

**Dynamics of phase oscillators with generalized frequency-weighted coupling**Can Xu,<sup>1,2</sup> Jian Gao,<sup>1,3</sup> Hairong Xiang,<sup>1</sup> Wenjing Jia,<sup>1</sup> Shuguang Guan,<sup>4</sup> and Zhigang Zheng<sup>2,\*</sup><sup>1</sup>*Department of Physics and the Beijing-Hong Kong-Singapore Joint Center for Nonlinear and Complex Systems (Beijing), Beijing Normal University, Beijing 100875, China*<sup>2</sup>*Institute of Systems Science and College of Information Science and Engineering, Huaqiao University, Xiamen 361021, China*<sup>3</sup>*Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen, P.O. Box 407, 9700 AK, The Netherlands*<sup>4</sup>*Department of Physics, East China Normal University, Shanghai 200241, China*

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Heterogeneous coupling patterns among interacting elements are ubiquitous in real systems ranging from physics, chemistry to biology communities, which have attracted much attention during recent years. In this paper, we extend the Kuramoto model by considering a particular heterogeneous coupling scheme in an ensemble of phase oscillators, where each oscillator pair interacts with different coupling strength that is weighted by a general function of the natural frequency. The Kuramoto theory for the transition to synchronization can be explicitly generalized, such as the expression for the critical coupling strength. Also, a self-consistency approach is developed to predict the stationary states in the thermodynamic limit. Moreover, Landau damping effects are further revealed by means of linear stability analysis and resonance poles theory below the critical threshold, which turns to be far more generic. Our theoretical analysis and numerical results are consistent with each other, which can help us understand the synchronization transition in general networks with heterogenous couplings.

DOI: [10.1103/PhysRevE.94.062204](https://doi.org/10.1103/PhysRevE.94.062204)**I. INTRODUCTION**

The emergence of spontaneous synchronization in a population of interacting elements is one important issue in the nonlinear dynamics and complex networks. Investigating the intrinsic microscopic mechanism of such phenomena provides insights for understanding the collective behaviors in a wide variety of fields, such as the flashing of fireflies, circadian rhythms, electrochemical and spin-torque oscillators, applause formation in a large audience, the power grids, and then again in other real systems [1–4]. Mathematically, the most successful model for studying the synchronization problem was introduced by Kuramoto [5], which stands for the classical paradigms for synchronization and turns out to be analytically solvable. During the last decades, the Kuramoto model with its generalizations have inspired and simulated extensive studies from several aspects, including the fundamental theory analyses and their relevance to practice [6,7].

The Kuramoto model describes the evolution of an ensemble of coupled phase oscillators by means of a set of time differential equations,

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i = 1, 2, \dots, N, \quad (1)$$

where  $\theta_i(t)$  is instantaneous phase of the  $i$ th oscillator,  $\omega_i$  is its natural frequency usually extracted from a certain probability density function  $g(\omega)$ , and  $K > 0$  is the global coupling strength. To characterize the degree of synchrony of system (1), it is useful to define the order parameter,

$$z(t) = r(t)e^{i\Psi(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j(t)}, \quad (2)$$

where  $z(t)$  is a complex valued vector on the complex plane,  $r(t)$  is the collective amplitude,  $\Psi(t)$  is its phase. It has been shown that with the increase of the coupling strength  $K$ , the system may experience a transition from the incoherent state, in which  $r = 0$ , and the oscillators evolve almost according to their natural frequency, into the coherent state, in which  $r > 0$ , and a number of oscillators became synchronized, sharing a common effective frequency  $\Omega$ . As already well-known results [8], Kuramoto showed that in the limit  $N \rightarrow \infty$ , when the natural frequency distribution function  $g(\omega)$  is unimodal and symmetric with respect to its center  $\hat{\omega}$ , a continuous second-order phase transition occurs at the critical coupling strength,

$$K_c = \frac{2}{\pi g(\hat{\omega})}, \quad (3)$$

and the collective amplitude  $r$  satisfies the 1/2 scale-law

$$r \propto \sqrt{\frac{K - K_c}{K_c}}, \quad (4)$$

close to the critical point  $K_c$ . With further increasing  $K$  above  $K_c$ , oscillators with natural frequencies sufficiently close to the collective frequency will become entrained by the synchronized group, forming a macroscopic oscillating cluster, and the size of the cluster becomes larger and larger. Eventually, when  $K$  is large enough, all oscillators coincide with each other and  $r$  approaches to 1, implying a complete synchrony of the system.

A typical heterogeneity of the classical Kuramoto model lies in the natural frequencies of oscillators, while the dispersion of frequencies competes with the attractive coupling  $K$ , in a way that a phase transition to synchronization takes place when the coupling strength is strong enough, or the frequency distribution  $g(\omega)$  is sufficiently narrow. A straightforward extension of the current model is to add a

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new component of heterogeneity into the coupling strength, which is a natural attribute in the realistic systems [9–14]. For example, when the phase oscillators are set on complex networks, the network properties strongly impact the route to synchronization, moreover, this structural heterogeneity is equivalent to the coupling heterogeneity in a mean-field form under suitable approximation [15,16]. In particular, recent studies of the traveling wave state, and the  $\pi$ -state in an ensemble of coupled phase oscillators is essentially a special case of a heterogeneous coupling pattern, where the interacting strength among elements has only two kinds of possible choices, either  $K$  or  $-K$  with different proportion coefficients [17,18]. Also it should be noted that most of the previous works are concerned with the topology of the coupling scheme, instead the role of oscillators themselves on synchrony, for it has been widely known that the interaction between two individuals would be influenced by the characteristics of their own. For example, a power grid network can be described as a network of Kuramoto oscillators, where the weighted coupling coefficient between two oscillators is related to their natural frequencies [19]. In social networks, different individuals may have different strengths of response to the same influence/stimuli, e.g., rumors from friends. If we roughly describe the human dynamics by Kuramoto phase oscillators, a weighted factor in terms of the natural frequencies of oscillators can reasonably characterize the heterogeneity of couplings in such networks. Furthermore, it has been shown that the Kuramoto model with frequency-weighted coupling can lead to non-trivial dynamical consequences, such as the discontinuous transition to collective synchronization in general networks [20–25], the chimera states [26,27], and Landau damping effects in conformist and contrarian oscillators [28].

The aim of this paper is to extend the Kuramoto theory by allowing for the heterogeneity in the coupling strength to the generalized frequency-weighted coupling, and present a complete framework to investigate the generalized frequency-weighted model. First, we establish rigorous self-consistency equations for the order parameter amplitude  $r$  and the synchronization frequency  $\Omega$ , through which all possible steady states of the system could be predicted. In contrast to the case of homogeneous coupling, the synchronization frequency here is not necessarily equal to the mean value of the natural frequency, actually, the critical frequency  $\Omega_c$  plays an important role in determining the critical coupling strength  $K_c$ . Second, we pay our particular attentions to the relaxation dynamics of the incoherent state when  $K < K_c$ , a detailed linear stability analysis of the incoherent state is performed. It is shown that the linearized operator has no discrete eigenvalues below the critical coupling. Furthermore, as the byproduct of the formulation, the explicit expression of the critical coupling strength is also obtained, which keeps the unified form for the classical Kuramoto model Eq. (3) and is consistent with the mean-field theory. The linear stability foretells that the incoherent state is only neutrally stable to perturbations. Nevertheless, we report a theoretical analysis and show that the relaxation to the incoherent state is indeed exponential by means of the resonance poles theory. Meanwhile, the relaxation rate can be solved in a general framework. Together with numerical simulations that verify

our theoretical analysis, in what follows, we report our main results, both theoretically and numerically.

## II. MEAN-FIELD THEORY

The Kuramoto model with generalized frequency-weighted coupling is described by the following dynamical equations:

$$\dot{\theta}_i = \omega_i + \frac{Kf(\omega_i)}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \quad (5)$$

where  $f(\omega_i)$  is a real value function of the natural frequency  $\omega_i$ . The most important characteristic of the current model is that we extend the frequency dependence to a generalized function, in contrast to the previous studies of the frequency-weight coupling, where  $f(\omega_i) = \omega_i^\beta$  or  $|\omega_i|$  [23–28]. Equation (5) defines a heterogeneous interaction pattern underlying the system, apart from the global coupling  $K > 0$ , and the coupling strength is weighted by the frequency  $\omega_i$  of the oscillators, which is a reasonable consideration in the role of realistic systems. The definition Eq. (2) allows us to rewrite Eq. (5) in the mean-field form, which yields

$$\dot{\theta}_i = \omega_i + Kr f(\omega_i) \sin(\Psi - \theta_i). \quad (6)$$

Here  $Krf(\omega_i)$  can be interpreted as an effective coupling strength, Eq. (6) reflects that the interaction to oscillator  $i$  is equivalent to a phase  $\Psi$ , and weighted by the effective coupling  $Krf(\omega_i)$ . It should be noted that the effective coupling here can be either positive or negative according to the sign of weighted function  $f(\omega_i)$ . As a matter of fact, the oscillators in the ensemble can be in general grouped into two populations, when  $f(\omega_i) > 0$ , the coupling to the mean-field is attractive, and synchronization is triggered by these oscillators, whereas  $f(\omega_i) < 0$ , the interaction to the mean-field turns out to be repulsive, as a result, synchronization between these oscillators is suppressed.

Since we are interested in the steady states of the system, the self-consistencies method turns out to be effective. In the long time limit, we assume Eq. (5) approaches to a stationary state, where the collective amplitude  $r$  is time independent and the mean-field phase  $\Psi$  rotates uniformly with a frequency, i.e.,  $\Psi(t) = \Omega t + \Psi_0$ , after an appropriate phase shift, we can set  $\Psi_0 = 0$ . By introducing the phase difference

$$\varphi_i = \theta_i - \Psi, \quad (7)$$

Eq. (6) can be transformed into

$$\dot{\varphi}_i = \omega_i - \Omega - Kr f(\omega_i) \sin(\varphi_i). \quad (8)$$

In the rotating frame, the evolution of each oscillator can be thought of as resulting from the interaction with the mean-field which is generated by the ensemble. In the thermodynamic limit  $N \rightarrow \infty$ , a density function  $\rho(\varphi, \omega, t)$  in the  $(\varphi, t)$  space, with dependence on the parameter  $\omega$  is needed, where  $\rho(\varphi, t, \omega)d\omega$  gives the fraction of oscillators with natural frequency  $\omega$ , and the phase deviation lying between  $\varphi$  and  $\varphi + d\varphi$  at time  $t$  with the normalization condition,

$$\int_0^{2\pi} \rho(\varphi, \omega, t) d\varphi = 1, \quad (9)$$

and  $2\pi$ -periodic in  $\varphi$ . Consequently, Eq. (8) is equivalent to the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}(\rho \cdot (\omega - \Omega - K \cdot r f(\omega) \sin \varphi)) = 0. \quad (10)$$

The stationary solutions of Eq. (10)  $\partial \rho / \partial t = 0$  should be discussed in two distinct cases, respectively,

$$\rho(\varphi, \omega) = \begin{cases} \delta(\varphi - \arcsin(\frac{\omega - \Omega}{K r f(\omega)})), & |\omega - \Omega| \leq |K r f(\omega)|, \\ \frac{\sqrt{(\omega - \Omega)^2 - (K r f(\omega))^2}}{2\pi|\omega - \Omega - K r f(\omega) \sin \varphi|}, & \text{otherwise.} \end{cases} \quad (11)$$

The first expression in Eq. (11) corresponds to the phase-locked oscillators, and is therefore a fixed point of Eq. (8), which stands for the time independent asymptotic phase deviation of an entrained phase oscillator of natural frequency  $\omega$  and with respect to the synchronization frequency  $\Omega$ .

Considering a weak perturbation away from the fixed point of Eq. (8) given by  $\sin \varphi_i = (\omega_i - \Omega) / K r f(\omega_i)$ , we obtain the linear perturbation equation about  $\delta \varphi_i$ , i.e.,

$$\dot{\delta \varphi}_i = -K r f(\omega_i) \cos \varphi_i \delta \varphi_i - K f(\omega_i) \sin \varphi_i \sum_{j=1}^N \frac{\partial r}{\partial \varphi_j} \delta \varphi_j. \quad (12)$$

According to the definition above, the amplitude of the order parameter  $z$  is

$$r = \frac{1}{N} \sum_{k=1}^N e^{i(\theta_k - \Psi)} = \frac{1}{N} \sum_{k=1}^N e^{i\varphi_k}, \quad (13)$$

and

$$\frac{\partial r}{\partial \varphi_j} = \frac{1}{N} \sum_{k=1}^N i e^{i\varphi_k} \delta_{jk} = \frac{i e^{i\varphi_j}}{N}. \quad (14)$$

In particular, we choose a specific perturbation direction for every trajectory  $\varphi_i$ , i.e., we set  $\delta \varphi_j = 0$  for  $i = 1, 2, \dots, N$  and  $j \neq i$ . In this case, the second term in Eq. (12) tends to 0 in the limit  $N \rightarrow \infty$ . Since  $K > 0$  and the  $r > 0$ , the stability of the fixed point of Eq. (8) is determined by the sign of  $f(\omega_i) \cos \varphi_i$ . As a consequence, the value of  $\cos \varphi_i$  of the stable fixed point of Eq. (8) should take the following form:

$$\cos \varphi_i = \text{sgn}(f(\omega_i)) \sqrt{1 - \left(\frac{\omega_i - \Omega}{K r f(\omega_i)}\right)^2}, \quad (15)$$

where  $\text{sgn}(x)$  is the sign function,  $\text{sgn}(x) = 1$  when  $x \geq 0$  and  $\text{sgn}(x) = -1$  when  $x < 0$ . The second term of Eq. (11) represents the drifting oscillators, where these oscillators could not be entrained by the mean-field Eq. (8). Equation (11) relates the stationary density function with the natural frequency  $\omega$  and  $f(\omega)$ , accordingly, the order parameter defined in Eq. (2) takes the integral form:

$$r = \int_{-\infty}^{\infty} \int_0^{2\pi} g(\omega) \rho(\varphi, \omega) e^{i\varphi} d\varphi d\omega. \quad (16)$$

It is easy to obtain that for the drifting oscillators, the integral

$$\langle \cos \varphi \rangle = \int_0^{2\pi} \cos \varphi \frac{\sqrt{(\omega - \Omega)^2 - (K r f(\omega))^2}}{2\pi|\omega - \Omega - K r f(\omega) \sin \varphi|} d\varphi = 0 \quad (17)$$

and

$$\begin{aligned} \langle \sin \varphi \rangle &= \int_0^{2\pi} \sin \varphi \frac{\sqrt{(\omega - \Omega)^2 - (K r f(\omega))^2}}{2\pi|\omega - \Omega - K r f(\omega) \sin \varphi|} d\varphi \\ &= \frac{\omega - \Omega}{K r f(\omega)} \left[ 1 - \sqrt{1 - \left(\frac{K r f(\omega)}{\omega - \Omega}\right)^2} \right]. \end{aligned} \quad (18)$$

Taking into account the contribution of both the phase-locked and the drifting oscillators to the order parameter. By inserting Eq. (11) into Eq. (16) and separating the real and imaginary parts, one can obtain the self-consistency equations,

$$\begin{aligned} r &= \int_{-\infty}^{\infty} d\omega g(\omega) \text{sgn}(f(\omega)) \sqrt{1 - \left(\frac{\omega - \Omega}{K r f(\omega)}\right)^2} \\ &\quad \times \Theta\left(1 - \left|\frac{\omega - \Omega}{K r f(\omega)}\right|\right), \end{aligned} \quad (19)$$

for the collective amplitude  $r$ , and

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\omega - \Omega}{K r f(\omega)} - \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\omega - \Omega}{K r f(\omega)} \\ &\quad \times \sqrt{1 - \left(\frac{K r f(\omega)}{\omega - \Omega}\right)^2} \Theta\left(\left|\frac{\omega - \Omega}{K r f(\omega)}\right| - 1\right), \end{aligned} \quad (20)$$

for the mean-field frequency  $\Omega$ , and here,  $\Theta(x)$  is the Heaviside function.

Equation (19) together with Eq. (20) provide closed equations for the dependence of the magnitude  $r$  and the frequency  $\Omega$  of the mean-field on  $K$ , in terms of the distribution  $g(\omega)$  and weighted function  $f(\omega)$ . Theoretically, an explicit expression for  $r(K)$  and  $\Omega(K)$  can be solved analytically or numerically for given  $g(\omega)$  and  $f(\omega)$ . However, a full analysis of the solution for an arbitrary form of  $g(\omega)$ , and any possible dependence on the weighted function  $f(\omega)$  is very difficult. Correspondingly, we focus on our attentions to the representative and generic properties of the self-consistency equations, and we look for the critical point with the onset of a non-vanishing mean-field. Hence, in the limit case  $r \rightarrow 0$ , by taking into account the Taylor expansion of Eqs. (19) and (20), one obtains the critical coupling strength  $K_c$  as

$$K_c = \text{sgn}[f(\Omega_c)] \frac{2}{\pi g(\Omega_c) f(\Omega_c)}, \quad (21)$$

and the critical mean-field frequency  $\Omega_c$  satisfies the balance equation

$$P \cdot \int_{-\infty}^{\infty} \frac{g(\omega) f(\omega)}{\omega - \Omega_c} d\omega = 0. \quad (22)$$

The symbol  $P$  means the principal-value integral along the real  $\omega$ .

The concise expression Eq. (21) is significant, which can be regarded as a straightforward extension of the classical Kuramoto model Eq. (3), together with the critical frequency  $\Omega_c$  determined by the balance equation Eq. (22). For the

previous studies of the Kuramoto model with heterogeneous coupling, the weighted coupling is uncorrelated with the natural frequency, and therefore the second integral in Eq. (20) vanishes, consequently, the mean-field frequency  $\Omega$  is always equal to the center  $\hat{\omega}$ , provided that  $g(\omega)$  is one-humped and  $g(\hat{\omega} - x) = g(\hat{\omega} + x)$ . For the current model, however, there is no guarantee that  $\Omega$  is necessarily equal to  $\hat{\omega}$  due to the rotational symmetry-breaking of the system [29,30]. Actually,  $\Omega_c$  plays a crucial role in determining the coupling strength, since  $K_c$  is inversely proportional to  $g(\Omega_c)$  and  $f(\Omega_c)$ .

### III. RELAXATION DYNAMIC OF THE INCOHERENT STATE

#### A. Linear stability analysis

The above analysis reveals that all the stationary states for the system as well as the stationary collective dynamical properties of the synchronization can be characterized in the frame of mean-field theory, the collective amplitude  $r$  and the group velocity  $\Omega$  can be formally solved for general weighted function through the self-consistency equations. However, a thorough stability of all the possible steady states is still elusive. In the following, we pay our particular attention to the incoherent state,  $r \equiv 0$ , and conduct a detailed linear stability analysis of it. As it will appear momentarily, the stability analysis can make up for the limitation of the mean-field theory [31].

Let us turn to the original phase reference, and introduce the phase density function  $F(\theta, \omega, t) \equiv \rho(\varphi + \Omega t, \omega, t)$ , which satisfies the continuity equation

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial \theta}(F \cdot v) = 0, \quad (23)$$

the velocity is given by

$$v = \omega + \frac{K}{2i} f(\omega)(z(t)e^{-i\theta} - z^*(t)e^{i\theta}), \quad (24)$$

where  $z(t)$  is the order parameter defined by Eq. (2) which has the integral form in the limit  $N \rightarrow \infty$ ,

$$z(t) = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} e^{i\theta} g(\omega) F(\theta, t, \omega) d\theta, \quad (25)$$

and “\*” denotes the complex conjugate of  $z(t)$ . Taking into account the  $2\pi$ -periodicity of  $\theta$  in  $F(\theta, \omega, t)$ , let

$$z_n(\omega, t) = \int_0^{2\pi} e^{ni\theta} F(\theta, t, \omega) d\theta, \quad n = 0, 1, 2, \dots, \quad (26)$$

be the  $n$ th Fourier coefficient of  $F(\theta, t, \omega)$ . Following this definition, we have  $z_0(t, \omega) \equiv 1$  due to the normalization of the density function, and the evolution of  $z_n(t, \omega)$  satisfies the differential equations,

$$\frac{dz_n}{dt} = ni\omega z_n + \frac{nK}{2} f(\omega)(z(t)z_{n-1} - z^*(t)z_{n+1}), \quad (27)$$

which must be solved self-consistently with an equation,

$$z(t) = \int_{-\infty}^{\infty} z_1(t, \omega) g(\omega) d\omega. \quad (28)$$

A careful examination of Eq. (27) reveals that  $z_n(t, \omega) \equiv 0$ , i.e., the incoherent state, where  $F(\theta, \omega) = 1/2\pi$  and  $r \equiv 0$  is

always a trivial fixed point of Eq. (27). To study the stability of this steady state, we can consider the evolution of a weak perturbation away from the incoherent state. By excluding second- and higher-order terms of  $\delta z_n$ , we obtain a set of linear equations for  $\delta z_n$  as

$$\begin{aligned} \frac{d\delta z_1}{dt} &= \left( i\omega + \frac{K}{2} f(\omega) \hat{P} \right) \delta z_1 = \hat{T} \cdot \delta z_1 \\ \frac{d\delta z_n}{dt} &= ni\omega \delta z_n, \quad n > 1. \end{aligned} \quad (29)$$

Here,  $\hat{P}$  is a linear operator defined as

$$\begin{aligned} \hat{P}q(\omega) &= \int_{-\infty}^{\infty} q(\omega)g(\omega)d\omega \\ &= (q(\omega), P_0), \end{aligned} \quad (30)$$

where  $q(\omega)$  is a function in the weighted-Lebesgue space,  $P_0 \equiv 1$ , and  $(,)$  is the concise inner product notation for the integral over  $\omega$  with respect to the weighted factor  $g(\omega)$ . Since higher Fourier harmonic components have no contribution to the order parameter  $z(t)$  Eq. (28), we only focus on the evolution of  $\delta z_1(t, \omega)$ . Let  $\lambda$  be the eigenvalue of linear operator  $\hat{T}$ , we have

$$\frac{d\delta z_1}{dt} = \hat{T} \cdot \delta z_1 = \lambda \delta z_1. \quad (31)$$

Substituting the expression of  $\hat{T}$  into Eq. (31), and multiplying both sides by the inverse operator  $(\lambda - i\omega)^{-1}$ , we obtain

$$\delta z_1 = (\lambda - i\omega)^{-1} \frac{K}{2} f(\omega)(\delta z_1, P_0). \quad (32)$$

Taking the inner product with  $P_0$  for both sides, then the self-consistent eigenvalue equation for the linear operator  $\hat{T}$  takes the form

$$\int_{-\infty}^{\infty} \frac{f(\omega)}{\lambda - i\omega} g(\omega) d\omega = \frac{2}{K}, \quad \lambda \in C \setminus i\omega, \quad (33)$$

where  $\lambda$  is on the complex plane except for those points  $i\omega$ . Notice that Eq. (33) relates implicitly the global coupling strength  $K$  with the eigenvalue  $\lambda$ . To simplify the discussion, we rewrite Eq. (33) into two equations by setting  $\lambda = x + iy$ , i.e.,

$$\int_{-\infty}^{\infty} \frac{x}{x^2 + (\omega - y)^2} f(\omega)g(\omega) d\omega = \frac{2}{K} \quad (34)$$

and

$$\int_{-\infty}^{\infty} \frac{\omega - y}{x^2 + (\omega - y)^2} f(\omega)g(\omega) d\omega = 0. \quad (35)$$

The sign of  $x$  determines the stability of the incoherent state, furthermore, it has been proven that [32,33], if the global coupling strength  $K > 0$  and is sufficiently small, the eigenvalue  $\lambda$  of the linear operator  $\hat{T}$  actually does not exist, provided that the product  $f(\omega)g(\omega)$  is analytic and has no singularity on the real axis  $\omega$ . Accordingly, the incoherent state is only neutrally stable to perturbations in the regime  $K < K_c$ , where the operator  $\hat{T}$  has only continuous spectrum  $i\omega$  on the whole imaginary axis. However, as  $K$  further increases, the discrete eigenvalues emerge with real part  $x \neq 0$  once  $K > K_c$ . Imposing the critical condition  $x \rightarrow 0^\pm$ ,  $y \rightarrow y_j$  for



TABLE I. Summary of the correspondence between the weighted function  $f(\omega_i)$ , the frequency distributions (FD)  $g(\omega)$ , and the balance equations, the critical mean-field frequencies  $\Omega_c$ , the critical coupling strength  $K_c$ .

Weight function	FD	Balance equation	$\Omega_c$	$K_c$
$ \omega_i $	$\frac{\gamma}{\pi} \frac{1}{\omega^2 + \gamma^2}$	$\frac{2\gamma}{\pi} \Omega_c \ln \frac{\gamma}{\Omega_c} / (\gamma^2 + \Omega_c^2) = 0$	$0, \pm \gamma$	4
$ \omega_i $	$\frac{1}{2a} \Theta(a -  \omega )$	$\frac{\Omega_c}{2a} \ln \frac{a^2 - \Omega_c^2}{\Omega_c^2} = 0$	$0, \pm \frac{a}{\sqrt{2}}$	$\frac{4\sqrt{2}}{\pi}$
$\omega_i$	$\frac{\gamma}{\pi} \frac{1}{(\omega - \Delta)^2 + \gamma^2}$	$\frac{\gamma^2 + \Delta(\Delta - \Omega_c)}{\gamma^2 + (\Delta - \Omega_c)^2} = 0$	$\frac{\gamma^2 + \Delta^2}{\Delta}$	$\text{sgn}(\Delta) \frac{2\gamma}{\Delta}$
$\omega_i$	$\frac{1}{2} \Theta(1 -  \omega )$	$1 - \Omega_c \text{arctanh}(\Omega_c) = 0$	$\pm 0.8335$	1.528
$\omega_i^2$	$\frac{\sqrt{2}\gamma^3}{\pi} \frac{1}{\omega^4 + \gamma^4}$	$\gamma^2(\gamma - \Omega_c)\Omega_c(\gamma + \Omega_c) = 0$	$0, \pm \gamma$	$\frac{2\sqrt{2}}{\gamma}$
$\omega_i^2$	$\frac{1}{2} \Theta(1 -  \omega )$	$\Omega_c - \Omega_c^2 \ln \left( \frac{1 + \Omega_c}{1 - \Omega_c} \right) / 2 = 0$	$0, \pm 0.8335$	1.8327
$\frac{1}{\omega_i}$	$\frac{1}{2a} \Theta(\omega - a) \Theta(2a - \omega)$	$-\ln \left( -2 - \frac{2a}{-2a + \Omega_c} \right) / 2a\Omega_c = 0$	$\frac{4a}{3}$	$\frac{16a^2}{3\pi}$
$\frac{1}{1 +  \omega_i }$	$\frac{\gamma}{\pi} \frac{1}{\omega^2 + \gamma^2}$	$-\frac{\gamma(\gamma^2 + \Omega_c)}{(\gamma + \gamma^3)(\gamma^2 + \Omega_c^2)} = 0$	$-\gamma^2$	$2\gamma(1 + \gamma^2)^2$
$\frac{1}{1 + \omega_i^2}$	$\frac{1}{\pi} \frac{1}{(\omega - \Delta)^2 + 1}$	$\frac{(\Delta - 2\Omega_c)(-3 + (\Delta - \Omega_c) \cdot \Omega_c)}{(4 + \Delta^2)(1 + (\Delta - \Omega_c)^2)(1 + \Omega_c^2)} = 0$	$\Delta/2$	$(4 + \Delta^2)^2/8$
			$(\Delta \pm \sqrt{-12 + \Delta^2})/2$	$2(4 + \Delta^2), \Delta^2 \geq 12$

Eq. (34), once again we obtain the critical coupling strength as

$$K_c = \text{sgn}[f(y_j)] \frac{2}{\pi \sup_j g(y_j) f(y_j)}, \quad (36)$$

where  $y_j$  are determined by Eq. (35) with the limit  $x \rightarrow 0^\pm$ . Just as mentioned in [30],  $\Omega_c$  is indeed the imaginary part of the eigenvalue of the operator  $\hat{T}$  at the boundary of stability. Since Eq. (35) may have more than one root in the limit  $x \rightarrow 0^\pm$ ,  $\sup_j$  means that we choose the  $j$ th root  $y_j$ , so that the absolute of the product  $g(y_j)f(y_j)$  is maximal, and  $K_c$  corresponds to the foremost critical point for the stability change of the incoherent state. Table I summarizes the balance equation, the critical mean-field frequency  $\Omega_c$ , and the critical coupling strength  $K_c$  with respect to different frequency distributions  $g(\omega)$  and weighted function  $f(\omega)$ , these analytical results were supported by the previous numerical simulations [23,25,28].

It should be pointed out that in the previous discussions, where  $f(\omega) > 0$  for any  $\omega$ , thus, it can be confirmed that  $x > 0$  once  $K > K_c$  according to Eq. (34). It means that the incoherent state is always linear unstable as long as the global coupling strength is sufficiently large. However, for the current model, when we release the restriction about  $f(\omega)$ , the real part of the eigenvalue  $\lambda$  may be negative even when  $K > K_c$ , which implies that the incoherent state can be linearly stable at some  $K > K_c$ , once the integral of  $f(\omega)g(\omega)/(x^2 + (\omega - y)^2) < 0$ , or equivalently, the repulsive terms dominate the coupling. Going further, the stability analysis also provides insight for the mean-field theory. In fact, when the repulsive terms ( $f(\omega) < 0$ ) prevail over the attractive terms ( $f(\omega) > 0$ ) underlying the system, the integral in Eq. (19) may be negative, which violates the precondition  $r > 0$ . As a result, in this situation, the system can never approach a stationary partially synchronized state

$r > 0$ . Accordingly, synchronization is inhibited and the only possible stationary state of the system is the incoherent state  $r = 0$  [18,28,43].

## B. Landau damping effects

According to the above linear stability analysis, the incoherent state of the system Eq. (5) is only neutrally stable when  $K < K_c$ , however, as an already well-known result [31], Strogatz *et al.* showed that the order parameter  $r(t)$  in this regime actually decays to zero in the long time limit ( $t \rightarrow \infty$ ) in an exponential form in the classical Kuramoto model. In this subsection, we show that such an effect is far more generic even in the system where phase oscillators are coupled to the mean-field weighted by their natural frequencies. Addressing this question is nontrivial, since it has already been shown that the relaxation to the equilibrium is related to susceptibility of the system to external stimulations in statistical physics [16,34–36]. In particular, the decaying mechanism is remarkably similar to the famous Landau damping in plasma physics [36,37]. To this end, we report a general framework to determine the decay exponent, together with extensive numerical simulations that support the theoretical predictions.

According to Eq. (31), a solution of  $\delta z_1(t, \omega)$  with an initial value  $\delta z_1(0, \omega)$  is given by

$$\delta z_1(t, \omega) = e^{\hat{T} \cdot t} \cdot \delta z_1(0, \omega), \quad (37)$$

where the operator  $e^{\hat{T} \cdot t}$  is calculated by means of the Laplace inversion formula, which yields

$$e^{\hat{T} \cdot t} = \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} e^{s \cdot t} (s - \hat{T})^{-1} ds, \quad (38)$$

for  $t > 0$  and  $x > 0$ . The resolvent  $(s - \hat{T})^{-1}$  is obtained as follows: let  $\phi(\omega)$  be an arbitrary function in the weighted Lebesgue space and note

$$\hat{R}(s)\phi = (s - \hat{T})^{-1}\phi = \left(s - i\omega - \frac{K}{2}f(\omega)\hat{P}\right)^{-1}\phi, \quad (39)$$

by multiplying  $(s - i\omega - \frac{K}{2}f(\omega)\hat{P})$  to both sides, we have

$$\begin{aligned} (s - i\omega)\hat{R}(s)\phi &= \phi + \frac{K}{2}f(\omega)\hat{P}\hat{R}(s)\phi \\ &= \phi + \frac{K}{2}f(\omega)(\hat{R}(s)\phi, P_0). \end{aligned} \quad (40)$$

This is rearranged as

$$\hat{R}(s)\phi = (s - i\omega)^{-1}\phi + \frac{K}{2}(\hat{R}(s)\phi, P_0)(s - i\omega)^{-1}f(\omega), \quad (41)$$

taking the inner product with  $P_0$ , we obtain

$$\begin{aligned} (\hat{R}(s)\phi, P_0) &= ((s - i\omega)^{-1}\phi, P_0) \\ &\quad + \frac{K}{2}(\hat{R}(s)\phi, P_0)((s - i\omega)^{-1}f(\omega), P_0). \end{aligned} \quad (42)$$

Reordering of the term then yields

$$(\hat{R}(s)\phi, P_0) = \frac{((s - i\omega)^{-1}\phi, P_0)}{1 - \frac{K}{2}((s - i\omega)^{-1}f(\omega), P_0)}. \quad (43)$$

Substituting Eq. (43) into Eq. (40) leads to

$$\begin{aligned} \hat{R}(s)\phi &= (s - i\omega)^{-1}\phi \\ &\quad + \frac{\frac{K}{2}((s - i\omega)^{-1}\phi, P_0)(s - i\omega)^{-1}f(\omega)}{1 - \frac{K}{2}((s - i\omega)^{-1}f(\omega), P_0)}. \end{aligned} \quad (44)$$

Then, the order parameter  $\delta z(t)$  reads

$$\begin{aligned} \delta z(t) &= (\delta z_1(t, \omega), P_0) = (e^{\hat{T}t}\delta z_1(0, \omega), P_0) \\ &= \lim_{y \rightarrow 0} \int_{x-iy}^{x+iy} e^{st} \frac{D(s)}{1 - KD'(s)/2} ds, \end{aligned} \quad (45)$$

where

$$D(s) = \int_{-\infty}^{\infty} \frac{g(\omega)\delta z_1(0, \omega)}{s - i\omega} d\omega, \quad (46)$$

and

$$D'(s) = \int_{-\infty}^{\infty} \frac{g(\omega)f(\omega)}{s - i\omega} d\omega. \quad (47)$$

To invert the Laplace transform Eq. (45), we have to find the poles of  $D(s)/(1 - KD'(s)/2)$ , and the calculation should be analytically continued to the left half-complex plane where  $\text{Re}(s) < 0$ . Figure 1 presents a series of results from the numerical resolution of Eq. (5). In fact, from Eq. (45) when  $K \rightarrow 0$ ,  $\delta z(t) = \int_{-\infty}^{\infty} e^{i\omega t} g(\omega) d\omega$ , which is the Fourier transform of  $g(\omega)$ , that is, the oscillators rotate at angular frequencies given by their own natural frequencies [38]. In terms of the Riemann-Lebesgue lemma, we obtain  $\delta z(t) \rightarrow 0$  in the limit  $t \rightarrow \infty$  as shown in Fig. 1(a). When the function  $g(\omega)f(\omega)$  is a rational case, such as  $g(\omega)$  is Lorentzian and  $f(\omega) = \omega$ , the resonance poles could be solved analytically Fig. 1(b). We have also conducted direct numerical simulations

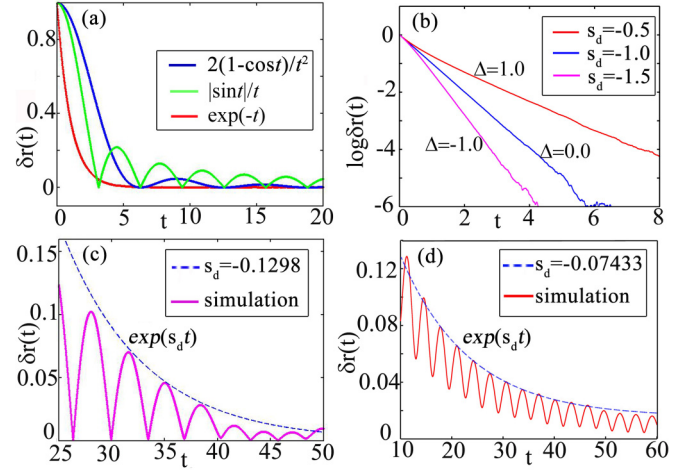


FIG. 1. Different scenarios of the decay of  $\delta r(t)$  with respect to different frequency distributions  $g(\omega)$  and frequency-weighted functions  $f(\omega)$  below the critical threshold ( $K < K_c$ ). The horizontal axes are time, solid lines refer to the direct numerical solutions of Eq. (5), and dashed lines are the fitted curve lines with exponent  $s_d$ . (a)  $K = 0$ , the red is  $g(\omega) = \frac{1}{\pi(\omega^2+1)}$ , the green is  $g(\omega) = \frac{1}{2}, \omega \in (-1, 1)$ , and the blue is  $g(\omega) = 1 - |\omega|, \omega \in (-1, 1)$ , the inset is the expression of  $\delta r(t)$ . (b)  $K = 1$ , with  $f(\omega) = \omega, g(\omega) = \frac{1}{\pi((\omega-\Delta)^2+1)}$ , the red  $\Delta = 1.0$ , the blue  $\Delta = 0$ , and the pink  $\Delta = -1.0$ , respectively, the inset is the decay exponent  $s_d$ . (c)  $K = 1.2$ , with  $f(\omega) = |\omega|, g(\omega) = \frac{1}{\pi(\omega^2+1)}$ , and  $s_d = -0.1298$  which is calculated numerically. (d)  $K = 0.8$ , with  $f(\omega) = \omega, g(\omega) = \frac{1}{2}, \omega \in (-1, 1)$ , and  $s_d = -0.07433$  calculated numerically. All curves belong to the neutrally stable regime of the incoherent state predicted by linear theory. In the numerical simulations, the initial states of the system are set in the fully coherent states, and the total number of oscillators is  $N = 100000$ .

for other typical cases, where the inverse Laplace transform could not be calculated analytically, and the resonance poles could be calculated numerically, as shown in Fig. 1(c) and (d). It is found that the envelope of the order parameter follows the form of exponential decay approximatively for the short time [28,36]. The above results suggest that the Landau damping effect is a generic phenomenon which is entirely due to the occurrence of resonance poles caused by analytic continuation, and the real parts of them control the exponential relaxation rate of the the order parameter  $\delta r(t)$  [31,33,36].

Recently, the Ott-Antonsen method has been proposed to obtain the low-dimensional dynamics of a large system of coupled oscillators [39,40], where the original set of differential equations can be reduced to the differential equations describing the temporal evolution of the order parameter  $z(t)$  alone, which makes it possible to depict the system in a global picture. However, the validity of this method needs several prerequisites. First, the first-order Fourier coefficient of the density function  $F(\theta, \omega, t)$  can be analytically continued from the real  $\omega$  into the complex  $\omega$ -plane such that the continuation has no singularities. Second, to avoid divergence of the density function, the evolution of  $z_1(t, \omega)$  must satisfy  $|z_1(t, \omega)| \leq 1$  at any time in the invariant manifold. In the Appendix, we include the Ott-Antonsen method to study the relaxation dynamics of

both the incoherent state and the coherent state [15], provided that the above restrictions can be satisfied. It turns out that such a method is consistent with our theoretical formulations. For a more general case, where the preconditions are violated, the Ott-Antonsen method fails to treat properly the system, and hence it does not provide more substantial information than the traditional analysis here. Further investigation of the system utilizes the amplitude equation theory, which reveals the local bifurcation behaviors close to the critical point when the system satisfies the  $O(2)$  group symmetry [41,42].

#### IV. CONCLUSION

To summarize, we extend the traditional Kuramoto model for the synchronization transition in an infinite large ensembles of globally coupled phase oscillators to the heterogeneously interacting scheme, where the mean-field coupling is weighted by their natural frequency characterized by a general function  $f(\omega_i)$ . Theoretically, the mean-field analysis, the linear stability analysis, the resonance poles method, and the Ott-Antonsen reduction have been carried out to obtain insights. Together with the numerical simulations, our study presented the following main results. First, we established the self-consistency equations that predict the steady states of the system. Second, the explicit expression of the critical coupling strength  $K_c$  was derived where the critical frequency  $\Omega_c$  plays a crucial role in determining  $K_c$ , and it must be obtained by solving a phase balance equation. Third, the relaxation dynamics of the incoherent state have been addressed, and we provided the evidence of a regime ( $K < K_c$ ) where the linear stability theory predicts neural stability but the order parameter decays exponentially, which resembles the phenomenon of Landau damping in plasma physics. Furthermore, the relaxation rate can be determined in the framework of the resonance pole theory. Finally, we provided the Ott-Antonsen method to capture the relaxation dynamics of general steady states, when the system satisfies the scope of applications. This work provided a complete framework to deal with the generalized frequency-weighted Kuramoto model, and the obtained results will enhance our understandings of the synchronization transition in networks with heterogeneous coupling schemes.

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#### APPENDIX: THE OTT-ANTONSEN METHOD FOR THE RELAXATION DYNAMICS

In this appendix, we provide the Ott-Antonsen method that allows to describe the generalized frequency-weighted Kuramoto model in the low-dimensional invariant manifold. Following this method, the density function  $F(\theta, t, \omega)$  is sought in the form

$$F(\theta, t, \omega) = \frac{g(\omega)}{2\pi} (1 + F_+ + F_-), \quad (\text{A1})$$

where

$$F_+ = \sum_{n=1}^{\infty} F_n(\omega, t) e^{in\theta}, \quad (\text{A2})$$

and  $F_- = F_+^*$ , with the additional ansatz

$$F_n = \alpha^n(\omega, t). \quad (\text{A3})$$

Substituting the ansatz into the continuity equation, one obtains the equation for  $\alpha(\omega, t)$

$$\frac{d\alpha}{dt} + i\omega\alpha + \frac{f(\omega)K}{2} (z \cdot \alpha^2 - z^*) = 0 \quad (\text{A4})$$

with the order parameter,

$$z(t) = \int g(\omega) \alpha^*(\omega, t) d\omega. \quad (\text{A5})$$

Similar to the analysis in the mean-field theory, we look for a steady state  $z(t) = r e^{i\Omega t}$  with a constant order parameter  $r$ , a group velocity  $\Omega$ . By introducing a suitable change of the reference frame  $\omega \rightarrow \omega + \Omega$  and set  $\dot{\alpha} = 0$ , we find  $\alpha_0(\omega)$  a solution:

$$\begin{aligned} & \pm \sqrt{1 - \left( \frac{\omega}{Krf(\omega + \Omega)} \right)^2} - \frac{i\omega}{Krf(\omega + \Omega)}, \quad |\omega| \leq Kr|f(\omega + \Omega)|, \\ & - \frac{i\omega}{Krf(\omega + \Omega)} \left[ 1 - \sqrt{1 - \left( \frac{Krf(\omega + \Omega)}{\omega} \right)^2} \right], \quad \text{otherwise.} \end{aligned} \quad (\text{A6})$$

Substituting Eq. (A6) into Eq. (A5) yields the self-consistency equations [Eqs. (19) and (20)] in the main text.

To investigate the relaxation dynamics, we consider a weak perturbation on the stationary state,

$$z(t) = r + \delta z(t), \quad \alpha(t) = \alpha_0(\omega) + \delta \alpha(\omega, t), \quad (\text{A7})$$

we obtain a linear equation for  $\delta \alpha(\omega, t)$

$$\frac{d\delta \alpha}{dt} + i\omega \delta \alpha + \frac{Kf(\omega)}{2} (2r\alpha_0 \delta \alpha + \alpha_0^2 \delta z - \delta z^*) = 0, \quad (\text{A8})$$

that must be solved self-consistently with

$$\delta z(t) = \int g(\omega) \delta \alpha^*(\omega, t) d\omega. \quad (\text{A9})$$

Taking the Laplace transform of both sides of Eq. (A8), and reordering the terms then yields

$$\delta \alpha(s, \omega) = \frac{\delta \alpha(t=0) + \frac{Kf(\omega)}{2}(\delta \alpha^*(s) - \alpha_0^2 \delta z(s))}{s + i\omega + Kf(\omega)r\alpha_0}. \quad (\text{A10})$$

Substituting Eq. (A10) into Eq. (A9) leads to

$$\delta z(s) = \frac{B(s)}{1 - KA(s)/2}, \quad (\text{A11})$$

where

$$B(s) = \int_{-\infty}^{\infty} g(\omega + \Omega) \frac{\delta \alpha(\omega, t=0)}{s + i\omega + Krf(\omega + \Omega)\alpha_0} d\omega \quad (\text{A12})$$

and

$$A(s) = \int_{-\infty}^{\infty} g(\omega + \Omega) \frac{f(\omega + \Omega)(1 - \alpha_0^2)}{s + i\omega + Krf(\omega + \Omega)\alpha_0} d\omega, \quad (\text{A13})$$

the order parameter  $\delta z(t)$  is the inverse Laplace transform of  $\delta z(s)$ . It is obvious that when the system is the incoherent state,  $\alpha_0 \equiv 0$ , and Eq. (A11) is the same as Eq. (45) in the main text.

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