Analytical expressions of collective excitations for trapped superfluid Fermi gases in a BCS-BEC crossover

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We investigate the collective excitations of a harmonically trapped superfluid Fermi gas at varying coupling strengths across a BCS-BEC crossover. Using a hydrodynamic approach, we solve analytically the eigenvalue problem of collective modes and provide explicit expressions for all eigenvalues and eigenfunctions, which are valid for both BCS and BEC limits and also for the whole crossover regime. Both spherical- and axial-symmetric traps are taken into account, and the features of these collective modes in the BCS-BEC crossover are discussed and compared with available experimental and numerical data.

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I. INTRODUCTION

The crossover from a Bose-Einstein condensate (BEC) to a Bardeen-Cooper-Schrieffer (BCS) superfluid has attracted considerable attention for decades [1]. Ultracold Fermi systems provide an excellent opportunity for exploring the property of BCS-BEC crossovers in a controllable way. Recently, investigation of the BCS-BEC crossover of two-component fermionic atoms—i.e., ^{6}Li or ^{40}K—has become a topic of much interest both experimentally and theoretically (for details, see Ref. [1] and references therein).

The theory of elementary excitations, pioneered by Landau, Bogoliubov, and Feynman, is of primary importance in quantum many-body physics [2]. After the remarkable experimental realization of condensed fermionic pairs in regimes of BEC [3], BCS [4], and the crossover [5], much attention has been paid to experimental and theoretical studies of excitations in harmonically trapped, interacting superfluid Fermi gases [6–17]. In the present experiments [3–9], the interaction of dilute fermion atoms, characterized by the s-wave scattering length a_{sc}, can be tuned by a magnetic-field-induced scattering resonance (known as the Feshbach resonance [18]), allowing one to manipulate the interaction strength over the range $-\infty < 1/(k_F a_{sc}) < \infty$, where k_F is the Fermi wave number, and hence giving a possibility to investigate the nature of the elementary excitations in different superfluid regimes.

There are several theoretical approaches to study the collective excitations of superfluid Fermi gases in the BCS-BEC crossover. One of them—microscopic theory (called resonance superfluid theory) based on a model Hamiltonian, which includes fermionic and bosonic degrees of freedoms and their coupling—has been proposed by several authors [19–21]. Because the fermion atomic pairs are trapped in a finite space, the inhomogeneous feature of the system makes the microscopic approach difficult. Notice that at very low temperature a low-frequency collective mode does not decay into fermions because of the existence of an energy gap. Since no thermal excitation is present, the system is a perfect superfluid and can be well described by the local particle density $\rho(\mathbf{r},t)$ and local velocity $\mathbf{v}(\mathbf{r},t)$. It is known that the interaction between particles in a homogeneous system is characterized by a chemical potential $\mu(\rho)$ (or called the equation of state), which can be obtained by using a quantum Monte Carlo simulation [10,22,23] or some other techniques [11–15,23]. For fermions confined in a trapping potential $V(\mathbf{r})$, the density profile $\rho(\mathbf{r},t)$ changes slowly in space if the particle number of the system is large enough. Under such conditions a local density approximation can be applied to the equation of state and hence one can suggest a hydrodynamic approach to investigate the physical properties of collective excitations at low temperatures. In fact, theoretical studies of the collective excitations in the BCS-BEC crossover have appeared recently based on the hydrodynamic approach [10–17], and all eigenspectra and eigenfunctions for the collective modes in an isotropic (spherical) trap have been obtained under a Thomas-Fermi (TF) approximation, valid for a large particle number. However, for an anisotropic trap, which is used in most of the recent experiments [3–9], only several particular eigenmodes (i.e., breathing modes) have been given in the literature [11–17,21,23]. Thus it is necessary to develop a general method to obtain all explicit solutions for the eigenspectra and eigenfunctions that are not only valid for an anisotropic trap but also for all superfluid regimes. It is this topic that will be addressed in this work. Note that an analytical method for finding various eigenvalues and eigenfunctions for condensed bosons [i.e., an atomic BEC with $\gamma=1$, where γ is a polytropic index defined in Eq. (4) below] has been proposed in Ref. [24]. In the present work, we generalize this method to the situation for any value of γ. We solve analytically the related eigenvalue problem for linear excitations and provide explicit solutions of the eigenspectra and eigenfunctions for entire collective modes of a superfluid Fermi system with the results being valid for different superfluid regimes.

The paper is arranged as follows. For completeness, in Sec. II we give a review of the chemical potential $\mu(\rho)$ in

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various superfluid regimes. The hydrodynamic form of the equation of motion for collective excitations is presented; in Sec. III we provide explicit solutions of the eigenspectra and eigenfunctions of the collective modes for both spheric and spheroidal symmetric traps. Section IV contains a brief discussion and a conclusion of our results.

II. EQUATION OF STATE AND HYDRODYNAMIC FORMULAS FOR COLLECTIVE EXCITATIONS

At temperature $T=0$, a Fermi gas is in superfluid state and all particles are paired with $\rho/2$ being the number density of these pairs [1,25,26]. In a study of ultracold Fermi gases, these pairs, referred as condensed fermionic atomic pairs, originate from two-component fermionic atomic systems (i.e., ^{6}Li or ^{40}K) with different internal states. By means of a magnetic-field-induced Feshbach resonance, the magnitude and sign of the s-wave scattering length a_{sc} can be tuned and hence provide the possibility of realizing the transition from the BCS to BEC regimes in a controllable way. It is known that, when $a_{sc}<0$ ($a_{sc}>0$), the system is in a BCS (BEC) regime. Passing through the BCS-BEC crossover, a smooth transition is expected theoretically from BCS superfluidity to BEC superfluidity [1,25,26]. By defining a dimensionless quantity $y=1/(k_F a_{sc})$, with $k_F=(3\pi^2\rho)^{1/3}$ being the Fermi wave vector, one can distinguish several different superfluidity regimes [22].

(i) *BEC regime* ($y>1$). In this case, bounded molecules (or called dimers) can form by two fermions at some high temperature T^* due to two-body interactions. These preformed fermionic pairs have a small size in space and undergo a BEC phase transition at $T=T_c(\ll T^*)$. When $T<T_c$, the system is in a BEC superfluid state. The particular case $y\gg 1$ is called the deep BEC regime.

(ii) *BCS regime* ($y<-1$). In this situation, the fermions of the system form weakly bounded Cooper pairs below a critical temperature T_c due to many-body effects of the system. These condensed fermionic pairs undergo a BCS transition when the temperature is lower than T_c. Note that both the formation of Cooper pairs and the condensation of these fermionic pairs occur simultaneously at $T=T_c$. In particular, $y\ll -1$ is called the deep BCS regime.

(iii) *BCS-BEC crossover regime* ($-1<y<1$). This is the regime intermediating between the BEC and the BCS superfluidity. The condensed fermionic pairs in this case have the character of both BEC molecules and BCS Cooper pairs, or their mixing. Especially, the point $y=0$ is called the unitarity limit, corresponding to $a_{sc}=\pm\infty$. At this limit, the average spacing of particles, $d_0=(3\pi^2 N_0)^{1/3}/k_F$, is still a typical length, with N_0 being the condensed particle number of the system.

At zero temperature, the ground-state energy per particle, ε, of a dilute Fermi gas can be written as

$$\varepsilon(\rho)=\frac{3}{5}\epsilon_F(\rho)\epsilon(y), \tag{1}$$

where $\epsilon_F(\rho)=\frac{\hbar^2 k_F^2}{2M}=\frac{\hbar^2}{2M}(3\pi^2\rho)^{2/3}$ is the Fermi energy with M the mass of fermions of the system. $\epsilon(y)$ is a yet unknown function of the interaction parameter $y=y(\rho)=1/[(3\pi^2\rho)^{1/3}a_{sc}]$. Some asymptotic expressions of $\epsilon(y)$ can be obtained by fitting the data from some calculating techniques (such as the quantum Monte Carlo method) [10,22,27]. Interpolating these asymptotic expressions for small and large $|y|$ one can obtain the following general formula:

$$\epsilon(y)=\alpha_1-\alpha_2\arctan\left(\alpha_3 y\frac{\beta_1+|y|}{\beta_2+|y|}\right). \tag{2}$$

The fitting parameters α_j ($j=1,2,3$) and β_l ($l=1,2$) in Eq. (2) have been given by Manini and Salasnich [15]. Note that these parameters are different in the regions $y<0$ and $y>0$. In terms of the $\varepsilon(\rho)$ given above, the chemical potential of the Fermi gas can be obtained by using Gibbs-Duhem relation [2], which reads

$$\mu(\rho)=\partial[\rho\varepsilon(\rho)]/\partial\rho. \tag{3}$$

Alternately, a simple approach for the equation of state is to take a polytropic approximation; i.e., one assumes [11,12]

$$\varepsilon(\rho)=c\rho^{\gamma}, \tag{4}$$

where c is a constant but γ (called effective polytropic index) is a function of y:

$$\gamma(y)=\frac{\frac{2}{3}\epsilon(y)-\frac{2y}{5}\epsilon'(y)+\frac{y^2}{15}\epsilon''(y)}{\epsilon(y)-\frac{y}{5}\epsilon'(y)}. \tag{5}$$

There are two well-known limits for the value of the polytropic index γ. One is $\gamma=2/3$ at $y=-\infty$ (BCS limit) and another one is $\gamma=1$ at $y=\infty$ (BEC limit). Note that at $y=0$ (unitarity limit) one has also $\gamma=2/3$. The minimum value of γ is approximately 0.6, located at $y=-0.55$. The maximum value of γ is around 1.35, located at $y=0.5$. Mathematically, the polytropic approximation, Eq. (4), is a little rough but it has the advantage of allowing one to get analytical expressions for the eigenfunctions and eigenfrequencies of collective modes [10–12,14] for all superfluid regimes in a unified way.

Assume that a dilute Fermi gas is confined in a trapping potential $V(\mathbf{r})$. At zero temperature, the evolution of the system can be described by the following nonlinear Schrödinger equation [28]:

$$i\hbar\dot{\Phi}(\mathbf{r},t)=\left[-\frac{\hbar^2}{2M}\nabla^2+V(\mathbf{r})+\mu(\rho(\mathbf{r},t))\right]\Phi(\mathbf{r},t), \tag{6}$$

where $\Phi(\mathbf{r},t)$ is superfluid wave function, $\mu(\rho)$ is the chemical potential given by Eq. (3), and the dot over Φ represents the derivative with respect to time. Notice that in writing Eq. (6), a local density approximation for the chemical potential is used [13–15]. Taking $\Phi(\mathbf{r},t)=\sqrt{\rho(\mathbf{r},t)}e^{i\phi(\mathbf{r},t)}$, Eq. (6) is transferred into the following hydrodynamic equations:

$$\dot{\rho}(\mathbf{r},t)+\boldsymbol{\nabla}\cdot[\rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)]=0, \tag{7}$$

$$M\dot{\mathbf{v}}(r,t) + \nabla\left[T_{qp} + V(\mathbf{r}) + \mu(\rho(\mathbf{r},t)) + \frac{1}{2M}|\mathbf{v}(\mathbf{r},t)|^2\right] = 0, \tag{8}$$

where $\mathbf{v}(\mathbf{r},t)=-(\hbar/M)\nabla\phi(\mathbf{r},t)$ is the superfluid velocity and $T_{qp}=-\hbar^2[\nabla^2\sqrt{\rho(\mathbf{r},t)}]/[2M\sqrt{\rho(\mathbf{r},t)}]$ is the quantum pressure.

For the ground state of the system, one has $\mathbf{v}(\mathbf{r},t)=0$ and $\rho(\mathbf{r},t)=\rho_0(\mathbf{r})$. Then Eqs. (7) and (8) are reduced to

$$-\frac{\hbar^2}{2M}\nabla^2\sqrt{\rho_0(\mathbf{r})} + [V(\mathbf{r}) + \mu(\rho_0(\mathbf{r}))]\sqrt{\rho_0(\mathbf{r})} = \mu_G\sqrt{\rho_0(\mathbf{r})}, \tag{9}$$

with μ_G the chemical potential at the ground state.

To study the linear excitations from the ground state, we write $\rho(\mathbf{r},t)=\rho_0(\mathbf{r})+\delta\rho(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)=\delta\mathbf{v}(\mathbf{r},t)$, where $\delta\rho(\mathbf{r},t)$ and $\delta\mathbf{v}(\mathbf{r},t)$ are small quantities. Differentiating Eq. (7) with respect to time yields

$$\delta\ddot{\rho}(\mathbf{r},t) + \nabla\cdot[\delta\dot{\rho}(\mathbf{r},t)\delta\mathbf{v}(\mathbf{r},t) + \rho(\mathbf{r},t)\delta\dot{\mathbf{v}}(\mathbf{r},t)] = 0. \tag{10}$$

Here $\delta\dot{\rho}=-\nabla(\rho\delta\mathbf{v})$ is obtained from Eq. (7) and $\delta\mathbf{v}$ is determined by Eq. (8). Using Eq. (9) and neglecting small quantities of higher order, such as $|\delta\mathbf{v}|^2$, $|\delta\mathbf{v}|\nabla\cdot(\rho\delta\mathbf{v})$ and $\delta\rho|\nabla\delta\rho|$, Eq. (10) becomes

$$-M\delta\ddot{\rho}(\mathbf{r},t) + \nabla\cdot[\rho_0(\mathbf{r})\nabla(\delta\mu + \delta T_{qp})] = 0, \tag{11}$$

where $\delta\mu=\mu-\mu_0$ and $\delta T_{qp}=T_{qp}-T_{qp}^{(0)}$ are the deviation of the chemical potential and quantum pressure from the ground state, respectively. Taking $\delta\rho(\mathbf{r},t)=\delta\rho(\mathbf{r})e^{i\omega t}$, under the TF approximation (i.e., neglecting the quantum pressure term) Eq. (11) becomes

$$M\omega^2\delta\rho(\mathbf{r}) + \nabla\cdot[\rho_0(\mathbf{r})(\nabla\delta\mu)] = 0. \tag{12}$$

This is the dynamic equation for the linear collective excitations with the density eigenfunction $\delta\rho$ and the square eigenvalue ω^2. Note that Eq. (12) can be reduced to the following form:

$$\hat{H}_0\delta\rho = \omega^2\delta\rho, \tag{13}$$

with "Hamiltonian" $\hat{H}_0=-\frac{1}{M}\nabla\cdot(\rho_0\nabla)\frac{\partial\mu}{\partial\rho}$.

Under the polytropic approximation, we have

$$\mu(\rho) = \partial[\rho\varepsilon(\rho)]/\partial\rho$$
$$= 3\hbar^2/(10M)(3\pi^2)^{2/3}[\rho(0)]^{2/3}(\gamma+1)\epsilon(y)[\rho/\rho(0)]^{\gamma},$$

with the expressions $\gamma(y)$ and $\epsilon(y)$ given by Eqs. (5) and (2), respectively. Then the $\delta\mu$ term in Eq. (11) becomes $\delta\mu=(\gamma+1)c(\rho^{\gamma}-\rho_0^{\gamma})\approx\gamma(\gamma+1)\mu_0\delta\rho/\rho_0$. The trapping potential in the experiments [3–9] is the harmonic one

$$V(\mathbf{r}) = \frac{1}{2}M\omega_\perp^2(s^2 + \lambda^2 z^2), \tag{14}$$

where $s^2=x^2+y^2$, $\omega_\perp$ is the harmonic oscillator frequency in the x and y directions, and $\lambda=\omega_z/\omega_\perp$ is the frequency ratio of the z-axis to the xy plane. In the TF limit (i.e., $N\to\infty$), the kinetic energy term can be neglected with respect to the interaction terms. Under this limit Eq. (9) admits the following solution:

$$\rho_0(\mathbf{r}) = \rho_0(0)(1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma}, \tag{15}$$

where we have used the dimensionless coordinates $(\bar{s},\bar{z})=(s,z)/R_\perp$, with $R_\perp=[2(\gamma+1)c\rho_0^{\gamma}(0)/(M\omega_\perp^2)]^{1/2}$ being the TF radius of the fermionic cloud in the ground state.

III. EIGENVALUE SOLUTIONS FOR LINEAR EXCITATIONS

We now consider the solutions of Eq. (13) under the polytropic approximation [29]. We shall give all eigenmodes for the trapping potentials with spherical [30] and axial symmetries. The solutions for a highly anisotropic trap will also be provided.

A. Spherically symmetrical solutions ($\lambda=1$)

For a spherically symmetrical trap, the spheroidal condensate reduces to a spherical one. In this case the orbital angular momentum l and its projection in the z axis, m, are two good quantum numbers. Therefore, the hydrodynamic equation (13) is separable in spherical polar coordinates (r,θ,φ), where $r^2=x^2+y^2+z^2$. The linear excitation of the system is determined by a radial expansion into a polynomial of order n_r. The eigenmodes of the excitation can be labeled by the entire quantum numbers n_r, l, and m, but the eigenfrequencies are independent of the axial quantum number m.

By introducing $\bar{\nabla}=R_\perp\nabla$, the eigenequation (13) is simplified to the dimensionless form

$$-\frac{2\bar{\omega}^2}{\gamma}\delta\rho = \bar{\nabla}\cdot[\bar{\rho}_0(\bar{\nabla}\bar{\rho}_0^{\gamma-1}\delta\rho)], \tag{16}$$

with the dimensionless density $\bar{\rho}_0=\rho_0/\rho_0(0)$ and dimensionless frequency $\bar{\omega}=\omega/\omega_\perp$. Its solutions have the form $\delta\rho(\mathbf{r})=\bar{r}^l\bar{\rho}_0^{1-\gamma}P(\bar{r})Y_{lm}(\theta,\varphi)$, where Y_{lm} are the spherical-harmonic functions and the radial function $P(\bar{r})$ can be written as $(1-x)^{1/\gamma-1}Q(x)$, where $x=\bar{r}^2=r^2/R_\perp^2$ and $Q(x)$ satisfies a hypergeometric differential equation

$$x(1-x)Q'' + \left[l+\frac{3}{2}-\left(l+\frac{3}{2}+\frac{1}{\gamma}\right)x\right]Q' + \frac{1}{2\gamma}(\bar{\omega}^2-l)Q = 0. \tag{17}$$

The solutions of Eq. (17) are a special hypergeometric series [11] $P_{n_r l}(x)=F(-n_r,n_r+l+1/\gamma+1/2,l+3/2,\bar{r}^2)$—i.e., a classical (n_s)th-order Jacobi polynomial $P_{n_r l}(x)=n_r B[n_r,l+3/2]P_{n_s}^{(l+1/2|,1/\gamma-1)}(1-2x)$ with $B[x,y]$ being a β function. The energy spectrum is given by

$$\bar{\omega}_{n_r l}^2 = l + n_r\gamma(2n_r+2l+1) + 2n_r, \tag{18}$$

with $n_r=0,1,2,\ldots$, and $l=0,1,2,\ldots$. The energy is independent of m but the eigenfunction has $(2l+1)$-fold degeneracy for a given l.

Notice that the above results cover several important cases studied before. In the deep BEC regime, one has $\gamma=1$ and thus Eq. (18) reduces to $\bar{\omega}_{n_r l}^2=l+n_r(2n_r+2l+3)$, which was obtained first by Stringari [31]. In the deep BCS region, one has $\gamma=2/3$. Equation (18) becomes $\bar{\omega}_{n_r l}^2=l$

$+\frac{4}{3}n_r(n_r+l+2)$, which is the same as that obtained by Baranov and Petrov [32] (also see Ref. [33]). Especially, in the unitarity limit where $a_{sc}=\pm\infty$ and $\gamma=2/3$, the spectrum also has the finite form of $\bar{\omega}^2_{n_r l}=l+\frac{4}{3}n_r(n_r+l+2)$ in the TF limit.

B. Axially symmetric solutions (λ≠1)

Solving Eq. (13) for the case of axial symmetry is not easy as it is for the case of spherical symmetry. However, the axially symmetric case is more important than the spherically symmetric one because axially symmetric traps are used in almost all experiments made recently on the collective excitations in superfluid Fermi gases [3–9]. For an axially symmetric trap, the axial component of the angular momentum, m, is still a good quantum number. One expects that there exists an additional conserved quantity [24] $\hat{B}$ which commutes $\hat{H}_0$ and $L_z=-i\hbar\partial/\partial\varphi$ since the system has three degrees of freedom. The coupling between the radial and axial degrees of freedom leads to quantum numbers of the excitation modes that depend on the choice of coordinates. In cylindrical coordinates $(\bar{s},\bar{z},\varphi)$, along the line of treatment for the case of $\gamma=1$ [24,34], we assume that the eigenfunction takes the following form:

$$\delta\rho^{(2n_s)}_{n_p}(\bar{z},\bar{s},\varphi)=(1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma-1}\bar{s}^{|m|}P^{(2n_s)}_{n_p}(\bar{z},\bar{s})e^{im\varphi}, \tag{19}$$

since $\bar{\rho}_0=(1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma}$ and the coupling occurs between the z axis and xy plane with fast varying γ in the different BCS-BEC crossovers. Here the coupled axial and radial function $P=P^{(2n_s)}_{n_p}(\bar{s},\bar{z})$ satisfies the two-dimensional differential equation

$$(1-\bar{s}^2-\lambda^2\bar{z}^2)\left[\frac{\partial^2}{\partial\bar{s}^2}+(1+2|m|)\frac{\partial^2}{\partial\bar{s}^2}+\frac{\partial^2}{\partial\bar{z}^2}\right] -\frac{2}{\gamma}\left(\bar{s}\frac{\partial}{\partial\bar{s}}+\lambda^2\bar{z}\frac{\partial}{\partial\bar{z}}\right)+\frac{2}{\gamma}(\bar{\omega}^2-|m|)P(\bar{s},\bar{z})=0, \tag{20}$$

and the corresponding eigenfunction takes the form

$$P^{(2n_s)}_{n_p}(\bar{z},\bar{s})=\sum_{k=0}^{n_p}\sum_{n=0}^{\mathrm{int}[k/2]}b_{k,n}\bar{z}^{k-2n}\bar{s}^{2n}, \tag{21}$$

where n_p is the principal quantum number for the total energy. From Eq. (21) we know that for a fixed n_p, n_s is the radial quantum number for the series of modes $2n_s$ and $n_s=0,1,2,\ldots,\mathrm{int}[n_p/2]$; n_z is correspondingly the axial quantum number satisfying the relation $n_z=n_p-2n_s$ in cylindrical coordinates.

By using techniques of decoupling and dimension reduction, one can generalize the solution obtained in Refs. [24,34] for $\gamma=1$ to any value of γ. The details of the calculation have been provided in the Appendix . For a fixed n_p $(=0,1,2,\ldots)$, we can label the excitation spectrum $\bar{\omega}^2=\bar{\omega}^2(n_p,|m|,\lambda^2,\gamma)$ by the modes $(n_z n_s m)$. Once the coefficients $b_{k,n}$ are obtained in Eq. (A2), one can get all eigenfrequencies and eigenfunctions of the linear collective excitations. In the following, we list some eigenmodes in low-excited states of the system.

(i) For $n_p=0,1$, one has $n_s=0$ and $n_z=0,1$, respectively. The eigenfrequencies read

$$\bar{\omega}^2_{n_z 0m}=|m|+\frac{1}{2}\lambda^2 n_z(\gamma n_z-\gamma+2), \tag{22}$$

and the eigenfunctions are given by $\delta\rho^{(0)}_{n_p}(\bar{z},\bar{s})=(1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma-1}e^{im\varphi}b_{n_p,0}\bar{z}^{n_p}\bar{s}^{|m|}$.

(ii) For $n_p=2$, we have $n_s=0,1$ and $n_z=2,0$, respectively. In this case one gets the following solutions for the eigenfrequencies and eigenfunctions:

$$\bar{\omega}^2_{n_z n_s m}=(1+\gamma)(1+|m|)+\left(1+\frac{\gamma}{2}\right)\lambda^2\pm\Bigg[(1+\gamma+\gamma|m|)^2 +[-2-3\gamma+\gamma^2+(-2+\gamma)\gamma|m|]\lambda^2 +\left(1+\frac{\gamma}{2}\right)^2\lambda^4\Bigg]^{1/2}, \tag{23}$$

$$\delta\rho^{(2n_s)}_2(\bar{z},\bar{s})=(1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma-1}e^{im\varphi}\times(b_{0,0}+b_{2,0}\bar{z}^2+b_{2,1}\bar{s}^2)\bar{s}^{|m|}, \tag{24}$$

with

$$b_{2,1}/b_{0,0}=-(|m|-\bar{\omega}^2)/(2+|m|-\bar{\omega}^2) \tag{25}$$

and

$$b_{2,0}/b_{0,0}=\left[2+2|m|+\frac{1}{\gamma}(2+|m|-\bar{\omega}^2)\right]\times(|m|-\bar{\omega}^2)/(2+|m|-\bar{\omega}^2). \tag{26}$$

In Eq. (23), + (−) represents either the $20m$ ($01m$) mode for $\lambda\geq 1$ or the $01m$ ($20m$) mode for $\lambda\leq 1$. In these modes, the ratio of the axial and radial amplitudes is $|b_{20}/b_{21}|=|2+2|m|+(2+|m|-\bar{\omega}^2)/\gamma|>(<)1$ for $\lambda>(<)1$. Therefore, the condensate oscillates along the z-axial (xy-plane) direction. Note that the special solutions found by Heiselberg [11] and Cozzini and Stringari [35] for the breathing modes are the special case here for $m=0$. These $m=0$ breathing modes have been investigated experimentally by Grimm's group and Thomas's group[6–8]. In Fig. 1 we show, respectively, the experimental and theoretical results of the radial and axial breathing modes for a highly elongated trap with $\lambda=0.05$, where γ as a function of the interaction parameter $1/(k_F a_{sc})$ is given by Eq. (5). In both figures, the dot-dashed lines are, respectively, the $1/(k_F a_{sc})$ dependence of the eigenfrequencies for the radial and axial breathing modes by using the theoretical result given above.

Of course, our general solution provided here may give more theoretical predictions of the character of eigenexcitations of the system. As examples, in Fig. 2 (Fig. 3) we plot the density fluctuation distributions of the radial (axial) breathing mode as a function of $1/(k_F a_{sc})$ for a highly elongated trap $\lambda=0.1$ and $b_{0,0}=1$ along the x (z) direction with $y=z=0$ ($\bar{x}=0.5, y=0$). It can be seen that, on the BCS side of the BCS-BEC crossover, the radial mode ω_{012} dominates the xy-plane oscillation and the axial mode ω_{202} has a strong coupling between the z direction and the xy plane. However, on the BEC side of the BCS-BEC crossover, the radial mode

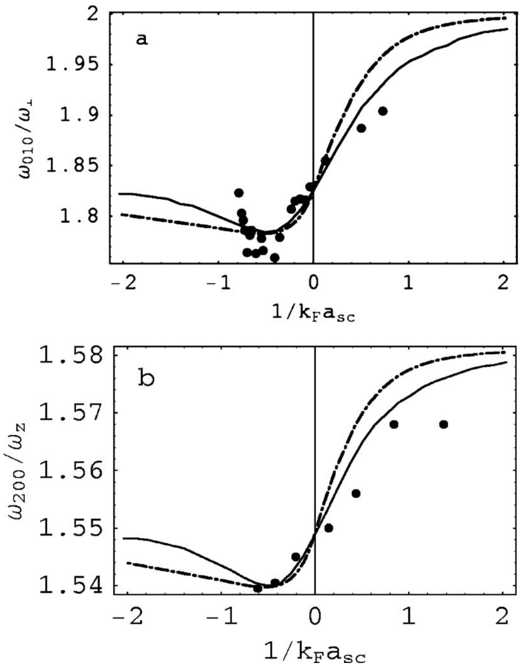

FIG. 1. The dimensionless interaction parameter $1/k_F a_{sc}$ dependence of the dimensionless frequency: (a) the radial breathing mode $\omega_{010}/\omega_\perp$ and (b) the axial breathing mode ω_{200}/ω_z. The dot-dashed line is plotted by using the theoretical result of this work for the eigenfrequency with anisotropic parameter $\lambda=0.05$, the value used in most of experiments. The solid line is the result of a parametrization based on Monte Carlo data [15]. Dots are the results from experiments [7] in (a) and Ref. [6] in (b).

ω_{012} mainly takes place in the xy plane and the axial mode ω_{202} dominates the z-direction oscillation with a coupling to the xy plane.

Shown in Fig. 4 is the $1/(k_F a_{sc})$ dependence of the radial mode ω_{202} [Fig. 4(a)] and axial mode ω_{012} [Fig. 4(b)] for different λ values (with $\lambda > 1$). Figure 5 shows the $1/(k_F a_{sc})$ dependence of the radial breathing mode ω_{012} for $\lambda < 1$. We see that the value of the anisotropic parameter λ in the trapping potential displays an obvious effect on the oscillating frequency of the collective excitations. We hope these interesting features found in this work can be explored in newly

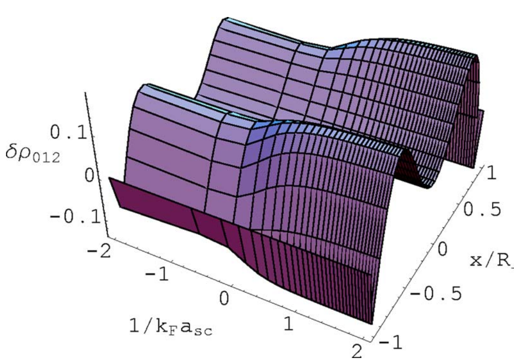

FIG. 2. (Color online). The theoretically predicted dimensionless density fluctuation of the breathing mode $\bar{\omega}_{012}$ in the xy plane vs dimensionless $1/k_F a_{sc}$ and $\bar{x}=x/R_\perp$ in the TF region for $\lambda=0.1$ and $b_{0,0}=1$ with $y=z=0$.

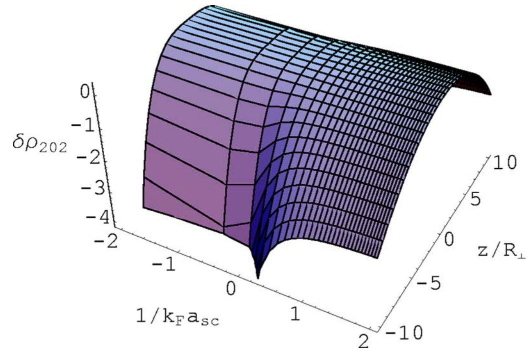

FIG. 3. (Color online). The theoretically predicted dimensionless density fluctuation of the breathing mode $\bar{\omega}_{202}$ in the z direction vs dimensionless $1/k_F a_{sc}$ and $\bar{z}=z/R_\perp$ in the TF region for $\lambda=0.1$ and $b_{0,0}=1$ with $\bar{x}=0.5$ and $y=0$.

designed experiments of superfluid Fermi gases in the BCS-BEC crossover.

(iii) For $n_p=3$, one has $n_s=0,1$ and $n_z=3,1$, respectively. The frequency spectrum is given by

$$\bar{\omega}^2_{n_z n_s m} = (1+\gamma)(1+|m|) + \left(2+\frac{3\gamma}{2}\right)\lambda^2 \pm \Bigg[(1+\gamma+\gamma|m|)^2 + [-2-5\gamma+3\gamma^2+(-2+3\gamma)\gamma|m|]\lambda^2 + \left(1+\frac{3\gamma}{2}\right)^2\lambda^4\Bigg]^{1/2}, \qquad (27)$$

and the corresponding functions are

$$\delta\rho_3^{(2n_s)}(\bar{z},\bar{s}) = (1-\bar{s}^2-\lambda^2\bar{z}^2)^{1/\gamma-1}e^{im\varphi} \times (b_{1,0}+b_{3,0}\bar{z}^2+b_{3,1}\bar{s}^2)\bar{z}\bar{s}^{|m|}, \qquad (28)$$

with $b_{3,1}/b_{1,0}=-(|m|+\lambda^2-\bar{\omega}^2)/(2+|m|+\lambda^2-\bar{\omega}^2)$ and

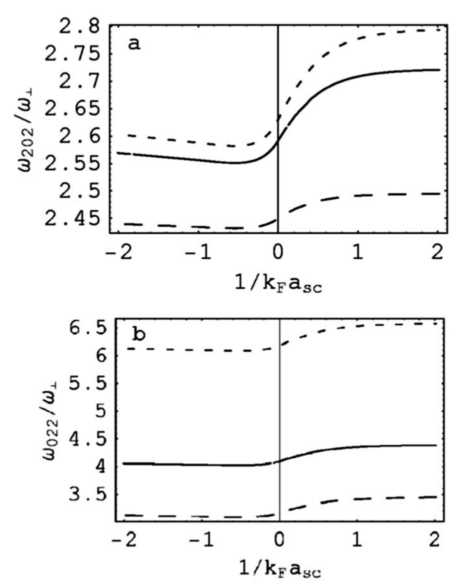

FIG. 4. The theoretically predicted dimensionless breathing mode frequencies vs the dimensionless interaction parameter $1/k_F a_{sc}$ for $\lambda > 1$. (a) $\bar{\omega}_{202}$ in the xy plane and (b) $\bar{\omega}_{012}$ in the z axis in the Thomas-Fermi limit with the long-dashed line, solid line, and short-dashed line for $\lambda=\sqrt{3}$, $\sqrt{8}$, and $\sqrt{24}$, respectively.

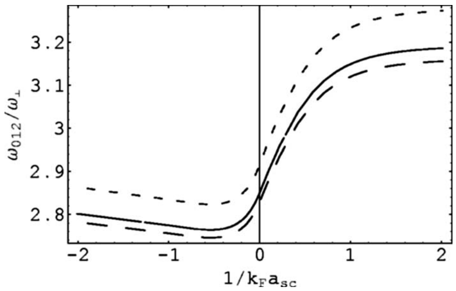

FIG. 5. The theoretically predicted dimensionless oscillating frequencies vs the dimensionless interaction parameter $1/k_F a_{sc}$ for the $\bar{\omega}_{012}$ mode in the case of $\lambda<1$. The long-dashed line, solid line, and short-dashed line correspond to $\lambda=0.1$, 0.5, and 0.9, respectively.

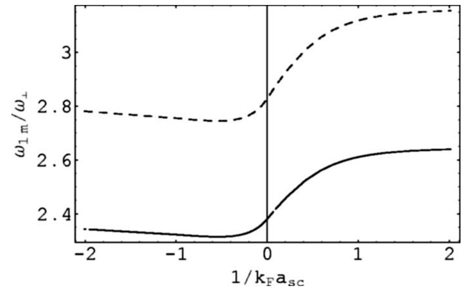

FIG. 6. The theoretically predicted dimensionless oscillating frequencies vs the dimensionless interaction parameter $1/k_F a_{sc}$ for the $\bar{\omega}_{01m}$ mode with $\lambda=0$. The solid line and short-dashed line correspond to $m=1$ and 2, respectively.

$$b_{3,0}/b_{1,0} = \tfrac{1}{3}\left[2+2|m|+\tfrac{1}{\gamma}(2+|m|+\lambda^2-\bar{\omega}^2)\right] \times(|m|+\lambda^2-\bar{\omega}^2)/(\bar{\omega}^2-2-|m|-\lambda^2).$$

In Eq. (27), + (−) represents either the $31m$ ($11m$) mode for $\lambda\geq 1$ or the $11m$ ($31m$) mode for $\lambda\leq 1$.

C. Circinally symmetric solutions ($\lambda\to 0$)

We now discuss the solutions for a highly anisotropic trap—i.e., $\omega_z\ll\omega_\perp$. In this case the hydrodynamic equation (13) in $\lambda\to 0$ limit has a solution of the form $\delta\rho_{n_s m}(\mathbf{s}) = \bar{s}^{|m|}e^{im\varphi}P(\bar{s}^2)$, expressed by plane polar coordinates $(\bar{s},\varphi)$, where the radial function $P(x)$ ($x=\bar{s}^2$) satisfies the differential equation

$$x(1-x)P'' + \left[(1+|m|)-\left(1+|m|+\frac{1}{\gamma}\right)x\right]P' + \frac{1}{2\gamma}(\bar{\omega}^2-|m|)P=0. \tag{29}$$

The eigenfunctions of Eq. (29) are a special hypergeometric series $P_{n_s m}(x)=F(-n_s,n_s+|m|+1/\gamma,1+|m|,\bar{s}^2)$—i.e., a classical ($n_s$)th-order Jacobi polynomial $P_{n_s m}(x)=n_s B[n_s,1+|m|]P_{n_s}^{(|m|,1/\gamma-1)}(1-2x)$. The eigenvalues are given by

$$\bar{\omega}^2_{n_s m} = 2n_s\gamma(n_s+|m|)+2n_s+|m|, \tag{30}$$

where the radial quantum number n_s takes the values $0,1,2,\ldots$. The energy levels are the same for $\pm m$. Figure 6 plots the $1/(k_F a_{sc})$ dependence of the radial ω_{11} and ω_{12} modes (i.e., $n_s=1$ and $m=1$ and 2, respectively, for $\lambda=0$). All of these plots are sensitive to the interaction parameter $1/(k_F a_{sc})$, which essentially originate from the relation $\gamma[1/(k_F a_{sc})]$ in the BCS-BEC crossover.

IV. CONCLUSION

We have investigated the collective excitations of a harmonically trapped superfluid Fermi gas in a BCS-BEC crossover. Starting from an equation of the superfluid wave function, we have solved analytically the linear eigenvalue problem for the collective modes of the system under a TF approximation. We have provided explicit analytical expressions for all eigenvalues and eigenfunctions, which are valid for both the BEC and BCS limits and also for the whole crossover regime. In our study, trapping potentials with both spherical and axial symmetries have been taken into account, and the solutions obtained cover some special solutions of breathing modes for an anisotropic trap, given by several authors previously. The features of these breathing modes in the BCS-BEC crossover have also been discussed and compared with related experimental and numerical results appearing recently. Further work is needed to go beyond the TF limit and consider a kinetic energy correction to both the ground state and collective excitations. Other work is to study the possible resonant interactions between the collective excitations based on the complete eigenmodes of collective excitations [36], which may improve our understanding of the physical properties of superfluid Fermi gases and lead to new experimental findings in the future.

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APPENDIX: EIGENVALUE SOLUTIONS FOR AN AXIALLY SYMMETRIC TRAPPING POTENTIAL ($\lambda\neq 1$)

It can be shown that a good operator

$$\hat{B} = -(1-\bar{s}^2)\frac{\partial^2}{\partial\bar{s}^2} - \left(\frac{1}{\bar{s}}-4\bar{s}\right)\frac{\partial}{\partial\bar{s}} - \frac{\partial^2}{\bar{s}^2\,\partial\varphi^2} - \left(\frac{1}{\lambda^2}-\bar{z}^2\right)\frac{\partial^2}{\partial\bar{z}^2} + 4\bar{z}\frac{\partial}{\partial\bar{z}} + 2\bar{s}\bar{z}\frac{\partial^2}{\partial\bar{s}\,\partial\bar{z}} \tag{A1}$$

commutes $\hat{H}_0$ and $L_z=-i\hbar\,\partial/\partial\varphi$, and $\hat{B}$ corresponds to $P_{n_p}^{(2n_s)}(\bar{z},\bar{s})$-like eigenfunctions, Eq. (21). By substituting Eq. (19) together with Eq. (21) into Eq. (13), an equation is rendered regarding the coefficients of the term $\bar{z}^{k-2n}\bar{s}^{2n+|m|}$:

$$4(n+1)(n+|m|+1)b_{k+2,n+1}+(k-2n+2)(k-2n+1)b_{k+2,n} = 4\lambda^2(n+1)(n+|m|+1)b_{k,n+1}+\left\{\frac{2}{\gamma}[2n+|m| + \lambda^2(k-2n)-\bar{\omega}^2]+\lambda^2(k-2n)(k-2n-1) + 4n(n+|m|)\right\}b_{k,n}+(k-2n+2)(k-2n+1)b_{k,n-1}. \tag{A2}$$

It is easy to prove that the series $b_{k,n}$ is divergent for $n_p \to \infty$. In order to obtain a convergent solution with the form of Eq. (21), one must use a cutoff condition—i.e., $b_{n_p+2,n+1} = b_{n_p+2,n} = 0$ for the fixed $n_p(=0,1,2,\ldots)$. The reason is that the expansion in Eq. (21) must be convergent at the pole $\lambda|\bar{z}|=1$ and that it can only be satisfied when the series ceases at a certain term of $\bar{z}$. We will not display explicitly the subscript k in the coefficient $b_{k,n}$ since we have taken its maximum value $k=n_p$. In order to get an explicit expression for b_n, we rewrite Eq. (A2) in the recurrence relation

$$b_{n+1}=\alpha_n b_n+\beta_{n-1}b_{n-1}, \tag{A3}$$

where all α_n and β_n can be easily obtained, which are given by

$$\alpha_n=[2\bar{\omega}^2-2|m|/\gamma-4n(n+|m|+1/\gamma) - \lambda^2(n_p-2n)(n_p-2n-1+2/\gamma)] \times[4\lambda^2(n+1)(n+|m|+1)]^{-1}, \tag{A4}$$

$$\beta_{n-1}=-\frac{(n_p-2n+1)(n_p-2n+2)}{4\lambda^2(n+1)(n+|m|+1)}. \tag{A5}$$

When $n_p \geq 2$, by defining $f_n=-\alpha_n b_n/\beta_{n-1}b_{n-1}$ and $g_n=\beta_n/\alpha_n\alpha_{n+1}$, the recurrence relation is reduced to the standard continued fraction form

$$f_n=\frac{1}{1+g_n f_{n+1}}. \tag{A6}$$

For a fixed n_p $(=0,1,2,\ldots)$, the expansion of $P_{n_p}^{(2n_s)}(\bar{z},\bar{s})$ terminates at $k=n_p$ for $\bar{z}$ and at $n=\text{int}[n_p/2]$ for $\bar{s}$. This leads to the condition $b_Z=0$ with $Z=1+\text{int}[n_p/2]$. After iterating Eq. (A3) for Z times and using $f_Z=0$, one finds a closed equation for determining the excitation spectrum $\bar{\omega}^2=\bar{\omega}^2(n_{p,}|m|,\lambda^2,\gamma)$ as

$$-1=\cfrac{g_0}{1+\cfrac{g_1}{1+\cdots \cfrac{\ddots}{1+g_{Z-2}}}}, \tag{A7}$$

where g_n has the form

$$g_n=-4\lambda^2(n+1)(n+|m|+1)(n_p-2n-1)(n_p-2n) \times\left\{\frac{2}{\gamma}[\bar{\omega}^2-2n-|m|-\lambda^2(n_p-2n)] - \lambda^2(n_p-2n)(n_p-2n-1)-4n(n+|m|)\right\}^{-1} \times\left\{\frac{2}{\gamma}[\bar{\omega}^2-2n-|m|-2-\lambda^2(n_p-2n-2)] - \lambda^2(n_p-2n-2)(n_p-2n-3)-4(n+1)(n+|m| + 1)\right\}^{-1}. \tag{A8}$$

Note that for any value of λ and γ and for the fixed value of $n_p(=2,3,\ldots)$, Eq. (A7) is an algebraic equation of order Z and it has possibly Z different solutions $\bar{\omega}^2_{n_z n_s m}=\omega^2_{n_z n_s m}/\omega^2_\perp$ for $n_s=0,1,2,\ldots,\text{int}[n_p/2]$, $n_z=n_p-2n_s$, and $m=0,\pm1,\pm2,\ldots$. Therefore, we can label the excitation spectrum by the modes $(n_z n_s m)$. The eigenfunctions $P_{n_p}^{(2n_s)}(\bar{z},\bar{s})$ are of the maximum order of n_p in $\bar{z}$ and of the maximum order of $2\,\text{int}[n_p/2]$ in $\bar{s}$. In general, the oscillating picture of the superfluid may be characterized by the axial parity of the corresponding eigenmodes. For even n_p, possible powers of $\bar{z}^{k-2n}$ are $0,2,\ldots,n_p$, $\mathbf{P}_z=[+]$, and the coefficients b_{k,n_s} are proportional to $b_{0,0}$ $(b_{1,0}\equiv 0)$. For odd n_p, possible powers of $\bar{z}^{k-2n}$ are $1,3,\ldots,n_p$, $\mathbf{P}_z=[-]$, and the coefficients b_{k,n_s} are proportional to $b_{1,0}$ $(b_{0,0}\equiv 0)$. In both cases, the coefficients b_{k,n_s} satisfy the N-dimensional linear algebraic equations with $N=\text{int}[\frac{1}{2}n_p]+\sum_{n_s=0}^{\text{int}[n_p/2]}n_s$. The existence of axial parity provides for a simple solution of Eq. (A7) for any values of λ and γ. Of course, the above results for $\gamma=1$ return to the one obtained in [24], and here we emphasize the role of $\gamma(y)$ in the BCS-BEC crossover.

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