Anomalous particle-number fluctuations in a three-dimensional interacting Bose-Einstein condensate

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The particle-number fluctuations originated from collective excitations are investigated for a three-dimensional, repulsively interacting Bose-Einstein condensate (BEC) confined in a harmonic trap. The contribution due to the quantum depletion of the condensate is calculated and the explicit expression of the coefficient in the formulas denoting the particle-number fluctuations is given. The results show that the particle-number fluctuations of the condensate follow the law $\langle \delta N_0^2 \rangle \sim N^{22/15}$ and the fluctuations vanish when temperature approaches the BEC critical temperature.

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The remarkable experimental realization of Bose-Einstein condensation in trapped, weakly interacting atomic gases has stimulated intensive theoretical and experimental researches on Bose-Einstein condensed gases [1]. In particular, much attention has been paid to the study of the particle-number fluctuations of Bose-Einstein condensates (BECs) [2]. This is mainly due to the fact that the particle-number fluctuations play a central role in the understanding of the statistical properties of BEC. In addition, the particle-number fluctuations of a condensate may result in the fluctuations of chemical potential and thus lead to the phase diffusion of condensate [3]. Therefore, the study on the particle-number fluctuations of BEC is not only of an intrinsic theoretical interest but also useful to understand and control the coherent property of BEC. On the other hand, a clear theoretical understanding will be helpful to guide a direct experimental observation on the particle-number fluctuations in a BEC.

Up to now there exist a lot of theoretical works exploring the property of the particle-number fluctuations in BECs. In an ideal Bose gas the particle-number fluctuations $\langle \delta N_0^2 \rangle \equiv \langle N_0^2 \rangle - \langle N_0 \rangle^2$ have been studied rather thoroughly in a homogeneous case (e.g., in a box) [4] and an inhomogeneous case (e.g., in a trap) [5]. The role of interatomic interaction in the particle-number fluctuations is an important theoretical problem and hence intensive theoretical investigations have been given [2,6–16]. The behavior of the particle-number fluctuations is described by the value of the power γ in the expression $\langle \delta N_0^2 \rangle \sim N^{\gamma}$ with N being the total number of particles. References [6,7,13] give the result of *anomalous* fluctuations with $\gamma = 4/3$, while Ref. [9] argues that the fluctuations should be normal with $\gamma = 1$.

For the temperature far below the critical temperature of BEC, the collective excitations created from condensate play an important role in the particle-number fluctuations. The physical reason is that due to the interatomic interaction the creation of the collective excitations induces a change of the particle number in the condensate. In the last few years, the particle-number fluctuations originated from collective excitations have been studied for three-dimensional (3D) [6–8,13,14] and 2D [15] weakly interacting Bose-Einstein condensed gases. In a pioneering work by Giorgini *et al.* [6], the particle-number fluctuations are investigated within a traditional particle-number-nonconserving Bogoliubov method. Kocharovsky *et al.* [7] extended the results of Ref. [6] by using a particle-number-conserving operator formalism. The scale behavior of the interacting condensate in a box was investigated in Ref. [8] with an arbitrary atomic interaction.

For a Bose-Einstein condensed gas confined in a magnetic trap, the total number of particles N of the system is conserved and hence a canonical ensemble is appropriate to investigate the particle-number fluctuations in the condensate. Directly from the canonical partition function of the system and by using a developed saddle-point method, a systematic approach was proposed by the present authors [13–16] for investigating the role of collective excitations on the particle-number fluctuations. Within the canonical ensemble a general method has been given recently for studying the thermodynamic properties of Bose-Einstein condensed gases based on the calculation of the probability distribution function Refs. [13,14]. In Ref. [15], the theory in Refs. [13,14] is developed to calculate the particle-number fluctuations due to collective excitations by including the effect of quantum depletion.

For a 3D Bose-Einstein condensed gas confined in a magnetic trap, although the effect of the quantum depletion of condensate is discussed by Giorgini *et al.* [6] within the particle-number-nonconserving Bogoliubov approach, explicit expression in the formula denoting the particle-number fluctuations by the quantum depletion is not provided. In the present work, we shall carefully calculate the particle-number fluctuations due to collective excitations within a canonical ensemble. We take into account the effect of quantum depletion and finite size of the condensate.

We first give a brief description for the method developed in Ref. [15] . For a condensate confined in a 3D magnetic trap, the collective excitations generated from the condensate can be described by three quantum numbers n (the number of radial modes), l (the magnitude of the total angular momentum), and m (the z component of the angular momentum). Assuming that N^B_{nlm} is the number of the collective excitations indexed by the quantum numbers nlm and N_0 is the number of particles in the condensate, the canonical partition function of the system takes the form

$$Z[N]=\sum{}' \exp\left[-\beta\left(E_0+\sum_{nlm\neq 0} N^B_{nlm}\varepsilon_{nlm}\right)\right], \tag{1}$$

where the prime in the summation represents the condition that the total number of atoms in the system should be conserved within the canonical ensemble, E_0 is the energy of the condensate which can be regarded as a ground-state energy of the system. Using the Bogoliubov theory [17,18] and a saddle-point method developed in Refs. [13–15], for the temperature below the BEC critical temperature the normalized probability distribution function reads [15]

$$G_n(N,N_0)=A_n\exp\left[-\frac{(N_0-N_0^p)^2}{2\Xi}\right], \tag{2}$$

where A_n is the normalization constant and N_0^p is the most probable value of the atomic number in the condensate. The quantity Ξ is given by

$$\Xi=\sum_{nlm\neq 0}\left[\left(\int u^2_{nlm}dV+\int v^2_{nlm}dV\right)^2\left(\frac{k_BT}{\varepsilon_{nlm}}\right)^2 +2\left(\int u^2_{nlm}dV+\int v^2_{nlm}dV\right)\int v^2_{nlm}dV\left(\frac{k_BT}{\varepsilon_{nlm}}\right) +\left(\int v^2_{nlm}dV\right)^2\right], \tag{3}$$

where u_{nlm}, v_{nlm}, and ε_{nlm} (the energy of the collective mode nlm) are determined by the following coupled equations:

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+2gn(\mathbf{r})\right)u_{nlm}+gn_0(\mathbf{r})v_{nlm} =\varepsilon_{nlm}u_{nlm}, \tag{4}$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V_{ext}(\mathbf{r})-\mu+2gn(\mathbf{r})\right)v_{nlm}+gn_0(\mathbf{r})u_{nlm} =-\varepsilon_{nlm}v_{nlm}, \tag{5}$$

where $V_{ext}(\mathbf{r})$ is the external potential confining the Bose gas, μ and g are, respectively, the chemical potential of the system and the coupling constant; $n(\mathbf{r})$ and $n_0(\mathbf{r})$ are the density distributions of the Bose gas and the condensate, respectively.

For temperature far below the critical temperature, we have $N_0^p \gg 1$. In this situation, from Eq. (2) the fluctuations of the condensate contributed from the collective excitations are given by

$$\langle\delta^2N_0\rangle=\langle N_0^2\rangle-\langle N_0\rangle^2=\Xi. \tag{6}$$

For Eqs. (4) and (5), u_{nlm} and v_{nlm} can be approximated as [19]

$$u_{nlm}\approx -v_{nlm}\approx i\sqrt{\frac{gn_0(\mathbf{r})}{2\varepsilon_{nlm}}}\chi_{nlm}. \tag{7}$$

For a Bose gas confined in an isotropic harmonic potential with angular frequency ω, χ_{nlm} and $\varepsilon_{nlm}(=\hbar\omega_{nlm})$ are determined by the eigenequation

$$-\frac{\omega^2}{2}\boldsymbol{\nabla}\cdot[(R^2-r^2)\boldsymbol{\nabla}\chi_{nlm}]=\omega^2_{nlm}\chi_{nlm}, \tag{8}$$

where R is the radius of the condensate, determined by the chemical potential μ of the system through $\mu=m\omega^2R^2/2$. In Eq. (8), ω_{nlm} and χ_{nlm} are found to be [20]

$$\omega_{nlm}=\omega(2n^2+2nl+3n+l)^{1/2}, \tag{9}$$

and

$$\chi_{nlm}=A_{nl}P_l^{(2nt)}\left(\frac{r}{R}\right)r^lY_{lm}(\theta,\varphi)\Theta(R-r), \tag{10}$$

where A_{nlm} is the normalization constant determined by $\int|\chi_{nlm}|^2dV=1$ and $\Theta(x)$ is a step function. In Eq. (10), $P_l^{(2n)}(x)=\sum_{k=0}^n\alpha_{2k}x^{2k}$ is a polynomial with coefficients satisfying the recurrence relation $\alpha_{2k+2}=-\alpha_{2k}(n-k)(2l+2k+2n+3)/(k+1)(2l+2k+3)$ with $\alpha_0=1$.

With the above formulas we now calculate the particle-number fluctuations in the condensate. Substituting the above results into Eqs. (3) and (6), we obtain the particle-number fluctuations due to collective excitations:

$$\langle\delta^2N_0\rangle=\mathfrak{R}_1+\mathfrak{R}_2+\mathfrak{R}_3, \tag{11}$$

where

$$\mathfrak{R}_1=\lambda_1\left(\frac{\mu}{\hbar\omega}\right)^2\left(\frac{k_BT}{\hbar\omega}\right)^2, \tag{12}$$

$$\mathfrak{R}_2=\lambda_2\left(\frac{\mu}{\hbar\omega}\right)^2\frac{k_BT}{\hbar\omega}, \tag{13}$$

and

$$\mathfrak{R}_3=\frac{\lambda_3}{4}\left(\frac{\mu}{\hbar\omega}\right)^2. \tag{14}$$

The coefficients λ_1, λ_2, λ_3 are given by

$$\lambda_1=\sum_{nl\neq 0}\frac{(2l+1)\vartheta^2_{nl}}{(2n^2+2nl+3n+l)^2},$$

$$\lambda_2=\sum_{nl\neq 0}\frac{(2l+1)\vartheta^2_{nl}}{(2n^2+2nl+3n+l)^{3/2}},$$

$$\lambda_3=\sum_{nl\neq 0}\frac{(2l+1)\vartheta^2_{nl}}{(2n^2+2nl+3n+l)}, \tag{15}$$

with

$$\vartheta_{nl}=\frac{\int_0^1 (1-x^2)[P_l^{(2n)}(x)]^2 x^{2l+2}dx}{\int_0^1 [P_l^{(2n)}(x)]^2 x^{2l+2}dx}. \tag{16}$$

The factor $(2l+1)$ in λ_1, λ_2, and λ_3 is due to the $(2l+1)$-fold degeneracy of the angular momentum. The numerical results of the above parameters are $\lambda_1=0.47$, $\lambda_2=0.94$, $\lambda_3=4.96$. Note that the coefficients in the particle-number fluctuations of the condensate are different from the result given in Refs. [6,13].

In Eq. (11), $\mathfrak{R}_1$ is the leading term of the particle-number fluctuations, while $\mathfrak{R}_3$ shows the particle-number fluctuations due to the quantum depletion of the condensate which do not vanish even at zero temperature.

For the Bose-Einstein condensed gas in the harmonic potential, the chemical potential is given by

$$\mu=\frac{\hbar\omega}{2}\left(\frac{15N_0 a}{a_{ho}}\right)^{2/5}, \tag{17}$$

with a being the s-wave scattering length and $a_{ho}=\sqrt{\hbar/m\omega}$ the harmonic-oscillator length. Using the expression of the critical temperature of an ideal Bose gas, $T_c^0=\hbar\omega[N/\zeta(3)]^{1/3}/k_B$, and introducing the dimensionless parameters $t=T/T_c^0$ and $\sigma=a/a_{ho}$, we obtain the leading term of $\mathfrak{R}_1$:

$$\mathfrak{R}_1=0.91t^2(1-t^3)^{4/5}\sigma^{4/5}N^{22/15}. \tag{18}$$

We see that the leading term of the particle-number fluctuations in the condensate exhibits an anomalous behavior of $\langle\delta^2 N_0\rangle\sim N^{22/15}$.

It is straightforward to obtain the results of $\mathfrak{R}_2$ and $\mathfrak{R}_3$, which are given by

$$\mathfrak{R}_2=1.93t(1-t^3)^{4/5}\sigma^{4/5}N^{17/15} \tag{19}$$

and

$$\mathfrak{R}_3=2.70(1-t^3)^{4/5}\sigma^{4/5}N^{4/5}. \tag{20}$$

We now make some remarks on the results obtained above:

(i) The anomalous behavior of the particle-number fluctuations of the condensate predicted by Eq. (18) is obtained when the particle number of the system is finite. In the thermodynamics limit of the Bose gas in the harmonic potential, i.e., letting $N\to\infty$ and $\omega\to 0$ while keeping $N\omega^3$ constant [21], the anomalous behavior of the form $\langle\delta^2 N_0\rangle\sim N^{4/3}$ given in Refs. [6,13] can be obtained.

(ii) Different from the result obtained in Ref. [6] where only the low-temperature behavior $T\ll T_c^0$ is taken into account, in the present work the temperature dependence of the fluctuations is valid for the temperature region below the critical temperature. When the temperature T approaches the BEC critical temperature T_c^0, the particle number in the condensate approaches zero and hence the fluctuations originated from the collective excitations vanish, as shown in Eq. (18).

(iii) For a higher temperature the collective excitations play only a second role and one must consider the contribution from single-particle excitations. The particle-number fluctuations contributed by the single-particle excitations show the behavior of $\langle\delta^2 N_0\rangle\sim N$ [13].

(iv) From Eq. (18), we see that the confining potential has the effect of increasing the particle-number fluctuations. It is easy to know from Eq. (18) that $\mathfrak{R}_1\sim\omega^{2/5}$.

In conclusion, we have investigated the particle-number fluctuations originated from collective excitations in a three-dimensional, repulsively interacting Bose-Einstein condensate confined in a harmonic potential. We have carefully calculated the contribution to the fluctuations due to the quantum depletion and provided the explicit expression of the coefficient in the formulas denoting the particle-number fluctuations. Our results show that the particle-number fluctuations of the condensate due to the collective excitations display an anomalous behavior of the form $\langle\delta N_0^2\rangle\sim N^{22/15}$. When $T\to T_c^0$ such fluctuations approach to zero and hence the fluctuations due to single-particle excitations become dominant. It is possible that the anomalous behavior predicted in this work could be observed experimentally by means of, e.g., the scattering of short and nonresonant laser pulses on a BEC [22].

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