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Quantum depletion of a soliton condensate

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Abstract

We present rigorous results on the diagonalization of Bogoliubov Hamiltonian for a soliton condensate. Using the complete and orthonormalized set of eigenfunction for the Bogoliubov–de Gennes equations, we calculate exactly the quantum depletion of the condensate and show that two degenerate zero-modes, which originate from a $U(1)$ gauge- and a translational-symmetry breaking of the system, induce the quantum diffusion and transverse instability of the soliton condensate.

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The study of matter waves and elementary excitations has received much attention because of the remarkable experimental realization of Bose–Einstein condensation in trapped, weakly interacting atomic gases [1–3]. In the past few years, several paths have been explored in studying elementary excitations in Bose–Einstein condensates (BECs). The most followed method is to use the Gross–Pitaevskii (GP) equation which is suitable to describe a zero-temperature BEC. One of the deficiency of this approach, in addition to the constraint of zero temperature, is the neglect of the quantum fluctuations of condensate, which is an important aspect of elementary excitation [4]. The eigenmodes of the elementary excitations thus obtained so far [1,2] are not complete and their orthonormalities have never been proved.

A different approach on elementary excitations is to use Bogoliubov theory [5] that was originally formulated for homogeneous Bose systems but is also valid for *inhomogeneous* ones. In this approach one makes a canonical transformation for boson operators to diagonalize the quantum Hamiltonian of system and hence the quantum fluctuations of condensate are taken into account [6]. An added advantage of the Bogoliubov theory is that it can be easily generalized to the case of finite temperature (with, e.g., the Hartree–Fock and Popov approximations) [1,2], therefore can be compared directly with experiments carried out at non-zero temperature environment.

In this Letter we apply the Bogoliubov theory to the investigation of the elementary excitations generated from a quasi-one-dimensional (1D) soliton condensate. It is known that a quasi-1D BEC can form in the presence of a transverse confinement, which introduces a cutoff for long wavelength fluctuations in the transverse directions and hence an off-diagonal-long-range order can establish, as have been reported in the case of low-dimensions BEC systems [7] and in the case of matter-wave bright solitons in quasi-1D systems [8,9] and the corresponding flourished theoretical activities [10–13]. The main contribution of our present work is the rigorous diagonalization of quantum Bogoliubov Hamiltonian and the exact calculation of quantum depletion of the soliton condensate and related conclusions on soliton stability.

We consider a quasi-1D attractive, weakly interacting Bose gas condensed in a trap with tight transverse confinement and negligible trapping potential in the axial direction. The grand

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canonical Hamiltonian of the system can be written as the following dimensionless from

$$\hat{K} = \hat{H} - \mu\hat{N} = \int dz\,\hat{\Psi}^\dagger(z,t)\left[-\frac{\partial^2}{\partial z^2} - \mu - \hat{\Psi}^\dagger(z,t)\hat{\Psi}(z,t)\right]\hat{\Psi}(z,t), \tag{1}$$

where $\hat{\Psi}(z,t)$ is the field operator satisfying $[\hat{\Psi}(z,t), \hat{\Psi}^\dagger(z',t)] = \delta(z-z')$ (with other commutators being zero). The chemical potential μ assures the conservation of the average particle number $\hat{N} = \int dz\,\hat{\Psi}^\dagger(z,t)\hat{\Psi}(z,t)$. In the expression (1) the coordinate, time, field operator, and energy (including chemical potential) are scaled by $l = \hbar^2/(m|U_0|)$, $2ml^2/\hbar$, $l^{-1/2}$, and $\hbar^2/(2ml^2)$, respectively. $U_0(<0)$ is the interatomic interaction constant and m is the atomic mass. The model (1) has been used in Ref. [12] to study the quantum phase transition from a homogeneous ground state to a bright-soliton ground state when the strength of the attractive interaction is increased.

Letting $\hat{\Psi}(z,t) = \psi_g(z) + \hat{\psi}(z,t)$ and assuming $\|\hat{\psi}\| \ll \psi_g$ we obtain the Bogoliubov form (quadratic for $\hat{\psi}$ and $\hat{\psi}^\dagger$) of the grand canonical Hamiltonian $\hat{K} = K_g + \hat{K}_2$ with $K_g = \int dz\,\psi_g(-\partial^2/\partial z^2 - \mu - \psi_g^2)\psi_g$ and $\hat{K}_2 = \int dz\,[\hat{\psi}^\dagger(-\partial^2/\partial z^2 - \mu - 4\psi_g^2)\hat{\psi} - \psi_g^2(\hat{\psi}\hat{\psi} + \hat{\psi}^\dagger\hat{\psi}^\dagger)]$, where the ground-state wave function satisfies the equation $(-\partial^2/\partial z^2 - \mu - 2\psi_g^2)\psi_g = 0$. For small $|U_0|$ the ground state is a homogeneous condensate and hence the Hamiltonian can be easily diagonalized and the elementary excitation exhibits a gapless phonon spectrum. For large $|U_0|$, however, this homogeneous condensate is dynamically unstable [12] and the system undergoes a quantum phase transition into the soliton ground state

$$\psi_g(z) = (N_0/2)\,\mathrm{sech}\big[(N_0/2)(z - z_0)\big] \tag{2}$$

with chemical potential $\mu = -N_0^2/4$, where z_0 is an arbitrary constant, indicating that the quantum phase transition results in a translational symmetry breaking of the system.

Our first objective is to search for a rigorous diagonalization of the Hamiltonian of the soliton condensate (2). Letting $(N_0/2)(z - z_0) \to z$ and $\hat{\psi} = (N_0/2)^{1/2}\hat{\phi}$ we get

$$\hat{K} = K_g + \left(N_0^2/4\right)\int dz\,\big[\hat{\phi}^\dagger\hat{L}\hat{\phi} - \mathrm{sech}^2 z\big(\hat{\phi}\hat{\phi} + \hat{\phi}^\dagger\hat{\phi}^\dagger\big)\big], \tag{3}$$

where $\hat{L} = -\partial^2/\partial z^2 - 4\,\mathrm{sech}^2 z + 1$ and $\hat{\phi}$ satisfies $[\hat{\phi}(z,t), \hat{\phi}^\dagger(z',t)] = \delta(z - z')$ (with other commutators being zero). To diagonalize $\hat{K}$ we make the canonical transformation

$$\hat{\phi} = \sum_j\big[u_j(z)\hat{c}_j + v_j^*(z)\hat{c}_j^\dagger\big] + \sum_{k\neq 0}\big[u_k(z)\hat{c}_k + v_k^*(z)\hat{c}_k^\dagger\big], \tag{4}$$

where $\hat{c}_j$ and $\hat{c}_k$ are usual boson operators. The first and second terms on the right-hand side (RHS) of (4) are respectively the contributions from the discrete (j) and continuum (k) spectra of the excitations generated from the soliton condensate. The key for diagonalizing $\hat{K}$ is to find a complete set of the eigenfunctions $\{u_q(z), v_q(z); q = k, j\}$ with which the expansion of $\hat{\phi}$ can be made. The discrete modes are in fact the eigenmodes with a zero eigenvalue, as shown below.

Substituting (4) into (3) and assuming that u_q and v_q fulfill the following eigenequations:

$$\hat{L}_2\hat{L}_1\phi_q(z) = E_q^2\phi_q(z), \qquad \hat{L}_1\hat{L}_2\psi_q(z) = E_q^2\psi_q(z), \tag{5}$$

where $\psi_q(z) = u_q(z) + v_q(z)$, $\phi_q(z) = u_q(z) - v_q(z)$ and $\hat{L}_j = d^2/dz^2 + (2\delta_{j1} + 6\delta_{j2})\,\mathrm{sech}^2 z - 1$ ($j = 1, 2$), by a detailed calculation we obtain a diagonalized $\hat{K} = K_g + (N_0^2/4)\hat{K}_{20}$, with

$$\hat{K}_{20} = \frac{2}{3} - \sum_{k\neq 0}\int dz\,E_k\big|v_k(z)\big|^2 - \hat{P}_1^2 + \hat{Q}_2^2 + \sum_{k\neq 0}E_k\hat{c}_k^\dagger\hat{c}_k, \tag{6}$$

where we have introduced the operators [6] $\hat{P}_j = (\hat{c}_j + \hat{c}_j^\dagger)/\sqrt{2}$ and $\hat{Q}_j = i(\hat{c}_j - \hat{c}_j^\dagger)/\sqrt{2}$. Eq. (5) is equivalent to the Bogoliubov–de Gennes (BdG) eigenvalue problem:

$$\hat{L}u_q(z) - 2\,\mathrm{sech}^2 z\,v_q(z) = E_q u_q(z), \tag{7}$$

$$\hat{L}v_q(z) - 2\,\mathrm{sech}^2 z\,u_q(z) = -E_q v_q(z). \tag{8}$$

The solutions of Eq. (5) has been known in *classical* soliton perturbation theory [14,15]. Thus we can use the result obtained in Refs. [14,15] to get the following solutions of the BdG equations (7) and (8):

$$u_k(z) = -\frac{1}{\sqrt{2\pi}(k^2+1)}\big[k^2 + 2ik\tanh z - 2\tanh^2 z\big] \times \exp(ikz), \tag{9}$$

$$v_k(z) = -\frac{1}{\sqrt{2\pi}(k^2+1)}\,\mathrm{sech}^2 z\exp(ikz), \tag{10}$$

with

$$E_k = k^2 + 1 \tag{11}$$

for $k \neq 0$ (continuous spectrum). Eqs. (7) and (8) admit also the following zero-mode solutions ($j = 1, 2$):

$$u_j(z) = \delta_{j1}\,\mathrm{sech}\,z\,\frac{2 - z\tanh z}{2} + \delta_{j2}\,\mathrm{sech}\,z\,\frac{z + \tanh z}{2}, \tag{12}$$

$$v_j(z) = -\delta_{j1}(z/2)\,\mathrm{sech}\,z\tanh z + \delta_{j2}\,\mathrm{sech}\,z\,\frac{\tanh z - z}{2}. \tag{13}$$

These zero-modes belong to the discrete spectrum of the system with a zero eigenvalue, which is two-fold degenerate. Physically, such degenerate zero-modes originate from a $U(1)$ gauge- as well as a translational-symmetry breaking of the system and thus open a gap in the excitation spectrum (see Eq. (11)). This is very different from the result obtained for homogeneous condensates. It can be shown that the eigenfunction set obtained above form a complete function set and they are also orthogonal.[1]

With the above results we find that $K_g = 2N_0^3/3$ and $\sum_{k\neq 0}E_k\int dz\,|v(z,k)|^2 = (\pi - 1)/(6\pi)$ and obtain diagonalized (dimensional) Bogoliubov quantum Hamiltonian

[1] A detailed discussion on the completeness and the orthonormalities of the eigenfunction set $\{u_q(z), v_q(z);\ q = k, j\}$ and their application will be given elsewhere.

$$\hat{H} = -\frac{1}{24}\frac{m|U_0|^2}{\hbar^2}N_0^3 + \frac{1}{8}\frac{m|U_0|^2}{\hbar^2}N_0^2 \times \left[\frac{2}{3} + \frac{\pi - 1}{6\pi} - \hat{P}_1^2 + \hat{Q}_2^2 + \sum_{k\neq 0} E_k \hat{c}_k^\dagger \hat{c}_k\right]. \quad (14)$$

Note that each term on the RHS of (14) has clear physical meaning. The first term describes the ground state energy of the system, which for large N_0 agrees with the result of Bethe ansatz solution [16]. The first term in the square bracket comes from the zero-modes whereas the second term is the contribution of the depletion of the condensate. The terms $-\hat{P}_1^2 + \hat{Q}_2^2$ are originated from the quantum fluctuations of the condensate and the last term represents the contribution from phonons. Because the continuous spectrum of the excitations E_k for the soliton condensate opens a gap, thus the phonons are *massive* due the breaking of translational symmetry of the system. This is very different from the case of homogeneous condensates where a phonon is non-massive.

Our next objective is to investigate the quantum dynamics of the soliton condensate. For the zero-modes we obtain the equations of motion

$$\frac{d\hat{P}_1}{d\tau} = 0, \qquad \frac{d\hat{Q}_1}{d\tau} = -2\hat{P}_1, \quad (15)$$

$$\frac{d\hat{P}_2}{d\tau} = -2\hat{Q}_2, \qquad \frac{d\hat{Q}_2}{d\tau} = 0, \quad (16)$$

where $\tau = (N_0^2/4)t$. The Hermitian operators $\hat{P}_j$ and $\hat{Q}_j$ satisfy the commutation relation $[\hat{Q}_j, \hat{P}_{j'}] = i\delta_{jj'}$, and they are associated with the collective motion of the solition condensate [6]. The exact solutions of the above equations read $\hat{P}_1(\tau) = \hat{P}_1(0)$, $\hat{Q}_1(\tau) = \hat{Q}_1(0) - 2\hat{P}_1(0)\tau$, $\hat{P}_2(\tau) = \hat{P}_2(0) - 2\hat{Q}_2(0)\tau$, and $\hat{Q}_2(\tau) = \hat{Q}_2(0)$. Using the zero-mode solutions and the relations between $\hat{c}_j$, $\hat{c}_j^\dagger$ and $\hat{P}_j$, $\hat{Q}_j$ we obtain

$$\hat{\Psi}(z,t) \approx \hat{A}(z,\tau)\,\mathrm{sech}\left[z - \frac{1}{\sqrt{N_0}}\hat{P}_2(\tau)\right] \times \exp\left\{-\frac{i}{\sqrt{N_0}}\left[\hat{Q}_1(\tau) + z\hat{Q}_2(\tau)\right]\right\}, \quad (17)$$

with $\hat{A}(z,\tau) \approx (N_0/2)[1 + (1 - z\tanh z)\hat{P}_1(\tau)/\sqrt{N_0}\,]$. Eq. (17) clearly shows quantum fluctuations in the amplitude, position and phase of the soliton condensate due to the existence of the zero-modes. This is again very different from the case of a homogeneous condensate.

The above results can be used to calculate the quantum depletion of the soliton condensate [17]. Let $|N_0, N_1, N_2, N_{\mathrm{exc}}\rangle$ denote the state with N_0 atoms in the condensate. N_1 and N_2 are number of atoms occupying the first and the second zero-modes, and N_{exc} are the number of atoms being in the continuous modes. Assume that the initial ($t = 0$) state of the system is a Bogoliubov (quasiparticle) vacuum $|N_0, 0, 0, 0\rangle$. We consider the time evolution of the particle numbers occupying in different modes. Noting that the particle number density operator of the system is given by $\hat{n}(z,t) = l^{-1}\hat{\Psi}^\dagger(z,t)\hat{\Psi}(z,t)$ and using the expression of the field operator, we get the particle-number density for the condensate mode $n_0(z) = \langle\hat{n}_0(z)\rangle = l^{-1}(N_0/2)^2\,\mathrm{sech}^2 z$, which gives rise naturally the total particle number N_0 in the condensate. Here $\langle\cdots\rangle$ denotes the average over the Bogoliubov vacuum. It is easy to show that the particle-number densities in the first and the second zero-modes are given by

$$n_j(z,\tau) = \frac{N_0}{2l}\left[v_j^2(z) + \left[u_j(z) + (-1)^j v_j(z)\right]^2\tau^2\right], \quad j = 1, 2. \quad (18)$$

The atoms in these zero-modes are *incoherent* ones, representing the quantum depletion of the soliton condensate. There are two features for the quantum depletion: (i) The spatial distributions of the incoherent atoms in the zero-modes are localized because $u_j(z)$ and $v_j(z)$ are localized functions of z; (ii) The distribution densities of the incoherent atoms are time-dependent and proportional to τ^2, which implies that the depleted atoms grow *algebraically* as time increases and hence *the soliton condensate loses atoms spontaneously even in the absence of thermal cloud*. This is a manifestation of the quantum diffusion of the soliton condensate implied in (17). The exact expressions of the total particle numbers depleted in the first and the second zero-mode are $N_1(t) = \int_{-\infty}^{\infty} dz\, n_1(z,t) = (12 + \pi)/72 + 2\tau^2$ and $N_2(t) = \int_{-\infty}^{\infty} dz\, n_2(z,t) = (2/3)(1 + \tau^2)$. The particle-number density in all continuous modes is given by

$$n_{\mathrm{exc}}(z,t) = \frac{N_0}{2l}\sum_{k\neq 0}\left|v_k(z)\right|^2 = \frac{N_0}{2l}\left(\frac{1}{4} - \frac{1}{2\pi}\right)\mathrm{sech}^4 z. \quad (19)$$

Thus, the total particle number depleted in the continuous modes is $N_{\mathrm{exc}}(t) = N_{\mathrm{exc}}(0) = \int_{-\infty}^{\infty} dz\, n_{\mathrm{exc}}(z,t) = (\pi - 2)/(3\pi)$ a small number in comparison with that depleted in the zero modes.

We can also obtain the exact expressions of the average values for $\hat{P}_j$ and $\hat{Q}_j$. They are given by $\langle\hat{P}_1(\tau)\hat{P}_1(\tau)\rangle = 1/2$, $\langle\hat{P}_2(\tau)\hat{P}_2(\tau)\rangle = (1/2)(1 + 4\tau^2)$, $\langle\hat{Q}_2(\tau)\hat{Q}_2(\tau)\rangle = (1/2)(1 + 4\tau^2)$ and $\langle\hat{Q}_2(\tau)\hat{Q}_2(\tau)\rangle = 1/2$. These expressions reflect again the diffusion property of the soliton condensate. It should be noted that the instability of the soliton condensate shown above is based on a linear Bogoliubov approach. The inclusion of the nonlinear effect between the quasiparticle excitations may modify these results.

Our third objective is to study the stability of the soliton condensate under a long wavelength transverse perturbation. Assuming the confinement in the x and y directions is relaxed the operator $\partial^2/\partial z^2$ in (1) and (3) is replaced by $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. To diagonalize the Hamiltonian in this case we use the canonical transformation $\sum_{\mathbf{k}_\perp}\sum_q [u_{q,\mathbf{k}_\perp}(z)\exp(i\mathbf{k}_\perp\cdot\mathbf{r}_\perp)\hat{c}_{q,\mathbf{k}_\perp} + v^*_{q,\mathbf{k}_\perp}(z)\exp(-i\mathbf{k}_\perp\cdot\mathbf{r}_\perp)\hat{c}^\dagger_{q,\mathbf{k}_\perp}]$, where $\mathbf{r}_\perp = (x, y)$ and $\mathbf{q} = (k_x, k_y, q) = (\mathbf{k}_\perp, q)$ with $q = k$ (for continuous modes) or j (for discrete modes). The eigenequations (5) are replaced by

$$\left(\hat{L}_2 - k_\perp^2\right)\left(\hat{L}_1 - k_\perp^2\right)\phi_{\mathbf{q}}(z) = \sigma_{\mathbf{q}}\phi_{\mathbf{q}}(z), \quad (20)$$

$$\left(\hat{L}_1 - k_\perp^2\right)\left(\hat{L}_2 - k_\perp^2\right)\psi_{\mathbf{q}}(z) = \sigma_{\mathbf{q}}\psi_{\mathbf{q}}(z), \quad (21)$$

where $\sigma_{\mathbf{q}} \equiv E_{\mathbf{q}}^2$. We consider a long wavelength perturbation in the transverse (x and y) directions, thus $k_\perp^2$ is a small quantity which can be taken as a small parameter to solve Eqs. (20) and (21) using a perturbation theory. Taking $\phi_{\mathbf{q}} = \phi_{\mathbf{q}}^{(0)} + k_\perp^2 \phi_{\mathbf{q}}^{(1)} + k_\perp^4 \phi_{\mathbf{q}}^{(2)} + \cdots$, $\psi_{\mathbf{q}} = \psi_{\mathbf{q}}^{(0)} + k_\perp^2 \psi_{\mathbf{q}}^{(1)} + k_\perp^4 \psi_{\mathbf{q}}^{(2)} + \cdots$ and $\sigma_{\mathbf{q}} = \sigma_{\mathbf{q}}^{(0)} + k_\perp^2 \sigma_{\mathbf{q}}^{(1)} + k_\perp^4 \sigma_{\mathbf{q}}^{(2)} + \cdots$, we get a hierarchy of the equations on $\phi_{\mathbf{q}}^{(l)}$, $\psi_{\mathbf{q}}^{(l)}$ and $\sigma_{\mathbf{q}}^{(l)}$ ($l = 0, 1, 2, \ldots$). The leading-order solutions, including the zero-modes and the continuous modes, are just those obtained given above. We are interested in what will happen for the zero-modes ($\sigma_{\mathbf{q}}^{(0)} = 0$) when the transverse perturbation is applied to the system. Thus in the following we consider the dynamics of one zero-mode (say $\phi_{\mathbf{q}}^{(0)}(z) = \phi_1(z) = u_1(z) - v_1(z) = \operatorname{sech} z$).[2] Because the leading-order solutions are a set of complete and orthonormalized functions, *all high-order solutions can be obtained exactly*. In the orders $l = 1$ and 2, for frequency correction we get $\sigma_{\mathbf{q}}^{(1)} = -2\int_{-\infty}^{\infty} \phi_1^2(z) = -4$ and $\sigma_{\mathbf{q}}^{(2)} = 1 + \int_{-\infty}^{\infty} dk\, a_k^{(1)} \int_{-\infty}^{\infty} dz\, [(k^2+1)\psi_1(z)\psi_k(z) - 2\phi_1(z)\phi_k(z)] = 3/2$. Here $a_k^{(1)} = -\sqrt{\pi/2}k/[(k^2+1)^2 \sinh(\pi k/2)]$. Thus under the long wavelength transverse perturbation the eigenfrequency corresponding to the zero-mode reads

$$E_{1,\mathbf{k}_\perp} = \sqrt{-4k_\perp^2 + 3k_\perp^4/2} = i2k_\perp\sqrt{1 - 3k_\perp^2/8}, \tag{22}$$

which is pure imaginary for $k_\perp^2 < 8/3$ and hence the time growth rate of the zero-mode is $\mathrm{Im}(E_{1,\mathbf{k}_\perp}) = 2k_\perp\sqrt{1 - 3k_\perp^2/8}$. Another zero-mode displays also similar behavior (see footnote 1). Consequently, if there is no confinement in x and y directions, the soliton condensate will be dynamically unstable under the long wavelength (i.e. for small $k_\perp$) transverse perturbation. The origin of this instability comes from the zero-modes. Thus to repress this instability a confinement in the x and y directions is necessary. This is why for observing a stable soliton condensate in experiment one must use a quasi-1D trapping potential.

We now briefly discuss the experimental aspects of quasi-1D condensate. Low dimensional BECs have been realized experimentally [7] and the dark solitons in the BECs have been investigated intensively [18–20]. Bright solitons and bright soliton trains for attractive atomic interaction have been observed recently [8,9]. These works will further stimulate intensive studies on elementary excitations in inhomogeneous Bose systems. Experimentally, a quasi-1D BEC, as those suitable for the present work, can be obtained by tightening the confinement in two transverse directions so that the energy-level spacings in those directions exceed the interaction energy. Consequently, the motion of the atoms in the transverse directions is frozen out and the system is essentially one-dimensional. Note that the results presented above can be applied to a quasi-1D Bose gas with attractive atomic interaction. By assuming the 3D field operator with the form $\hat{\Phi}(\mathbf{r}, t) = \Phi_0(\mathbf{r}_\perp)\hat{\Psi}(z, t)$, where $\Phi_0(\mathbf{r}_\perp)$ is the single-particle ground state wave function in the transverse directions, one can obtain a quasi-1D Hamiltonian with the same form of (1) but the coordinate z should be scaled by $l = \hbar^2/(mI_0|U_0'|)$ with $U_0' = U_0 \int d\mathbf{r}_\perp |\Phi_0(\mathbf{r}_\perp)|^4$.

In conclusion, we have presented rigorous results on the diagonalization of a Bogoliubov quasiparticle Hamiltonian for a inhomogeneous soliton condensate. Based on the set of complete and orthonormalized eigenfunction of the Bogoliubov–de Gennes equations, we have calculated exactly the quantum depletion of the condensate. We have shown explicitly that two degenerate zero-modes, appearing due to the $U(1)$ gauge- and a translational-symmetry breaking of the system, are the origin of the quantum diffusion and the transverse instability of the soliton condensate. We should point out that, in our study presented above, a negligible axial trapping potential has been assumed thus the results are valid only for a very long cigar-shaped condensate. If the axial trapping potential, which breaks also the translational symmetry of the system, cannot be neglected and taken as a slowly-varying function, a rigorous result on the diagonalization of the Bogoliubov Hamiltonian and quantum depletion of a BEC remains to be a challenge.

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