Outline

- Introduction
- Signal, random variable, random process and spectra
- Analog modulation
- Analog to digital conversion
- Digital transmission through baseband channels
- **Signal space representation**
- Optimal receivers
- Digital modulation techniques
- Channel coding
- Synchronization
- Information theory

Signal space representation

- Concepts
  - The key to analyzing and understanding the performance of digital transmission is the realization that **signals used in communications can be expressed and visualized graphically (constellation)**
  - Thus, we need to understand signal space concepts applied to digital communications

Chapter 8.1

Signal space representation

- Traditional bandpass signal representation
  - Baseband signals are the message signal generated at the source
  - Passband (Bandpass) signals refer to the signals after modulating with a carrier. The bandwidth of these signals are usually small compared with the carrier frequency f_c
  - Passband signals can be represented in three forms

1. Magnitude and phase representation
2. Quadrature representation
3. Complex envelope representation

Signal space representation

- Magnitude and phase representation

$$s(t) = a(t)\cos\left[2\pi f_c t + \theta(t)\right]$$

where $a(t)$ is the magnitude of $s(t)$

$\theta(t)$ is the phase of s(t)

Signal space representation

- Quadrature (I/Q) representation

$$s(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

where x(t) and y(t) are real-valued baseband signals called the **in-phase** and **quadrature** components of s(t)

Signal space is a more convenient way than I/Q representation to study modulation scheme in digital communications

Signal space representation

- Vectors and space
  - Consider an n-dimensional space with unity basis vectors

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

  - Any vector **a** in the space can be written as

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i \quad \Rightarrow \quad \mathbf{a} = (a_1, a_2, \ldots, a_n)$$

$n \triangleq$ Dimension = Minimum number of vectors that is necessary and sufficient for representation of any vector in space

Signal space representation

- Vectors and space
  - Inner product $$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$$
  - **a** and **b** are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$
  - Norm $$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^{n} a_i^2}$$
  - A set of vectors are orthonormal if they are mutually orthogonal and all have unit norm

Signal space representation

- Vectors and space
  - The set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of a space are chosen such that

1. Should be complete or span the vector space, i.e., any vector a can be expressed as a linear combination of these vectors
2. Should be orthonormal vectors

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, \ \forall i \neq j \qquad\qquad \|e_i\| = 1, \ \forall i$$

  - A set of basis vectors satisfying these properties is also said to be a complete orthonormal basis

Signal space representation

- Signal space
  - Basic idea: If a signal can be represented by n-tuple, then it can be treated in much the same way as a n-dim vector.
  - Let $\phi_1(t), \phi_2(t), ..., \phi_n(t)$ be n signals
  - Consider a signal x(t) and suppose that

$$x(t) = \sum_{i=1}^{n} x_i \phi_i(t)$$

  - If every signal can be written as above, then

$\{\phi_1(t), \ldots, \phi_n(t)\}$ ~ **basis functions (基函数)**

Signal space representation

- Orthonormal basis
  - Signal set $\{\phi_k(t)\}$ is an orthogonal set if

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)\,dt = \begin{cases} 0 & j \neq k \\ c_j & j = k \end{cases}$$

  - If $c_j \equiv 1 \forall j \to \{\phi_k(t)\}$ is an orthonormal set.
  - In this case,

$$x_k = \int_{-\infty}^{\infty} x(t)\phi_k(t)dt$$

$$x(t) = \sum_{i=1}^{n} x_i \phi_i(t)$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Signal space representation

- Orthonormal basis
  - Given the set of the orthonormal basis

$$\{\phi_1(t), \ldots, \phi_n(t)\}$$

  - Let x(t) and y(t) be represented as

$$x(t) = \sum_{i=1}^{n} x_i \phi_i(t) \,, \qquad y(t) = \sum_{i=1}^{n} y_i \phi_i(t)$$

  with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

  - Then the inner product of **x** and **y** is

$$\boxed{\mathbf{x} \cdot \mathbf{y} = \int_{-\infty}^{\infty} x(t) y(t) dt}$$

Signal space representation

- Orthonormal basis
  - Proof

$$\int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} x_i\phi_i(t)\right]\left[\sum_{j=1}^{n} y_j\phi_j(t)\right] dt$$

$$= \sum_{k=1}^{n} x_k y_k \triangleq \mathbf{x}\cdot\mathbf{y}$$

Since $\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

E_x = Energy of $x(t)$ = $\int_{-\infty}^{\infty} x^2(t)dt$

$$\Rightarrow \quad \boxed{E_x = \mathbf{x}\cdot\mathbf{x} = \|\mathbf{x}\|^2}$$

Signal space representation

- Basis functions for a signal set
  - Consider a set of M signals (M-ary symbol) $\{s_i(t), i = 1, 2, ..., M\}$ with finite energy, i.e.,

$$\int_{-\infty}^{\infty} s_i^2(t)dt < \infty$$

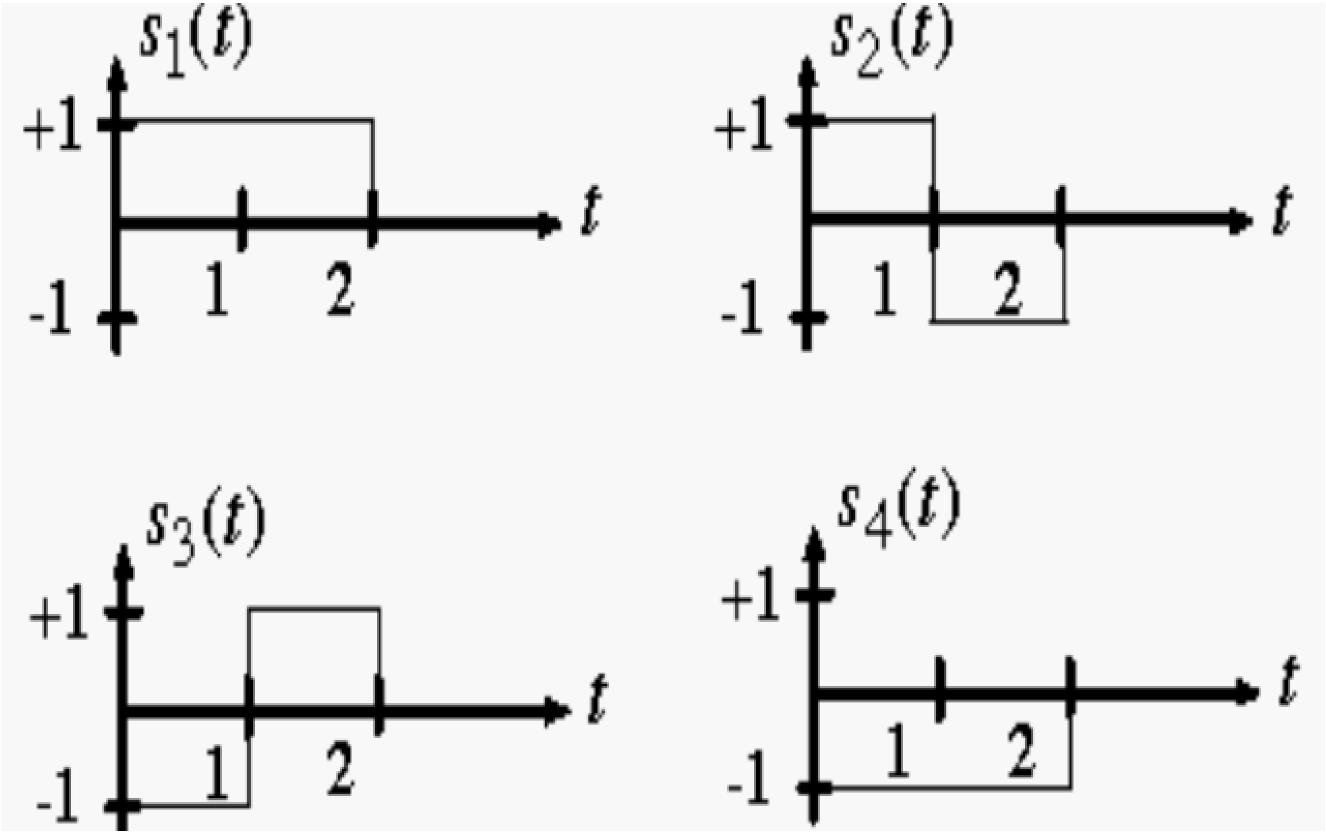
  - Then, we can express each of these waveforms as weighted linear combination of orthonormal signals:

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \qquad \text{for } i = 1, \ldots, M$$

where $N \leq M$ is the dimension of the signal space and $\{\phi_j(t)\}_1^N$ are called the orthonormal basis functions

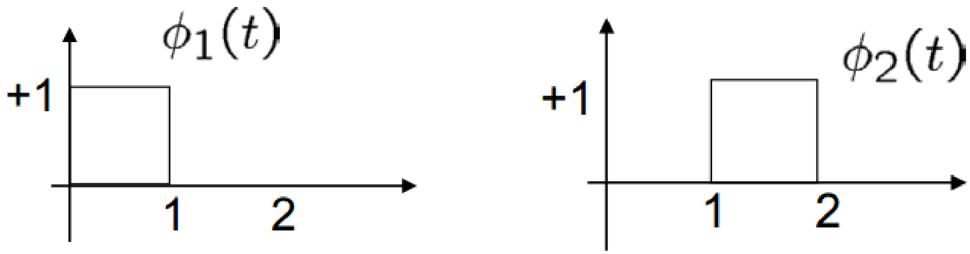
Signal space representation

- Basis functions for a signal set
  - Consider the following signal set



Signal space representation

- Basis functions for a signal set
  - By inspection, the signals can be expressed in terms of the following two basis functions:



$$s_1(t) = 1 \cdot \phi_1(t) + 1 \cdot \phi_2(t) \qquad s_3(t) = -1 \cdot \phi_1(t) + 1 \cdot \phi_2(t)$$

$$s_2(t) = 1 \cdot \phi_1(t) - 1 \cdot \phi_2(t) \qquad s_4(t) = -1 \cdot \phi_1(t) - 1 \cdot \phi_2(t)$$

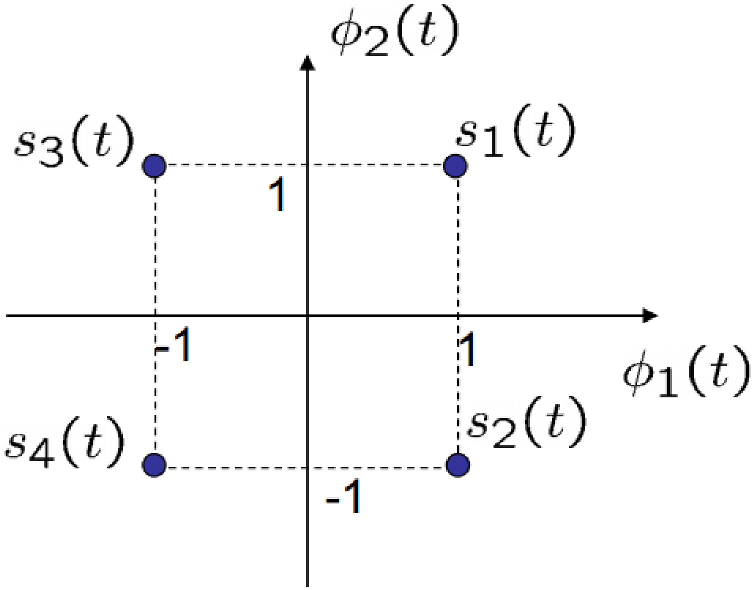
  - Note that the basis is orthonormal

$$\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = 0$$

$$\int_{-\infty}^{\infty} |\phi_1(t)|^2 dt = \int_{-\infty}^{\infty} |\phi_2(t)|^2 dt = 1$$

Signal space representation

- Basis functions for a signal set
  - Constellation diagram is a representation of a **digital modulation** scheme in the **signal space**
  - The axes are labeled with $\phi_1(t)$ and $\phi_2(t)$
  - Possible signals are plotted as points, called **constellation points**



Signal space representation

- Basis functions for a signal set
  - Suppose our signal set can be represented in the following form

$$s(t) = \pm\sqrt{\frac{2}{T}}\cos(2\pi f_c t) \pm \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

with $t \in [0, T)$ and $f_c T >> 1$

  - We can choose the basis functions as follows

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \qquad \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$t \in [0, T)$

Since

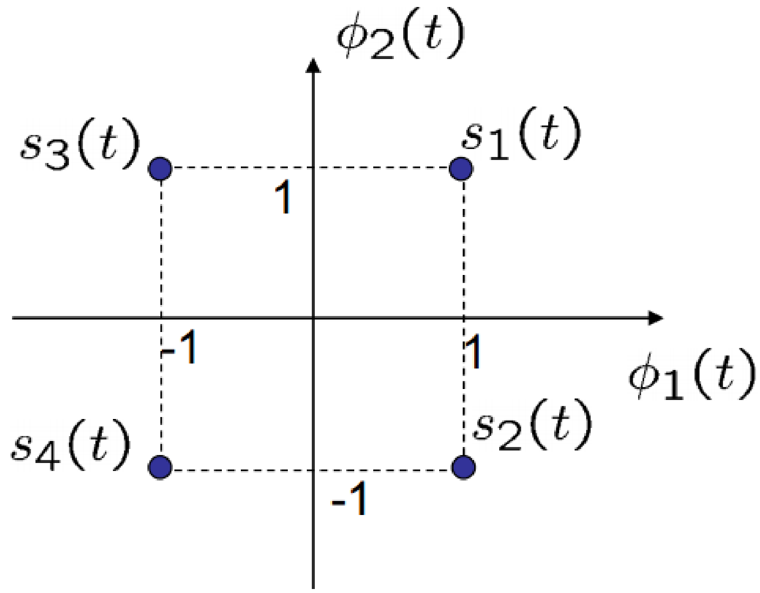
$$\begin{aligned}\int_0^T \phi_1(t)\phi_2(t)dt &= \int_0^T \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t)\sqrt{\tfrac{2}{T}}\sin(2\pi f_c t)dt \\ &= \tfrac{2}{T}\int_0^T \tfrac{1}{2}[\sin(0) + \sin(4\pi f_c t)]dt \\ &= -\frac{1}{4\pi f_c T}[\cos(4\pi f_c t)]_0^T \approx 0, \text{for } f_c T \ll 1\end{aligned}$$

and

$$\int_0^T |\phi_1(t)|^2 dt = \int_0^T |\phi_2(t)|^2 dt = \tfrac{2}{T}\int_0^T \tfrac{1}{2}[1 + \cos(4\pi f_c t)]dt \approx 1$$

Signal space representation

- Basis functions for a signal set
  - The above example is exactly **QPSK** modulation. Its constellation diagram is also



Widely used in commercial communication systems, including 4G-LTE, Wifi, and incoming 5G.

Signal space representation

- Basis functions for a signal set
  - Two entirely different signal sets can have the same **geometric representation**.
  - The underlying geometry will determine the **performance** and the **receiver structure**
  - In general, is there any method to find a complete orthonormal basis for an arbitrary signal set?

Gram-Schmidt Orthogonalization (GSO) Procedure

Signal space representation

- GSO procedure
  - Suppose we are given a signal set $\{s_1(t), \ldots, s_M(t)\}$
  - Find the orthogonal basis functions for this signal set

    $\{\phi_1(t), \ldots, \phi_K(t)\}$ with $K \leq M$
  - **Step 1**: Compute the energy in signal 1

$$E_1 = \int_{-\infty}^{\infty} s_1^2(t)dt$$

  - The first basis function is just a normalized version of $s_1(t)$

$$\boxed{\phi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)}$$

$$\Rightarrow \quad s_1(t) = s_{11}\phi_1(t) = \sqrt{E_1}\phi_1(t)$$

$$s_{11} = \int_{-\infty}^{\infty} s_1(t)\phi_1(t)dt = \sqrt{E_1}$$

Signal space representation

- GSO procedure
  - **Step 2**: Compute the correlation between signal 2 and basic function 1

$$s_{21} = \int_{-\infty}^{\infty} s_2(t)\phi_1(t)dt$$

  - Subtract off the correlation portion

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \Rightarrow g_2(t) \text{ is orthogonal to } \phi_1(t)$$

  - Compute the energy in the remaining portion

$$E_{g_2} = \int_{-\infty}^{\infty} [g_2(t)]^2 \, dt$$

  - Normalize the remaining portion

$$\boxed{\phi_2(t) = \frac{1}{\sqrt{E_{g_2}}} g_2(t)}$$

$$\Rightarrow s_{22} = \int_{-\infty}^{\infty} s_2(t)\phi_2(t)dt = \sqrt{E_{g_2}}$$

Signal space representation

- GSO procedure
  - **Step 3**: For signal $s_k(t)$, compute

$$s_{ki} = \int_{-\infty}^{\infty} s_k(t)\phi_i(t)dt$$

  - Define

$$g_k(t) = s_k(t) - \sum_{i=1}^{k-1} s_{ki}\phi_i(t)$$

  - Compute the energy of $g_k(t)$

$$E_{g_k} = \int_{-\infty}^{\infty} [g_k(t)]^2\, dt$$

  - k-th basis function

$$\boxed{\phi_k(t) = \frac{1}{\sqrt{E_{g_k}}} g_k(t)}$$

$$\Rightarrow \quad s_{kk} = \int_{-\infty}^{\infty} s_k(t)\phi_k(t)dt = \sqrt{E_{g_k}}$$

Signal space representation

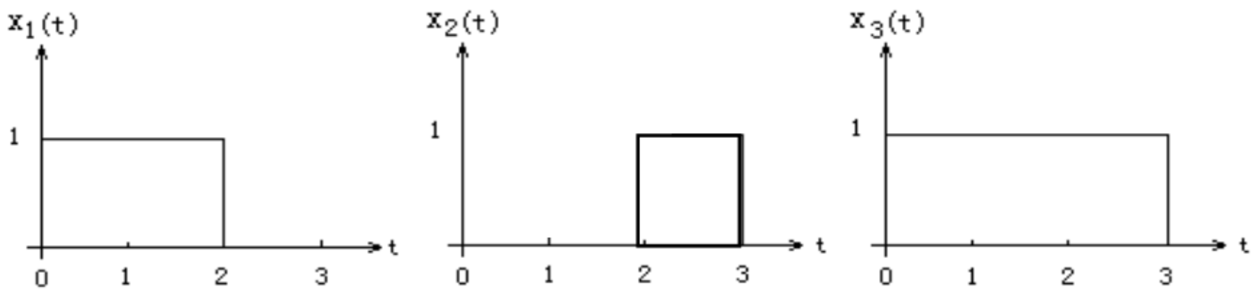
- GSO procedure
  - **Summary**

1. 1st basis function is normalized version of the first signal
2. Successive basis functions are found by removing portions of signals that are correlated to previous basis functions and normalizing the result
3. The procedure is repeated until all basis functions are found (if $g_k(t)=0$, no new basis functions is added)

The order in which signals are considered is arbitrary

Signal space representation

- GSO procedure
  - Consider an example.
  - Use the Gram-Schmidt procedure to find a set of orthonormal basis functions corresponding to the signals shown below



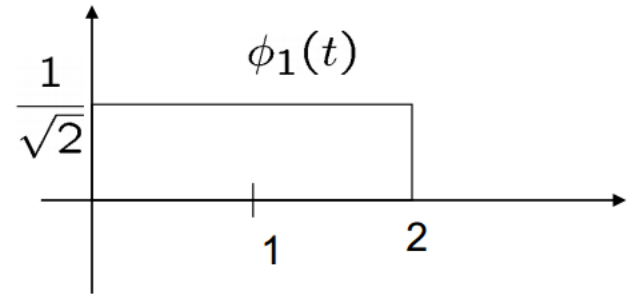
  - Express x_1, x_2, x_3, in terms of the orthonormal basis functions found previously
  - Draw the constellation diagram for this signal set

- GSO procedure
  - Example

Step 1: $$E_1 = \int_{-\infty}^{\infty} x_1^2(t)dt = 2$$

$$\phi_1(t) = \frac{1}{\sqrt{2}} x_1(t)$$

$$x_{11} = \sqrt{2}$$

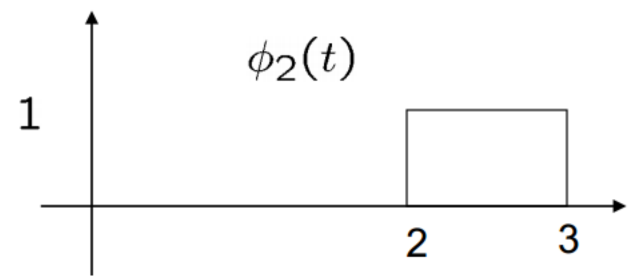


Step 2: $$x_{21} = \int_{-\infty}^{\infty} x_2(t)\phi_1(t)dt = 0$$

$g_2(t) = x_2(t)$ and $E_{g_2} = 1$

$$\phi_2(t) = x_2(t)$$

$$x_{22} = 1$$



Signal space representation

- GSO procedure
  - Example

Step 3: $$x_{31} = \int_{-\infty}^{\infty} x_3(t)\phi_1(t)dt = \sqrt{2}$$

$$x_{32} = \int_{-\infty}^{\infty} x_3(t)\phi_2(t)dt = 1$$

$$g_3(t) = x_3(t) - x_{31}f_1(t) - x_{32}f_2(t) = 0$$

=> No more new basis functions

Procedure completes

$$\begin{cases} \phi_1(t) = \dfrac{1}{\sqrt{2}}x_1(t) \\ \phi_2(t) = x_2(t) \end{cases}$$

Signal space representation

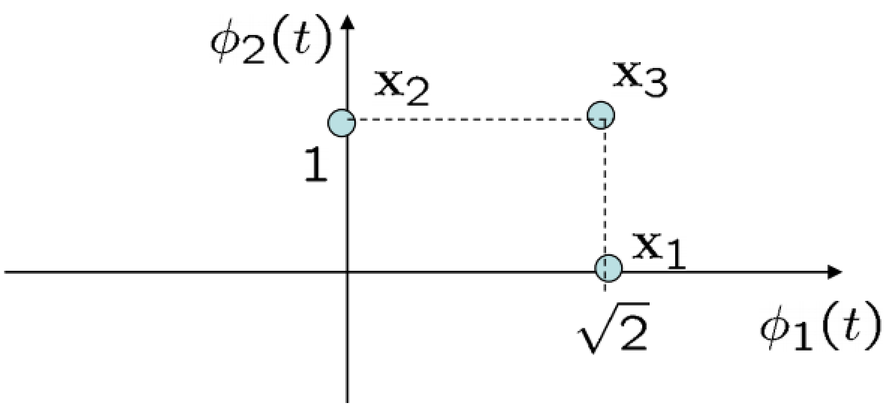
- GSO procedure
  - Example

Express x_1, x_2, x_3 in basis functions

$$x_1(t) = \sqrt{2}\phi_1(t), \quad x_2(t) = \phi_2(t)$$

$$x_3(t) = \sqrt{2}\phi_1(t) + \phi_2(t)$$

Constellation diagram



Signal space representation

- GSO procedure
  - Exercise

Given a set of signals (8PSK modulation)

$$s_i(t) = A\cos\left(2\pi f_c t + \frac{\pi}{4} i\right)$$

$$i = 0, 1, \ldots, 7 \quad \text{and } 0 \leq t < T$$

- Find the orthonormal basis functions using Gram Schmidt procedure
- What is the dimension of the resulting signal space ?
- Draw the constellation diagram of this signal set

Signal space representation

- GSO procedure
  - A signal set may have many different sets of basis functions
  - A change of basis functions is essentially a rotation of the signal points around the origin
  - The order in which signals are used in GSO procedure affects the resulting basis functions
  - The choice of basis functions does not affect the performance of the modulation scheme