Vol 15 No 8, August 2006
1009-1963/2006/15(08)/1871-08

Chinese Physics

Landau damping of collective modes in a harmonically trapped Bose–Einstein condensate*

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(Received 26 February 2006; revised manuscript received 20 March 2006)

This paper proposes a method for calculating the Landau damping of a low-energy collective mode in a harmonically trapped Bose–Einstein condensate. Based on the divergence-free analytical solutions for ground-state wavefunction of the condensate and eigenvalues and eigenfunctions for thermally excited quasiparticles, obtained beyond Thomas–Fermi approximation, this paper calculates the coupling matrix elements describing the interaction between the collective mode and the quasiparticles. With these analytical results this paper evaluates the Landau damping rate of a monopole mode in a spherical trap and discusses its dependence on temperature, particle number and trapping frequency of the system.

Keywords: Bose–Einstein condensation, collective modes, Landau damping
PACC: 6740D, 0530J, 4265K

1. Introduction

Due to the remarkable experimental realization of Bose–Einstein condensation in weakly interacting atomic gases, much effort has been devoted to the linear and nonlinear collective excitations in trapped Bose–Einstein condensates (BECs).[1−5] One of the challenging problems in this research direction is the temperature dependence of the damping rate and frequency shift of a collective mode, which has been received considerable attention in recent years in both experiment.[6−12] and theory.[13−21]

The dynamics of collective excitations in a BEC displays very rich behaviours, depending on the temperature and density of system. At high temperature and high density, the system is in a collisional regime and thus in a local thermodynamic equilibrium. The damping mechanism of a collective excitation is of dissipative type and the dynamics of the system may be described by a theory of two-fluid hydrodynamics.[13] In contrast, if the system is very dilute and at very low temperature the collisions between the excitations play a minor role. The damping mechanism in this collisionless regime is not related to thermalization processes but related to coupling between the excitations and thus can be described by a mean-field approach.[1,13−20] Up to now, most experiments with trapped Bose-condensed gases have been performed in this regime.

The present work is focused on the damping of a low-energy collective mode in the collisionless regime. The main damping mechanism in the regime is Landau damping, which arises by the process of a collective mode being absorbed by a quasiparticle (thermal excitation), and then turned into another quasiparticle. In recent years many theoretical techniques have been put forward to calculate the Landau damping of collective modes in trapped BECs.[1,14−20] Among them the time-dependent mean-field theory is widely employed since it gives an accurate description on the coupled dynamics of condensate and non-condensate components.[16] For evaluating Landau damping rate various coupling matrix elements describing the interaction between the collective mode and quasiparticles must be calculated, which, however, requires to solve Gross–Pitaevskii (GP) equation and Bogoliubov–de Gennes (BdG) equations in order to get the ground state wavefunction of condensate and the eigenvalues and eigenfunction of quasiparticles respectively. Due to the existence of trapping potential, it is very difficult to obtain the analytical solutions of these eigenfunctions. As far as we know, up to now nearly all works on this problem are based on numerical

*Project supported by the National Nature Science Foundation of China (Grant Nos 90403008 and 10434060), and State Key Development Program for Basic Research of China (Grant No 2005CB724508).

simulations.[14,15,17,18] Analytical work can be done for repulsive atomic interaction and a large particle number by using a Thomas-Fermi approximation (TFA), but the results obtained for the Bogoliubov amplitudes of quasiparticles and the coupling matrix elements display uncontrollable divergence.[22] Thus the Landau damping cannot be calculated based on such simple approach. In this work, we propose a new method for investigating the Landau damping of a low-energy collective mode in a harmonically trapped BEC. By using the divergence-free analytical solutions of ground-state wavefunction of condensate and eigenvalues and eigenfunctions of quasiparticles, obtained recently beyond TFA,[23,24] we can get divergence-free coupling matrix elements. For testing our method we calculate the Landau damping rate of a monopole mode in a spherical trap and discuss its dependence on temperature, particle number and trapping frequency of the system.

2. Time-dependent mean-field theory for Landau damping

We consider an interacting Bose gas trapped in an external potential $V_{\rm ext}(\boldsymbol{r})$. The grand-canonical Hamiltonian of the system in terms of bosonic creation and annihilation $\psi(\boldsymbol{r},t)$ and $\psi^\dagger(\boldsymbol{r},t)$ takes the form $H=\int {\rm d}\boldsymbol{r}\psi^\dagger(\boldsymbol{r},t)H_0\psi(\boldsymbol{r},t)+(g/2)\int {\rm d}\boldsymbol{r}\psi^\dagger(\boldsymbol{r},t)\psi^\dagger(\boldsymbol{r},t)\psi(\boldsymbol{r},t)\psi(\boldsymbol{r},t)$, where $H_0=-\hbar^2\nabla^2/(2m)+V_{\rm ext}(r)-\mu$ with μ being the chemical potential, $g=4\pi\hbar^2a_{\rm sc}/m$ with $a_{\rm sc}$ being the s-wave scattering length. The Heisenberg equation of motion for $\psi(\boldsymbol{r},t)$ reads

$$\begin{aligned}{\rm i}\hbar\partial\psi(\boldsymbol{r},t)/\partial t=&H_0\psi(\boldsymbol{r},t)\\&+g\psi^\dagger(\boldsymbol{r},t)\psi(\boldsymbol{r},t)\psi(\boldsymbol{r},t).\end{aligned}\qquad(1)$$

For a Bose-condensed gas one can use self-consistent time-dependent Hartree–Fock–Bogoliubov mean-field approximation, which corresponds to taking: (i) $\psi(\boldsymbol{r},t)=\varPhi(\boldsymbol{r},t)+\tilde\psi(\boldsymbol{r},t)$, where $\varPhi=\langle\psi\rangle$ and $\tilde\psi$ represent condensate and noncondensate components, respectively, the symbol $\langle\cdots\rangle$ ($\langle\cdots\rangle_0$) represents nonequilibrium (equilibrium) average; (ii) $\langle\tilde\psi^\dagger(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\rangle=\tilde n(\boldsymbol{r},t)$, $\langle\tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\rangle=\tilde m(\boldsymbol{r},t)$, where $\tilde n(\boldsymbol{r},t)$ and $\tilde m(\boldsymbol{r},t)$ denote normal and anomalous (thermal) particle densities respectively; (iii) $\tilde\psi^\dagger(\boldsymbol{r},t)\tilde\psi^\dagger(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)=4\tilde n(\boldsymbol{r},t)\tilde\psi^\dagger(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)+\tilde m(\boldsymbol{r},t)\tilde\psi^\dagger(\boldsymbol{r},t)\tilde\psi^\dagger(\boldsymbol{r},t)+\tilde m^*(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)$; (iv) $\langle\tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\rangle=\langle\tilde\psi^\dagger(\boldsymbol{r},t)\ \tilde\psi(\boldsymbol{r},t)\tilde\psi(\boldsymbol{r},t)\rangle=0$. Note that in (iv) we have set all averages of cubic products of the noncondensate operators to be zero. This is expected to be a good approximation for a dilute system[16]. Using above prescription we get the equation of motion for the condensate wavefunction

$$\begin{aligned}{\rm i}\hbar\partial\varPhi/\partial t=&H_0\varPhi+g|\varPhi|^2\varPhi\\&+2g\varPhi\tilde n(\boldsymbol{r},t)+g\varPhi^*\tilde m(\boldsymbol{r},t).\end{aligned}\qquad(2)$$

We see that there is a dynamical coupling between the condensate and noncondensate particles. The above equation reduces to GP equation if neglecting the effect of the thermal particles.

Take the Bogoliubov transformation

$$\tilde\psi(\boldsymbol{r},t)=\sum_j[u_j(\boldsymbol{r})\alpha_j(t)+v_j^*(\boldsymbol{r})\alpha_j^+(t)]$$

and

$$\tilde\psi^\dagger(\boldsymbol{r},t)=\sum_j[u_j^*(\boldsymbol{r})\alpha_j^\dagger(t)+v_j(\boldsymbol{r})\alpha_j(t)],$$

where the quasiparticle operator α_j, $\alpha_j^\dagger$ satisfy Bose commutation relations and the Bogoliubov amplitudes u_j and v_j satisfy the normalization condition

$$\int{\rm d}r[u_i^*(\boldsymbol{r})u_j(\boldsymbol{r})-v_i^*(\boldsymbol{r})v_j(\boldsymbol{r})]=\delta_{ij}\,.$$

The time evolutions of the normal and anomalous densities $\tilde n(\boldsymbol{r},t)$ and $\tilde m(\boldsymbol{r},t)$ are reflected by the following equations of motion:

$${\rm i}\hbar\partial f_{ij}(t)/\partial t=\langle[\alpha_i^\dagger(t)\alpha_j(t),H]\rangle\,,\qquad(3a)$$

$${\rm i}\hbar\partial g_{ij}(t)/\partial t=\langle[\alpha_i(t)\alpha_j(t),H]\rangle\,,\qquad(3b)$$

where $f_{ij}(t)=\langle\alpha_i^\dagger(t)\alpha_j(t)\rangle$ and $g_{ij}(t)=\langle\alpha_i(t)\alpha_j(t)\rangle$, f_i^0 is the equilibrium density of quasiparticles, whose explicit expression will be given below.

To study the damping of a collective mode of the condensate we assume $\varPhi=\varPhi_0(\boldsymbol{r})+\delta\varPhi(\boldsymbol{r},t)$, where $\delta\varPhi(\boldsymbol{r},t)$ is a small fluctuation. Then from Eq. (2) we get

$$(H_0+gn_0(\boldsymbol{r}))\varPhi_0(\boldsymbol{r})=0,\qquad(4a)$$

$$\begin{aligned}{\rm i}\hbar\partial\delta\varPhi/\partial t=&(H_0+2gn_0)\delta\varPhi+gn_0\delta\varPhi^*\\&+g\varPhi_0\sum_{ij}\{2[u_i^*u_j+v_i^*+v_i^*u_j]f_{ij}(t)\\&+[2v_iu_j+u_iu_j]g_{ij}(t)\\&+[2u_i^*v_j^*+v_i^*v_j^*]g_{ij}^*(t)\},\end{aligned}\qquad(4b)$$

where $n_0=|\varPhi_0(\boldsymbol{r})|^2$. Equation (4a) is a time-independent GP equation determining the ground state wavefunction $\Phi_0(\boldsymbol{r})$ of the condensate. In above equations the static distributions of the normal and anomalous particle densities have been assumed to be zero. This is reasonable because for the excitations at low temperature the static normal and anomalous particle densities are negligible.[17] If u_j and v_j are chosen to satisfy the BdG equations

$$Lu_j(\boldsymbol{r}) + gn_0(\boldsymbol{r},t)v_j(\boldsymbol{r}) = \epsilon_j u_j(\boldsymbol{r})\,, \tag{5a}$$

$$Lv_j(\boldsymbol{r}) + gn_0(\boldsymbol{r},t)u_j(\boldsymbol{r}) = -\epsilon_j v_j(\boldsymbol{r})\,, \tag{5b}$$

where $L = -\hbar^2\nabla^2/(2m) + V_{\rm ext}(\boldsymbol{r}) - \mu + 2gn_0(\boldsymbol{r})$, the Hamiltonian of the system can be expressed as the form

$$H = \text{constant} + \sum_j \epsilon_j \alpha_j^\dagger \alpha_j + H',$$

where ϵ_j is the energy of the quasiparticles and H' represents their interaction. Equations (3a) and (3b) now become

$$\begin{aligned}{\rm i}\hbar\partial f_{ij}(t)/\partial t =&(\epsilon_j - \epsilon_i)f_{ij}(t) + 2g(f_i^0 - f_j^0)\\ &\times\int {\rm d}\boldsymbol{r}\,\Phi_0\{[\delta\Phi + \delta\Phi^*]\times[u_iu_j^* + v_iv_j^*]\\ &+ \delta\Phi v_iu_j^* + \delta\Phi^* u_iv_j^*\},\end{aligned} \tag{6a}$$

$$\begin{aligned}{\rm i}\hbar\partial g_{ij}(t)/\partial t =&(\epsilon_j + \epsilon_i)g_{ij}(t) + 2g(1 + f_i^0 + f_j^0)\\ &\times\int {\rm d}\boldsymbol{r}\,\Phi_0\{[\delta\Phi + \delta\Phi^*]\times[u_i^*u_j^* + v_i^*v_j^*]\\ &+ \delta\Phi u_i^*u_j^* + \delta\Phi^* v_i^*v_j^*\}\,.\end{aligned} \tag{6b}$$

where $f_j^0 = \langle\alpha_j^+\alpha_j\rangle_0 = [\exp(\epsilon_j/(k_{\rm B}T)) - 1]^{-1}$.

Equations (4a), (6a) and (6b) can be solved by Fourier transform. We suppose that a collective mode of the condensate with oscillating frequency ω_0 is excited, i.e. $\delta\Phi(\boldsymbol{r},t) = u_{\rm osc}(\boldsymbol{r})\exp(-{\rm i}\omega_0 t)$ and $\delta\Phi^*(\boldsymbol{r},t) = v_{\rm osc}(\boldsymbol{r})\exp(-{\rm i}\omega_0 t)$. It is easy to show that $(u_{\rm osc}, v_{\rm osc})$ obeys also the BdG Eq.(5). Then by using the Fourier-transformed form of Eqs.(4b), (6a) and (6b), we can get the frequency correction of the collective mode, which includes real and imaginary parts, i.e. $\omega = \omega_0 + \eta - {\rm i}\gamma$. According to the expression of the imaginary part, one can readily get the Landau damping rate as

$$\gamma_L = \sum_{ij}\gamma_{ij}\delta(\omega_0 + \omega_i - \omega_j), \tag{7}$$

where $\omega_j = \epsilon_j/\hbar$, and

$$\gamma_{ij} = (4\pi g^2/\hbar^2)|A_{ij}|^2(f_i^0 - f_j^0), \tag{8}$$

which is called damping strength of the transition from state $|i\rangle$ to state $|j\rangle$,[17] where A_{ij} are the coupling matrix elements describing the energy transfer between the collective modes and quasiparticles, given by

$$\begin{aligned}A_{ij} = \int {\rm d}\boldsymbol{r}\,\Phi_0[&u_{\rm osc}(u_iu_j^* + v_iv_j^* + v_iu_j^*)\\ &+ v_{\rm osc}(u_iu_j^* + v_iv_j^* + u_iv_j^*)].\end{aligned} \tag{9}$$

From the expression[7] we see that only the three-mode resonant interactions which fulfil the resonant condition (i.e. phase-matching condition) $\omega_0 + \omega_i - \omega_j = 0$ will contribute to the Landau damping rate of the collective mode.

3. Divergence-free solutions of the BdG equations

In order to evaluate the Landau damping rate[7] one must calculate the coupling matrix elements A_{ij}, which, however, requires to solve the GP Eq.(4a) and the BdG Eqs.(5a) and (5b) for determining the ground state wavefunction Φ_0 and the eigenvalues and eigenfunctions u_j and v_j of the quasiparticles respectively. Exact analytical solutions for them are not available because of the existence of trapping potential. Up to now most results obtained are based on TFA, which however can not avoid the appearance of divergence in the Bogoliubov amplitudes and hence also in the coupling matrix elements.[22] In recent works this problem has been investigated in detail and divergence-free solutions have been obtained beyond the TFA[23,24]. Here we apply these analytical results to calculate the Landau damping rate of a collective mode in a trapped BEC.

For simplicity we consider a trapping potential of spherical symmetry with the form $V_{\rm ext}(\boldsymbol{r}) = m\omega_{\rm ho}^2 r^2/2$, where $r^2 = x^2 + y^2 + z^2$. We rescale the variables by introducing $\bar{r} = r/R_0$, $\bar{\nabla} = R_0\nabla$, and $\zeta = \hbar\omega_{\rm ho}/2\mu$, where $R_0 = \sqrt{2\mu/M\omega_{\rm ho}^2}$ is the characteristic radius of the condensate. Then the GP equation (4a) takes the dimensionless form

$$\zeta^2\sigma(\bar{\boldsymbol{r}}) + \bar{r}^2 - 1 + \mid \Phi_0(\bar{\boldsymbol{r}})/\Phi_0(0)\mid^2 = 0, \tag{10}$$

where $\sigma(\bar{\boldsymbol{r}}) = -[\bar{\nabla}^2\Phi_0(\bar{\boldsymbol{r}})]/\Phi_0$ is a quantity proportional to zero-point pressure. With the definitions $\phi_j^\pm = u_j \pm v_j$ and $\varpi = \omega_j/\omega_{\rm ho}$, the BdG Eqs.(5a) and (5b) become

$$\begin{aligned}&-\bar{\nabla}^2(1-\bar{r}^2)\phi_j^+ - (1-\bar{r}^2)\sigma\phi_j^+\\ &+\frac{\zeta^2}{2}[\bar{\nabla}^4 + 3\bar{\nabla}^2\sigma + \sigma\bar{\nabla}^2 + 3\sigma^2]\phi_j^+ = 2\bar{\omega}_j^2\phi_j^+,\end{aligned} \tag{11a}$$

$$-\bar{\nabla}^2(1-\bar{r}^2)\phi_j^- - (1-\bar{r}^2)\sigma\phi_j^-$$

$$+\frac{\zeta^2}{2}[\bar{\nabla}^4+\bar{\nabla}^2\sigma+3\sigma\bar{\nabla}^2+3\sigma^2]\phi_j^- = 2\bar{\omega}_j^2\phi_j^- . \quad (11b)$$

We solve Eq.(10) beyond TFA by using a Fetter-like variational wavefunction for ground state as[23,24] $\Phi_0(\bar{\mathbf{r}}) = C_0(1-\bar{r}^2)^{(q+1)/2}\Theta(1-\bar{r})$, where $C_0 = [N_0/(2\pi R_0^3 B(3/2,2+q))]^{1/2}$ is a normalized constant, with $B(p,q)$ being Beta function and $N_0 = \int \mathrm{d}\boldsymbol{r}|\Phi_0(\boldsymbol{r})|^2 = R_0^3\int \mathrm{d}\bar{\boldsymbol{r}}|\Phi_0(\bar{\boldsymbol{r}})|^2$ being the particle number in the condensate. The ratio $R_0/a_{\rm ho}$ and the chemical potential take the simple forms $R_0/a_{\rm ho} = [4P/B(3/2,2+q)]^{1/5}$ and $\mu = \hbar\omega_{\rm ho}[4P/B(3/2,2+q)]^{2/5}/2$ in TF regime, where $P = N_0 a_{\rm sc}/a_{\rm ho}$ is dimensional atom-atom interaction strength. The variational parameter q is chosen by minimizing the ground state energy of the system.[23,24]

Using the analytical solution of the ground state wavefunction $\Phi_0(\bar{\boldsymbol{r}})$ one can get the expression of $\sigma(\bar{\boldsymbol{r}})$. Then solving Eqs. (11a) and (11b) by taking ζ^2 as a small parameter we obtain the leading-order solution:

$$\phi_{nlm}^{\pm}(\bar{\boldsymbol{r}}) = [2/(I_{nl}R_0^3)]^{1/2}(\zeta\bar{\omega}_{nl})^{\pm\frac{1}{2}} (1-\bar{r}^2)^{\frac{q\mp1}{2}}\times\bar{r}^l P_{nl}(\bar{r}^2)Y_{lm}(\theta,\varphi), \quad (12)$$

where $Y_{lm}(\theta,\varphi)$ is spherical harmonic function, $I_{nl} = \int_0^1 \mathrm{d}x\, x^{l+1/2}(1-x)^q P_{nl}^2(x)$ is normalized integration, and the function $P(x)$ satisfies the hypergeometric differential equation

$$2x(1-x)\mathrm{d}^2P/dx^2+[2l+3-(2l+5+2q)x]\mathrm{d}P/\mathrm{d}x + [(\bar{\omega}_{nl})^2-l-lq]P=0 . \quad (13)$$

The solutions of Eq.(13) are nth-order Jacob polynomials $P_n^{(l+1/2,q)}(1-2x)$ which form an orthonormal function set in the interval $0\le x\le 1$ with weight $x^{l+1/2}(1-x)^q$. The eigenvalues are given by

$$(\bar{\omega}_{nl})^2 = (\bar{\omega}_{nl}^{\rm TF})^2+(2n+l)q, \quad (14)$$

where $(\bar{\omega}_{nl}^{\rm TF})^2 = 2n^2+2nl+3n+l$ are TFA eigenfrequencies. Here the quantum numbers $n=0,1,\ldots$; $l=0,1,\ldots$; and $m=0,\pm1,\ldots\pm l$. It should be noticed that for the mode with quantum number (n,l) there is $(2l+1)$-fold degeneracies. Note that in the present problem the eigenvalues and eigenfunctions are labelled by $j=(n,l,m)$, where n is principal quantum number denoting the number of nodes in the radial direction, l is orbital angular momentum number with m being its projection.

4. Results for Landau damping rate

In this section we apply the analytical results presented in the last two sections to investigate the Landau damping rate of a collective mode excited in a BEC.

4.1. Expressions of dimensionless coupling matrix elements and Landau damping

For the convenience of later calculation, it is useful to write the coupling matrix elements and the Landau damping rate in dimensionless forms. Because A_{ij} has a dimension of inverse volume and γ_{ij} has dimension of frequency square, we define $\bar{A}_{ij} = A_{ij}/a_{\rm ho}^3$ and $\bar{\gamma}_{ij} = \gamma_{ij}/\omega_{\rm ho}^2$, both of them are thus dimensionless. Then one obtains the dimensionless Landau damping rate as

$$\bar{\gamma}_{\rm L} \equiv \gamma_{\rm L}/\omega_{\rm ho} = \sum_{ij}\bar{\gamma}_{ij}\delta(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j) \quad (15)$$

with

$$\bar{\gamma}_{ij} = 4\pi(4\pi a_{\rm sc}/a_{\rm ho})^2|\bar{A}_{ij}|^2(f_i^0-f_j^0). \quad (16)$$

$$\bar{A}_{ij} = \frac{[4P/B(3/2,2+q)]^{1/10}}{4\pi[I_0I_iI_j\bar{\omega}_0\bar{\omega}_i\bar{\omega}_j]^{1/2}}\Big(\frac{N_0}{P}\Big)^{1/2} \times\int \mathrm{d}\bar{\boldsymbol{r}}\{(1-\bar{r}^2)^{2q}W_0W_iW_j^* \times[3\zeta^2\frac{\bar{\omega}_0\bar{\omega}_i\bar{\omega}_j}{1-\bar{r}^2}+(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)(1-\bar{r}^2)]\}, \quad (17)$$

where $W_j(\bar{r}) = \bar{r}^l P_{nl}(\bar{r}^2)Y_{lm}(\theta,\varphi)$ and f_j^0 is expressed as $f_j^0 = [\exp(2\zeta\bar{\omega}_j/\bar{T})-1]^{-1}$, where $\bar{T} = k_{\rm B}T/\mu$ is the dimensionless temperature. Note that, different from the numerical approaches in Refs.[14,15,17,18], in our present method $\bar{A}_{ij}$ can be calculated analytically based on the variational ground state wavefunction of the condensate and the eigenfunctions of quasiparticle given in the last section.

In a practical calculation, one must has a way to calculate the Dirac delta function appearing in Eq.(15). Noting that $\delta(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j) = (1/\pi)\lim_{\bar{\Delta}\to0}(\bar{\Delta}/2)/[(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)^2+(\bar{\Delta}/2)^2]$, we have

$$\bar{\gamma}_L = \lim_{\bar{\Delta}\to0}\bar{\gamma}_{\rm L}(\bar{\Delta}), \quad (18)$$

where

$$\bar{\gamma}_{\rm L}(\bar{\Delta}) = \frac{1}{\pi}\sum_{ij}\bar{\gamma}_{ij}\frac{\bar{\Delta}/2}{[(\bar{\omega}_0+\bar{\omega}_i-\bar{\omega}_j)^2+(\bar{\Delta}/2)^2]}. \quad (19)$$

4.2. Damping strength for various transitions

We consider a gas of ^{87}Rb atoms ($a_{\rm sc} = 5.82 \times 10^{-9}$ m) in a spherically symmetric harmonic trap with trapping frequency $\omega_{\rm ho} = 1000$ Hz, thus $a_{\rm ho} = 8.52 \times 10^{-7}$ m. The collective excitation under study is the monopole mode $(n, l, m) = (1, 0, 0)$ and hence we have $u_{\rm osc} = (\phi^{+}_{100} + \phi^{-}_{100})/2$ and $v_{\rm osc} = (\phi^{+}_{100} - \phi^{-}_{100})/2$. From the expression of coupling matrix elements Eq.(17) we see that there is a selection rule $\Delta l = 0$ and $\Delta m = 0$ for quasiparticle transitions and thus the related integration in Eq.(17) involves only in radial part. The particle number in the condensate at temperature T is given by $N_0(T) = N[1 - (T/T_c^0)^3]$, where N is the total particle number of the system and T_c^0 is the critical temperature of the BEC transition. Since practically the frequency of the collective mode has a finite line-width, the phase-matching condition for three-mode resonant interactions, $\omega_0 + \omega_i - \omega_j = 0$, can not be exactly satisfied. Thus a small mis-match for the three-mode resonant condition should be introduced. Under such consideration, we assume that the resonances contributing the Landau damping occur in the interval $0.82\,\omega_0 < \omega_{ij} < 1.18\,\omega_0$.[17] Therefore, the quantum numbers n and l should be chosen by all possible transitions allowed in this interval.

Noting that the eigenfunctions of quasiparticles for the levels with large quantum numbers n and l have very fast oscillations and their maxima are far away from the centre of the condensate, we find that the coupling matrix elements for the transition between levels with large n and l are small. In addition, the levels with larger n and l have larger energy and hence have smaller Bose occupation factor f_j^0. Therefore, the contribution to the damping strength by the energy levels of large n and l is not significant. By a suitable estimation, in our calculation we choose $n = 0, 1, 2, 3$, and consider the transitions $(n = 1, l) \longrightarrow (n = 0, l)$ for l from 2 to 8, $(n = 2, l) \longrightarrow (n = 1, l)$ for l from 6 to 16, and $(n = 3, l) \longrightarrow (n = 2, l)$ for l from 10 to 12. In Fig.1 we show that the histogram of the damping strength γ_{ij} (in unit of $\omega^2_{\rm ho}$), which is taken as a function of the transition frequencies ω_{ij} (in unit of $\omega_{\rm ho}$), is allowed by monopole selection rules. We take $N_0 = 2 \times 10^4$ for the spherical trap with $\omega_{\rm ho} = 1000$ Hz and for $k_{\rm B}T = 1.0\,\mu$ (corresponding to $T = 101.8$ nK). The monopole frequency is $\omega_0 = 2.24$ (in unit of $\omega_{\rm ho}$) and the variational parameter for the ground state wavefunction is given by $q = 1.05$. The arrow in the figure points to the frequency of the collective mode ω_0 (in unit of $\omega_{\rm ho}$). The position of bars corresponds to allowed transition frequency ω_{ij} (in unit of $\omega_{\rm ho}$), whereas their heights define the calculating values of γ_{ij}. The relatively large values of γ_{ij} correspond to the transitions between the lowest levels ($n = 1 \longrightarrow n = 0$) for different values of l (l from 2 to 8 from left to right). The transitions between the higher levels ($n = 2 \longrightarrow n = 1$) and ($n = 3 \longrightarrow n = 2$) give the relatively small value of γ_{ij}.

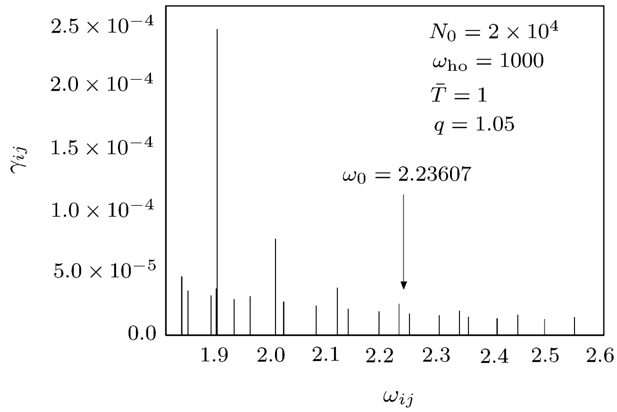

Fig.1. Histogram of the damping strength γ_{ij} (in unit of $\omega^2_{\rm ho}$) as a function of resonance frequencies ω_{ij} (in unit of $\omega_{\rm ho}$), allowed by monopole selection rules for $N_0 = 2 \times 10^4$ in a spherical harmonic trap with $\omega_{\rm ho} = 1000$ Hz, at $k_{\rm B}T = 1.0\,\mu$. The monopole frequency is $\omega_0 = 2.24$ (in unit of $\omega_{\rm ho}$). The variational parameter for the ground state wavefunction is $q = 1.05$.

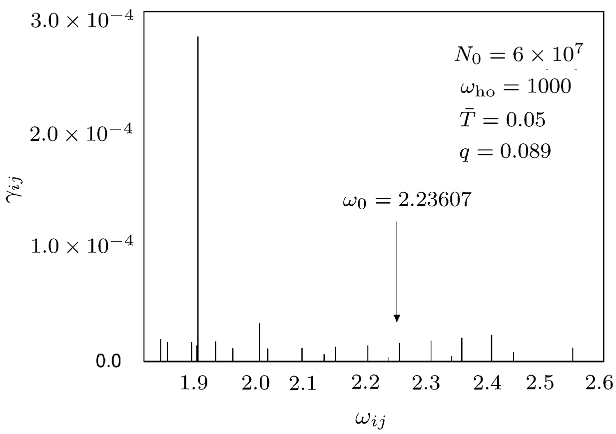

Fig.2. Same as Fig.1 but for $N_0 = 6 \times 10^7$ and $k_{\rm B}T = 0.05\,\mu$. The variational parameter in this case is $q = 0.089$.

Figure 2 shows the result for the damping strength for $N_0 = 6 \times 10^7$ at $k_{\rm B}T = 0.05\,\mu$ (corresponding to $T = 101.6$ nK) with other parameters which are the same as those in Fig.1. We see that in this case the difference of the damping strength between strong and weak transitions is more impressive than that for a smaller condensate (Fig.1). The strong transitions comes mainly from $n = 1$ to $n = 0$ for l=2 and 3. Obviously, these strong transitions give significant contributions to the Landau damping rate.

4.3. The result for Landau damping

In order to evaluate the Landau damping rate of the collective mode, one must first calculate $\bar{\gamma}_{\rm L}(\bar{\varDelta})$, which is given by Eq.(19). If the variation of $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ with respect to $\bar{\varDelta}$ is weak, an extrapolation back to $\bar{\varDelta} \to 0$ can be made and hence the value of $\bar{\gamma}_L$ can thus be obtained.[17]

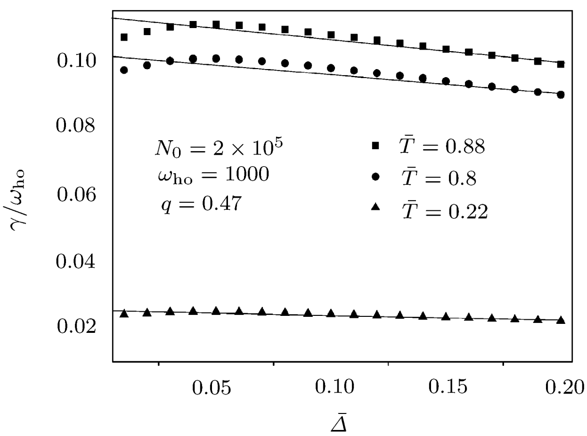

Fig.3. Landau damping rate $\gamma_{\rm L}$ (in unit of $\omega_{\rm ho}$) as a function of the Lorentzian width $\bar{\varDelta}$, for $N_0 = 2 \times 10^5$ atoms in the spherical trap with $\omega_{\rm ho} = 1000$ Hz for different temperatures. Triangles, solid dots, and squares are numerical values calculated at $k_{\rm B}T/\mu = 0.22$, 0.8, 0.88 respectively. The variational parameter is $q = 0.47$.

Figure 3 shows the result of $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ for $N_0 = 2\times10^5$ at $k_{\rm B}T = 0.22\,\mu$, $0.8\,\mu$, and $0.88\,\mu$, respectively. One can see that the variation of $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ is weak when $\bar{\varDelta}$ lies between 0.04 and 0.20. In fact, under the condition $\bar{\varDelta}\omega/\omega_{\rm ho} << \bar{\varDelta} << 1$, $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ has only a weak dependence on $\bar{\varDelta}$, where $\bar{\varDelta}\omega$ is the average distance of the transitions. $\bar{\varDelta}\omega$ is indeed small because of finite lifetime of the quasiparticles. In addition, a real system cannot be exactly isotropic and hence the $(2l+1)$-fold degeneracy of the energy-levels is broken, it results in small energy-level separation. By fitting the data of $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ to a straight line and extrapolating it back to $\bar{\varDelta} = 0$, we can obtain the Landau damping rate of the collective mode.

In Fig.3 we have shown $\bar{\gamma}_{\rm L}(\bar{\varDelta})$ for three different dimensionless temperature values, i.e. $\bar{T}\ (= k_{\rm B}T/\mu) =$ 0.22, 0.8 and 0.88, which correspond to $T = 50$ nK, 182 nK, and 200 nK respectively. By extrapolation we obtain $\bar{\gamma}$=0.02, 0.1, and 0.11 (in unit of $\omega_{\rm ho}$), and hence the dimensional Landau damping rates are given by $\gamma_{\rm L} = 20\ {\rm s}^{-1}$, $100\ {\rm s}^{-1}$, and $110\ {\rm s}^{-1}$ respectively. Our results are larger than experimental values for an anisotropic trap[6] but more close to the theoretical ones obtained in Ref.[17].

4.4. N_0- and $\omega_{\rm ho}$-dependence of the Landau Damping

With above results we can also investigate the N_0- and $\omega_{\rm ho}$-dependence of the Landau damping.

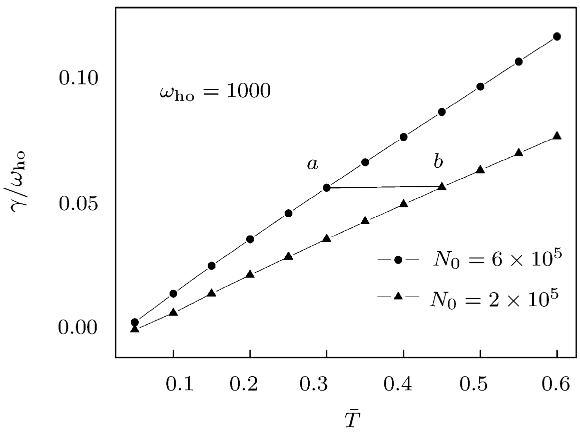

Fig.4. Landau damping rate $\gamma_{\rm L}$ (in unit of $\omega_{\rm ho}$) as a function of $k_{\rm B}T/\mu$ for $N_0 = 2 \times 10^5$ and $N_0 = 6 \times 10^5$ respectively. The variational parameter is $q = 0.47$ ($q = 0.34$) for $N_0 = 2 \times 10^5$ ($N_0 = 6 \times 10^5$).

In Fig.4, we have plotted the damping rate as a function of $k_{\rm B}T/\mu$ for $\omega_{\rm ho} = 1000$ Hz. Two cases for different atomic number in the condensate, i.e. $N_0 = 2 \times 10^5$ and $N_0 = 6 \times 10^5$, are considered. We see that, as expected, the Landau damping rates increase with temperature. The reason is that the number of quasiparticles available becomes larger when T increases. In addition, the damping rates for different atomic numbers at the same temperature display no significant difference. This can be seen, for example, by looking at the points a and b in Fig.4, which represent the case of different N_0 but with the same temperature. Note that with the expression of the chemical potential ($\mu = \hbar\omega_{\rm ho}[4P/B(3/2, 2+q)]^{2/5}/2$) one has $T = \bar{T}\hbar(\omega_{\rm ho}/2k_{\rm B})[4P/B(3/2, 2+q)]^{2/5}$.

Figure 5 show the damping rates versus $k_{\rm B}T/\mu$ but for different trapping frequencies. Three different cases for $N_0 = 2 \times 10^5$, i.e. $\omega_{\rm ho} = 300$ Hz, 1000 Hz, and 3000 Hz, have been taken into account. We see that the Landau damping rates increase with the trapping frequency. The reason is that the atomic density increases as the trapping frequency becomes larger.

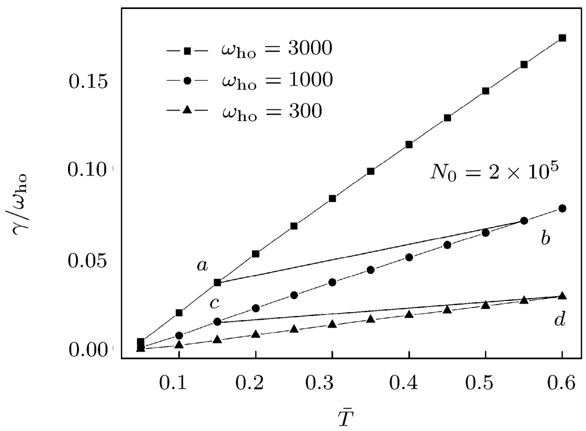

Fig.5. Landau damping rate $\gamma_{\rm L}$ (in unit of $\omega_{\rm ho}$) as a function of $k_{\rm B}/\mu$ for $N_0 = 2\times10^5$ for different trapping frequencies. Triangles, solid dots, and squares correspond respectively to $\omega_{\rm ho} = 300$ Hz, 1000 Hz, and 3000 Hz. The variational parameters for the ground state wavefunction are $q = 0.58$, $q = 0.47$, and $q = 0.40$ respectively.

In the figure, the points a and b (also c and d) are of the same temperature, which means that the damping rate varies almost linearly with respect to $\omega_{\rm ho}$.

5. Conclusion

In this paper we have proposed a method for calculating the Landau damping rate of a low-energy collective mode excited in a BEC with a harmonical trap. By using the divergence-free analytical solutions for the ground-state wavefunction of the condensate and the eigenvalues and eigenfunctions for quasiparticles, which are obtained beyond TFA, we have calculated the coupling matrix elements describing the three-mode resonant interactions among the collective mode and the quasiparticles. In terms of these analytical results we have evaluated the Landau damping rate of a monopole mode in a spherical trap and discussed its dependence on temperature, particle number and trapping frequency of the system. These results are instructive for further studies on Landau damping rates of other low-energy collective modes and for the cases of anisotropic traps.

Acknowledgment

The authors thank Professor Weiping Zhang for useful discussions.

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