15] G. Huang, *et al.*, Phys. Rev. **A65** (2002) 053605.
[16] G. Huang, *et al.*, Phys. Rev. **A67** (2003) 023604.
[17] O. Zobay, E.V. Goldstein, and P. Meystre, Phys. Rev. **A60** (1999) 3999.
[18] M.G. Moore and P. Meystre, Phys. Rev. Lett. **86** (2001) 4199; W. Ketterle and S. Inouye, Phys. Rev. Lett. **86** (2001) 4203.
[19] S.L. Rolston and W.D. Phillips, Nature (London) **416** (2002) 219.
[20] C.J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases*, Cambridge University Press, Cambridge (2002).
[21] P.A. Ruprecht, *et al.*, Phys. Rev. **A54** (1996) 4178.
[22] F. Dalfovo, *et al.*, Phys. Rev. **A56** (1997) 4855.
[23] S.A. Morgan, *et al.*, Phys. Rev. **A57** (1998) 3818.
[24] G. Hechenblaikner, *et al.*, Phys. Rev. Lett. **85** (2000) 692.
[25] G. Hechenblaikner, *et al.*, Phys. Rev. **A65** (2002) 033612.
[26] U. Al Khawaja and H.T.C. Stoof, Phys. Rev. **A65** (2002) 013605.
[27] D. McPeake and J.F. McCann, arXiv:cond-mat/0303058.
[28] E. Hodby, *et al.*, Phys. Rev. Lett. **86** (2001) 2196.
[29] J. Steinhauer, *et al.*, Phys. Rev. Lett. **88** (2002) 120407.
[30] R. Ozeri, *et al.*, Phys. Rev. Lett. **90** (2003) 170401.
[31] C. Sun and G. Huang, J. Phys. **B**, submitted.
[32] G. Huang, *et al.*, J. Phys. A: Math. Gen. **83** (2000) 8477.
[33] M.H. Anderson, *et al.*, Science **269** (1995) 198.
[34] D.S. Jin, *et al.*, Phys. Rev. Lett. **77** (1996) 420.
[35] N. Katz, *et al.*, Phys. Rev. Lett. **89** (2002) 220401.