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One-Dimensional Optimal System and Similarity Reductions of Wu–Zhang Equation*

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Abstract *The one-dimensional optimal system for the Lie symmetry group of the (2+1)-dimensional Wu–Zhang equation is constructed by the general and systematic approach. Based on the optimal system, the complete and inequivalent symmetry reduction systems are presented in the form of table. It is noteworthy that a new Painlevé integrable equation with constant coefficient is in the table besides the classic Boussinesq equation and the steady case of the Wu–Zhang equation.*

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1 Introduction

The (2+1)-dimensional Wu–Zhang (WZ) equation,[1] describing the (2+1)-dimensional dispersive long waves is

$$u_t + uu_x + vu_y + w_x = 0\,, \quad v_t + uv_x + vv_y + w_y = 0\,,$$
$$w_t + (uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) = 0\,, \tag{1}$$

where w is the total water depth ($w-1$ being the wave elevation), (u, v) is the horizontal projection of the surface velocity of water particle. Equation (1) is derived from the Euler equation with a perturbation scheme under the assumption that the amplitude of wave elevation is small and the wave is long compared with the water depth (scaled to be 1). It can be used to model the three-dimensional behavior of solitary waves on the uniform layer of water, such as oblique interaction, oblique reflection from a vertical wall and turning in a cured channel. If the waves propagate in only one-dimensional, e.g., along y coordinate, Eq. (1) can be reduced to the classical Boussinesq equation,[1−2] which is known to be integrable and be equivalent to Broer–Kaup (BK) system[3] and a member of Ablowitz–Kaup–Newell–Segur (AKNS) system.[4] The investigation of the classical Boussinesq equation is abundant.[5−8] However, Chen *et al.*[2] demonstrated that the WZ equation has no Painlevé property. Moreover, to our best knowledge, no integrable properties (such as Lax Pair, Bäcklund transformation, Bi-Hamiltonian structure) of the WZ equation have been reported. Therefore, the classical method for solving integrable systems can not be employed to study the WZ equation.

The symmetry method is a general method for treating different equations, no matter whether the equation is integrable or not. The Lie symmetry analysis, basic similarity reductions and some exact solutions of the WZ equation (1) are provided in Ref. [9]. However, they only considered the similarity reduction from the eight simple infinitesimal generators. In order to obtain all inequivalent reduction systems, the general linear combinations of the infinitesimal generators should be considered. Thus, among these combinations, which are suitable for reducing the original equation? The effective, systematic approach of dealing with this problem is to construct optimal system.

The concept of optimal system can trace back to the classification of group invariant solutions, which can reduce to the problem of classifying subgroups of full symmetry group under conjugation or the same classification problems of subalgebras. A list of subalgebras forms an optimal system if every subalgebra of a Lie algebra is equivalent to a unique member of the list under some element of the adjoint representation.[10] Consequently, the problem of determining the optimal system of group invariant solutions is converted to the classification problems of the adjoint representation of the symmetry group on its Lie algebra. Ovsiannikov[11] proposed the construction of optimal system for the Lie algebra. The method has been received extensive progress by Patera, Winternitz, and Zassenhaus and many examples of optimal systems of subgroups for the important Lie groups of mathematical physics were obtained.[12] For the one-dimensional

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optimal systems, the Ovsiannikov's technique has been used until Olver[10] gives a slightly different and elegant technique. Olver constructed a table of adjoint operators to simplify a general element in Lie algebra as much as possible and applied the technique to the KdV equation and the heat equation. Some examples of optimal systems can also be found in Refs. [13–21]. As we know, if a list of $\{v_\alpha\}_{\alpha\in\mathcal{A}}$ is a one-dimensional optimal system, it must satisfy two conditions: (i) Completeness — any one-dimensional subalgebra is equivalent to some v_α; (ii) Inequivalence — v_α and v_β are inequivalent for distinct α and β. Recently, Hu and two of the authors[22] have proposed a valid method for computing all the general invariants of one-dimensional Lie algebra, which include the well-known Killing form. Based on the invariants, they put forward a direct and effective algorithm to construct one-dimensional optimal system, which can guarantee the inequivalence as well as the completeness.

On the basis of the systematic algorithm,[22] we find an optimal system of one-dimensional subalgebras of the (2+1)-dimensional WZ equation, which is spanned by 15 inequivalent infinitesimal generators. The eight simple infinitesimal generators considered in Ref. [9] are proven to be only equivalent to the four elements of our optimal system. The complete and inequivalent symmetry reduction systems are listed in Table 3, which include not only the classic Boussinesq equation and the steady case of the WZ equation (both of which have been derived in Ref. [9]), but also a new Painlevé integrable equation. The Painlevé integrable equation can be reduced to Painlevé II equation by way of the travelling wave method. There are also 12 variable coefficient reduction systems obtained.

The outline of this paper is as follows. In Sec. 2, briefly reviewing the direct algorithm of one-dimensional optimal system for the general symmetry algebra. In Sec. 3, we apply the algorithm to the WZ equation, and construct its one-dimensional optimal systems step by step. Then comparing with the results in Ref. [9], Sec 4 is devoted to list (1+1)-dimensional similarity reduction systems. Some conclusions and discussions are given in the last section.

2 Review of Direct Algorithm of One-Dimensional Optimal System

According to the process of Ref. [22], the algorithm involves three steps:

Step 1 Calculation of the invariants

A real function ϕ on the Lie algebra $\mathfrak{g}$ is called an invariant if $\phi(\mathrm{Ad}_g(v)) = \phi(v)$ for all $v \in \mathfrak{g}$ and $g \in G$. If two vectors v and w are equivalent under the adjoint action, it is necessary that $\phi(v) = \phi(w)$ for any invariant ϕ.

For the n-dimensional symmetry algebra $\mathfrak{g}$, taking any subgroup $g = \mathrm{e}^{\varepsilon w}$ $(w = \sum_{j=1}^n b_j v_j)$ to act on $v = \sum_{i=1}^n a_i v_i$, we have

$$\mathrm{Ad}_{\mathrm{e}^{\varepsilon w}}(v) = \mathrm{e}^{-\varepsilon w} v\, \mathrm{e}^{\varepsilon w} = (a_1 v_1 + \cdots + a_n v_n) + \varepsilon(\Theta_1 v_1 + \cdots + \Theta_n v_n) + o(\varepsilon)\,, \tag{2}$$

where $\Theta_i \equiv \Theta_i(a_1, \ldots, a_n, b_1, \ldots, b_n)$ $(i = 1, \ldots, n)$ can be easily obtained from commutator table.

Equivalently, omitting v_i one can rewrite Eq. (2) as

$$(a_1, a_2, \ldots, a_n) \longrightarrow (a_1 + \varepsilon\Theta_1, a_2 + \varepsilon\Theta_2, \ldots, a_n + \varepsilon\Theta_n) + o(\varepsilon)\,. \tag{3}$$

According to the definition of the invariant, it is necessary that

$$\phi(a_1, a_2, \ldots, a_n) = \phi(a_1 + \varepsilon\Theta_1 + o(\varepsilon), a_2 + \varepsilon\Theta_2 + o(\varepsilon), \ldots, a_n + \varepsilon\Theta_n + o(\varepsilon))\,. \tag{4}$$

Taking the right of Eq. (4) Taylor expansion, there must be

$$\Theta_1 \frac{\partial\phi}{\partial a_1} + \Theta_2 \frac{\partial\phi}{\partial a_2} + \cdots + \Theta_n \frac{\partial\phi}{\partial a_n} = 0 \tag{5}$$

for any b_i. Then extracting the coefficients of all b_i, N $(N \leqslant n)$ linear differential equations of ϕ are obtained. By solving these equations, all invariants can be derived.

Step 2 Calculation of the adjoint transformation matrix

The second task is to construct the general adjoint transformation matrix A, which is the product of the matrix of the separate adjoint actions $A_1, A_2, \ldots, A_n$.

Firstly, applying the adjoint action of v_1 to $v = \sum_{i=1}^n a_i v_i$ and with the help of adjoint representation table, we get

$$\mathrm{Ad}_{\mathrm{e}^{\varepsilon_1 v_1}}(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) \doteq R_1 v_1 + R_2 v_2 + \cdots + R_n v_n\,, \tag{6}$$

where $R_i \equiv R_i(a_1, a_2, \ldots, a_n, \varepsilon_1)$ $(i = 1, \ldots, n)$. To be intuitive, the formula (6) can be rewritten as the following matrix form

$$v \doteq (a_1, a_2, \ldots, a_n) \longrightarrow (R_1, R_2, \ldots, R_n) = (a_1, a_2, \ldots, a_n)A_1\,. \tag{7}$$

Similarly, we can construct the matrices $A_2, A_3, \ldots, A_n$ of the separate adjoint actions of $v_2, v_3, \ldots, v_n$, respectively. Then the general adjoint transformation matrix A is the product of $A_1, A_2, \ldots, A_n$ taking in any order

$$A \equiv A(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = A_{i_1} A_{i_2} \cdots A_{i_n}\,, \tag{8}$$

where $i_1, i_2, \ldots, i_n$ is any permutation of $1, 2, \ldots, n$. That is to say, applying the most general adjoint action $\mathrm{Ad}_{\mathrm{e}^{\varepsilon_{i_n} v_{i_n}}} \cdots \mathrm{Ad}_{\mathrm{e}^{\varepsilon_{i_1} v_{i_1}}}$ to v, we get

$$v \doteq (a_1, a_2, \ldots, a_n) \longrightarrow (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)\,. \tag{9}$$

At this moment,

$$(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (a_1, a_2, \ldots, a_n)A\,, \tag{10}$$

or

$$(a_1, a_2, \ldots, a_n) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)A\,. \tag{11}$$

In fact, Eq. (10) (or Eq. (11)) can be regarded as n algebraic equations with respect to $\varepsilon_1, \ldots, \varepsilon_n$. If it has solutions with respect to $\varepsilon_1, \ldots, \varepsilon_n$, which means that

$v=\sum_{i=1}^n a_iv_i$ is equivalent to $\tilde{v}=\sum_{i=1}^n \tilde{a}_iv_i$ under the adjoint action; Otherwise, which shows that $v=\sum_{i=1}^n a_iv_i$ and $\tilde{v}=\sum_{i=1}^n \tilde{a}_iv_i$ are inequivalent.

Step 3 Classification of the finite-dimensional Lie algebra $\mathfrak{g}$

(i) Scale the invariants

If two vectors v and w are adjoint equivalent, it is necessary that $\phi(v)=\phi(w)$ for any invariant ϕ. However, if $v=cw$, where v and w are also equivalent, their corresponding invariants satisfy $\phi(v)=c'\phi(w)$ and it is usually $\phi(v)\neq\phi(w)$. To avoid the latter case, we firstly make a scale to the invariant by adjusting the coefficients of generators. Without loss of generality, one just needs to consider values of the invariants to be 1, -1 and 0. To illustrate the point more clearly, three remarks are given as follow:

Remark 1 If the degree of the invariant is odd, $\phi(v)=c^{2k+1}\phi(w)$ with $v=cw$ are obtained, then the right c can be selected to transform the positive (negative) invariant into the negative (positive) one. So we only need to consider two cases: $\phi=0$ and $\phi\neq 0$ (for simplicity scaling it to 1 or -1);

Remark 2 If the degree of the invariant is even (excluding zero), there is $\phi(v)=c^{2k}\phi(w)$ with $v=cw$, then we cannot choose the right c to transform the positive (negative) invariant into the negative (positive) one. So we should consider three cases: $\phi=0$, $\phi>0$ and $\phi<0$. Without loss of generality, we let $\phi=0$, $\phi=1$ and $\phi=-1$;

Remark 3 Once one of the invariants is scaled (not zero), the other invariants (if any) cannot be adjusted.

(ii) Select the representative elements

According to the different values of the invariants given in step 1, we can choose the corresponding representative element in the simplest form named $\tilde{v}=\sum_{i=1}^n \tilde{a}_iv_i$. Then solving the adjoint transformation equation (10). If Eq. (10) has solutions with respect to $\varepsilon_1,\ldots,\varepsilon_n$, it indicated that the choose of representative element is right, otherwise, we need reselect the proper one. Repeat the process until all the cases are finished in step 1.

3 One-Dimensional Optimal System of WZ Equation

The Lie algebra of infinitesimal symmetries for the WZ equation is spanned by eight vector fields

$$
\begin{aligned}
&v_1=\partial_x\,,\quad v_2=\partial_y\,,\quad v_3=\partial_t\,,\quad v_4=\partial_u+t\partial_x\,,\\
&v_5=\partial_v+t\partial_y\,,\quad v_6=y\partial_x-x\partial_y+v\partial_u-u\partial_v\,,\\
&v_7=x\partial_x+y\partial_y+2t\partial_t-u\partial_u-v\partial_v-2w\partial_w\,,\\
&v_8=xt\partial_x+yt\partial_y+t^2\partial_t+(x-ut)\partial_u+(y-vt)\partial_v-2wt\partial_w\,. \qquad (12)
\end{aligned}
$$

Now consider the 8-dimensional symmetry algebra $\mathfrak{g}$ generated by $\{v_1,v_2,\ldots,v_8\}$ in Eq. (12). Their commutator table is given in Table 1.

Table 1 The commutator table; the (i,j)-th entry is $[v_i,v_j]=v_iv_j-v_jv_i$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0	0	0	0	$-v_2$	v_1	v_4
v_2		0	0	0	0	v_1	v_2	v_5
v_3			0	v_1	v_2	0	$2v_3$	v_7
v_4				0	0	$-v_5$	$-v_4$	0
v_5					0	v_4	$-v_5$	0
v_6						0	0	0
v_7							0	$2v_8$
v_8								0

Substituting $v=\sum_{i=1}^8 a_iv_i$ and $w=\sum_{i=1}^8 b_jv_j$ into Eq. (2) yields

$$
\begin{aligned}
&\Theta_1=a_1b_7-a_7b_1+a_2b_6-a_6b_2+a_3b_4-a_4b_3\,,\\
&\Theta_2=-a_1b_6+a_6b_1+a_2b_7-a_7b_2+a_3b_5-a_5b_3\,,\\
&\Theta_4=a_1b_8-a_8b_1-a_4b_7+a_7b_4+a_5b_6-a_6b_5\,,\\
&\Theta_5=a_2b_8-a_8b_2-a_4b_6+a_6b_4-a_5b_7+a_7b_5\,,\\
&\Theta_3=2(a_3b_7-a_7b_3)\,,\quad \Theta_6=0\,,\\
&\Theta_7=a_3b_8-a_8b_3\,,\quad \Theta_8=2(a_7b_8-a_8b_7)\,. \qquad (13)
\end{aligned}
$$

Substituting Eq. (13) into Eq. (5) and extracting the coefficients of all b_i, the equations about $\phi(a_1,a_2,\ldots,a_8)$ are derived as

$$
\begin{aligned}
&a_3\frac{\partial\phi}{\partial a_1}+a_7\frac{\partial\phi}{\partial a_4}+a_6\frac{\partial\phi}{\partial a_5}=0\,,\\
&a_3\frac{\partial\phi}{\partial a_2}-a_6\frac{\partial\phi}{\partial a_4}+a_7\frac{\partial\phi}{\partial a_5}=0\,,\\
&a_6\frac{\partial\phi}{\partial a_1}+a_7\frac{\partial\phi}{\partial a_2}+a_8\frac{\partial\phi}{\partial a_5}=0\,,\\
&a_7\frac{\partial\phi}{\partial a_1}-a_6\frac{\partial\phi}{\partial a_2}+a_8\frac{\partial\phi}{\partial a_4}=0\,,\\
&a_4\frac{\partial\phi}{\partial a_1}+a_5\frac{\partial\phi}{\partial a_2}+2a_7\frac{\partial\phi}{\partial a_3}+a_8\frac{\partial\phi}{\partial a_7}=0\,,\\
&a_2\frac{\partial\phi}{\partial a_1}-a_1\frac{\partial\phi}{\partial a_2}+a_5\frac{\partial\phi}{\partial a_4}-a_4\frac{\partial\phi}{\partial a_5}=0\,,\\
&a_1\frac{\partial\phi}{\partial a_4}+a_2\frac{\partial\phi}{\partial a_5}+a_3\frac{\partial\phi}{\partial a_7}+2a_7\frac{\partial\phi}{\partial a_8}=0\,,\\
&a_1\frac{\partial\phi}{\partial a_1}+a_2\frac{\partial\phi}{\partial a_2}+2a_3\frac{\partial\phi}{\partial a_3}-a_4\frac{\partial\phi}{\partial a_4}\\
&\qquad -a_5\frac{\partial\phi}{\partial a_5}-2a_8\frac{\partial\phi}{\partial a_8}=0\,. \qquad (14)
\end{aligned}
$$

Solving the above equations, one can obtain two basic common invariants:

$$\Delta_1\equiv\phi(a_1,a_2,\ldots,a_8)=a_6\,, \qquad (15)$$

$$\Delta_2\equiv\phi(a_1,a_2,\ldots,a_8)=a_7^2-a_3a_8\,. \qquad (16)$$

The Killing form of WZ equation is $3(a_7^2-a_3a_8)-a_6^2$, which is just the combination of invariants Δ_1 and Δ_2.

The adjoint representation table is given in Table 2.

Table 2 The adjoint representation table; the (i,j)-th entry gives $\mathrm{Ad}_{\exp(\varepsilon v_i)}(v_j)$.

Ad	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	v_1	v_2	v_3	v_4	v_5	$v_6+\varepsilon v_2$	$v_7-\varepsilon v_1$	$v_8-\varepsilon v_4$
v_2	v_1	v_2	v_3	v_4	v_5	$v_6-\varepsilon v_1$	$v_7-\varepsilon v_2$	$v_8-\varepsilon v_5$
v_3	v_1	v_2	v_3	$v_4-\varepsilon v_1$	$v_5-\varepsilon v_2$	v_6	$v_7-2\varepsilon v_3$	$v_8-\varepsilon v_7+\varepsilon^2 v_3$
v_4	v_1	v_2	$v_3+\varepsilon v_1$	v_4	v_5	$v_6+\varepsilon v_5$	$v_7+\varepsilon v_4$	v_8
v_5	v_1	v_2	$v_3+\varepsilon v_2$	v_4	v_5	$v_6-\varepsilon v_4$	$v_7+\varepsilon v_5$	v_8
v_6	$\mathcal{C}v_1-\mathcal{S}v_2$	$\mathcal{S}v_1+\mathcal{C}v_2$	v_3	$\mathcal{C}v_4-\mathcal{S}v_5$	$\mathcal{S}v_4+\mathcal{C}v_5$	v_6	v_7	v_8
v_7	$\mathrm{e}^{\varepsilon}v_1$	$\mathrm{e}^{\varepsilon}v_2$	$e^{2\varepsilon}v_3$	$e^{-\varepsilon}v_4$	$e^{-\varepsilon}v_5$	v_6	v_7	$e^{-2\varepsilon}v_8$
v_8	$v_1+\varepsilon v_4$	$v_2+\varepsilon v_5$	$v_3+\varepsilon v_7+\varepsilon^2 v_8$	v_4	v_5	v_6	$v_7+2\varepsilon v_8$	v_8

with

$$\mathcal{C}=\cos(\varepsilon)\,,\quad \mathcal{S}=\sin(\varepsilon)\,.$$

Applying the adjoint action of v_i $(i=1,\ldots,8)$ to $v=\sum_{i=1}^{8}a_iv_i$ respectively, we can easily obtain the matrices $A_1,\ldots,A_8$, which are shown in Appendix. Then the general adjoint transformation matrix A of the WZ equation is:

$$\begin{aligned}A&=(a_{ij})_{8\times 8}=A_6A_1A_2A_3A_4A_5A_7A_8\\&=\begin{pmatrix}\cos(\varepsilon_6) & -\sin(\varepsilon_6) & 0 & \varepsilon_8\cos(\varepsilon_6) & -\varepsilon_8\sin(\varepsilon_6) & 0 & 0 & 0\\ \sin(\varepsilon_6) & \cos(\varepsilon_6) & 0 & \varepsilon_8\sin(\varepsilon_6) & \varepsilon_8\cos(\varepsilon_6) & 0 & 0 & 0\\ 0 & \varepsilon_5 & 1 & 0 & \varepsilon_5\varepsilon_8 & 0 & \varepsilon_8 & \varepsilon_8^2\\ -\varepsilon_3\cos(\varepsilon_6) & \varepsilon_3\sin(\varepsilon_6) & 0 & \Omega\cos(\varepsilon_6) & -\Omega\sin(\varepsilon_6) & 0 & 0 & 0\\ -\varepsilon_3\sin(\varepsilon_6) & -\varepsilon_3\cos(\varepsilon_6) & 0 & \Omega\sin(\varepsilon_6) & \Omega\cos(\varepsilon_6) & 0 & 0 & 0\\ -\varepsilon_2 & \Lambda & 0 & -(\varepsilon_5+\varepsilon_2\varepsilon_8) & \varepsilon_8\Lambda & 1 & 0 & 0\\ -\Lambda & -\Gamma & -2\varepsilon_3 & -\varepsilon_8\Lambda & \varepsilon_5-\varepsilon_8\Gamma & 0 & 1-2\varepsilon_3\varepsilon_8 & 2\varepsilon_8\Omega\\ \varepsilon_3\Lambda & \varepsilon_3(\varepsilon_2+\varepsilon_3\varepsilon_5) & \varepsilon_3^2 & -\Omega\Lambda & -\Omega(\varepsilon_2+\varepsilon_3\varepsilon_5) & 0 & -\varepsilon_3\Omega & \Omega^2\end{pmatrix},\end{aligned}\tag{17}$$

with

$$\Lambda=\varepsilon_1+\varepsilon_4+\varepsilon_7\,,\quad \Omega=1-\varepsilon_3\varepsilon_8\,,\quad \Gamma=\varepsilon_2+2\varepsilon_3\varepsilon_5\,.\tag{18}$$

The adjoint transformation equation of the WZ equation is

$$(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_8)=(a_1,a_2,\ldots,a_8)A\,,\tag{19}$$

with Eq. (17).

In the following, two invariants Δ_1 and Δ_2 will be made full use of to give a classification of the algebra $\mathfrak{g}$. Since the degree of Δ_1 is one, we can scale it to two cases by Remarks:

Case 1 $\Delta_1=1$, $\Delta_2=c_2$.

Here c_2 is an arbitrary real constant.

Case 1.1 $c_2>0$.

Under the above condition, choose a representative element: $\tilde{v}=v_6+\sqrt{c_2}v_7$.

From $\Delta_2=a_7^2-a_3a_8>0$, we know that a_3, a_7, a_8 can not be all zeros simultaneously. Without loss of generality, we only consider $a_8\neq 0$. For $a_8=0$ ($a_3\neq 0$ or $a_7\neq 0$), one can transform it into the case of $a_8\neq 0$ by selecting the appropriate $\varepsilon_i\,(i=1,\ldots,8)$ which are shown in Eq. (10).

When $a_8\neq 0$, the general solution of $\Delta_1=1$, $\Delta_2=c_2$ is

$$a_6=1\,,\quad a_3=\frac{a_7^2-c_2}{a_8}\,,\tag{20}$$

where a_1, a_2, a_4, a_5, a_7, a_8 are arbitrary real constants.

Substituting $\tilde{a}_6=1,\tilde{a}_7=\sqrt{c_2},\tilde{a}_1=\tilde{a}_2=\tilde{a}_3=\tilde{a}_4=\tilde{a}_5=\tilde{a}_8=0$ with Eq. (20) into Eq. (19), one can find the solution

$$\begin{aligned}&\varepsilon_1=-\varepsilon_4-\varepsilon_7+\frac{a_2a_4-a_1a_5}{\sqrt{F}}+\frac{a_4^2+a_5^2}{a_8\sqrt{F}}-\frac{\sqrt{F}}{a_8(c_2+1)}\,,\quad \varepsilon_3=\frac{a_7-\sqrt{c_2}}{a_8}\,,\\&\varepsilon_2=\frac{a_1a_4+a_2a_5}{\sqrt{F}}-\frac{a_7(a_4^2+a_5^2)}{a_8\sqrt{F}}+\frac{\sqrt{c_2F}}{a_8(c_2+1)}\,,\quad \varepsilon_5=\frac{\sqrt{F}}{c_2+1}\,,\\&\varepsilon_6=\mathrm{atan2}\Big(\frac{a_5+a_4a_7-a_1a_8}{\sqrt{F}},\frac{a_4+a_2a_8-a_5a_7}{\sqrt{F}}\Big)\,,\quad \varepsilon_8=-\frac{a_8}{2\sqrt{c_2}}\,,\end{aligned}$$

with

$$F = (a_4 + a_2a_8 - a_5a_7)^2 + (a_5 + a_4a_7 - a_1a_8)^2\,, \quad (21)$$

which obviously indicates $F \geqslant 0$. But $F \not\equiv 0$, if not, there must be all $a_i = 0$ $(i = 1, 2, \ldots, 8)$, which is meaningless.

Case 1.2 $c_2 = 0$.

From $\Delta_2 = a_7^2 - a_3a_8 = 0$, we can divide it into the following two cases:

(i) $a_3 = a_7 = a_8 = 0$.

Now the general algebra becomes $a_1v_1 + a_2v_2 + a_4v_4 + a_5v_5 + v_6$, it can be easily converted into v_6 for the solutions

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_1a_5 - a_2a_4}{a_4\sqrt{1+(a_5/a_4)^2}}\,,$$

$$\varepsilon_2 = \frac{a_1a_4 + a_2a_5 - \varepsilon_3(a_4^2 + a_5^2)}{a_4\sqrt{1+(a_5/a_4)^2}}\,,$$

$$\varepsilon_5 = \frac{a_4^2 + a_5^2}{a_4\sqrt{1+(a_5/a_4)^2}}\,, \quad \varepsilon_6 = \arctan\Big(\frac{a_5}{a_4}\Big)\,.$$

(ii) Not all a_3, a_7 and a_8 are zeros. Without loss of generality, let $a_8 \neq 0$.

When $a_8 > 0$, choose the representative element: $\tilde{v} = v_3 + v_6$ and Eq. (19) has the solutions

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{F}} + \frac{a_4^2 + a_5^2}{a_8\sqrt{F}} - \frac{\sqrt{F}}{a_8}\,,$$

$$\varepsilon_2 = \frac{a_1a_4 + a_2a_5}{\sqrt{F}} - \frac{a_7({a_4}^2 + {a_5}^2)}{a_8\sqrt{F}} - \sqrt{\frac{F}{a_8}}\,,$$

$$\varepsilon_3 = \frac{a_7 + \sqrt{a_8}}{a_8}\,, \quad \varepsilon_5 = \sqrt{F}\,, \quad \varepsilon_8 = \sqrt{a_8}\,,$$

$$\varepsilon_6 = \operatorname{atan2}\Big(\frac{a_5 + a_4a_7 - a_1a_8}{\sqrt{F}}, \frac{a_4 + a_2a_8 - a_5a_7}{\sqrt{F}}\Big)\,.$$

When $a_8 < 0$, choose the representative element: $\tilde{v} = -v_3 + v_6$ and Eq. (19) has the solutions

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{F}} + \frac{a_4^2 + a_5^2}{a_8\sqrt{F}} - \frac{\sqrt{F}}{a_8}\,,$$

$$\varepsilon_2 = \frac{a_1a_4 + a_2a_5}{\sqrt{F}} - \frac{a_7({a_4}^2 + {a_5}^2)}{a_8\sqrt{F}} + \sqrt{\frac{F}{-a_8}}\,,$$

$$\varepsilon_3 = \frac{a_7 + \sqrt{-a_8}}{a_8}\,, \quad \varepsilon_5 = \sqrt{F}\,, \quad \varepsilon_8 = -\sqrt{-a_8}\,,$$

$$\varepsilon_6 = \operatorname{atan2}\Big(\frac{a_5 + a_4a_7 - a_1a_8}{\sqrt{F}}, \frac{a_4 + a_2a_8 - a_5a_7}{\sqrt{F}}\Big)\,.$$

Case 1.3 $c_2 < 0$.

From $\Delta_2 = {a_7}^2 - a_3a_8 < 0$, a_3 and a_8 must be the same sign and not zero, now we have

$$a_6 = 1\,, \quad a_3 = \frac{a_7^2 - c_2}{a_8}\,. \quad (22)$$

(i) When $a_8 < 0$ and $c_2 \neq -1$

Under the above condition, choose a representative element: $\tilde{v} = -(c_2/\chi)v_3 + v_6 + \chi v_8$, χ is an arbitrary constant.

Firstly, we proof that when $\chi_1/\chi_2 \geqslant 1$, $-(c_2/\chi_1)v_3 + v_6 + \chi_1 v_8$ and $-(c_2/\chi_2)v_3 + v_6 + \chi_2 v_8$ are equivalent.

In fact, owing to the definition of equivalent under the adjoint action about vectors, as long as

$$\Big(0, 0, -\frac{c_2}{\chi_2}, 0, 0, 1, \chi_2\Big) = \Big(0, 0, -\frac{c_2}{\chi_1}, 0, 0, 1, \chi_1\Big)A$$

has solutions with respect to $\varepsilon_1, \ldots, \varepsilon_n$, which means that the former two vectors are equivalent. It is easy to find that when $\chi_1/\chi_2 \geqslant 1$, the solutions are

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7\,, \quad \varepsilon_2 = \varepsilon_5 = 0\,,$$

$$\varepsilon_3 = \frac{\sqrt{c_2\chi_2(\chi_2 - \chi_1)}}{\chi_1\chi_2}\,, \quad \varepsilon_8 = \frac{\sqrt{c_2\chi_2(\chi_2 - \chi_1)}}{c_2}\,,$$

which mean that when χ takes different values, representative elements $\tilde{v} = -(c_2/\chi)v_3 + v_6 + \chi v_8$ are all equivalent. In particular, let $\chi = c_2$, the representative element is $\tilde{v} = v_3 + v_6 + c_2v_8$.

Substituting $\tilde{a}_3 = -1$, $\tilde{a}_6 = 1$, $\tilde{a}_8 = c_2$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = 0$ with Eq. (22) into Eq. (19), one can find the solution

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{F}} + \frac{a_4^2 + a_5^2}{a_8\sqrt{F}} - \frac{\sqrt{F}}{a_8(c_2+1)}\,,$$

$$\varepsilon_2 = \frac{a_1a_4 + a_2a_5}{\sqrt{F}} - \frac{a_7(a_4^2 + a_5^2)}{a_8\sqrt{F}} - \frac{\sqrt{F(c_2 - a_8)}}{a_8(c_2+1)}\,,$$

$$\varepsilon_3 = \frac{a_7 + \sqrt{c_2 - a_8}}{a_8}\,, \quad \varepsilon_5 = \frac{\sqrt{F}}{c_2+1}\,, \quad \varepsilon_8 = -\sqrt{c_2 - a_8}\,,$$

$$\varepsilon_6 = \operatorname{atan2}\Big(\frac{a_5 + a_4a_7 - a_1a_8}{\sqrt{F}}, \frac{a_4 + a_2a_8 - a_5a_7}{\sqrt{F}}\Big)\,.$$

(ii) When $a_8 > 0$ and $c_2 \neq -1$

Similar to (i), choose a representative element: $\tilde{v} = v_3 + v_6 - c_2v_8$ and the solutions are

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{F}} + \frac{a_4^2 + a_5^2}{a_8\sqrt{F}} - \frac{\sqrt{F}}{a_8(c_2+1)}\,,$$

$$\varepsilon_2 = \frac{a_1a_4 + a_2a_5}{\sqrt{F}} - \frac{a_7(a_4^2 + a_5^2)}{a_8\sqrt{F}} + \frac{\sqrt{F(c_2 + a_8)}}{a_8(c_2+1)}\,,$$

$$\varepsilon_3 = \frac{a_7 - \sqrt{c_2 + a_8}}{a_8}\,, \quad \varepsilon_5 = \frac{\sqrt{F}}{c_2+1}\,, \quad \varepsilon_8 = -\sqrt{c_2 + a_8}\,,$$

$$\varepsilon_6 = \operatorname{atan2}\Big(\frac{a_5 + a_4a_7 - a_1a_8}{\sqrt{F}}, \frac{a_4 + a_2a_8 - a_5a_7}{\sqrt{F}}\Big)\,.$$

(iii) When $c_2 = -1$, we have

$$a_6 = 1\,, \quad a_3 = \frac{a_7^2 + 1}{a_8}\,. \quad (23)$$

Substituting Eq. (23) into Eq. (19), then the equation is as follows,

$$\Theta_1\frac{\partial\Phi}{\partial a_1} + \Theta_2\frac{\partial\Phi}{\partial a_2} + \Theta_4\frac{\partial\Phi}{\partial a_4} + \Theta_5\frac{\partial\Phi}{\partial a_5} + \Theta_7\frac{\partial\Phi}{\partial a_7} + \Theta_8\frac{\partial\Phi}{\partial a_8} = 0, \quad (24)$$

for any b_i, where a_3, a_6 of Θ_1, Θ_2, Θ_4, Θ_5, Θ_7, Θ_8 are replaced by Eq. (23). Extracting the coefficients of all b_i of Eq. (24), the equations about $\Phi(a_1, a_2, a_4, a_5, a_7, a_8)$ are derived as

$$
\begin{aligned}
&\frac{\partial\Phi}{\partial a_1}+a_7\frac{\partial\Phi}{\partial a_2}+a_8\frac{\partial\Phi}{\partial a_5}=0\,,\\
&a_1\frac{\partial\Phi}{\partial a_2}-a_2\frac{\partial\Phi}{\partial a_1}-a_5\frac{\partial\Phi}{\partial a_4}+a_4\frac{\partial\Phi}{\partial a_5}=0\,,\\
&a_4\frac{\partial\Phi}{\partial a_1}+a_5\frac{\partial\Phi}{\partial a_2}+a_8\frac{\partial\Phi}{\partial a_7}=0\,,\\
&\frac{(1+a_7^2)}{a_8}\frac{\partial\Phi}{\partial a_2}-\frac{\partial\Phi}{\partial a_4}+a_7\frac{\partial\Phi}{\partial a_5}=0\,,\\
&\frac{\partial\Phi}{\partial a_2}-a_7\frac{\partial\Phi}{\partial a_1}-a_8\frac{\partial\Phi}{\partial a_4}=0\,,\\
&a_1\frac{\partial\Phi}{\partial a_4}+a_2\frac{\partial\Phi}{\partial a_5}+\frac{(1+a_7^2)}{a_8}\frac{\partial\Phi}{\partial a_7}+2a_7\frac{\partial\Phi}{\partial a_8}=0\,.
\end{aligned}
\tag{25}
$$

Solving the above equations (25), one obtains a new invariant

$$
\Delta_3=\frac{F}{a_8}\,. \tag{26}
$$

Let $\Delta_3=c_3$, owing to $F\geqslant 0$ always holds, the sign of c_3 and a_8 are same, we have

$$
a_1=\frac{a_5+a_4a_7\pm\sqrt{H}}{a_8}\,, \tag{27}
$$

with

$$
H=c_3a_8-(a_4+a_2a_8-a_5a_7)^2\,. \tag{28}
$$

(i) When $c_3>0$, choose a representative element: $\tilde v=\sqrt{c_3}v_1+v_3+v_6+v_8$ and the solutions are

$$
\begin{aligned}
&\varepsilon_3=\frac{a_7+\delta_1}{a_8}\,,\quad \varepsilon_8=\delta_1\,,\quad \varepsilon_6=\operatorname{atan2}\Big(\frac{a_4+a_2a_8-a_5a_7+\delta_1\sqrt{H}}{a_8\sqrt{c_3}},\frac{\sqrt{H}-(a_4+a_2a_8-a_5a_7)\delta_1}{a_8\sqrt{c_3}}\Big)\,,\\
&\varepsilon_2=(\varepsilon_1+\varepsilon_4+\varepsilon_7)\delta_1-\frac{\delta_1^2\sqrt{c_3}}{a_8}-\frac{(a_5a_8-2a_5+2a_4\delta_1)\sqrt{H}+(a_4+a_2a_8-a_5a_7)(-a_4a_8+2a_4+2a_5\delta_1)}{a_8^2\sqrt{c_3}}\,,\\
&\varepsilon_5=\delta_1\sqrt{c_3}-a_8(\varepsilon_1+\varepsilon_4+\varepsilon_7)+\frac{(a_4+a_5\delta_1)\sqrt{H}-(a_4+a_2a_8-a_5a_7)(a_4\delta_1-a_5)}{a_8\sqrt{c_3}}\,,
\end{aligned}
$$

with $\delta_1=\sqrt{a_8-1}$.

(ii) When $c_3=0$, choose a representative element: $\tilde v=v_3+v_6+v_8$ and the solutions are

$$
\begin{aligned}
&\varepsilon_3=\frac{a_7+\delta_1}{a_8}\,,\quad \varepsilon_8=\delta_1\,,\quad \varepsilon_5=-a_8(\varepsilon_1+\varepsilon_4+\varepsilon_7)+\frac{a_8(a_1a_7-a_2)\cos(\varepsilon_6)+a_8(a_1+a_2a_7)\sin(\varepsilon_6)}{1+a_7^2}\,,\\
&\varepsilon_2=(\varepsilon_1+\varepsilon_4+\varepsilon_7)\delta_1-\frac{[(a_1a_7-a_2)\delta_1-a_1-a_2a_7)]\cos(\varepsilon_6)+[(a_1+a_2a_7)\delta_1-a_2+a_1a_7]\sin(\varepsilon_6)}{1+a_7^2}\,.
\end{aligned}
$$

Notice Obviously, the above case (ii) is the special case of (i).

(iii) When $c_3<0$, choose a representative element: $\tilde v=-\sqrt{-c_3}v_1-v_3+v_6-v_8$ and the solutions are

$$
\begin{aligned}
&\varepsilon_3=\frac{a_7+\delta_2}{a_8}\,,\quad \varepsilon_8=-\delta_2\,,\quad \varepsilon_6=\operatorname{atan2}\Big(-\frac{a_4+a_2a_8-a_5a_7+\delta_2\sqrt{H}}{a_8\sqrt{-c_3}},\frac{(a_4+a_2a_8-a_5a_7)\delta_2\sqrt{H}}{a_8\sqrt{-c_3}}\Big)\,,\\
&\varepsilon_2=(\varepsilon_1+\varepsilon_4+\varepsilon_7)\delta_2-\frac{{\delta_2}^2\sqrt{-c_3}}{a_8}+\frac{(a_4+a_2a_8-a_5a_7)(a_4a_8+2a_4+2a_5\delta_2)-(a_5a_8+2a_5-2a_4\delta_2)\sqrt{H}}{a_8^2\sqrt{-c_3}}\,,\\
&\varepsilon_5=\delta_2\sqrt{-c_3}-a_8(\varepsilon_1+\varepsilon_4+\varepsilon_7)+\frac{(a_4+a_2a_8-a_5a_7)(a_4\delta_2-a_5)-(a_4+a_5\delta_2)\sqrt{H}}{a_8\sqrt{-c_3}}\,,
\end{aligned}
$$

with $\delta_2=\sqrt{-a_8-1}$.

Case 2 $\Delta_1=0,\Delta_2=c_2$.

Since the degree of $\Delta_2=a_7^2-a_3a_8$ is two, we can scale it to three cases:

Case 2.1 $c_2=1$.

Owing to $a_7^2-a_3a_8=1$, not all a_3,a_7 and a_8 are zeros. Without loss of generality, let $a_8\neq 0$, we have

$$
a_6=0\,,\quad a_3=\frac{a_7^2-1}{a_8}\,. \tag{28}
$$

Under the above condition, choose the representative element: $\tilde v=v_7$.

Substituting $\tilde a_7=1$, $\tilde a_1=\tilde a_2=\tilde a_3=\tilde a_4=\tilde a_5=\tilde a_6=\tilde a_8=0$ with Eq. (28) into these equations, one can find the solution

$$
\begin{aligned}
&\varepsilon_1=-\varepsilon_4-\varepsilon_7+\frac{a_2a_4-a_1a_5}{\sqrt{G}}\,,\quad \varepsilon_3=\frac{a_7-1}{a_8}\,,\\
&\varepsilon_5=\sqrt{G}\,,\quad \varepsilon_8=-\frac{a_8}{2}\,,\\
&\varepsilon_2=\frac{a_1a_4+a_2a_5}{\sqrt{G}}-\frac{a_7(a_4^2+a_5^2)}{a_8\sqrt{G}}+\frac{\sqrt{G}}{a_8}\,,\\
&\varepsilon_6=\operatorname{atan2}\Big(\frac{a_4a_7-a_1a_8}{\sqrt{G}},\frac{a_2a_8-a_5a_7}{\sqrt{G}}\Big)\,,
\end{aligned}
$$

with

$$
G=(a_1a_8-a_4a_7)^2+(a_2a_8-a_5a_7)^2\,. \tag{29}
$$

Case 2.2 $c_2=-1$.

Owing to $a_7^2 - a_3a_8 = -1$, let $a_8 \neq 0$, we have

$$a_6 = 0\,, \quad a_3 = \frac{a_7^2+1}{a_8}\,. \tag{30}$$

Similar to Case 1.3(i),

(i) when $a_8 > 0$, choose a representative element: $\tilde{v} = v_3 + v_8$, one can find the solution

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{G}}\,, \quad \varepsilon_3 = \frac{a_7 + \sqrt{a_8-1}}{a_8}\,,$$

$$\varepsilon_5 = -\sqrt{G}\,, \quad \varepsilon_8 = \sqrt{a_8-1}\,,$$

$$\varepsilon_2 = \frac{a_1a_4+a_2a_5}{\sqrt{G}} - \frac{a_7(a_4^2+a_5^2)}{a_8\sqrt{G}} + \frac{\sqrt{a_8-1}\sqrt{G}}{a_8}\,,$$

$$\varepsilon_6 = \mathrm{atan2}\Big(\frac{a_4a_7 - a_1a_8}{\sqrt{G}}, \frac{a_2a_8 - a_5a_7}{\sqrt{G}}\Big)\,.$$

(ii) when $a_8 < 0$, choose a representative element: $\tilde{v} = -v_3 - v_8$, one can find the solution

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 + \frac{a_2a_4 - a_1a_5}{\sqrt{G}}\,, \quad \varepsilon_3 = \frac{a_7 + \sqrt{-a_8-1}}{a_8}\,,$$

$$\varepsilon_5 = -\sqrt{G}\,, \quad \varepsilon_8 = \sqrt{-a_8-1}\,,$$

$$\varepsilon_2 = \frac{a_1a_4+a_2a_5}{\sqrt{G}} - \frac{a_7(a_4^2+a_5^2)}{a_8\sqrt{G}} + \frac{\sqrt{-a_8-1}\sqrt{G}}{a_8}\,,$$

$$\varepsilon_6 = \mathrm{atan2}\Big(\frac{a_4a_7 - a_1a_8}{\sqrt{G}}, \frac{a_2a_8 - a_5a_7}{\sqrt{G}}\Big)\,.$$

It is not difficult to find that $v_3 + v_8$ and $-v_3 - v_8$ are equivalent.

Case 2.3 $c_2 = 0$.

From $a_7^2 - a_3a_8 = 0$, it can be divided into two cases:

Case 2.3.1 $a_3 = a_7 = a_8 = 0$.

Like Case 1.3(iii), we get the new invariant

$$\Delta_4 = a_1a_5 - a_2a_4\,. \tag{31}$$

Since the degree of Δ_4 is two, there are three cases as follows:

(i) $\Delta_4 = 1$.

Now the general algebra becomes $a_1v_1 + a_2v_2 + a_4v_4 + a_5v_5$, it can be easily converted into $v_1 + v_5$ for the solutions

$$\varepsilon_3 = \frac{a_4 + a_2(a_4^2+a_5^2) - \sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5(a_4^2+a_5^2)}\,, \quad \varepsilon_8 = -\frac{\sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5}\,,$$

$$\varepsilon_6 = \mathrm{atan2}\left(\frac{-a_4 + \sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_4^2+a_5^2}, \frac{a_5^2 + a_4\sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5(a_4^2+a_5^2)}\right)\,.$$

(ii) $\Delta_4 = -1$.

Now the general algebra becomes $a_1v_1 + a_2v_2 + a_4v_4 + a_5v_5$, it can be easily converted into $v_1 - v_5$ for the solutions

$$\varepsilon_3 = \frac{-a_4 + a_2(a_4^2+a_5^2) - \sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5(a_4^2+a_5^2)}\,, \quad \varepsilon_8 = -\frac{\sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5}\,,$$

$$\varepsilon_6 = \mathrm{atan2}\left(\frac{a_4 + \sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_4^2+a_5^2}, \frac{-a_5^2 + a_4\sqrt{a_5^2(a_4^2+a_5^2-1)}}{a_5(a_4^2+a_5^2)}\right)\,.$$

(iii) $\Delta_4 = 0$.

Not all a_1, a_4, a_7, a_8 are zeros, the general algebra becomes $a_1v_1 + a_2v_2 + a_4v_4 + a_5v_5$ can be easily converted into v_1 for the solutions

$$\varepsilon_3 = \frac{a_2(a_4^2+a_5^2) - a_5\sqrt{a_4^2+a_5^2}}{a_5(a_4^2+a_5^2)}\,, \quad \varepsilon_8 = -\sqrt{a_4^2+a_5^2}\,, \quad \varepsilon_6 = \mathrm{atan2}\left(\frac{a_5}{\sqrt{a_4^2+a_5^2}}, \frac{a_4}{\sqrt{(a_4^2+a_5^2)}}\right)\,.$$

Case 2.3.2 Not all a_3, a_7 and a_8 are zeros. Without loss of generality, supposing $a_8 \neq 0$, we have

$$a_6 = 0\,, \quad a_3 = \frac{a_7^2}{a_8}\,. \tag{32}$$

Like Case 1.3(iii), we get the new invariant

$$\Delta_5 = \frac{G}{a_8}\,. \tag{33}$$

Owing to $G \geqslant 0$, the sign of Δ_5 and a_8 are the same.

Since the degree of Δ_5 is three, we can scale it to two cases:

(i) $\Delta_5 = 1$. This time $a_8 > 0$ and we derive

$$a_1 = \frac{a_4a_7 \pm \sqrt{a_8 - (a_2a_8 - a_5a_7)^2}}{a_8}\,, \tag{34}$$

choose the representative element: $\tilde{v} = \pm v_1 + v_3 + v_7 + v_8$ and the solutions are

$$\varepsilon_1 = -\varepsilon_4 - \varepsilon_7 \pm \frac{1 - a_2a_5}{\sqrt{a_8}} \pm \frac{1}{a_8} \pm \frac{a_5^2a_7 \mp a_4\sqrt{a_8 - (a_5a_7 - a_2a_8)^2}}{a_8^{3/2}}\,,$$

$$\varepsilon_3 = \frac{1}{\sqrt{a_8}} + \frac{a_7}{a_8}\,,$$

$$\varepsilon_5 = -\varepsilon_2\sqrt{a_8} \pm a_2a_4 \mp \frac{a_5(a_4a_7 \pm \sqrt{a_8 - (a_5a_7 - a_2a_8)^2})}{a_8}\,,$$

$$\varepsilon_6 = \mathrm{atan2}\left(\frac{a_5a_7 - a_2a_8}{\sqrt{a_8}}, -\sqrt{\frac{a_8 - (a_5a_7 - a_2a_8)^2}{a_8}}\right)\,,$$

$\varepsilon_8 = 1 + \sqrt{a_8}\,,$

and we can easily prove that $v_1 + v_3 + v_7 + v_8$ and $-v_1 + v_3 + v_7 + v_8$ are equivalent.

(ii) $\Delta_5 = 0$.

Owing to $a_8 \neq 0$, we obtain

$$a_3 = \frac{a_7^2}{a_8}\,, \quad a_1 = \frac{a_4 a_7}{a_8}\,, \quad a_2 = \frac{a_5 a_7}{a_8}\,, \tag{35}$$

when $a_8 > 0$, the general algebra can be converted into $\tilde{v} = v_3$ with

$$\varepsilon_1 = \frac{a_4\cos(\varepsilon_6) + a_5\sin(\varepsilon_6) - a_8(\varepsilon_4 + \varepsilon_7)}{a_8}\,,$$

$$\varepsilon_3 = \frac{a_7 \pm \sqrt{a_8}}{a_8}\,,$$

$$\varepsilon_2 = \frac{a_5\cos(\varepsilon_6) - a_4\sin(\varepsilon_6) \mp \sqrt{a_8}\varepsilon_5}{a_8}\,, \quad \varepsilon_8 = \pm\sqrt{a_8}\,.$$

Meanwhile, the case of $a_8 < 0$ is equivalent to $\tilde{v} = -v_3$ for the solution

$$\varepsilon_1 = \frac{a_4\cos(\varepsilon_6) + a_5\sin(\varepsilon_6) - a_8(\varepsilon_4 + \varepsilon_7)}{a_8}\,,$$

$$\varepsilon_3 = \frac{a_7 \pm \sqrt{-a_8}}{a_8}\,,$$

$$\varepsilon_2 = \frac{a_5\cos(\varepsilon_6) - a_4\sin(\varepsilon_6) \mp \sqrt{-a_8}\varepsilon_5}{a_8}\,, \quad \varepsilon_8 = \mp\sqrt{-a_8}\,.$$

It is easy to find that v_3 and $-v_3$ are equivalent.

To summarize, an optimal system of one-dimensional subalgebras of the (2+1)-dimensional WZ equation is provided by those generated by

$$\begin{aligned}
&\text{(i)}\quad v_6 + \sqrt{c_2}v_7\,(c_2{>}0)\,, \quad \text{(ii)}\quad v_6\,, \quad \text{(iii)}\quad v_3 + v_6\,,\\
&\text{(iv)}\quad -v_3 + v_6\,, \quad \text{(v)}\quad -v_3 + v_6 + c_2 v_8\,(c_2{<}0, c_2{\neq}-1)\,,\\
&\text{(vi)}\quad v_3 + v_6 - c_2 v_8 (c_2{<}0, c_2{\neq}-1)\,,\\
&\text{(vii)}\quad \sqrt{c_3}v_1 + v_3 + v_6 + v_8\,(c_3{>}0)\,,\\
&\text{(viii)}\quad \sqrt{-c_3}v_1 + v_3 - v_6 + v_8\,(c_3{<}0)\,,\\
&\text{(ix)}\quad v_7\,, \quad \text{(x)}\quad v_3 + v_8\,, \quad \text{(xi)}\quad v_1 + v_5\,, \quad \text{(xii)}\quad v_1 - v_5\,,\\
&\text{(xiii)}\quad v_1\,, \quad \text{(xiiii)}\quad v_1 + v_3 + v_7 + v_8\,, \quad \text{(xv)}\quad v_3\,.
\end{aligned} \tag{36}$$

Obviously, four infinitesimal generators v_2, v_4, v_5 and v_8 do not appear in our optimal system. In fact, such generators are equivalent to one of Eq. (36) via the adjoint transformation equation (19)

$$\text{For } v_2:\ v_2 \sim v_1, \ \Bigl(\varepsilon_6 = -\frac{\pi}{2}, \varepsilon_8 = 0\Bigr)\,, \tag{37}$$

$$\begin{aligned}
\text{For } v_4 \text{ and } v_5:\ v_5 &\sim v_4, \ \Bigl(\varepsilon_3 = 0, \varepsilon_6 = \frac{\pi}{2}\Bigr)\\
&\sim -v_1 - v_4, \ (\varepsilon_3 = 1, \varepsilon_6 = \pi)\\
&\sim v_1, \ (\varepsilon_3 = 1, \varepsilon_6 = \pi, \varepsilon_8 = 1)\,,
\end{aligned} \tag{38}$$

$$\begin{aligned}
\text{For } v_8:\ v_8 &\sim v_3 + v_7 + v_8, \ (\varepsilon_1 = -\varepsilon_4 - \varepsilon_7,\\
&\qquad \varepsilon_2 = \varepsilon_5, \varepsilon_3 = -1, \varepsilon_8 = 0)\\
&\sim v_3, \ (\varepsilon_1 = -\varepsilon_4 - \varepsilon_7,\\
&\qquad \varepsilon_2 = \varepsilon_5, \varepsilon_3 = 0, \varepsilon_8 = -1)\,.
\end{aligned} \tag{39}$$

Therefore, the similarity reductions, which are reduced from eight simple infinitesimal generators in Ref. [9], are only equivalent to the ones generated by four generators v_1, v_3, v_6, and v_7.

4 (1+1) Similarity Reduction

Based on the optimal system, the similarity variables can be obtained by solving the characteristic equations. Taking the similarity variables as new variables, the WZ equation can be reduced to a system of PDEs with two independent variables ξ and η and three dependent variables P, Q and R. Apart from the symmetry reductions of generators v_j $(j = 1, \ldots, 8)$, the complete and inequivalent similarity reduction systems are presented in the following table.

Compared to the reduction results in Ref. [9], there are more reductions as well as four cases (v_1, v_3, v_6 and v_7) as we show in Sec. 3. This fact intensively suggests the necessity of finding the optimal system for a given differential system, and more importantly, the basic structure of Lie algebra can be derived.

For the purpose of illustration, we only focus on three sets of constant coefficient differential equations generated by v_1, v_3 and $v_1 + v_3 + v_7 + v_8$ in Table 3. The reduction equation from v_1 (to which v_4 is equivalent) is the classical Boussinesq equation, and v_3 (to which v_8 is equivalent) corresponds to the steady case of WZ equation. Some exact special solutions for both reduction systems have been discussed in Ref. [9].

In the following, we consider the reduction system generated by the vector $v_1 + v_3 + v_7 + v_8$,

$$\begin{aligned}
&PP_\xi + QP_\eta + R_\xi = 0\,,\\
&PQ_\xi + QQ_\eta + R_\eta = 0\,,\\
&(PR)_\xi + (QR)_\eta + \frac{1}{3}(P_{\xi\xi\xi} + P_{\xi\eta\eta} + Q_{\eta\eta\eta} + Q_{\xi\xi\eta}) = 0\,.
\end{aligned} \tag{40}$$

With Weiss–Tabor–Carnevale (WTC) method, we demonstrate that Eqs. (40) passes the Painlevé test, which is, apart from the classical Boussinesq equation and the steady case of the WZ equation, another new reduction equation with the property of Painlevé integrability. We further study the traveling wave reduction of Eqs. (40), namely, the similarity variable read $z = \xi + k\eta$ with arbitrary nonzero constant k. Then Eqs. (40) become a system of ODEs as follows:

$$\begin{aligned}
&(P - kQ)P' + R' - 1 = 0\,, \quad (P - kQ)Q' - kR' = 0\,,\\
&\frac{k^2+1}{3}(P''' - kQ''') + (P' - kQ')R + (P - kQ)R' = 0\,,
\end{aligned} \tag{41}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$. After the simple calculation, Eqs. (41) are written as

$$R = -\frac{F^2 + 2(\lambda_1 - z)}{2(1+k^2)}\,, \quad Q' = \frac{k}{k^2+1}\Bigl(\frac{1}{F} - F'\Bigr)\,, \tag{42}$$

$$F'' - \frac{3F^3}{2(k^2+1)^2} + \frac{3(z - \lambda_1)F}{(k^2+1)^2} + \frac{\lambda_2}{k^2+1} = 0\,, \tag{43}$$

with $F = P - kQ$ and λ_1, λ_2 are integrable constants.

Table 3 Reduction of the WZ equation.

Case	Similarity variables	Reduced equations
(i) $v_6+\sqrt{c_2}v_7$ $(c_2>0)$	$\xi=-\frac{y\cos(s)+x\sin(s)}{\sqrt{t}},\quad \eta=\frac{x\cos(s)-y\sin(s)}{\sqrt{t}},$ $P=(v\cos(s)+u\sin(s))\sqrt{t},\quad s=\frac{\ln t}{2\sqrt{c_2}},$ $Q=(u\cos(s)-v\sin(s))\sqrt{t},\quad R=tw$	$Q-\xi P_\eta+\eta P_\xi+\sqrt{c_2}[(\xi+2P)P_\xi+(\eta-2Q)P_\eta+2R_\xi+P]=0,$ $P+\xi Q_\eta-\eta Q_\xi-\sqrt{c_2}[(\xi+2P)Q_\xi+(\eta-2Q)Q_\eta-2R_\eta+Q]=0,$ $\frac{1}{2}(\xi R_\eta-\eta R_\xi)-\frac{\sqrt{c_2}}{2}(\xi R_\xi+\eta R_\eta)-\sqrt{c_2}[R+(PR)_\xi-(QR)_\eta]$ $-\frac{\sqrt{c_2}}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}-Q_{\eta\eta\eta}-Q_{\xi\xi\eta})=0$
(ii) v_6	$\xi=x^2+y^2,\quad \eta=t,\quad R=w,$ $P=uy-vx,\quad Q=ux+vy$	$P_\eta+2QP_\xi=0,\quad \xi(Q_\eta+2QQ_\xi+2\xi R_\xi)-(P^2+Q^2)=0,$ $R_\eta+2(RQ)_\xi+\frac{8}{3}(Q_{\xi\xi}+\xi Q_{\xi\xi\xi})=0$
(iii) v_3+v_6	$\xi=x\sin(t)+y\cos(t),\quad \eta=x\cos(t)-y\sin(t),$ $P=u\sin(t)+v\cos(t), Q=u\cos(t)-v\sin(t),$ $R=w$	$(P+\eta)Q_\xi+(Q-\xi)Q_\eta+R_\eta+P=0,$ $(P+\eta)P_\xi+(Q-\xi)P_\eta+R_\xi-Q=0,$ $(PR)_\xi+(QR)_\eta+\eta R_\xi-\xi R_\eta$ $+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}+Q_{\eta\eta\eta}+Q_{\xi\xi\eta})=0$
(iv) $-v_3+v_6$	$\xi=y\cos(t)-x\sin(t),\quad \eta=x\cos(t)+y\sin(t),$ $P=v\cos(t)-u\sin(t), Q=u\cos(t)+v\sin(t),$ $R=w$	$(P-\eta)Q_\xi+(Q+\xi)Q_\eta+R_\eta-P=0,$ $(P-\eta)P_\xi+(Q+\xi)P_\eta+R_\xi+Q=0,$ $(PR)_\xi+(QR)_\eta-\eta R_\xi+\xi R_\eta$ $+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}+Q_{\eta\eta\eta}+Q_{\xi\xi\eta})=0$
(v) $-v_3+v_6+c_2v_8$ $(c_2<0, c_2\neq-1)$	$\xi=\frac{y\cos(s)-x\sin(s)}{\sqrt{1-c_2t^2}},\quad \eta=\frac{y\sin(s)+x\cos(s)}{\sqrt{1-c_2t^2}},$ $P=\frac{[u+c_2t(x-ut)]\cos(s)+[v+c_2t(y-vt)]\sin(s)}{\sqrt{1-c_2t^2}},$ $Q=\frac{[v+c_2t(y-vt)]\cos(s)-[u+c_2t(x-ut)]\sin(s)}{\sqrt{1-c_2t^2}},$ $R=w(1-c_2t^2),\quad s=\frac{\arctan(\sqrt{-c_2}t)}{\sqrt{-c_2}}$	$(Q-\eta)P_\xi+(P+\xi)P_\eta+R_\eta-Q-c_2\eta=0,$ $(Q-\eta)Q_\xi+(P+\xi)Q_\eta+R_\xi+P-c_2\xi=0,$ $(QR)_\xi+(PR)_\eta+\xi R_\eta-\eta R_\xi$ $+\frac{1}{3}(P_{\eta\eta\eta}+P_{\xi\xi\eta}+Q_{\xi\xi\xi}+Q_{\xi\eta\eta})=0$
(vi) $v_3+v_6-c_2v_8$ $(c_2<0, c_2\neq-1)$	$\xi=\frac{y\cos(s)+x\sin(s)}{\sqrt{1-c_2t^2}},\quad \eta=\frac{x\cos(s)-y\sin(s)}{\sqrt{1-c_2t^2}},$ $P=\frac{[u+c_2t(x-ut)]\cos(s)-[v+c_2t(y-vt)]\sin(s)}{\sqrt{1-c_2t^2)}},$ $Q=\frac{[v+c_2t(y-vt)]\cos(s)+[u+c_2t(x-ut)]\sin(s)}{\sqrt{1-c_2t^2}},$ $R=w(1-c_2t^2),\quad s=\frac{\arctan(\sqrt{-c_2}t)}{\sqrt{-c_2}}$	$(Q+\eta)P_\xi+(P-\xi)P_\eta+R_\eta+Q-c_2\eta=0,$ $(Q+\eta)Q_\xi+(P-\xi)Q_\eta+R_\xi-P-c_2\xi=0,$ $(QR)_\xi+(PR)_\eta-\xi R_\eta+\eta R_\xi$ $+\frac{1}{3}(P_{\eta\eta\eta}+P_{\xi\xi\eta}+Q_{\xi\xi\xi}+Q_{\xi\eta\eta})=0$
(vii) $\sqrt{c_3}v_1+v_3+v_6+v_8$ $(c_3>0)$	$\xi=\frac{2y+2xt-\sqrt{c_3}t^2}{2(1+t^2)},\quad R=w(1+t^2),$ $\eta=\frac{\sqrt{c_3}}{2}\arctan(t)+\frac{-2x+2yt+\sqrt{c_3}t}{2(1+t^2)},$ $P=-u+vt+\frac{y+2xt-(y+\sqrt{c_3})t^2}{\sqrt{1+t^2}},$ $Q=v+ut+\frac{\sqrt{c_3}}{2}\arctan(t)-\frac{(2y+2xt+\sqrt{c_3})t}{\sqrt{2(1+t^2)}}$	$(Q-\eta)Q_\xi+(P+\sqrt{c_3})Q_\eta+R_\xi+P=0,$ $(Q-\eta)P_\xi+(P+\sqrt{c_3})P_\eta+R_\eta-2(Q-\eta)=0,$ $(QR)_\xi+(PR)_\eta-\eta R_\xi+\sqrt{c_3}R_\eta$ $+\frac{1}{3}(P_{\eta\eta\eta}+P_{\xi\xi\eta}+Q_{\xi\xi\xi}+Q_{\xi\eta\eta})=0$
(viii) $\sqrt{-c_3}v_1+v_3-v_6+v_8$ $(c_3<0)$	$\xi=\frac{2y-2xt+\sqrt{-c_3}t^2}{2(1+t^2)},\quad R=w(1+t^2),$ $\eta=\frac{\sqrt{-c_3}}{2}\arctan(t)+\frac{2x+2yt-\sqrt{-c_3}t}{2(1+t^2)},$ $P=u+vt+\frac{y-2xt-(y-\sqrt{-c_3})t^2}{\sqrt{1+t^2}},$ $Q=v-ut+\frac{\sqrt{c_3}}{2}\arctan(t)-\frac{(2y+2xt+\sqrt{c_3})t}{\sqrt{2(1+t^2)}}$	$(Q-\eta)Q_\xi+(P-\sqrt{-c_3})Q_\eta+R_\xi+P=0,$ $(Q-\eta)P_\xi+(P-\sqrt{-c_3})P_\eta+R_\eta-2(Q-\eta)=0,$ $(QR)_\xi+(PR)_\eta-\eta R_\xi-\sqrt{-c_3}R_\eta$ $+\frac{1}{3}(P_{\eta\eta\eta}+P_{\xi\xi\eta}+Q_{\xi\xi\xi}+Q_{\xi\eta\eta})=0$
(ix) v_7	$\xi=\frac{x}{\sqrt{t}},\quad \eta=\frac{y}{\sqrt{t}},\quad P=u\sqrt{t},$ $Q=v\sqrt{t},\quad R=wt$	$(\xi-2P)P_\xi+(\eta-2Q)P_\eta+P-2R_\eta=0,$ $(\xi-2P)Q_\xi+(\eta-2Q)Q_\eta+Q-2R_\eta=0,$ $(P_\xi+Q_\eta-1)R-\frac{1}{2}[(\eta-2Q)R_\eta+(\xi-2P)R_\xi]$ $+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}+Q_{\eta\eta\eta}+Q_{\xi\xi\eta})=0$
(x) v_3+v_8	$\xi=\frac{x}{\sqrt{t^2+1}},\quad \eta=\frac{y}{\sqrt{t^2+1}},\quad R=w(t+1)^2,$ $P=\frac{u+ut^2-xt}{\sqrt{t^2+1}},\quad Q=\frac{v+vt^2-yt}{\sqrt{t^2+1}}$	$\xi+R_\xi+PP_\xi+QP_\eta=0,\quad \eta+R_\eta+PQ_\xi+QQ_\eta=0,$ $(P-\eta)R_\xi+(Q+\xi)R_\eta+(P_\xi+Q_\eta)R$ $+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}+Q_{\eta\eta\eta}+Q_{\xi\xi\eta})=0$
(xi) v_1+v_5	$\xi=t,\quad \eta=y-tx,$ $P=u,\quad Q=v-x,\quad R=w$	$(1+Q-\xi P)P_\xi-\xi R_\xi=0,\quad (1+Q-\xi P)Q_\xi+R_\xi+P=0,$ $(Q_\eta-\xi P_\eta)R+(Q-\xi P)R_\eta+R_\xi-\frac{\xi^2+1}{3}(\xi P_{\eta\eta\eta}+Q_{\eta\eta\eta})=0$
(xii) v_1-v_5	$\xi=t,\quad \eta=y+tx,$ $P=u,\quad Q=v+x,\quad R=w$	$P_\xi+(\xi P+Q)P_\eta+\xi R_\eta=0,\quad Q_\xi+(\xi P+Q)Q_\eta+R_\eta-P=0,$ $(Q_\eta+\xi P_\eta)R+(Q+\xi P)R_\eta+R_\xi+\frac{\xi^2+1}{3}(\xi P_{\eta\eta\eta}+Q_{\eta\eta\eta})=0$
(xiii) v_1	$\xi=y,\quad \eta=t,$ $P=u,\quad Q=v,\quad R=w$	$P_\eta+QP_\xi=0,\quad Q_\eta+QQ_\xi+R_\xi=0,$ $R_\eta+(QR)_\xi+\frac{1}{3}Q_{\xi\xi\xi}=0$
(xiiii) $v_1+v_3+v_7+v_8$	$\xi=\frac{1+2x+2xt}{2(t+1)^2},\ \eta=-\frac{y}{t+1},\ R=w(t+1)^2,$ $P=x-u(t+1)+\frac{1}{t+1},\quad Q=y-v(t+1)$	$P_\xi+QQ_\xi-RQ_\eta-1=0,\quad QR_\xi-RR_\eta-P_\eta=0,$ $(PR)_\xi-(QR)_\eta+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}-Q_{\eta\eta\eta}-Q_{\xi\xi\eta})=0$
(xv) v_3	$\xi=x,\ \eta=y,$ $P=u,\quad Q=v,\quad R=w$	$PP_\xi+QP_\eta+R_\xi=0,\ PQ_\xi+QQ_\eta+R_\eta=0,$ $(PR)_\xi+(QR)_\eta+\frac{1}{3}(P_{\xi\xi\xi}+P_{\xi\eta\eta}+Q_{\eta\eta\eta}+Q_{\xi\xi\eta})=0$

Note that the last equation (43) is nothing but Painlevé II equation.

Another 12 variable coefficient reduction systems may have many interesting properties, which we do not discuss here.

5 Conclusion and Discussion

On the basis of the general and systematic approach which is based on commutator relations, adjoint matrix and the invariants, we construct the optimal system of one-dimensional subalgebras of the full symmetry algebra for (2+1)-dimensional WZ equation. This optimal system consists of 15 inequivalent subalgebras. Based on the optimal system, the complete and inequivalent symmetry reduction systems are presented in the form of table, which consist of 3 constant coefficient reduction systems and 12 variable coefficient equations. Apart from the classic Boussinesq equation reduced from v_1 and the steady case of the WZ equation reduced from v_3 have been derived in Ref. [9], a new Painlevé integrable equation with constant coefficient reduced from $v_1+v_3+v_7+v_8$ are obtained, which can be reduced to Painlevé II equation by means of the traveling wave reduction.

In the future work, we will consider the following two aspects:

(i) For (2+1)-dimensional PDEs, two-dimensional optimalizations exist. The construction of two-dimensional optimal systems of the (2+1)-dimensional WZ equation should be discussed. Such optimal systems can directly reduce the original equation to the ordinary differential equation.

(ii) Many (2+1)-dimensional PDEs have symmetries with arbitrary functions, i.e. their symmetries form infinite-dimensional Lie algebra. We will discuss the optimal systems of the infinite-dimensional algebra in the next work.

Appendix: The Adjoint Representation

With the help of the adjoint representation Table 2, applying the adjoint action of v_1 to

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7 + a_8v_8\,, \tag{44}$$

we have

$$\begin{aligned}&\mathrm{Ad}_{\mathrm{e}^{\varepsilon_1 v_1}}(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7 + a_8v_8)\\&= (a_1-\varepsilon_1a_7)v_1 + (a_2+\varepsilon_1a_6)v_2 + a_3v_3 + (a_4-\varepsilon_1a_8)v_4 + a_5v_5 + a_6v_6 + a_7v_7 + a_8v_8\\&= (a_1, a_2, \ldots, a_8)\cdot A_1\cdot(v_1, v_2, \ldots, v_8)^{\mathrm{T}}\,.\end{aligned} \tag{45}$$

According to Eq. (10), it is easy to obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & \varepsilon_1 & 0 & 0 & 0 & 1 & 0 & 0\\ -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\varepsilon_1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{46}$$

Similarly, the other matrices of the separate adjoint actions of the vectors $v_2, \ldots, v_8$ are found as follows:

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ -\varepsilon_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & -\varepsilon_2 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -\varepsilon_2 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ -\varepsilon_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & -\varepsilon_3 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -2\varepsilon_3 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & \varepsilon_3^2 & 0 & 0 & 0 & -\varepsilon_3 & 1 \end{pmatrix}, \tag{47}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & \varepsilon_4 & 0 & 0 & 0 & 1 & 0 & 0\\ -\varepsilon_4 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\varepsilon_4 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \varepsilon_5 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\varepsilon_5 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & \varepsilon_5 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{48}$$

$$A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \varepsilon_7 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\varepsilon_7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon_7 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 1 & 0 & 0 & \varepsilon_8 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \varepsilon_8 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \varepsilon_8 & \varepsilon_8^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\varepsilon_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{49}$$

$$A_6 = \begin{pmatrix} \cos(\varepsilon_6) & -\sin(\varepsilon_6) & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin(\varepsilon_6) & \cos(\varepsilon_6) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\varepsilon_6) & -\sin(\varepsilon_6) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(\varepsilon_6) & \cos(\varepsilon_6) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

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