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THREE-WAVE SOLITON EXCITATIONS IN A DISK-SHAPED BOSE–EINSTEIN CONDENSATE

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A three-wave resonant interaction of collective modes and related soliton excitations in a disk-shaped Bose–Einstein condensate are investigated. The phase-matching conditions for the resonant interaction are satisfied by suitably choosing the wavevectors and the frequencies of the collective modes. A set of nonlinearly coupled envelope equations describing the spatio-temporal evolution of the three-wave resonant interaction are derived by using a method of multiple-scales, and some explicit (2+1)-dimensional three-wave soliton solutions are also presented and discussed.

Keywords: Bose–Einstein condensation; solitons; three-wave interaction.

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1. Introduction

Since the successful experimental realization of Bose–Einstein condensation in weakly interacting atomic gases,[1] much progress has been made on the study of linear (i.e. small-amplitude) collective excitations (or called Bogoliubov quasi-particles) created in Bose–Einstein condensates (BECs). At the same time, the research on nonlinear (i.e. large-amplitude) collective excitations in BECs have also received intensive attention.[2–18] For a large-amplitude excitation, the nonlinear effect resulting from the interaction among the excitations cannot be neglected and many interesting nonlinear phenomena can appear. Up to now, the investigation on the nonlinear collective excitations in BECs can be classified into two types. One of them is the excitations with the size the same as that of the condensate. The eigen-frequencies of such excitations are discrete, i.e. they are oscillating (i.e. standing wave) modes. The nonlinear frequency shift, mode coupling, and harmonic generation have been explored both theoretically and experimentally by Oxford group[4–8] and some other authors.[9–12] The other type of excitations explored are the ones with the size much smaller than that of the condensate. In this case, the eigen-frequencies of the excitations are continuous (or quasi-continuous). The most typical nonlinear excitations explored in BECs are solitary excitations, including dark[2] and bright[3] solitons. Recently, Ozeri *et al.*[13] investigated a three-wave mixing of energy down-conversion in a homogeneous Bose-condensed gas and observed the oscillations of excitation numbers between different momentum modes (i.e. plane-waves) involved in the mixing.[19]

In this work, we investigate a three-wave resonant interaction (TWRI) of collective excitations and related three-wave solitons in a disk-shaped BEC. In a disk-shaped condensate, the confinement is much stronger in one spatial direction comparing with other two directions so the motion of atoms is almost two dimensional. The three-wave resonance condition for the excitations in such condensate (which is necessary for an effective TWRI) can easily be fulfilled (see the next section). We consider an energy up-conversion of the excitations through the TWRI at zero temperature.[20] In this situation, the time-dependent Gross–Pitaevskii (GP) equation, which controls the linear and nonlinear evolution of the condensate, is a natural starting point.[1,8] By suitably choosing the wavevectors and frequencies of the excitations, the three-wave resonant conditions for the TWRI can be fulfilled. We derive the nonlinearly coupled envelope (or amplitude) equations describing the TWRI among three modulated plane-wave modes by using a method of multiple-scales. Three-wave soliton excitations in the TWRI process are shown to be possible in the condensate and their stability is checked by a numerical simulation.

2. Order Parameter Equation and TWRI Conditions

The dynamics of a weakly interacting Bose gas at zero temperature is governed by the time-dependent Gross–Pitaevskii (GP) equation[1,21]

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + g|\Psi|^2\right]\Psi\,, \tag{1}$$

where Ψ is an order parameter, $\int d\mathbf{r}|\Psi|^2 = N$ is the atomic number in the condensate, $g = 4\pi\hbar^2 a_s/m$ is the interaction constant with a_s the s-wave scattering length ($a_s > 0$ for a repulsive interaction). We consider a disk-shaped harmonic trap of the form $V_{\rm ext}(\mathbf{r}) = \frac{m}{2}[\omega_\perp^2(x^2+y^2) + \omega_z^2 z^2]$ with $\omega_\perp \ll \omega_z$, where $\omega_\perp$ and ω_z are the frequencies of the trap in the transverse (x and y) directions and the z-direction, respectively. Expressing the order parameter in terms of its modulus and phase, $\Psi = \sqrt{n}\exp(i\phi)$, we get a set of coupled equations for n and ϕ. By introducing $(x, y, z) = a_z(x', y', z')$, $t = \omega_z^{-1}t'$, $n = n_0 n'$ with $a_z = [\hbar/(m\omega_z)]^{1/2}$ and $n_0 = N/a_z^3$, we obtain the following dimensionless equations of motion after dropping the primes:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \nabla \phi) = 0\,, \tag{2}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}z^2 + V_{\|}(x,y) + Qn + \frac{1}{2}\left[(\nabla\phi)^2 - \frac{1}{\sqrt{n}}\nabla^2\sqrt{n}\right] = 0\,, \tag{3}$$

with $Q = 4\pi N a_s/a_z$ and $\int d\mathbf{r}n = 1$. $V_{\|}(x,y) = (\omega_\perp/\omega_z)^2(x^2+y^2)/2$ is the dimensionless trapping potential in the x and y directions.

We consider the dynamics of the excitations generated in the condensate with a thin disk-shaped trap. The thin disk-shaped trap here means that the conditions $a_z \ll l_0$ and $\hbar\omega_\perp \ll n_0 g \ll \hbar\omega_z$ can be fulfilled, where $l_0 = (4\pi n_0 a_s)^{-1/2}$ is the healing length. In this situation, we can make the quasi-2D approximation[22] $\sqrt{n} = P(x,y,t)G_0(z)$, $\phi = -\mu t + \varphi(x,y,t)$, where $G_0(z) = \exp(-z^2/2)$ is the ground-state wavefunction of the 1D harmonic oscillator with the harmonic potential $z^2/2$ in the z-direction, μ is the chemical potential of the condensate and φ is a phase function due to the existence of the excitation, which is assumed to be a function of x and y because as mentioned above the generated excitation can propagate only in the x and y directions. Thus Eqs. (2) and (3) are reduced to

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{P}{2}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = 0\,, \tag{4}$$

$$\begin{aligned} &-\frac{1}{2}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) - \left(\mu - \frac{1}{2}\right)P \\ &\quad + \left[\frac{\partial\varphi}{\partial t} + V_{\|}(x,y) + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]P + Q'P^3 = 0\,, \end{aligned} \tag{5}$$

where $Q' = I_0 Q$ is an effective interaction constant with $I_0 = \int_{-\infty}^{\infty} dz G_0^4(z)/\int_{-\infty}^{\infty} dz G_0^2(z) = 1/\sqrt{2}$. The reduction to Eqs. (4) and (5) from Eq. (1) can be taken as a projection process. In principle, one can take into account the contribution of the higher-order eigen-modes of the harmonic oscillator in the z-direction, as done in Ref. 23 for a cigar-shaped trap. However, since we have assumed $n_0 g \ll \hbar\omega_z$ here, the contribution from these higher-order eigen-modes is small and can thus be safely neglected. On the other hand, for the thin disk-shaped trap ($\omega_\perp/\omega_z \ll 1$), the trapping potential in the (x,y) plane is a slowly-varying function of x and y and hence the size of the condensate in the radial direction is much larger than the size of the excitations (with the order of the healing length) considered below. In the propagation of the excitations for short times, the boundary of the condensate does not come into play and we can therefore take the condensate as uniform in the (x,y) plane (i.e. neglecting the affect from $V_{||}(x,y)$). The effect of the condensate boundary will be considered elsewhere.

The linear dispersion relation of an excitation is obtained by assuming in Eqs. (4) and (5) that $P = u_0 + a(x,y,t)$ ($u_0 > 0$). Here $(a,\varphi) = (a_0,\varphi_0)\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] + c.c.$

with $\mathbf{k} = (k_x, k_y)$ and $\mathbf{r} = (x, y)$ (u_0, a_0 and φ_0 are constants). The result reads

$$\omega(\mathbf{k}) = \frac{1}{2} k (4c^2 + k^2)^{1/2}\,, \tag{6}$$

where $k^2 = k_x^2 + k_y^2$ and $c = \sqrt{Q'} u_0$ is the sound speed of the system. Equation (6) is a Bogoliubov-type linear excitation spectrum of quasi-particles in two dimensions. Such excitation spectrum has been measured in BECs recently.[24]

Recently, two-dimensional solitons created in the BEC with the disk-shaped trap have been considered.[22] Here, we are interested in a resonant interaction among three collective modes, i.e. a TWRI of the collective excitations created in the condensate. To obtain an efficient TWRI, the phase-matching conditions

$$\omega_1 + \omega_2 = \omega_3\,, \tag{7}$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3\,, \tag{8}$$

($\omega_j \equiv \omega(\mathbf{k}_j)$) are required. From Eq. (6) it is easy to show that these conditions can be satisfied if we choose

$$\mathbf{k}_1 = (k_1 \cos\vartheta, k_1 \sin\vartheta)\,, \tag{9}$$

$$\mathbf{k}_3 = (k_3, 0)\,, \tag{10}$$

$$\mathbf{k}_2 = \mathbf{k}_3 - \mathbf{k}_1 = (k_3 - k_1 \cos\vartheta, -k_1 \sin\vartheta)\,, \tag{11}$$

where k_1 and k_2 are positive and the angle ϑ satisfies the relation[13]

$$\cos\vartheta = \frac{1}{2k_3 k_1}\left\{ k_3^2 + k_1^2 + 2c^2 - 2\left[c^4 + \left(k_3\sqrt{c^2 + \frac{1}{4}k_3^2} - k_1\sqrt{c^2 + \frac{1}{4}k_1^2} \right)^2 \right]^{\frac{1}{2}} \right\}. \tag{12}$$

It is easy to show that, for any nonvanishing k_1 and k_3, one has $0 < \cos\vartheta \leq 1$ and hence $-\pi/2 < \vartheta \leq \pi/2$. Another necessary condition for the TWRI is that the nonlinearity describing the interaction between the collective modes must be quadratic, similar to a $\chi^{(2)}$ nonlinearity in a nonlinear optical medium. From Eqs. (4) and (5), we see that the equations describing the excitations $(P - u_0, \varphi)$ are of not only quadratic but also cubic nonlinearities. Consequently, in the disk-shaped condensate, a TWRI for the collective excitations is indeed possible if the angle ϑ is chosen according to Eq. (12).

3. Envelope Equations for the TWRI

We know that an optical TWRI occurs in active media with no inversion symmetry. For trapped atoms this symmetry is not broken and hence the TWRI in the BEC is possible only when the ground state (condensate) is not depleted by the excitations. This imposes a constraint that the amplitude of the excitations cannot be too large.

Here, we develop a weak nonlinear theory[25] for the TWRI in the BEC by making the asymptotic expansion

$$P - u_0 = \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \varepsilon^3 a^{(3)} + \cdots , \tag{13}$$

$$\varphi = \varepsilon \varphi^{(1)} + \varepsilon^2 \varphi^{(2)} + \varepsilon^3 \varphi^{(3)} + \cdots , \tag{14}$$

where ε is a small parameter characterizing the relative amplitude of the excitation, and $a^{(j)}$, $\varphi^{(j)}$ $(j = 1, 2, 3, \ldots)$ are the functions of the multiple-scale variables x, y, t, εx, εy, and εt. Note that the expansion here is different from that used in Refs. 22, 23 and 26 because at present we are considering envelope-type excitations in the condensate. Substituting the above expansion into Eqs. (4) and (5) we get

$$\frac{\partial a^{(j)}}{\partial t} + \frac{1}{2} u_0 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi^{(j)} = \alpha^{(j)} , \tag{15}$$

$$-\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) a^{(j)} + 2c^2 a^{(j)} + u_0 \frac{\partial}{\partial t} \varphi^{(j)} = \beta^{(j)} , \tag{16}$$

$j = 1, 2, 3, \ldots$. The explicit expressions of $\alpha^{(j)}$ and $\beta^{(j)}$ are omitted here.

At leading order $(j = 1)$, the solution reads

$$\varphi^{(1)} = A_0 + [A \exp(i\theta) + c.c] , \tag{17}$$

$$a^{(1)} = \frac{i}{2} \frac{u_0 k^2}{\omega} A \exp(i\theta) + c.c. , \tag{18}$$

where A_0 (real) and A (complex) are yet to be determined envelope functions of the slow variables εx, εy and εt, introduced necessarily to eliminate the secular terms appearing in the higher-order approximations. $\theta = \mathbf{k} \cdot \mathbf{r} - \omega t$ with $\omega(\mathbf{k})$ being just the linear dispersion relation given by Eq. (6), and *c.c.* represents a corresponding complex conjugate term.

In the process of the TWRI, three wave modes are involved and hence the leading-order solution should be taken as

$$\varphi^{(1)} = A_0 + \sum_{j=1}^{3} [A_j \exp(i\theta_j) + c.c.] , \tag{19}$$

$$a^{(1)} = \sum_{j=1}^{3} B_j \exp(i\theta_j) + c.c. , \tag{20}$$

where $B_j = [i u_0 k_j^2 / (2\omega_j)] A_j$, $\theta_j = \mathbf{k}_j \cdot \mathbf{r} - \omega_j t$, and A_j is the envelope of jth wave mode. $\mathbf{k}_j$ and ω_j $(j = 1, 2, 3)$ are chosen according to the TWRI phase-matching conditions (7) and (8).

At the next order $(j = 2)$ the solvability conditions of Eqs. (15) and (16) give rise to the closed equations governing the evolution of the envelopes B_j. After making the transformation $b_j = \varepsilon B_j$ $(j = 1, 2, 3)$, these equations read

$$\frac{\partial b_1}{\partial t} + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}} b_1 = \gamma_1 b_2^* b_3^* , \tag{21}$$

$$\frac{\partial b_2}{\partial t} + \mathbf{v}_2 \cdot \nabla_{\mathbf{r}} b_2 = \gamma_2 b_1^* b_3^* \,, \tag{22}$$

$$\frac{\partial b_3}{\partial t} + \mathbf{v}_3 \cdot \nabla_{\mathbf{r}} b_3 = \gamma_3 b_1^* b_2^* \,, \tag{23}$$

when returning to the original variables, where $\nabla_{\mathbf{r}} = (\partial/\partial x, \partial/\partial y)$, and $\mathbf{v}_j = \nabla_{\mathbf{k}_j} \omega_j$ is the group velocity of the jth wave mode. The explicit expressions of the nonlinear coupling coefficients γ_j, which carry the signs of the wave energy, are omitted here.

Equations (21)–(23) are (2+1)-dimensional TWRI equations with b_j describing the envelope of the jth wave mode. Such a set of nonlinear coupled envelope equations appear also in nonlinear optics, fluid physics and other fields.[27,28]

4. Three-Wave Soliton Solutions

Now we discuss the three-wave soliton solutions of Eqs. (21)–(23). For simplicity, we set $X = x - ut$, and $Y = y - vt$ with u and v being constants. Then the Eqs. (21)–(23) are transformed into the following (1+1)-dimensional form:

$$\frac{\partial b_j}{\partial X} + c_j \frac{\partial b_j}{\partial Y} = \bar{\gamma}_j b_k^* b_i^* \,, \tag{24}$$

where i, j, k are cyclic and equal to $1, 2$ and 3, $c_j = (v_j - v)/(u_j - u)$ and $\bar{\gamma}_j = \gamma_j/(u_j - u)$ with $(u_j, v_j) \equiv \mathbf{v}_j$.

(i) The first type of solution: Using a transformation the coupling coefficients $\bar{\gamma}_j$ in Eq. (24) can be scaled to unity magnitude (i.e. $|\bar{\gamma}_j| = 1$). Then based on the method of inverse scattering transform[29] we obtain the following soliton solutions of Eq. (24) under the condition that $c_2 > c_3 > c_1$ and the sign of $\bar{\gamma}_3$ is different from the signs of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ (i.e. $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) = (-,-,+)$ or $(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) = (+,+,-)$):

$$b_1 = \frac{4\eta_1(\beta_{12}\beta_{13})^{\frac{1}{2}}}{D} \left(\frac{-1}{\bar{\gamma}_2\bar{\gamma}_3} \right)^{\frac{1}{2}} \exp[-i(\phi_1 - 2\xi_1 z_1)] \times \left[\exp(2\eta_2 z_2) + \frac{\zeta_1^* - \zeta_2^*}{\zeta_1^* - \zeta_2} \exp(-2\eta_2 z_2) \right], \tag{25}$$

$$b_3 = \frac{-16i\eta_1\eta_2\beta_{12}}{D(\zeta_1 - \zeta_2^*)(\beta_{13}\beta_{32})^{\frac{1}{2}}} \left(\frac{1}{\bar{\gamma}_1\bar{\gamma}_2} \right)^{\frac{1}{2}} \exp[-i(\phi_1 + \phi_2 - 2\xi_1 z_1 - 2\xi_2 z_2)] \,, \tag{26}$$

$$b_2 = \frac{4\eta_2(\beta_{12}\beta_{32})^{\frac{1}{2}}}{D} \left(\frac{-1}{\bar{\gamma}_1\bar{\gamma}_3} \right)^{\frac{1}{2}} \exp[-i(\phi_2 - 2\xi_2 z_2)] \times \left[\exp(-2\eta_1 z_1) + \frac{\zeta_1 - \zeta_2}{\zeta_1^* - \zeta_2} \exp(2\eta_1 z_1) \right], \tag{27}$$

with

$$D = \exp(2z_1\eta_1 + 2z_2\eta_2) + \exp(2z_2\eta_2 - 2z_1\eta_1) + \left|\frac{\zeta_1 - \zeta_2}{\zeta_1^* - \zeta_2}\right| \exp(2z_1\eta_1 - 2z_2\eta_2) + \exp(-2z_1\eta_1 - 2z_2\eta_2)\,, \tag{28}$$

where $z_1 = Y - c_1X - Y_{10}$, $z_2 = Y - c_2X - Y_{20}$, $\zeta_1 = 2(\xi_1 + i\eta_1)/\beta_{32}$, $\zeta_2 = 2(\xi_2 + i\eta_2)/\beta_{13}$, $\beta_{ij} = c_j - c_i$. ξ_j, η_j, ϕ_j and Y_{j0} $(j = 1, 2)$ are integral constants.

In the leading-order approximation the modulus of the order parameter is given by

$$\begin{aligned} |\Psi| = P &= (u_0 + \varepsilon a^{(1)}) \exp(-z^2) \\ &= \exp(-z^2)\{u_0 + [b_1 \exp(i\theta_1) + b_2 \exp(i\theta_2) + b_3 \exp(i\theta_3) + c.c]\}. \end{aligned} \tag{29}$$

Note that in the solution (25)–(27), η_1 and η_2 are two important parameters controlling the magnitude of soliton amplitude. Shown in Fig. 1 is the modulus $|\Psi|$ when the parameters in the solution are chosen as $u_0 = 1$, $k_1 = 5$, $k_3 = 6$, $u = 8$, $v = -1$, $\xi_1 = 5$, $\xi_2 = 1$, $\eta_1 = 0.2$, $\eta_2 = 0.3$, $\phi_1 = 1$, $\phi_2 = 1$, $Y_{10} = 1$, and $Y_{20} = 1$ at time $t = 1$. For an illustration, we have taken the value of $|\Psi|$ on the $z = 0$ plane. We see that the solution represents an interaction between two envelope solitons, which are excited from the condensate background $u_0 = 1$. Both the wave-mode with the wavevector $\mathbf{k}_1$ (called k_1-soliton) and the wave-mode with the wavevector $\mathbf{k}_2$ (called k_2-soliton) are plane-wave solitons, which are localized only in their propagating directions. When the k_1-soliton and the k_2-soliton (which have different velocities) collide, the third one (i.e. k_3-soliton) appears. The k_3-soliton then

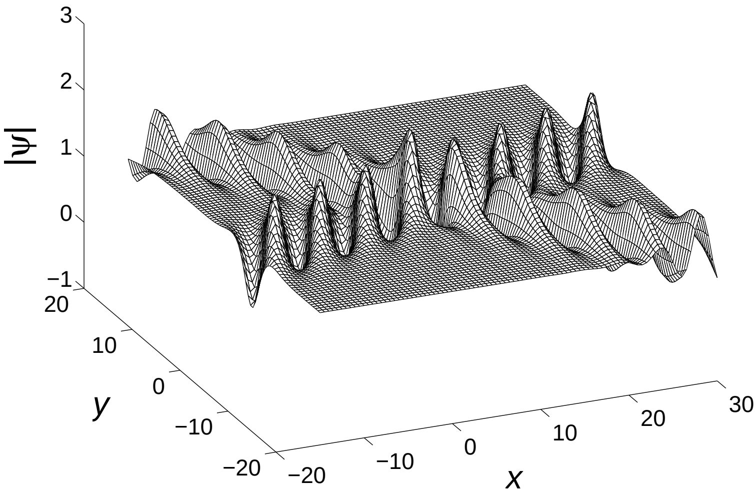

Fig. 1. The modulus $|\Psi|$ of the order parameter in the case of the three-wave soliton excitation given by the solution (25)–(27), when the parameters are chosen as $u_0 = 1$, $k_1 = 5$, $k_3 = 6$, $u = 8$, $v = -1$, $\xi_1 = 5$, $\xi_2 = 1$, $\eta_1 = 0.2$, $\eta_2 = 0.3$, $\phi_1 = 1$, $\phi_2 = 1$, $Y_{10} = 1$, and $Y_{20} = 1$ at time $t = 1$.

decays, giving back to k_1- and k_2-solitons, which maintain their initial shape and duration after the collision. The interaction results in a phase (or position) shift to their initial propagating directions and a phonon radiation which is too small to be seen in the figure.

(ii) The second type of solution: Under the same conditions, i.e. the sign of $\bar{\gamma}_3$ is different from the signs of $\bar{\gamma}_1$ and $\bar{\gamma}_2$, and $c_2 > c_3 > c_1$, Eq. (24) admits another type of three-wave soliton solution:[29]

$$\begin{aligned} b_1 = &-2i\nu\beta_{32}(\beta_{13}\beta_{12})^{\frac{1}{2}}(-\bar{\gamma}_3\bar{\gamma}_2)^{-\frac{1}{2}}\Delta^{-1}\exp[i\omega\beta_{32}(Y - c_1X - \phi_1) \\ &- \nu\beta_{32}(Y - c_1X - \eta_1)]\,, \end{aligned} \tag{30}$$

$$\begin{aligned} b_3 = &\,2i\nu\beta_{12}(\beta_{13}\beta_{32})^{\frac{1}{2}}(\bar{\gamma}_1\bar{\gamma}_2)^{-\frac{1}{2}}\Delta^{-1}\exp[-i\omega\beta_{12}(Y - c_3X - \phi_3) \\ &- \nu\beta_{12}(Y - c_3X - \eta_3)]\,, \end{aligned} \tag{31}$$

$$b_2 = (2i\nu\beta_{32}\beta_{12})^{-1}(-\bar{\gamma}_1\bar{\gamma}_3)^{-\frac{1}{2}}b_1^*b_3\Delta\,, \tag{32}$$

where

$$\Delta = 1 + \exp[-2\nu\beta_{12}(Y - c_3X - \eta_3)] + \exp[2\nu\beta_{32}(Y - c_1X - \eta_1)]\,, \tag{33}$$

where ω, ν, η_1, η_3, ϕ_1 and ϕ_3 are arbitrary constants.

Figure 2 shows the configuration of modula $|\Psi|$ at time $t = 1$ for the case of the three-wave soliton excitation given by the solution (30)–(32). The parameters chosen in the figure are $u_0 = 1$, $k_1 = 5$, $k_3 = 6$, $u = 7.8$, $v = -0.2$, $\nu = 0.5$, $\omega = 10$,

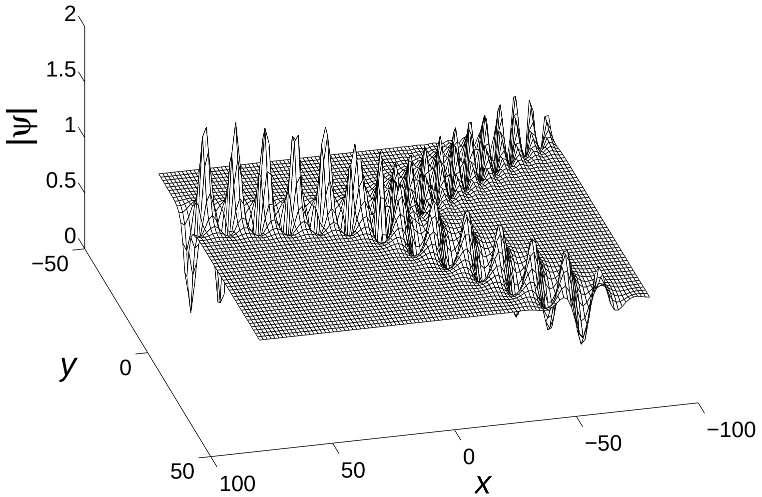

Fig. 2. The modulus of the order parameter Ψ in the case of the three-wave soliton excitation represented by the solution (30)–(32). The parameters are chosen as $u_0 = 1$, $k_1 = 5$, $k_3 = 6$, $u = 7.8$, $v = -0.2$, $\nu = 0.5$, $\omega = 10$, $\eta_1 = 0$, $\eta_2 = 0$, $\phi_1 = 0$ and $\phi_2 = 0$ at time $t = 1$.

$\eta_1 = 0$, $\eta_2 = 0$, and $\phi_1 = 0$. We see that this is a typical three-wave mixing process in which two solitons interact and disappear, and then a new soliton is created.

5. Numerical Simulations

In this section, we give numerical evidence for the existence and stability of the solutions presented in the preceding section. We apply a time-splitting Fourier spectral method to numerically solve Eq. (1), which was used recently by Bao *et al.*[30] to solve the Schrödinger and GP equations. We adapt the combination of both time-splitting discretization and Fourier spectral method to the GP equation and infer the computational domain and mesh size based on the analytical results obtained in previous sections. The merit of this method is that it is unconditionally stable, time reversible, time-transverse invariant and conserves the total particle number. Compared with other methods (e.g. Crank–Nicolson finite difference and spectral methods), the time-splitting Fourier spectral method is more powerful for solving the GP-like equation.

As pointed out in Sec. 2, because for a disk-shaped trap the atoms are strongly confined in the z-direction, one can project the (3+1)-dimensional GP equation (1) into a (2+1)-dimensional one. The reasonableness of such a kind of reduction from high dimensions to low dimensions has been proved rigorously in Ref. 30 and is shown to be very effective in our numerical simulation. We include in the numerical simulation the slowly-varying trapping potential $V_{\|}(x, y)$ in the x and y directions, which has been neglected in the analytical approach in the last two sections. The ground state wavefunction u_0 of the condensate in this situation is a slowly-varying function of x and y, which for a large condensate (i.e. the condensate with a large enough particle number), can be expressed by the Thomas–Fermi wavefunction:[22]

$$u_0 = \left\{ \frac{1}{I_0 Q} \left[\frac{\omega_\perp}{\omega_z} \left(\frac{I_0 Q}{\pi^{3/2}} \right)^{1/2} - \frac{1}{2} \left(\frac{\omega_\perp}{\omega_z} \right)^2 r^2 \right] \right\}^{1/2}, \tag{34}$$

with $r^2 = x^2 + y^2$, $r \leq l$. Here l is the radius of the condensate, decided by the solvability condition of u_0. On the boundary $x^2 + y^2 = l^2$, the order parameter can be approximately taken as zero.[22] Accordingly, at the boundary one can take $\Psi = \partial\Psi/\partial x = \partial\Psi/\partial y = 0$.

We choose the spatial mesh size $h = \Delta x = \Delta y = 0.5$ and the number of grid points as 230×230. The time step is selected as $p = \Delta t = 10^{-5}$. Taking the disk-shaped BEC of ^{23}Na realized in Ref. 31 as an example, we have the parameters $a_s = 2.75$ nm, $\omega_z/(2\pi) = 790$ Hz, $\omega_\perp/(2\pi) = 10$ Hz, and particle number $N = 2.9 \times 10^5$. We obtain $a_z = 0.74$ μm and thus $l = 57$ (dimensionless).

Figures (3a)–(3c) show the numerical results of modula $|\Psi|$ for the BEC of ^{23}Na atoms on the $z = 0$ plane at time $t = 0$, $t = 0.85$ and $t = 1.2$ ms, respectively. The solution given by (25)–(27) is taken as the initial condition in the simulation with these parameters at different times. The three snapshots shown in the figures are the projection of the soliton amplitude on the ground-state background, which is a large

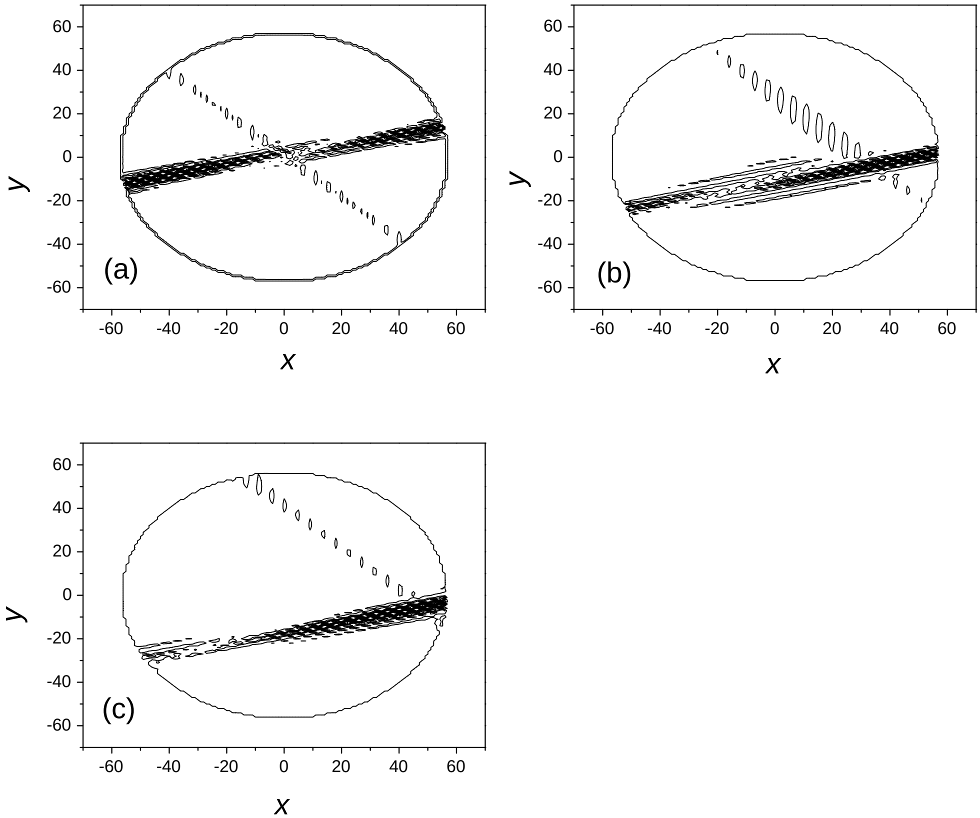

Fig. 3. Numerical results for modula $|\Psi|$ of the disk-shaped BEC for ^{23}Na atoms on the $z = 0$ plane at (a) $t = 0$ ms, (b) $t = 0.85$ ms and, (c) $t = 1.2$ ms, respectively. The solution given by (25)–(27) is taken as the initial condition.

hemisphere in 3D space. The large circle in each figure denotes the exact shape of the boundary of the ground-state background, which has been approximated as an infinite plane in the analytical study in the preceding sections (i.e. the ground state solution u_0 is taken as a constant). The boundary is coarse since in the numerical discretization we use a non-continuous lattice grid. Two trips inside the large circle represent a two soliton excitation on the ground-state background. Near the cross region of the trips there is a Mach stem, which is too small to be seen in the figures, representing the third soliton produced by resonant interaction. Each soliton is localized in its propagating direction. The darker regions in the figures denote that the excitation in those regions has a smaller value of amplitude. Since each soliton in the excitation is an envelope one, trips in the figures are composed of many small circles or small trips. Comparing with Fig. 1, we see that the result of the numerical experiment is basically similar to the analytical predication and the three-wave soliton excitation in the system is fairly stable and hence feasible for an experimental observation. We have also made a numerical simulation for other types of three-wave soliton excitations presented in the previous section and obtained similar results.

6. Discussion and Summary

We have investigated the three-wave resonant interaction and related soliton excitations of the Bogoliubov quasi-particle excited in a disk-shaped Bose–Einstein condensate. By suitably choosing the wavevectors and frequencies of the plane-wave modes involved, the phase-matching conditions of the three-wave mixing process with an energy up-conversion can be fulfilled. Based on a method of multiple-scales we have derived a set of nonlinearly coupled (2+1)-dimensional envelope equations describing the spatio-temporal evolution of the three-wave resonant interaction. Three-wave soliton solutions have been presented by using the results from the method of inverse scattering transform developed in soliton theory. We have also made a numerical simulation for checking the stability for the three-wave soliton solutions.

To observe experimentally the TWRI in a Bose–Einstein condensate predicted above, one should first prepare a disk-shaped condensate similar to that realized by Görlitz *et al.*[31] With such a condensate, one can apply the two-photon Bragg transition to excite the condensate and populate different plane-wave modes, like that proposed in Ref. 24. In this way the condensate is excited with excitations of momentum k and q which fulfill the Bragg condition. Caused by the atom-atom interaction, the k momentum excitation is separated into two excitations with momenta q and $k-q$ and a new momentum mode $k-q$ can populate through the TWRI process. One can further consider a multi-mode interaction of the excitations in Bose–Einstein condensates, such as a second-harmonic generation, difference-frequency mixing, and parametric amplification, etc. The results provided in this work may be useful for understanding the nonlinear property of large-amplitude excitations and as a guide for new experimental findings in the study of Bose–Einstein condensates.

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19. There exists much interest on the excitations in molecular BECs through magnetic-field Feshbach resonance, see, e.g. J. Kinast *et al.*, *Phys. Rev. Lett.* **92**, 150402 (2004); M. Bartenstein *et al.*, *ibid.* **92**, 203201 (2004); C. Chin *et al.*, *Science* **305**, 1128 (2004); M. Greiner *et al.*, *Phys. Rev. Lett.* **94**, 073403 (2005).
20. We consider here the TWRI of the excitations created in BEC at very low temperatures (e.g. 10^{-9} K), the finite temperature effect is small and hence can be neglected. If this is not the case one must work beyond the zero-temperature approximation. However, this is the topic beyond our present work.
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