Construction of Soliton-Cnoidal Wave Interaction Solution for the (2+1)-Dimensional Breaking Soliton Equation

2015 Commun. Theor. Phys. 63 549

(http://iopscience.iop.org/0253-6102/63/5/549)

Construction of Soliton-Cnoidal Wave Interaction Solution for the (2+1)-Dimensional Breaking Soliton Equation*

CHENG Wen-Guang (程文广),[1] LI Biao (李彪),[1,†] and CHEN Yong (陈勇)[2]

[1]Ningbo Collabrative Innovation Center of Nonlinear Harzard System of Ocean and Atmosphere, Ningbo university, Ningbo 315211, China

[2]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

(Received December 25, 2014; revised manuscript received February 13, 2015)

Abstract *In this paper, the truncated Painlevé analysis and the consistent tanh expansion (CTE) method are developed for the (2+1)-dimensional breaking soliton equation. As a result, the soliton-cnoidal wave interaction solution of the equation is explicitly given, which is difficult to be found by other traditional methods. When the value of the Jacobi elliptic function modulus $m = 1$, the soliton-cnoidal wave interaction solution reduces back to the two-soliton solution. The method can also be extended to other types of nonlinear evolution equations in mathematical physics.*

PACS numbers: 02.30.Ik, 05.45.Yv

Key words: (2+1)-dimensional breaking soliton equation, soliton-cnoidal wave interaction solution, CTE method, truncated Painlevé analysis

1 Introduction

The investigation of exact solutions for nonlinear evolution equations (NLEEs) arising from many science fields is of an important significance. The exact solutions of NLEEs can be constructed by many powerful methods, such as the inverse scattering transformation,[1] the Darboux transformations (DT),[2] the Bäcklund transformation (BT),[3] Hirota's bilinear method,[4] Lie group method,[5–6] Painlevé analysis[7] and various function expansion methods,[8–15] etc. Many kinds of nonlinear waves such as the solitons, cnoidal periodic waves, Painlevé waves are found by various effective methods. However, except for the interactions among solitons, finding the interactions among these nonlinear waves is very difficult. Recently, according to the result of the symmetry reductions with nonlocal symmetries,[16–18] Lou *et al.*[19–24] proposed the consistent tanh expansion (CTE) method, which is a simple but effective method to look for interaction solutions between solitons and other types of nonlinear excitations and possible new physical properties.

We consider the (2+1)-dimensional breaking soliton equation[25]

$$v_t + v_{xxy} - 4vv_y - 2v_x\partial_x^{-1}v_y = 0\,, \tag{1}$$

with $\partial_x^{-1} = \int \cdot \mathrm{d}x$. Setting $v = u_x$, Eq. (1) becomes

$$u_{xt} + u_{xxxy} - 4u_xu_{xy} - 2u_{xx}u_y = 0\,, \tag{2}$$

which decribes the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis.[26] For $y = x$, Eq. (1) is reduced to the KdV equation. The Painlevé property, the Lax pair, the Hamiltonian structure, and various exact solutions have been studied.[27–36] In this paper, the analytic interaction solution between the soliton and the cnoidal periodic wave for the (2+1)-dimensional breaking soliton equation is shown by means of the truncated Painlevé analysis and CTE method.

The paper is arranged as follows. In Sec. 2, for the (2+1)-dimensional breaking equation, we derive explicit interaction solution between the soliton and the cnoidal periodic wave with the help of the truncated Painlevé analysis and the CTE method. In the last section, some conclusions and discussions are given.

2 Soliton-Cnoidal Wave Interaction Solution for the (2+1)-Dimensional Breaking Soliton Equation

It is well known that painlevé test is a systematic method to identify the integrability of NLEEs. Moreover, the painlevé test can be used to solve special solutions for NLEEs. Balancing the nonlinear and dispersive terms in Eq. (2), we have the truncated Painlevé expansion in the form

$$u = \frac{u_0}{\phi} + u_1\,, \tag{3}$$

where u_0, u_1 and ϕ are arbitrary functions with respect to x, y, and t. Substituting Eq. (3) into Eq. (2) and vanishing coefficients of the different powers $1/\phi$, we obtain

$$u_0 = -2\phi_x\,, \quad u_1 = \frac{1}{2}\frac{\phi_{xx}}{\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\,\mathrm{d}x\,, \tag{4}$$

which yields the solution of Eq. (2) as follows:

$$u = -\frac{2\phi_x}{\phi} + \frac{1}{2}\frac{\phi_{xx}}{\phi_x} + \frac{1}{4}\int \frac{\phi_{xx}^2}{\phi_x^2}\,\mathrm{d}x\,, \tag{5}$$

*Supported by National Natural Science Foundation of China under Grant Nos. 11271211, 11275072, 11435005, and K.C. Wong Magna Fund in Ningbo University

†Corresponding author, E-mail: libiao@nbu.edu.cn

with the associated compatibility condition of ϕ

$$\phi_{xt}+\phi_{xxxy}-\frac{3\phi_{xx}\phi_{xxy}}{\phi_x}-\frac{\phi_{xx}\phi_t}{\phi_x}-\frac{\phi_{xy}\phi_{xxx}}{\phi_x}+\frac{3\phi_{xy}\phi_{xx}^2}{\phi_x^2}=0\,. \tag{6}$$

Therefore,

$$v=u_x=\frac{2\phi_x^2}{\phi^2}-\frac{2\phi_{xx}}{\phi}+\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}-\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2} \tag{7}$$

is a solution of Eq. (1).

Compatibility condition (6) can be written as Schwarzian form of Eq. (2)

$$C_x+KS_x+2SK_x+K_{xxx}=0\,, \tag{8}$$

where

$$C=\frac{\phi_t}{\phi_x}\,,\quad K=\frac{\phi_y}{\phi_x}\,,\quad S=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}\,. \tag{9}$$

The above Schwartzian equation (8) is consistent with the result of Ref. [37]. Due to all the quantities C, K, and S are Möbious transformation

$$\phi\to\frac{a+b\phi}{c+d\phi}\,,\quad (ad\neq bc)\,, \tag{10}$$

invariants, the Schwarzian equation (8) is invariant under the Möbious transformation (10).

Using the following straightening transformation

$$\phi=\frac{2}{\tanh(w)-1}\,, \tag{11}$$

the solution (7) can be rewritten as

$$v=2w_x^2\tanh^2(w)-2w_{xx}\tanh(w)-w_x^2+\frac{1}{2}\frac{w_{xxx}}{w_x}-\frac{1}{4}\frac{w_{xx}^2}{w_x^2}\,, \tag{12}$$

with the equivalent compatibility condition of w

$$\mathcal{C}_x+\mathcal{K}\mathcal{S}_x+2\mathcal{S}\mathcal{K}_x+\mathcal{K}_{xxx}-4w_xw_{xy}=0\,, \tag{13}$$

where

$$\mathcal{C}=\frac{w_t}{w_x}\,,\quad \mathcal{K}=\frac{w_y}{w_x}\,,\quad \mathcal{S}=\frac{w_{xxx}}{w_x}-\frac{3}{2}\frac{w_{xx}^2}{w_x^2}\,. \tag{14}$$

It is seen that solution (12) can be considered as the generalization of the usual tanh function expansion method. Here we can obtain the solution (12) via the CTE method.[19–24] By using the leading order analysis for Eq. (2), we may take the following generalized truncated tanh function expansion

$$u=u_0+u_1\tanh(w)\,, \tag{15}$$

where u_0, u_1, and w are functions of (x,y,t) to be determined later. Substituting Eq. (15) into Eq. (2) and setting zero the coefficients of $\tanh^i(w)$, we have six over-determined equations for only three undetermined functions. It is fortunate to find that these over-determined equations are consistent and possess the following solution

$$u_1=-2w_x\,,\quad u_0=\frac{1}{2}\frac{w_{xx}}{w_x}+\int w_x^2\mathrm{d}x+\frac{1}{4}\int\frac{w_{xx}^2}{w_x^2}\mathrm{d}x\,, \tag{16}$$

then we deduce the same solution (12) of Eq. (1) with the compatibility condition (13).

Due to the fact that the single soliton solution of the Eq. (1) is only the straightened solution $w=k_0x+l_0y+\omega_0t$ of Eq. (13), the interaction solutions between solitons and other nonlinear excitations of Eq. (1) can be constructed by solving Eq. (13). In order to obtain the interaction solution of Eq. (1), we consider w in the form

$$w=k_0x+l_0y+\omega_0t+g\,, \tag{17}$$

where g is a function of x, y and t. In this study, we only discuss the solutions with the form

$$w=k_0x+l_0y+\omega_0t+W(X)\,,\quad X=k_1x+l_1y+\omega_1t\,. \tag{18}$$

Substituting Eq. (18) into Eq. (13), we can find that $W_1(X)$ satisfies

$$W_1(X)_X^2=4W_1(X)^4+2a_1W_1(X)^3+2a_2W_1(X)^2+a_3W_1(X)+a_4\,, \tag{19}$$

with

$$\begin{aligned}
&W(X)_X=W_1(X)\,,\\
&a_1=\frac{4k_0-C_1k_1^3}{k_1}\,,\\
&a_2=\frac{2k_0^2-3C_1k_0k_1^3+C_2k_1^3}{k_1^2}\,,\\
&a_3=\frac{k_1\omega_0-k_0\omega_1-6C_1k_0^2k_1^3l_1+4C_2k_0k_1^3l_1}{k_1^3l_1}\,,\\
&a_4=\frac{k_0(k_1\omega_0-k_0\omega_1-2C_1k_0^2k_1^3l_1+2C_2k_0k_1^3l_1)}{k_1^4l_1}\,,
\end{aligned} \tag{20}$$

and C_1, C_2 are arbitrary constants.

It is known that the general solution of Eq. (19) can be written out in terms of Jacobi elliptic functions. To show more clearly of this kind of solution, two special cases are listed.

Case 1 A special solution of Eq. (19) reads

$$W(X)=cE_\pi(\mathrm{sn}(X,m),n,m)\,, \tag{21}$$

which leads to the soliton-cnoidal wave interaction solution of Eq. (1):

$$\begin{aligned}
v&=\Big[2k_0^2-\frac{4ck_0k_1}{nS^2-1}+\frac{2c^2k_1^2}{(nS^2-1)^2}\Big]\tanh^2(k_0x+l_0y+\omega_0t+cE_\pi(S,n,m))\\
&\quad-\frac{4ck_1^2nSCD}{(nS^2-1)^2}\tanh(k_0x+l_0y+\omega_0t+cE_\pi(S,n,m))-\frac{[ck_1-k_0(nS^2-1)]^2}{(nS^2-1)^2}\\
&\quad-\frac{ck_1^3nD^2\{k_0[n^2S^4(2C^2+1)-2nS^2-2C^2+1]+ck_1[-nS^2(C^2+1)-2C^2+1]\}}{[ck_1-k_0(nS^2-1)]^2(nS^2-1)^2}\\
&\quad+\frac{ck_1^3nm^2S^2C^2}{(nS^2-1)[ck_1-k_0(nS^2-1)]}\,,
\end{aligned} \tag{22}$$

where $\{k_1, l_0, l_1, \omega_1, m, n\}$ are arbitrary constants, $S \equiv \mathrm{sn}(k_1x + l_1y + \omega_1t, m)$, $C \equiv \mathrm{cn}(k_1x + l_1y + \omega_1t, m)$, $D \equiv \mathrm{dn}(k_1x + l_1y + \omega_1t, m)$, and

$$c^2 = -\frac{(n-1)(m^2-n)}{n}\,, \quad k_0 = -ck_1\,, \quad \omega_0 = 4ck_1^2l_1n - c\omega_1\,. \tag{23}$$

In solution (22), $E_\pi(\zeta, n, m)$ is the third type of incomplete elliptic integral.

The solution given in Eq. (22) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave. The simulation of soliton-cnoidal wave solution (22) is illustrated in Figs. 1 and 2 at two different choices of the arbitrary parameters. In Fig. 1, we plot the interaction solution between the solitary wave and the cnoidal wave when the value of the Jacobi elliptic function modulus $m \neq 1$. We can see that a soliton propagates on a cnoidal wave background instead of on the plane continuous wave background. This kind of solution can be easily applicable to the analysis of physically interesting processes. If setting the modulus $m = 1$, the soliton-cnoidal wave interaction solution reduces back to the two-soliton solution, whose interaction behaviors are displayed in Fig. 2.

Case 2 Another special solution of Eq. (19) is given as

$$W(X) = A\mathrm{arctanh}(\mathrm{sn}(X, m))\,, \tag{24}$$

which yields the soliton-cnoidal wave interaction solution of Eq. (1):

$$\begin{aligned} v = {} & \frac{k_1^2[S^4(m^4+6m^2+1) - 4S^2(m^2+1)(CD+2) + 8(CD+1)]}{2(1-S^2+CD)^2}\tanh^2\Big(k_0x + l_0y + \omega_0t + \frac{1}{2}\mathrm{arctanh}(S)\Big) \\ & + \frac{k_1^2\big[S^3(1-m^4) + 2S(m^2-1)(CD+1)\big]}{(1-S^2+CD)^2}\tanh\Big(k_0x + l_0y + \omega_0t + \frac{1}{2}\mathrm{arctanh}(S)\Big) \\ & - \frac{k_1^2[S^4(1+8m^2-m^4) - S^2(m^2+3)(2CD+m^2+3) + 2(m^2+3)(CD+1)]}{4(1-S^2+CD)^2}\,, \end{aligned} \tag{25}$$

where $\{k_1, l_0, l_1, \omega_1, m\}$ are arbitrary constants, $S \equiv \mathrm{sn}(k_1x+l_1y+\omega_1t, m)$, $C \equiv \mathrm{cn}(k_1x+l_1y+\omega_1t, m)$, $D \equiv \mathrm{dn}(k_1x+l_1y+\omega_1t, m)$, and

$$A = \frac{1}{2}, \quad k_0 = \frac{k_1}{2}, \quad \omega_0 = k_1^2l_1(m^2-1) + \frac{\omega_1}{2}\,. \tag{26}$$

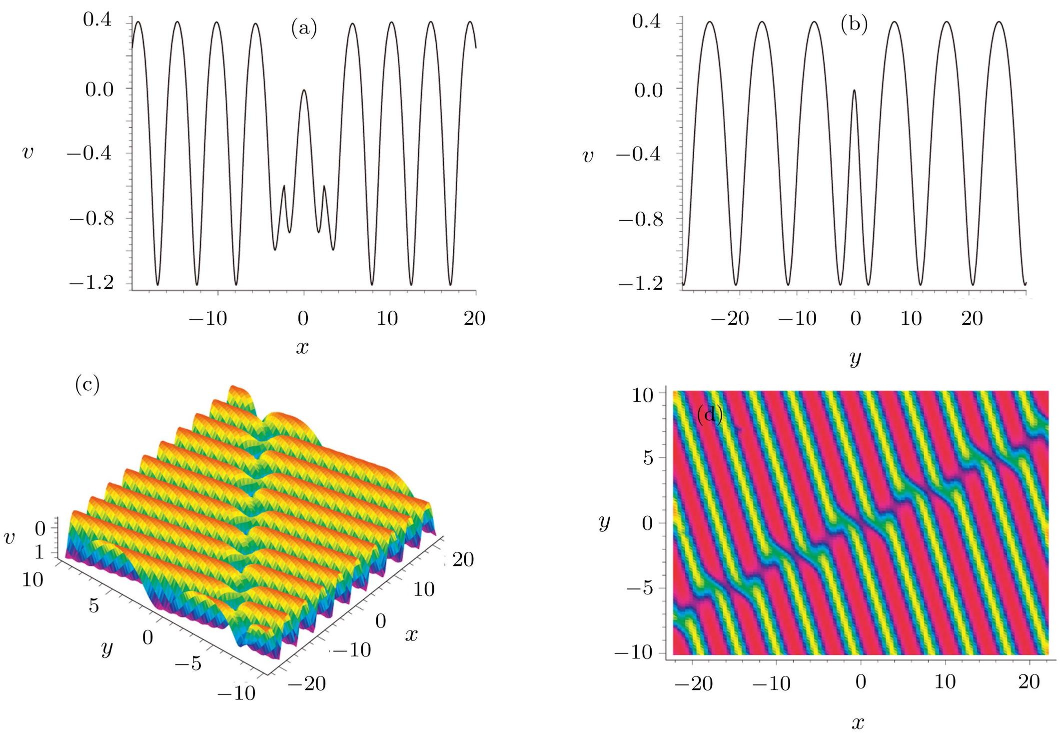

Fig. 1 Soliton-cnoidal wave interaction solution for Eq. (1) given by Eq. (22), with the parameters $k_1 = 1$, $l_0 = 1.2$, $l_1 = 0.5$, $\omega_1 = 0.8$, $m = 0.9$, and $n = 0.5$. (a) One-dimensional image at $t = 0$ and $y = 0$; (b) One-dimensional image at $t = 0$ and $x = 0$; (c) The two-dimensional perspective view of the corresponding solution.

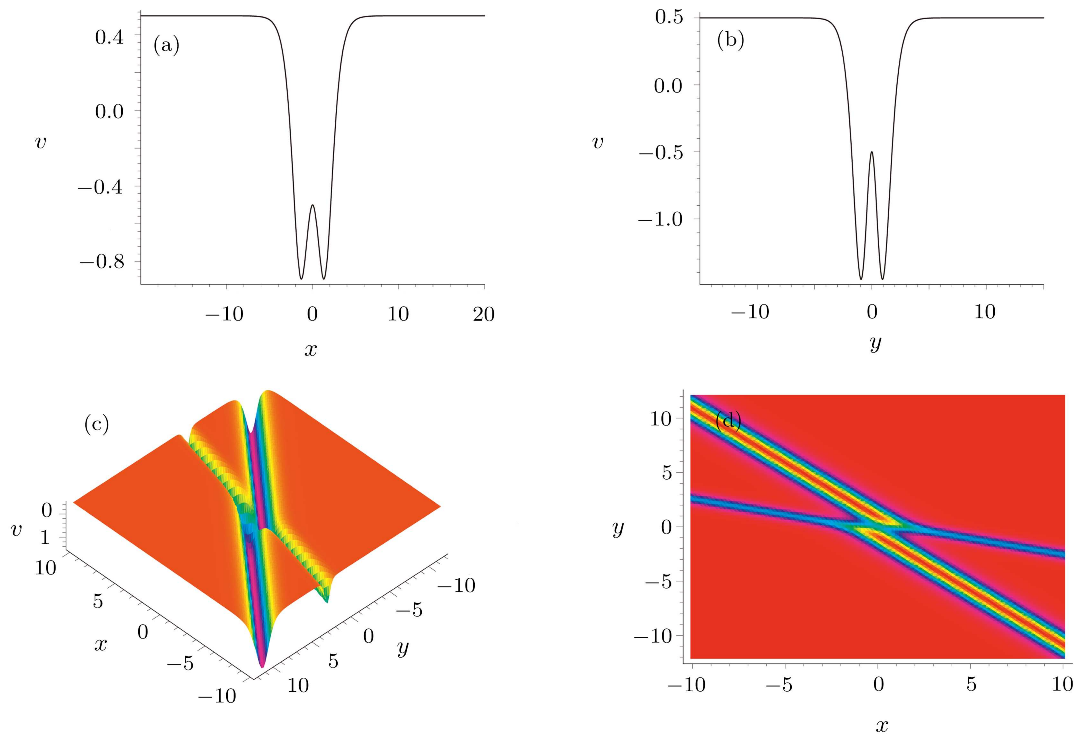

Fig. 2 Two-soliton solution for Eq. (1) given by Eq. (22), with the parameters $k_1 = l_0 = l_1 = \omega_1 = 1$, $m = 1$, and $n = 0.5$. (a) One-dimensional image at $t = 0$ and $y = 0$; (b) One-dimensional image at $t = 0$ and $x = 0$; (c) The two-dimensional perspective view of the corresponding solution.

3 Summary and Discussion

In conclusion, the (2+1)-dimensional breaking soliton equation is investigated by using the truncated Painlevé analysis and the CTE method to find the soliton-cnoidal wave solution. This kind of solution can be easily applicable to the analysis of physically interesting processes. Despite the simplicity of the CTE method, it did provide us with the result which is quite nontrivial and difficult to be obtained by other traditional approaches.

The method presented here could be applied to other kinds of integrable models, especially for supersymmetric models and discrete ones. The details on the CTE method and other methods to solve interaction solutions among different kinds of nonlinear waves will be discussed in our future research work.

References

[1] C.S. Gardner, J.M. Green, M.D. Kruskal, and R.M. Miura, Phys. Rev. Lett. **19** (1967) 1095.

[2] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).

[3] R.M. Miura, *Bäcklund Transformation*, Springer-Verlag, Berlin (1978).

[4] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.

[5] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, Berlin (1986).

[6] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York (1989).

[7] J. Weiss, M. Taboe, and G. Carnevale, J. Math. Phys. **24** (1983) 522.

[8] Z.Y. Yan and H.Q. Zhang, Phys. Lett. A **252** (1999) 291.

[9] E.G. Fan, Phys. Lett. A **277** (2000) 212.

[10] S.K. Liu, Z.T. Fu, S.D Liu, and Q. Zhao, Phys. Lett. A **289** (2001) 69.

[11] B. Li, Y. Chen, and H.Q. Zhang, J. Phys. A: Math. Gen. **35** (2002) 8253.

[12] A.M. Wazwaz, Appl. Math. Comput. **154** (2004) 713.

[13] Y. Chen, Q. Wang, and B. Li, Chaos, Solitons & Fractals **26** (2005) 231.

[14] J.H. He and X.H. Wu, Chaos, Solitons & Fractals **30** (2006) 700.

[15] A. Bekir and A.C. Cevikel, Chaos, Solitons & Fractals **41** (2009) 1733.

[16] X.R. Hu, S.Y. Lou, and Y. Chen, Phys. Rev. E **85** (2012) 056607.

[17] S.Y. Lou, X.R. Hu, and Y. Chen, J. Phys. A: Math. Theor. **45** (2012) 155209.

[18] X.N. Gao, S.Y. Lou, and X.Y. Tang, J. High Energy Phys. **05** (2013) 029.

[19] S.Y. Lou, Stud. Appl. Math. **134** (2015) 372.

[20] Y. Lei and S.Y. Lou, Chin. Phys. Lett. **30** (2013) 060202.

[21] D. Yang, S.Y. Lou, and W.F. Yu, Commun. Theor. Phys. **60** (2013) 387.

[22] C.L. Chen and S.Y. Lou, Chin. Phys. Lett. **30** (2013) 110202.

[23] C.L. Chen and S.Y. Lou, Commun. Theor. Phys. **61** (2014) 545.

[24] S.Y. Lou, X.P. Cheng, and X.Y. Tang, Chin. Phys. Lett. **31** (2014) 070201.

[25] F. Calogero and A. Degnsperis, Nuovo Cimento B **32** (1976) 201.

[26] O.I. Bogoyavlenskii, Usp. Mat. Nauk. **45** (1990) 17.

[27] R. Radha and M. Lakshmanan, Phys. Lett. A **197** (1995) 7.

[28] T. Alagesan, Y. Chung, and K. Nakkeeran, Chaos, Solitons & Fractals **26** (2005) 1203.

[29] S.Y. Lou, X.Y. Tang, Q.P. Liu, and T. Fukuyama, Z. Naturforsch A **57** (2002) 737.

[30] Y.S. Li, Theor. Math. Phys. **99** (1994) 710.

[31] Z.Y. Yan and H.Q. Zhang, Comput. Math. Appl. **44** (2002) 1439.

[32] B. Li, Y. Chen, H.N. Xuan, and H.Q. Zhang, Chaos, Solitons & Fractals **17** (2003) 885.

[33] X.G. Geng and C.W. Cao, Chaos, Solitons & Fractals **22** (2004) 683.

[34] E.G. Fan, Phys. Rev. E **78** (2008) 036607.

[35] A.M. Wazwaz, Phys. Scr. **81** (2010) 035005.

[36] E.G. Fan and K.W. Chow, J. Math. Phys. **52** (2011) 023504.

[37] K. Toda and S.J. Yu, J. Math. Phys. **41** (2000) 4747.