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Citation: Chaos **26**, 123117 (2016); doi: 10.1063/1.4972117
View online: http://dx.doi.org/10.1063/1.4972117
View Table of Contents: http://aip.scitation.org/toc/cha/26/12
Published by the American Institute of Physics

CHAOS 26, 123117 (2016)

Intermittent Bellerophon state in frequency-weighted Kuramoto model

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(Received 29 September 2016; accepted 29 November 2016; published online 19 December 2016)

Recently, the Bellerophon state, which is a quantized, time dependent, clustering state, was revealed in globally coupled oscillators [Bi *et al.*, Phys. Rev. Lett. **117**, 204101 (2016)]. The most important characteristic is that in such a state, the oscillators split into multiple clusters. Within each cluster, the instantaneous frequencies of the oscillators are not the same, but their average frequencies lock to a constant. In this work, we further characterize an intermittent Bellerophon state in the frequency-weighted Kuramoto model with a biased Lorentzian frequency distribution. It is shown that the evolution of oscillators exhibits periodical intermittency, following a synchronous pattern of bursting in a short period and resting in a long period. This result suggests that the Bellerophon state might be generic in Kuramoto-like models regardless of different arrangements of natural frequencies. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4972117]

In the study of synchronization, the classical Kuramoto model and its many generalizations turn out to be prototypes to offer important insight. So far, by extensive investigations, a variety of coherent states have been found in such systems, including the partially synchronous state, the traveling wave state, the standing wave state, and the Chimera state. Typically, these coherent states can be classified into two types: stationary and nonstationary. Here, the stationary state refers to such an asymptotic state of the dynamical system in which the probability density function of oscillators is time-independent, and nonstationary state otherwise. Since these asymptotic states characterize the long-term dynamics of the system, they are of fundamental importance for understanding the collective behaviors of these systems. Recently, several new Kuramoto-like models, such as the frequency-weighted Kuramoto model, have been investigated, which have been shown that transitions in such systems can be discontinuous, i.e., the first-order. Thus, it is important to identify and characterize possible coherent states in these models. Recently, a new quantized, time dependent, clustering state was revealed in the frequency-weighted Kuramoto model with the bimodal Lorentzian frequency distribution (FD). Along this line, in this paper, we further report an intermittent Bellerophon state in this model with the biased Lorentzian frequency distribution. We provide a detailed characterization of the dynamical features of such state, which is helpful for us to understand the complicated collective behaviors in coupled oscillators.

I. INTRODUCTION

Synchronization refers to the collective behaviors self-organized in dynamical systems with a large number of interacting components. Under certain circumstances, the motions of vast degrees of freedom may dissipate and the dynamics of such a system then can be governed by a few degrees of freedom. Mathematically, the original high-dimensional dynamical space collapses into a subspace, typically low-dimensional, known as synchronization manifold. Synchronization has been extensively observed in various fields, and typical examples include the firing of fireflies, pacemaker cells in heart, human circadian rhythms, Josephson junction arrays, and laser networks.[1]

Theoretically, synchronization has been successfully studied in models of coupled oscillators. In the simplest form, the dynamics of an oscillator is described by only a phase variable. The coupling of N phase oscillators leads to the famous Kuramoto model, whose dynamical equation reads[2]

$$\dot{\theta}_j = \omega_j + \frac{\kappa}{N}\sum_{n=1}^{N} \sin\left(\theta_n - \theta_j\right), \quad j = 1, ..., N. \qquad (1)$$

Here, θ_j (ω_j) are the instantaneous phase (the natural frequency) of the jth oscillator. Dot denotes the temporal derivative and κ is the global coupling strength. The set of N natural frequencies $\{\omega_j\}$ is drawn from the certain frequency distribution (FD) $g(\omega)$. In this system, the coupling strength serves as the control parameter. When it is small, the system is in the incoherent state, in which oscillators rotate almost according to their natural frequencies. However, when the coupling strength is increased to exceed certain threshold, synchronization occurs and the system goes into the (partially) coherent state, in which part oscillators have been entrained by the mean field and become phase-locked.

The Kuramoto model and its various generalizations have been extensively investigated for more than four decades.[3,4] So far, several typical coherent states have been observed and characterized, such as the partially coherent state,[2] the standing wave state,[5,6] the traveling wave state,[6–10] and the Chimera state.[11,12] In all these coherent states, oscillators inside the coherent cluster are typically frequency-locked. Recently, a quantized, time dependent, clustering state, named as the Bellerophon state, was revealed in globally coupled

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oscillators,[13,14] which is essentially different from the coherent states previously observed in Kuramoto-like models. In such a state, the oscillators organize into multiple coherent clusters. By coherent here we mean that within each cluster, the instantaneous frequencies of the oscillators are not equal to each other, but their average frequencies lock to a constant. In this paper, we further show that such a state also occurs with a different arrangement of FD, i.e., a biased Lorentzian distribution. Interestingly, the observed nonstationary state turns out to be periodically intermittent, occurring in the intermediate regime between incoherence and fully synchrony.

The dynamical system we study is a variant of Kuramoto model, namely, the frequency-weighted Kuramoto model

$$\dot{\theta}_j = \omega_j + \frac{\kappa|\omega_j|}{N}\sum_{n=1}^{N}\sin\left(\theta_n - \theta_j\right), \quad j = 1, ..., N. \tag{2}$$

Recently, this model has been intensively investigated in terms of explosive synchronization.[15–20] For example, in Ref. 20, it has been studied with a biased Lorentzian distribution

$$g(\omega) = \frac{\Delta}{\pi[(\omega-\omega_0)^2 + \Delta^2]}, \tag{3}$$

where ω_0 corresponds to the peak of the distribution that is generally not zero, and 2Δ is the width at half maximum. It is shown that as ω_0 increases, the synchronization type in the system converts from discontinuous (first-order) to continuous (second-order). Interestingly, several types of coherent states have been observed in these systems, including two types of phase-locking states and one type of oscillatory coherent state in which the order parameter turns out to be time-dependent. In fact, such an oscillatory state has also been observed in noise-driven active rotators in the Shinomoto-Kuramoto model that is relevant to many physical contexts.[21–23] In this work, we further investigate the observed oscillatory coherent state in Ref. 20 and show that it is a special, i.e., intermittent, Bellerophon state.

In our numerical simulations, the coupled ordinary differential equations are integrated by the fourth-order Runge-Kutta method with time step 0.01. The initial conditions for the phase variables are chosen from $[-\pi, \pi]$ at random. To simplify the situation, we keep $\Delta = 1$ as a constant throughout this work. In both forward and backward transitions, the coupling strength is increased/decreased in an adiabatic way with a step of 0.02. For each κ, the order parameters are averaged in a time window after the transient stage. Typically, the total number of oscillators is $N = 10\,000$.

To characterize the phase synchronization or collective behavior in the model, an order parameter can be defined as

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \tag{4}$$

where r and ψ are the module and argument of the mean field, respectively. Physically, the complex order parameter can be regarded as a vector on the complex plane. According to its definition, r is between 0 and 1. Typically, $r \approx 0$ indicates a totally random phase distribution, i.e., the incoherent state, while $r > 0$ indicates a (partially) phase-locking state, i.e., the coherent or synchronized state. As the system becomes more coherent, r will gradually approach 1, corresponding to fully synchrony in the system. In addition, to characterize the coherent states, especially the microscopic perspectives, we can take snapshots of it and plot the following distributions, such as the instantaneous phases vs the natural frequencies, the instantaneous frequencies (the angular speeds of oscillators along unit circle) vs the natural frequencies, and the averaged instantaneous frequencies (the average angular speeds) vs the natural frequencies.

II. RESULTS

Now we report the new coherent state observed in the frequency-weighted Kuramoto model and characterize its main properties. A typical example has been illustrated in Fig. 1. In fact, this state is observed in a regime below synchrony during the backward process of the first-order transition.[20] As discussed in Ref. 20, when κ is sufficiently large, the coherent oscillators, though split into two clusters, are frequency-locking; i.e., they rotate in one speed along the unit circle. This situation belongs to the traveling wave state. However, when κ gradually decreases to below the synchronous regime, the two phase-locking clusters further decompose into multiple phase-coherent clusters as shown in Fig. 1(a). However, interestingly, it is found that the oscillators in these coherent clusters are no longer frequency-locked. As shown

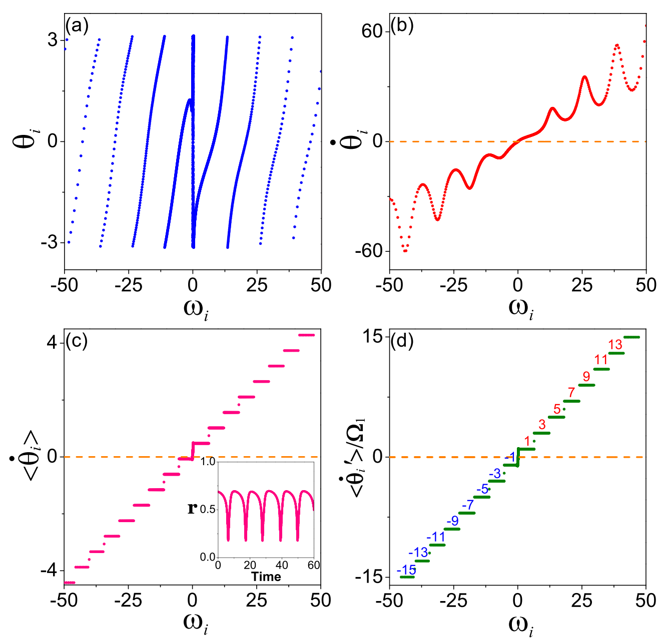

FIG. 1. Characterizing the Bellerophon state observed in the model. $\omega_0 = 0.5$ and $\kappa = 2.08$. (a)–(c) The phases (a), the instantaneous frequencies (speeds) (b), and the average frequencies (average speeds) (c) vs the natural frequencies of oscillators. (d) The counterpart of (c) in the rotating frame. Note that the average frequencies are normalized by the fundamental frequency Ω_1. In (a) and (b), the snapshots are taken after integrating Eq. (2) to $t = 1000$. In (c) and (d), the averages are calculated in the time window $[1000, 1000 + T_1]$, where $T_1 = 2\pi/\Omega_1$ (see text for the definition of Ω_1). Such a numerical scheme is also used for Fig. 5.

in Fig. 1(b), their instantaneous frequencies are generally different from each other.

Naturally, one may ask a question: as long as the instantaneous frequencies of oscillators are not locked, how can their phases become seemly coherent as shown in Fig. 1(a)? By carefully examining the dynamics of oscillators, we have understood this question as follows. The key point here is that although the instantaneous frequencies of oscillators in each cluster are different, their average frequencies (average angular speeds) are the same! As shown in Fig. 1(c), the instantaneous frequencies of synchronous clusters exhibit a structure like staircases. Of course, there are also drifting oscillators between two coherent zones [those between staircases in Fig. 1(c)]. This implies that although the instantaneous speeds of oscillators in each cluster are different from each other, they do correlate in certain form so that their average speeds can still maintain the same. The existence of this coherent state shows that the dynamical system can achieve a weaker form of coherence, which is between the nonsynchronous state (full incoherence) and the synchronous state (full coherence). Typically, as shown in Fig. 1(c), the order parameter oscillates in this state. Following Refs. 13 and 14, such a nonstationary clustering state belongs to the Bellerophon state. Compared with the previously found coherent states, the observed Bellerophon state may share some features with the standing wave state, where two clusters rotate in opposite directions along the unit circle. However, it has two essential differences. First, in the standing wave state, oscillators in each coherent cluster are frequency-locking; but in the Bellerophon state, oscillators in each coherent cluster are not frequency-locking [Fig. 1(b)]. Second, in the standing wave state, there are only two clusters; while in the Bellerophon state, there are multiple pairs of clusters and on average each pair of clusters rotates with different speeds.

As shown in Fig. 1(c), the distribution of frequency staircases is not symmetric with respect to zero. This is because in the model the effective coupling strength is frequency-weighted and a biased Lorentzian distribution, i.e., Eq. (3), is used, where the center of the distribution has been shifted to ω_0. To further reveal the characteristics of the Bellerophon state, it is convenient to transform the dynamical system into a rotating frame by setting $\theta_j = \theta_j' + \omega' t$ and $\omega_j = \omega_j' + \omega'$. In Fig. 1(d), we replot Fig. 1(c) in an appropriate rotating frame. Remarkably, some quantitative properties for the averaged frequencies of coherent clusters in the Bellerophon state can be identified. It is shown that the staircases of the average frequencies distribute symmetrically with respect to zero now. The average frequencies of coherent oscillators have a fundamental (the lowest) frequency and all the higher frequencies are odd times of it,[24] i.e., $\Omega_{\pm n} = \pm(2n-1)\Omega_1$ with $n = 1, 2, 3, \ldots$. Thus, the gap between two neighboring frequency staircases is twice of the fundamental frequency. So the Bellerophon state can be characterized by a series of coherent clusters $C^{\pm(2n-1)}$ with $n = 1, 2, 3, \ldots$. Based on the above analysis, now we can explain the physical picture of such a state. It is a weak coherence achieved by the coupled oscillators when the coupling strength is in the intermediate regime.[20] In this regime, multiple coherent clusters coexist in the sense that their average frequencies inside any clusters are identical. However, unlike previously observed coherent states, the oscillators in each coherent cluster still have certain degree of freedom; i.e., their instantaneous frequencies are generally not locked. To demonstrate this important dynamical feature, we provide two animated movies in Figs. 2 and 3 (Multimedia view), which help visualizing the evolution of phases, speeds, and collective rotations of oscillators on the unit circle.

Then we reveal another interesting characteristic of the observed Bellerophon state in this model: the intermittency. To this end, we choose 12 coherent clusters and sample one oscillator in each cluster. In Fig. 4, we plot the time series of the instantaneous frequencies of these oscillators. As shown in Fig. 4(a), it is shown that the motions of coherent oscillators have two stages: the bursting stage and the resting stage. At the first stage, which occupies relatively shorter period, they burst to rotate with highly heterogeneous speeds along the unit circle. Then in the following stage, which occupies most of time, they almost keep static. This dynamical behavior reminds us of the firing of neurons and excitable media. Such a pattern is repeated again and again as time increases. Interestingly, this is a synchronous periodic intermittency. So it is different from the intermittency observed in chaotic systems, where the bursting is intrinsically stochastic and unpredictable. To be precise, we can call this pattern "the periodic intermittency." In Figs. 4(c) and 4(d), we provide more details for these two stages during evolution. It is seen that during the bursting stage, the instantaneous frequencies of oscillators in different clusters show different patterns. Of course, their average frequencies also differ significantly. On the contrary, during the resting stage, all coherent oscillators

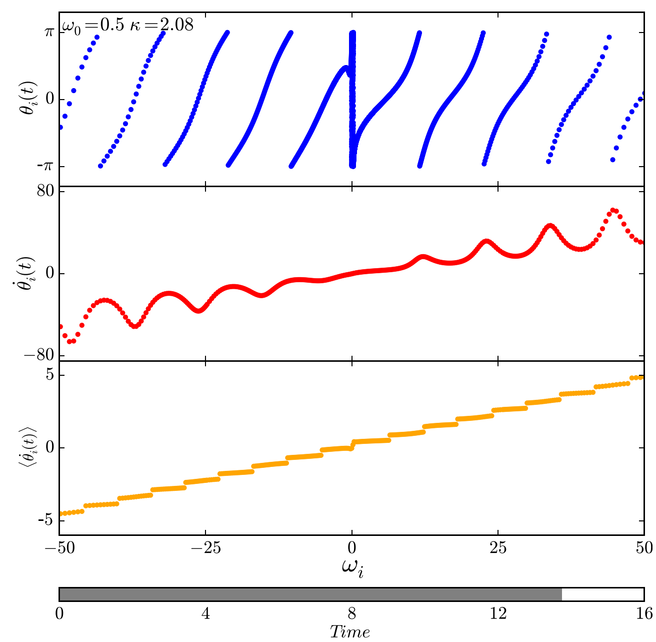

FIG. 2. Evolution of the instantaneous phases $\theta_i(t)$ (top), the instantaneous speeds $\dot\theta_i(t)$ (middle), and the accumulated average of instantaneous speeds $\langle\dot\theta_i(t)\rangle = \frac{1}{t}\int_0^t \dot\theta_i(\tau)d\tau$ (bottom) of all oscillators in state Fig. 1. The used time window is about two periods of $T_1 = 2\pi/\Omega_1$. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4972117.1]

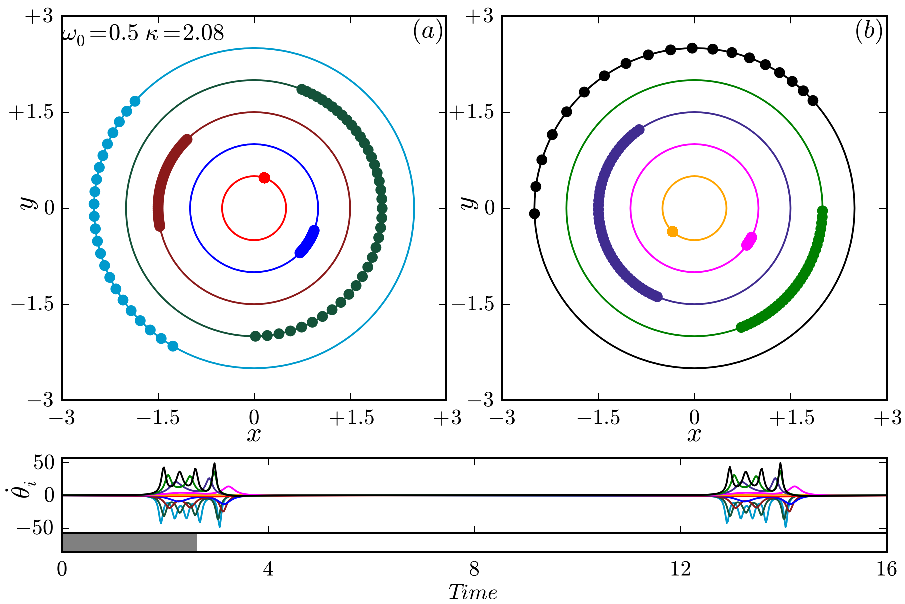

FIG. 3. Collective motion of the oscillators forming C^1, C^3, C^5, C^7, C^9 in state Fig. 1 (from inner to outer circle). For better visualization, we show the motions of oscillators with negative frequencies (i.e., those rotating in the clockwise direction) and positive frequencies (i.e., those rotating in the counterclockwise direction) in two panels (a) and (b), respectively. In fact, all oscillators (corresponding to $n = 1, 2, \ldots$) with both positive and negative frequencies rotate along the unit circle. As shown in the movie, oscillators in C^1, C^3, C^5, … rotate 1, 3, 5, … loops, respectively, within the period T_1. Oscillators in each cluster generally have heterogeneous speeds. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4972117.2]

have almost the same instantaneous frequencies, which are very small.

In Fig. 5, we further illustrate the bursting stage for oscillators inside the same coherent cluster. It is shown that although the average frequencies of oscillators in one cluster are the same, their instantaneous frequencies are generally different, following highly heterogeneous patterns. For example, as shown in Fig. 5, although on average all oscillators in one coherent cluster rotate along one direction (with positive speeds), there are some oscillators which can rotate reversely against the mainstream (with negative speeds) for a while. This is because that the average speeds of oscillators inside one cluster must be the same, so those oscillators who run too fast in the beginning have to go back to wait for the others to catch up. This is a very interesting behavior. Compared with the simple collective behaviors such as phase-locking or frequency-locking, the Bellerophon state is a high-order, time-dependent coherence, which is essentially different from other coherent states previously observed in the Kuramoto-like models.

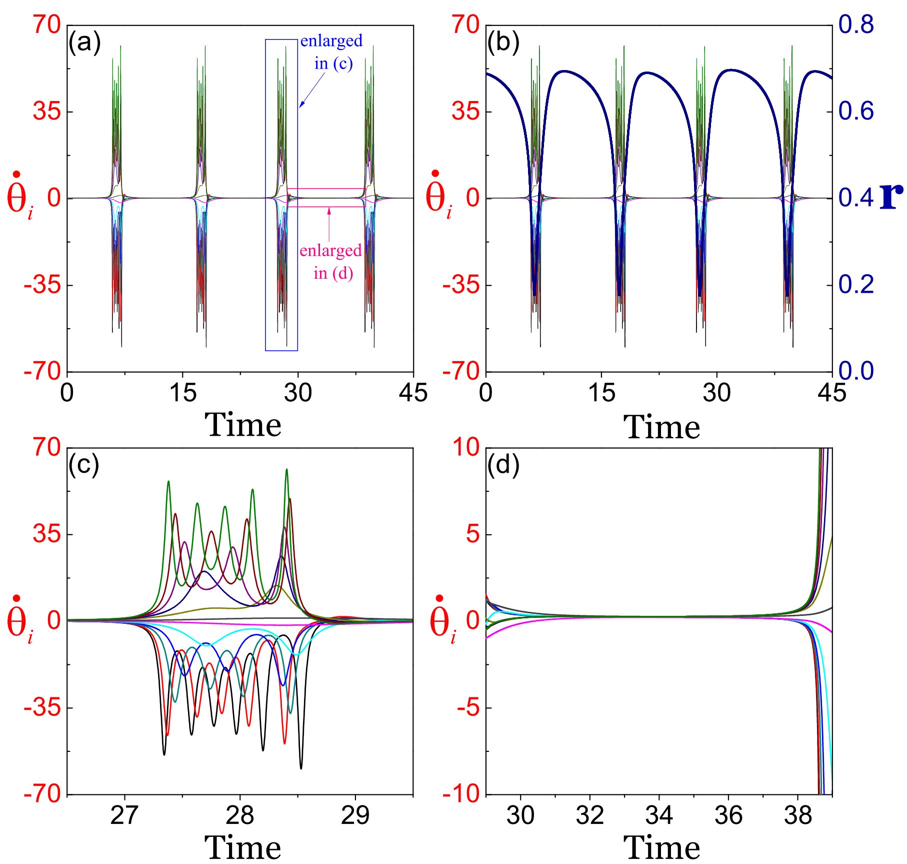

FIG. 4. The intermittency of nonstationary clustering state for $\omega_0 = 0.5$ and $\kappa = 2.08$. (a) Evolution of the instantaneous speeds of all oscillators. (b) Evolution of the instantaneous speeds and the order parameter. (c) and (d) Enlargement of the boxes in (a) respectively. For better visualization, we only choose 12 clusters ($n = 1$–6) and sample one oscillator in each cluster. Note that a transient stage of $t = 1000$ has been discarded before plotting the time series. Same treatment is also adopted in Figs. 3, 4, and 6.

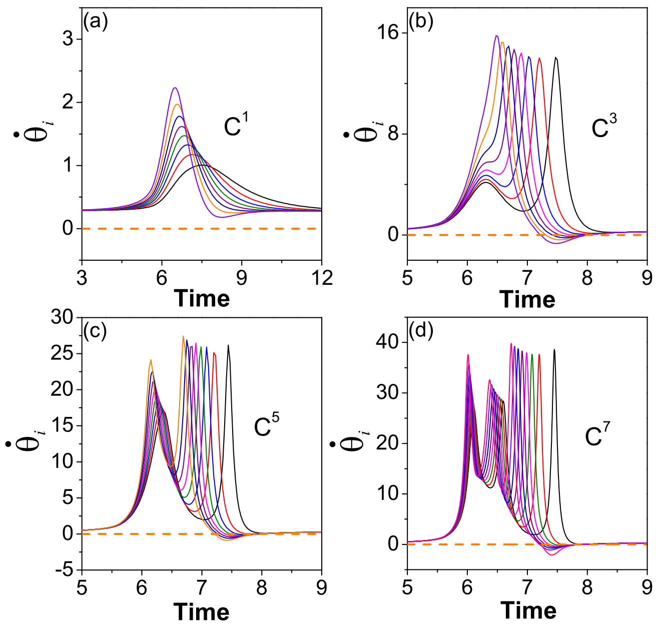

FIG. 5. (a)–(d) Evolutions of the instantaneous frequencies (speeds) in 4 coherent clusters C^1, C^3, C^5, and C^7, respectively. $\omega_0 = 0.5$ and $\kappa = 2.08$. For better visualization, we only choose 8 oscillators in each cluster. Note that the instantaneous frequencies of oscillators inside one coherent cluster are not frequency-locking. Instead, they follow heterogeneous patterns. In (b)–(d), it is seen that although on average oscillators are supposed to rotate toward positive direction, some oscillators turn out to rotate inversely for a while during evolution.

The formation of this intermittency can be heuristically understood as follows. Since the Bellerophon state occurs when the coupling strength is moderately large, on the one hand, the oscillators can be entrained to certain extent by the mean field. So they can be approximately synchronized as in the resting stage. On the other hand, the coupling strength is not large enough to maintain this coherence permanently. So due to the difference of natural frequencies, oscillators try to escape from the synchronization and manifest as the bursting behavior. In fact, the Bellerophon state, which is a weak form of synchronization, can be understood as a transitional state between the incoherent state and the fully coherent state. Moreover, based on the above analysis, it is not difficult to understand the oscillatory pattern in the order parameter [Fig. 1(c)]. As shown in Fig. 4(b), in the resting stage, where oscillators are almost synchronous, the order parameter could achieve a relatively large value due to the high coherence; while in the bursting stage, it takes a relatively small value due to the low coherence among oscillators. We emphasize that even in the bursting stage, oscillators still maintain certain correlation rather than complete disorder.

The Bellerophon state is a partially coherent state, in which both coherent oscillators (on the frequency staircases) and drifting oscillators (between the frequency staircases) coexist. How do the drifting oscillators behave? We answer this question by illustrating an example in Fig. 6. It is shown that the evolution of the instantaneous frequencies of drifting oscillator still consists of the bursting stage and the resting stage. However, it is not periodic though it does have a characteristic time scale of bursting. Interestingly, although during most time the oscillator moves along one direction, it

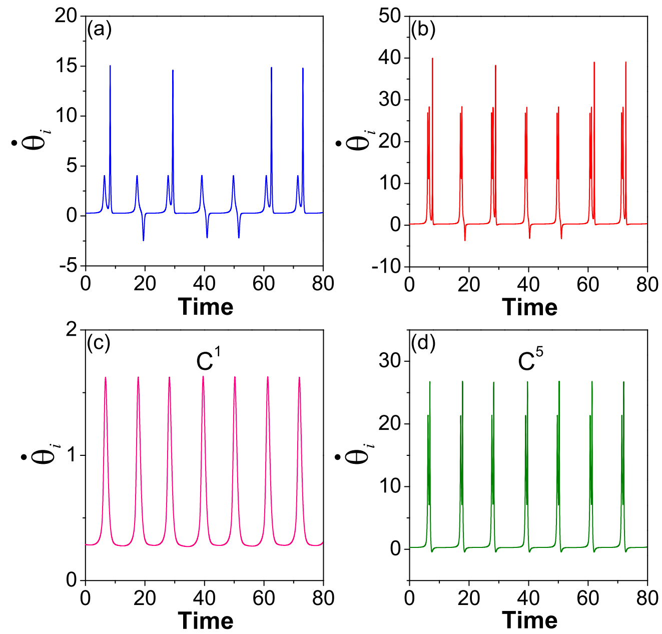

FIG. 6. Characterizing the instantaneous frequencies of drifting oscillators outside the coherent clusters. (a) and (b) Two drifting oscillators between C^1 and C^3, and C^5 and C^7, respectively. (c) and (d) Two oscillators in coherent clusters C^1 and C^5 for comparison. $\omega_0 = 0.5$ and $\kappa = 2.08$.

occasionally reverses its direction of rotation. This is a manifestation that the coupling strength is not enough to fully entrain the oscillators.

Finally, in Figs. 7 and 8, we characterize another Bellerophon state observed in model (2). Different from the state discussed above, this one is observed in the process of the second-order transition.[20] However, qualitatively, this state shares all the dynamical features of the previous example. Certainly, they belong to the same type of coherent state. In Figs. 9 and 10 (Multimedia view), we also provide two animated movies to demonstrate the properties of this state.

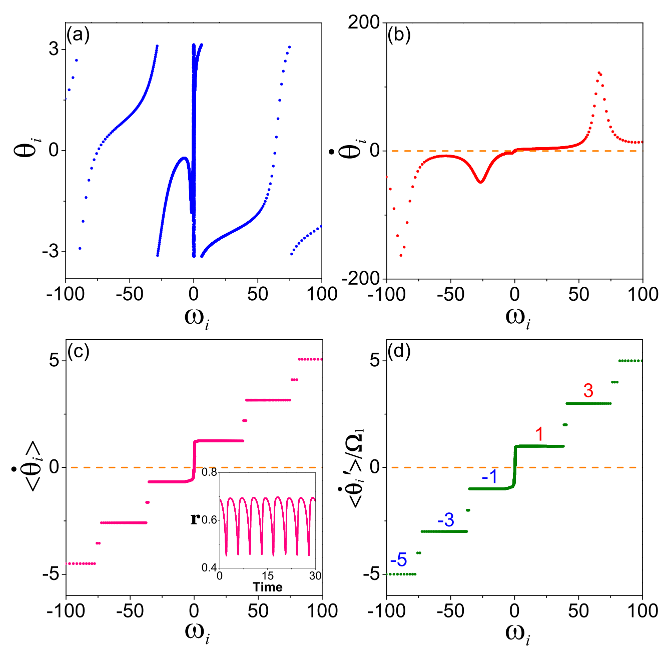

FIG. 7. Characterizing the Bellerophon state observed in the model. $\omega_0 = 1.2$ and $\kappa = 1.9$. The figure caption is the same as in Fig. 1.

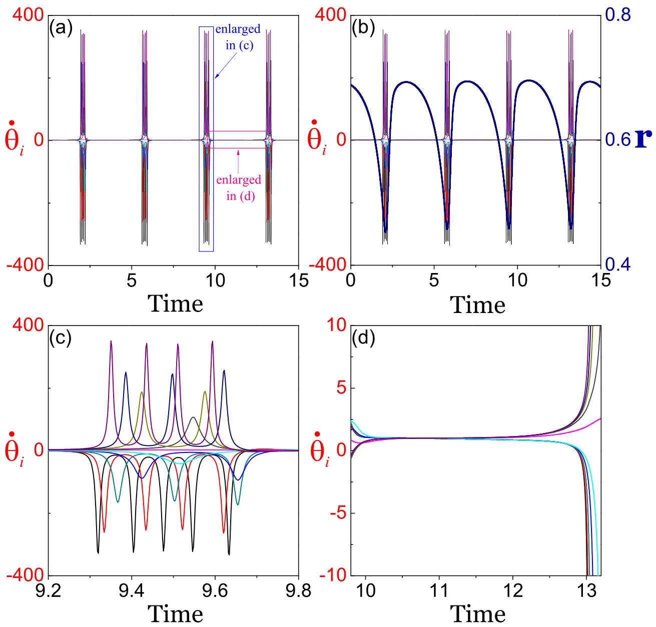

FIG. 8. The intermittency of nonstationary clustering state for $\omega_0 = 1.2$ and $\kappa = 1.9$. The figure caption is the same as in Fig. 4. Here, we only choose 10 clusters ($n = 1$–5) and sample one oscillator in each cluster.

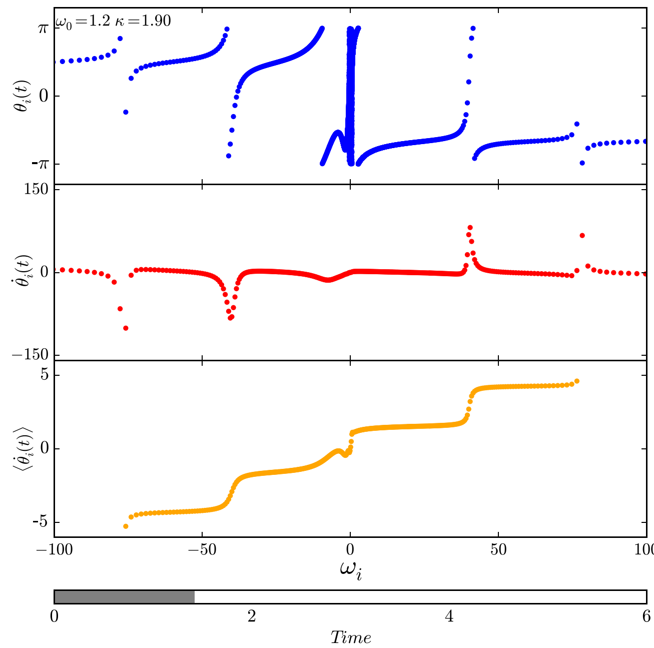

FIG. 9. Evolution of the instantaneous phases $\theta_i(t)$ (top), the instantaneous speeds $\dot{\theta}_i(t)$ (middle), and the accumulated average of instantaneous speeds $\langle\dot{\theta}_i(t)\rangle = \frac{1}{t}\int_0^t \dot{\theta}_i(\tau)d\tau$ (bottom) of all oscillators in state Fig. 7. The used time window is about two periods of $T_1 = 2\pi/\Omega_1$. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4972117.3]

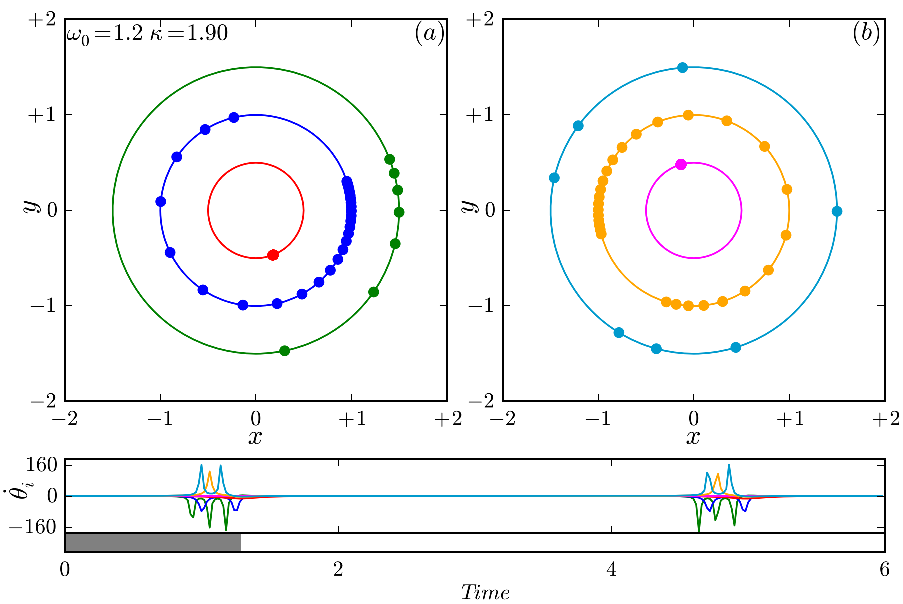

FIG. 10. Collective motion of the oscillators forming C^1, C^3, C^5 in state Fig. 7 (from inner to outer circle). For better visualization, we show the motions of oscillators with negative frequencies (i.e., those rotating in the clockwise direction) and positive frequencies (i.e., those rotating in the counterclockwise direction) in two panels (a) and (b), respectively. In fact, all oscillators (corresponding to $n = 1, 2, \ldots$) with both positive and negative frequencies rotate along the unit circle. As shown in the movie, oscillators in C^1, C^3, C^5, … rotate 1, 3, 5, … loops, respectively, within the period T_1. Oscillators in each cluster generally have heterogeneous speeds. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4972117.4]

III. CONCLUSION

In conclusion, we have identified an intermittent Bellerophon state in the frequency-weighted Kuramoto model. Such state is a transitional state between the incoherence and the full synchrony when the coupling strength is in the intermediate regime. In this state, the oscillators split into multiple coherent clusters. Inside each cluster, the instantaneous frequencies of oscillators follow a highly heterogeneous pattern; however, the average frequencies lock to a constant. In terms of the average frequency, this state is characterized by staircases structure with an equal gap between neighboring clusters. Remarkably, the motions of oscillators turn out to be periodically intermittent, following a synchronous pattern of bursting and resting. Our work revealed that there exists complicated, time dependent collective behavior in coupled phase oscillators in Kuramoto-like models.

ACKNOWLEDGMENTS

This work was supported by the NSFC Grant Nos. 11135001, 11405059, and 11305062.

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