Optics Letters

Localization of light in a parity-time-symmetric quasi-periodic lattice

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Received 26 March 2015; accepted 5 May 2015; posted 21 May 2015 (Doc. ID 236895); published 8 June 2015

We address the propagation of light in a parity-time ($\mathcal{PT}$) symmetric quasi-periodic optical lattice. We show that the $\mathcal{PT}$-symmetry breaking threshold of the system depends on the relative strength of two simple lattices forming quasi-periodic structure. The increase of the imaginary part of such structure leads to a dynamical phase transition from delocalization to re-localization for all eigenmodes. The nontrivial interplay between $\mathcal{PT}$-symmetry breaking and the delocalization-localization transition in the system opens new prospects for the manipulation of dynamical evolution of light beams.

OCIS codes: (260.2710) Inhomogeneous optical media; (050.1940) Diffraction.

http://dx.doi.org/10.1364/OL.40.002758

Propagating beams exhibit fundamentally different dynamics for one-dimensional (1D) periodic and disordered media. In the former case, the eigenstates of the beams are delocalized Bloch modes, hence delocalized input beams that excite the whole spectrum of the eigenmodes gradually spread out upon evolution due to the dephasing of such modes. The opposite situation is observed in disordered systems, where all eigenmodes are localized, so that initial excitation causes beating between multiple localized modes within a finite spatial domain. This phenomenon is known as Anderson localization [1]. In a 1D disordered system, the modification of system parameters leads to only a change of the degree of localization, i.e., no dynamical phase transition from localization to delocalization occurs.

The situation changes drastically if there is a modulation of system parameters characterized by two different periods, with the ratio of the two periods evolving between a periodic system (when the ratio is a rational number) and a quasi-periodic one (when the ratio is irrational number). In this case, a localization-delocalization transition (LDT) can occur in a 1D system. This phenomenon was first described on the basis of the Harper [2] or Aubry and Andre [3] semi-discrete model (below referred to as H-AA model), where a long-periodic potential was added to the tight-binding approximation describing a quantum particle in a short-periodic potential. In the H-AA model, a LDT was shown to occur for some specific values of the large period.

Since LDT is a ubiquitous wave phenomenon that can be encountered in various areas of physics, it has attracted a considerable attention during last several years (see e.g., [4–8] and references therein). In particular, a number of rigorous mathematical results [9] as well as experimental validation [10] were presented.

Although quasi-periodic structures exhibit no disorder, they allow observation of LDT, thereby realizing an intermediate situation between ordered and disordered systems. Being essentially deterministic, quasi-periodic structures are promising for various applications, such as controllable steering and diffraction engineering of light beams. From this viewpoint, they are desirable for the management of system properties *in situ*.

Notice that, in contrast to conservative systems whose parameters are usually fixed by fabrication process, dissipative systems allow relatively simple dynamical control of their gain and loss by using external pump fields. If only gain is present in such a system, the amplitude of the wavepacket will unavoidably grow upon its evolution. However, the spectrum of eigenmodes may remain purely real for a very specific class of dissipative system, where inhomogeneous gain and loss are properly balanced to ensure that complex refractive index satisfies the condition $n(x) = n^*(-x)$, known as the parity-time ($\mathcal{PT}$) symmetry [11]. Such systems were already realized experimentally in optics [12]. $\mathcal{PT}$-symmetric systems may exhibit entirely real spectrum even though the underlying potential is complex (typically this occurs if the imaginary part of potential is below certain threshold value). This situation is referred to as unbroken $\mathcal{PT}$-symmetry [13]. In the state with broken $\mathcal{PT}$-symmetry, usually achieved when imaginary part of potential becomes high enough, the spectrum becomes complex. The point between broken and unbroken phases represents an exceptional point.

A $\mathcal{PT}$-symmetric photonic quasi-periodic lattice allows one to investigate the interplay between $\mathcal{PT}$-symmetry breaking and LDT transition. Our goal in this Letter is to show that in a complex quasi-periodic potential, the $\mathcal{PT}$-symmetry breaking point is affected by the ratio of amplitudes of two periodic lattices forming quasi-periodic structure, and that the imaginary part of the potential strongly affects localization of eigenmodes and can cause their delocalization or re-localization.

To be specific we address the Schrödinger equation

$$i\frac{\partial q}{\partial \xi}+\frac{1}{2}\frac{\partial^2 q}{\partial \eta^2}+U(\eta)q=0, \quad (1)$$

for the dimensionless field amplitude q, where ξ is the propagation distance, and η is the transverse coordinate. Equation (1) describes propagation of a beam in the external complex potential

$$U(\eta)=U_0\{\cos(s_1\eta)-iw\,\sin(s_1\eta)+\epsilon[\cos(s_2\eta)-iw\,\sin(s_2\eta)]\}, \quad (2)$$

where U_0 characterizes the potential depth, s_1 and s_2 are spatial frequencies of two $\mathcal{PT}$-symmetric sub-lattices creating the quasi-periodic structure, ϵ is the relative amplitude of the second sub-lattice, and w is the relative depth of the imaginary part in each sub-lattice, which is selected to be identical for both sub-lattices. In what follows, we consider $s_1 \geq s_2$ and set $U_0=1$. In order to observe LDT phenomenon, one should operate with incommensurate frequencies. Therefore, below we select $s_1=\sqrt{5}+1$ and $s_2=2$ and vary the remaining free parameters in the system, i.e., the depth of the imaginary part w and relative lattice strength ϵ. One should mention that the case $w=0$ corresponds to conservative quasi-periodic lattice, the case of $\epsilon=0$ corresponds to a simple $\mathcal{PT}$-symmetric lattice [14], and in the limit $s_2 \ll s_1$ and $\epsilon \gg 1$ and in the tight-binding approximation, the model can be reduced to the discrete $\mathcal{PT}$-symmetric H-AA model studied in [15,16].

Turning to possible experimental implementation of the model, we notice that the effects reported below are purely linear, and hence it is possible to use passive waveguiding configurations, such as those implemented in [17], or active waveguides with proper arrangement of domains with gain and losses [12]. Alternatively, one can exploit a $\mathcal{PT}$-symmetric refractive index profile imprinted in a cold gas of two atomic isotopes with a Λ-type energy-level configuration loaded in an atomic cell [18]. In this last case, the dielectric susceptibility of the cold atomic gas is determined by two external control fields that allow to fine-tune all the parameters of the imprinted lattices, including their periods.

In order to characterize the LDT of the system quantitatively, we introduce the integral form-factor $\chi=\int|q|^4 d\eta/[\int|q|^2 d\eta]^2$ which is inversely proportional to the width of the eigenmode. Therefore, spatially localized (delocalized) distribution is characterized by a large (small) value of χ. For excitations in the form of a single eigenmode $q(\eta,\xi)=A(\eta)\exp(i\lambda\xi)$, where A is the complex mode profile, and λ is the propagation constant, the form-factor χ remains constant, but it is ξ-dependent for a wavepacket consisting of several eigenmodes.

Propagation constants λ of the eigenmodes are determined by solving the stationary Schrödinger eigenvalue problem $(H_1+\epsilon H_2)q=\lambda q$ with $H_j=[(1/2)d^2/d\eta^2+U_j(\eta)]$ and $U_j(\eta)=\cos(s_j\eta)-iw\,\sin(s_j\eta)$. For each operator H_j, the exceptional point is $w_{\mathcal{PT}}=1$ [14], but, importantly, this is not the case for their sum. This follows from the fact that at $w=w_{\mathcal{PT}}$ generally speaking, H_1 and H_2 are not reducible to a Jordan form simultaneously at arbitrary nonzero ϵ. In other words, in the case of quasi-periodic lattice, the exceptional point at $\epsilon>0$ differs from $w_{\mathcal{PT}}$, and one may expect the possibility of the $\mathcal{PT}$-symmetry breaking already at $w<w_{\mathcal{PT}}$.

Among all eigenmodes of the quasi-periodic lattice (2) below we follow mainly the eigenmodes with maximal real or maximal imaginary parts of the propagation constant λ, since the former mode is usually the most localized one, while the latter mode experiences fastest growth upon evolution if the $\mathcal{PT}$-symmetry is broken, and thus determines the $\mathcal{PT}$-symmetry breaking.

In Figs. 1(a) and 1(c) we show the evolution of the field modulus distributions $|q|$ of these two modes as functions of η and ϵ for a relatively small imaginary part of the lattice ($w=0.5$). Like in a conservative system, the increase of ϵ results in a transition from delocalized to localized eigenmodes around $\epsilon=0.62$. Since we use large, but finite integration window that contains more than 100 periods for each of the sub-lattices, the LDT is slightly smoothed. Upon calculation of these dependencies, we dropped out edge modes that may arise at the edge of integration window due to lattice truncation and considered only modes residing in the bulk of the lattice.

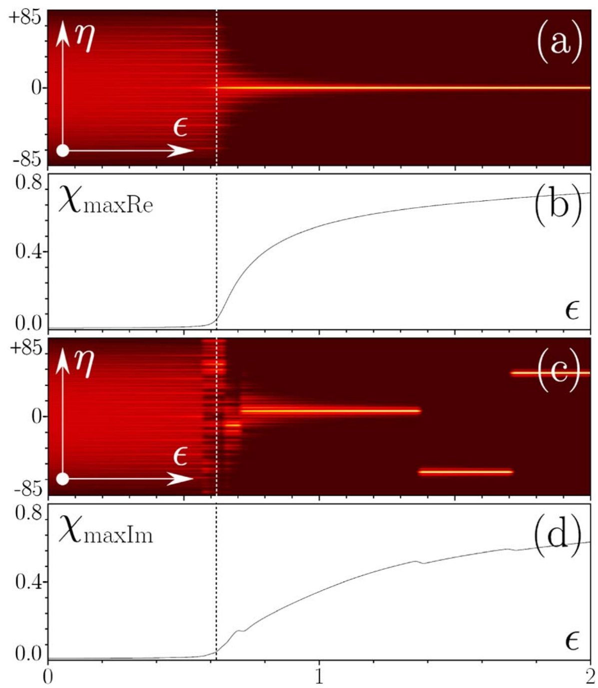

Fig. 1. Field modulus distribution $|q|$ as a function of η and ϵ [panel (a)] and integral form-factor $\chi_{\max\,\mathrm{Re}}$ (i.e., χ for the mode whose eigenvalue has maximal real part) as a function of ϵ [panel (b)] for $w=0.5$. Panels (c) and (d) show the same dependencies, but for the mode $|q|$ corresponding to the eigenvalue with maximal imaginary part. Vertical dashed lines indicate approximate localization and symmetry breaking threshold.

Above the transition point, the integral form-factors $\chi_{\max\,\mathrm{Re}}$ and $\chi_{\max\,\mathrm{Im}}$ (they denote χ for the modes with maximal real and imaginary parts of the eigenvalue) grow rapidly with increase of ϵ, but eventually they saturate for large values of ϵ when the width of the mode becomes comparable with periods of the sub-lattices [Fig. 1(b) and 1(d)]. The spectrum of the system becomes complex, i.e., $\mathcal{PT}$-symmetry breaking occurs (recall that here $w = 0.5 < w_{\mathcal{PT}}$), practically at the same value of ϵ at which LDT occurs for the mode with maximal real part of λ [Fig. 1(c)]. It is at this ϵ value, the mode with maximal imaginary part of the eigenvalue becomes spatially localized too [cf. vertical dashed lines in Figs. 1(b) and 1(d)]. Figure 1 illustrates an important conjecture that symmetry breaking in this system can be achieved even at $w = 0.5$, i.e., below the symmetry breaking point $w_{\mathcal{PT}} = 1$ for strictly periodic potentials.

The second important result of this Letter is that the imaginary part of the lattice also affects LDT. In Fig. 2, we plot the evolution of mode shape $|q|$ [panels (a) and (c)] and integral form-factor χ [panels (b) and (d)] upon increase of the imaginary part of quasi-periodic lattice w for fixed relative amplitude $\epsilon = 1.4$ that corresponds to fully localized eigenmodes in the conservative limit. Now the $\mathcal{PT}$-symmetry breaking threshold is $w = 0$. Increasing imaginary part of the lattice leads to simultaneous delocalization of all modes, both with maximal real and maximal imaginary parts of eigenvalues. Relatively sharp LDT occurs around $w = 0.9$, when all modes of the system become delocalized nearly simultaneously. However, further increase of the gain/loss coefficient results in the unexpected restoration of localization even for growing modes. The localization is restored first for modes with maximal real part of the eigenvalue and a bit later for modes with maximal imaginary part of the eigenvalue. This dependence of the localization degree on w suggests that delocalization is induced by the symmetry breaking point at $w_{\mathcal{PT}}$, i.e., the LDT transition for the entire complex quasi-periodic lattice seems to be directly connected with the symmetry breaking point for each individual lattice. The width of delocalization domain strongly depends on ϵ and decreases when ϵ grows. Also, for $\epsilon < 0.6$, when all eigenmodes are delocalized at $w = 0$, one can observe only one transition from delocalized to localized states with increase of w from zero value.

In Fig. 3, we show the form-factors [panels (a) and (c)] and the imaginary parts of the eigenvalues [panels (b) and (d)] of modes having maximal real and imaginary parts of the eigenvalue on the (ϵ, w) plane. From Figs. 3(a) and 3(c), we see that at $w < w_{\mathcal{PT}} = 1$, the above mentioned modes remain delocalized for $\epsilon < \epsilon_{\mathrm{loc}}$, where ϵ_{loc} is the localization threshold in the conservative quasi-periodic lattice. The spectrum remains purely real in this domain as it follows from Fig. 3(d). Above ϵ_{loc}, the spectrum simultaneously becomes complex and localization occurs [cf. the position of the left domain with nonzero $\lambda_{\max\,\mathrm{Im}}$ in Fig. 3(d) indicating on the existence of growing modes with position of left domains in Figs. 3(a) and 3(c), showing form-factors of the eigenmodes]. At $w < w_{\mathcal{PT}}$, the domains where form-factors are high (modes are localized) coincide for most localized mode (the mode with maximal real part of the eigenvalue) and for mode that experiences fastest growth (the mode with maximal imaginary part of the eigenvalue). The LDT point shifts toward larger w values with increase of ϵ in the region where $\epsilon > 0.6$. For $w > w_{\mathcal{PT}}$, the increase of the imaginary part of the lattice results in transition from delocalization to localization irrespectively of the

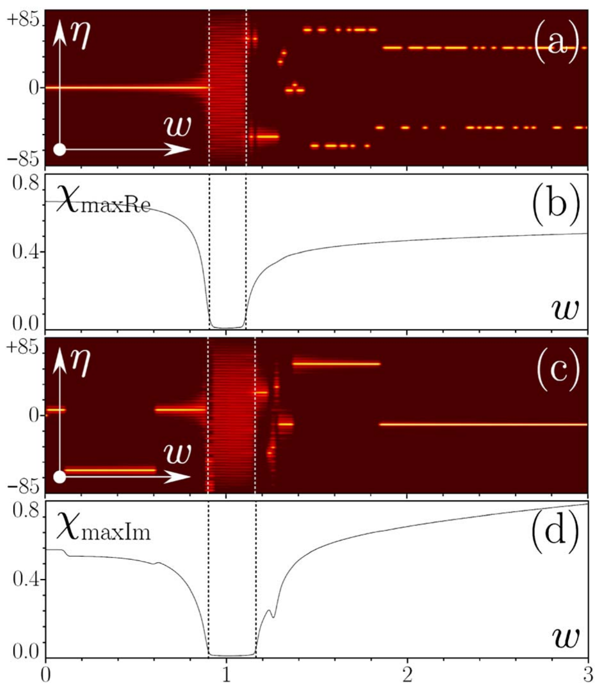

Fig. 2. Field modulus distribution $|q|$ as a function of η and w [panel (a)] and integral form-factor $\chi_{\max\,\mathrm{Re}}$ of the mode as a function of w [panel (b)] for $\epsilon = 1.4$. Panels (c) and (d) show the same dependencies, but for the mode $|q|$ corresponding to the eigenvalue with maximal imaginary part. Vertical dashed lines indicate approximate localization/delocalization thresholds.

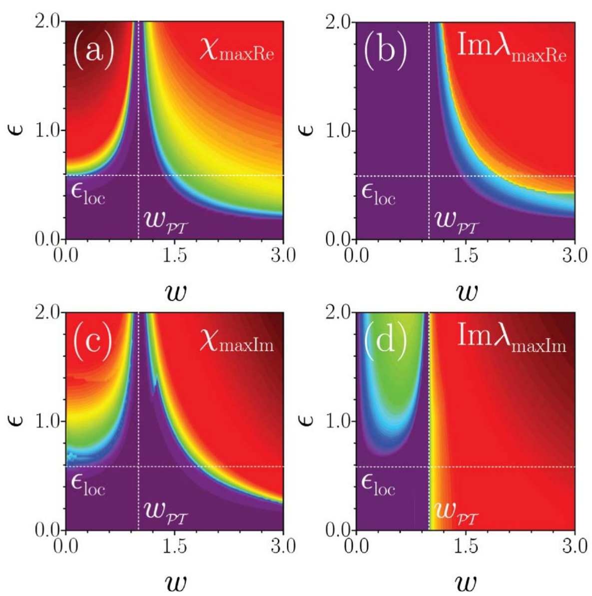

Fig. 3. Integral form-factor (a) and imaginary part of the eigenvalue (b) for the mode corresponding to eigenvalue with maximal real part on the (ϵ, w) plane. Panels (c) and (d) show the same dependencies, but for the mode corresponding to the eigenvalue with maximal imaginary part. $\mathrm{Im}\,\lambda_{\max\,\mathrm{Re}}$ and $\mathrm{Im}\,\lambda_{\max\,\mathrm{Im}}$ are plotted in the logarithmic scale. Vertical and horizontal dashed lines denote $\mathcal{PT}$ breaking point for the periodic lattice $w_{\mathrm{cr}} \approx 1$ and localization threshold for the real lattice $\epsilon_{\mathrm{loc}} \approx 0.6$, respectively.

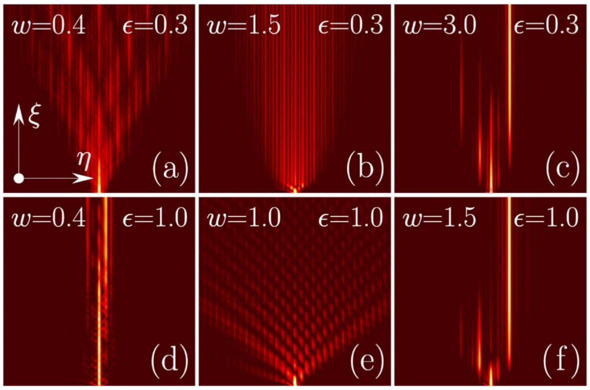

Fig. 4. Propagation of the localized input beam of the form $q = \mathrm{sech}(\eta/1.5)$ in the complex quasi-periodic potential (2) for various sets of (ϵ, w) indicated in the plots. In order to illustrate the detail of diffraction patterns even in the regime with broken symmetry, at each propagation distance ξ, the power was renormalized. Propagation distance $\xi = 150$ is chosen for all panels.

value of ϵ. This actually means that imaginary part of potential can cause localization even below localization threshold $\epsilon = \epsilon_{\mathrm{loc}}$ in the conservative case. The threshold value of ϵ required for localization in this regime monotonically decreases with increase of w [see Figs. 3(a) and 3(c)]. Finally, for mode with maximal real part of the eigenvalue, the symmetry breaking occurs nearly in the same point where increasing w (i.e., the gain and loss of the system) results in re-localization of this mode [cf. the edges of right bright domains in Figs. 3(a) and 3(b)]. However, for the mode with maximal imaginary part of the eigenvalue, re-localization and symmetry breaking points drastically differ. Symmetry breaking for this mode (that indicates also on global symmetry breaking in the system) takes place exactly at $w = w_{\mathcal{PT}}$ [Fig. 3(d)]. However, this mode becomes spatially localized at substantially larger w value [see the border of right domain in Fig. 3(c)].

In Fig. 4, we show the examples of beam propagation obtained by direct simulations using Eq. (1). Below localization threshold in conservative lattice $\epsilon < \epsilon_{\mathrm{loc}}$, one observes usual diffraction without any growth of the energy flow $P = \int |q|^2 d\eta$ at $w < w_{\mathcal{PT}} = 1$ [Fig. 4(a)]. When w increases above the symmetry breaking threshold $w_{\mathcal{PT}}$ but does not reach yet LDT point, one also observes the expansion of the pattern, although with a smaller rate [Fig. 4(b)]. Above the localization threshold the growing localized modes are excited [Fig. 4(c)]. At $\epsilon > \epsilon_{\mathrm{loc}}$, the modes are localized even at small w, but they grow upon evolution [Fig. 4(d)]. Increasing gain-losses initially cause rapid delocalization [Fig. 4(e)]. In this regime, the energy flow remains nearly constant, since the imaginary parts of all eigenvalues also decrease. This delocalization is replaced by re-localization (accompanied by fast growth of the energy flow) for large w values [Fig. 4(f)].

Finally, we have verified numerically that the LDT can also be observed in a lattice where only the imaginary part of the periodic potential is incommensurable, while the real part is periodic, i.e., in the complex potential of the form $U(\eta) = U_0[\cos(s_1\eta) - iw\,\sin(s_1\eta) - iw\epsilon\,\sin(s_2\eta)]$.

In conclusion, we have studied the LDT of light in a $\mathcal{PT}$-symmetric quasi-periodic optical lattice. We have shown that growing imaginary part of the lattice may induce delocalization around $\mathcal{PT}$-symmetry breaking threshold of the individual constituents of quasi-periodic potential. This delocalization may be followed by an unexpected re-localization of the eigenmodes of the system. Moreover, the $\mathcal{PT}$-symmetry breaking threshold in this system depends on the strength of second lattice forming the quasi-periodic structure.

Fundação para a Ciência e a Tecnologia (FCT) (PTDC/FIS-OPT/1918/2012); National Natural Science Foundation of China (NSFC) (11474099, 11475063); Program of Introducing Talents of Discipline to Chinese Universities (B12024).

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