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An integrable semi-discretization of the coupled Yajima–Oikawa system

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Abstract

An integrable semi-discrete analogue of the one-dimensional coupled Yajima–Oikawa system, which is comprised of multicomponent short waves and one component long wave, is proposed by using a bilinear technique. Based on the reductions of the Bäcklund transformations of the semi-discrete BKP hierarchy, both the bright and dark soliton solutions in terms of pfaffians are constructed.

Keywords: integrable semi-discretization, coupled Yajima–Oikawa system, BKP hierarchy reductions, bright and dark soliton

1. Introduction

It is well known that there are two typical mathematical models in nonlinear waves: one is a long wave (LW) model such as the Korteweg–de Vries equation and the other is a short wave (SW) model such as the nonlinear Schrödinger (NLS) equation. A resonant interaction between LW and SW occurs when the phase speed of the LW is equal to the group velocity of the SW [1]. The Yajima–Oikawa (YO) system, which is also known as the long wave–short wave resonance interaction system

$$\mathrm{i}S_t - S_{xx} - LS = 0, \tag{1}$$

$$L_t = 2(|S|^2)_x, \tag{2}$$

is a fundamental model describing resonant interaction of LWs and SWs [2–8]. The YO system has been derived as a model equation for the interaction of a Langmuir wave with an ion-sound wave in a plasma [2], for the interaction of a long gravity wave and a capillary-gravity wave [3], for the interaction of a long internal wave and a short internal wave [4], for the interaction of a long internal wave and a short surface wave in a two layer fluid [5], and for the interaction of a LW and a SW in nonlinear negative refractive index media [6]. From the mathematical point of view, it is known that the NLS equation and the YO system can be derived from the so-called k-constrained KP hierarchy [9]. Specifically, the NLS equation corresponds to the k-constrained KP hierarchy with $k = 1$, while the YO system is associated with the case of $k = 2$ [10]. Multi-dimensional and multi-component generalizations of the YO system have been investigated and their various interesting solutions have been found [11–20]. Very recently, the rogue wave solutions to the YO system and the two-component YO system have been presented [21–24], and it has attracted much attention of physicists in recent years.

A discrete NLS equation

$$\mathrm{i}q_{n,t} = q_{n+1} - 2q_n + q_{n-1} + \sigma\,|q_n|^2 q_n \tag{3}$$

appears in many physical applications including nonlinear optics [25], molecular biology [26] and condensed matter physics [27]. The discrete NLS equation (3) is not integrable although its continuous limit goes to the integrable NLS equation [28–30]. The discrete NLS equation (3) admits discrete solitary waves (discrete solitons), and they have been observed experimentally in a nonlinear optical array [31, 32]. Understanding properties of discrete solitons in the discrete NLS equation is very important for physical applications, but it is hard to gain a thorough understanding because of the lack of integrability of the discrete NLS equation (3).

The integrable space-discretization of the NLS equation

$$\mathrm{i}q_{n,t} = (1 + \sigma\,|q_n|^2)(q_{n+1} + q_{n-1}) \tag{4}$$

was originally derived by Ablowitz and Ladik [33, 34], and it is often called the Ablowitz–Ladik (AL) lattice. After the discovery of an integrable space-discretization of the NLS equation, many mathematical properties of the AL lattice have been investigated. Similar to the continuous case, the AL lattice admits bright soliton solution for the focusing case ($\sigma = 1$) [35, 36], dark soliton solution for the defocusing case ($\sigma = -1$) [37] and rogue wave solutions [38, 39]. It should be noted that mathematical studies of the AL lattice provide useful insights into the studies of discrete solitons in the non-integrable discrete NLS equation.

The semi-discrete coupled NLS equation

$$\mathrm{i}q_{n,t}^{(1)} = (1 + \sigma_1\,|q_n^{(1)}|^2 + \sigma_2\,|q_n^{(2)}|^2)(q_{n+1}^{(1)} + q_{n-1}^{(1)}), \tag{5}$$

$$\mathrm{i}q_{n,t}^{(2)} = (1 + \sigma_1\,|q_n^{(1)}|^2 + \sigma_2\,|q_n^{(2)}|^2)(q_{n+1}^{(2)} + q_{n-1}^{(2)}), \tag{6}$$

where $\sigma_i = \pm 1 (i = 1, 2)$, is of importance both mathematically and physically. It can be solved by the inverse scattering transform [28, 40–42]. The general multi-soliton solution in terms of pfaffians was found recently in [43], which is of bright type for the focusing-focusing case ($\sigma_1 = \sigma_2 = 1$), is of dark type for the defocusing-defocusing case ($\sigma_1 = \sigma_2 = -1$), and could be of mixed type for the focusing-defocusing case ($\sigma_1 = 1, \sigma_2 = -1$).

In contrast with the studies of the NLS equation, the studies of discrete analogues of the YO system are missing although resonant interactions of LWs and SWs in discrete settings are possible to be realized physically. For instance, the discrete YO system is expected to model the resonant interaction of solitary waves (LW) and discrete breathers (SWs) in lattice models such as the Fermi–Pasta–Ulam lattice. On the other hand, it is important to construct an integrable semi-discrete analogue of the YO system from both mathematical and physical point of view. In this paper, we consider an integrable semi-discrete analogue of the one-dimensional (1D) coupled YO system:

$$\mathrm{i}S_t^{(\mu)} - S_{xx}^{(\mu)} - LS^{(\mu)} = 0, \quad \mu = 1, 2, \cdots, M, \tag{7}$$

$$L_t = 2\sum_{\mu=1}^{M} c^{(\mu)} (|S^{(\mu)}|^2)_x, \tag{8}$$

where $c^{(\mu)}$ are arbitrary real constants, $S^{(\mu)}$ and L indicate the μth SW and LW components, respectively.

The goal of the present paper is to construct an integrable semi-discrete analogue of the coupled YO system via a bilinear approach, and derive both the bright and dark soliton (for the SW components) solutions by using a pfaffian technique and a reduction method. The remainder of this paper is organized as follows. In section 2, we present an integrable semi-discrete version of the coupled YO system. In sections 3 and 4, the bright and dark soliton solutions in terms of pfaffians of the semi-discrete coupled YO system are constructed based on two types of Bäcklund transformation of the BKP hierarchy. Section 5 is concluded by some comments and discussions.

2. Integrable semi-discrete coupled YO system

Through the dependent variable transformation

$$S^{(\mu)} = \frac{G^{(\mu)}}{F}, \quad L = 2(\ln F)_{xx}, \quad \mu = 1, 2, \cdots, M, \tag{9}$$

the 1D coupled YO system (7) and (8) can be cast into the bilinear form

$$(\mathrm{i}D_t - D_x^2) G^{(\mu)} \cdot F = 0, \quad \mu = 1, 2, \cdots, M, \tag{10}$$

$$D_x D_t F \cdot F - 2cF \cdot F = 2\sum_{\mu=1}^{M} c^{(\mu)} G^{(\mu)} \bar{G}^{(\mu)}, \tag{11}$$

where c is an integral constant and $\bar{}$ means complex conjugate. The Hirota's D-operator is defined by

$$D_x^n D_t^m (a \cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x, t) b(x', t') \bigg|_{x=x', t=t'}.$$

By discretizing the spacial part of the above bilinear equations

$$D_x^2 G^{(\mu)} \cdot F \to \frac{1}{\varepsilon^2} (G_{n+1}^{(\mu)} F_{n-1} - 2G_n^{(\mu)} F_n + G_{n-1}^{(\mu)} F_{n+1}), \tag{12}$$

$$F_x \to \frac{1}{\varepsilon} (F_{n+1} - F_n), \tag{13}$$

one can obtain

$$\mathrm{i}D_t G_n^{(\mu)} \cdot F_n - \frac{1}{\varepsilon^2}(G_{n+1}^{(\mu)}F_{n-1} - 2G_n^{(\mu)}F_n + G_{n-1}^{(\mu)}F_{n+1}) = 0, \ \ \mu = 1, 2, \cdots, M, \tag{14}$$

$$\frac{1}{\varepsilon}D_t F_{n+1} \cdot F_n - cF_n \cdot F_n = \sum_{\mu=1}^{M} c^{(\mu)} G_n^{(\mu)} \bar{G}_n^{(\mu)}. \tag{15}$$

Furthermore, we require that the discretized bilinear forms are invariant under the gauge transformation

$$F_n \to F_n \exp(q_0 n), \ \ G_n^{(\mu)} \to G_n^{(\mu)} \exp(q_0 n),$$

then one gets the gauge invariant semi-discrete bilinear YO system

$$\mathrm{i}D_t G_n^{(\mu)} \cdot F_n - \frac{1}{\varepsilon^2}(G_{n+1}^{(\mu)}F_{n-1} - 2G_n^{(\mu)}F_n + G_{n-1}^{(\mu)}F_{n+1}) = 0, \ \ \mu = 1, 2, \cdots, M, \tag{16}$$

$$\frac{1}{\varepsilon}D_t F_{n+1} \cdot F_n - cF_{n+1} \cdot F_n = \sum_{\mu=1}^{M} \frac{c^{(\mu)}}{2}(G_{n+1}^{(\mu)} \bar{G}_n^{(\mu)} + G_n^{(\mu)} \bar{G}_{n+1}^{(\mu)}). \tag{17}$$

Letting

$$S_n^{(\mu)} = \frac{G_n^{(\mu)}}{F_n}, \ \ L_n = \frac{2}{\varepsilon^2}\left(\frac{F_{n+1}F_{n-1}}{F_n^2} - 1\right), \ \ \hat{\hat{L}}_n = 2\ln F_n, \tag{18}$$

the bilinear equations (16) and (17) are transformed into

$$\mathrm{i}\frac{\mathrm{d}S_n^{(\mu)}}{\mathrm{d}t} - \frac{1}{\varepsilon^2}(S_{n+1}^{(\mu)} + S_{n-1}^{(\mu)} - 2S_n^{(\mu)}) - \frac{1}{2}L_n(S_{n+1}^{(\mu)} + S_{n-1}^{(\mu)}) = 0, \ \ \mu = 1, 2, \cdots, M, \tag{19}$$

$$\frac{1}{\varepsilon}(\hat{\hat{L}}_{n+1,t} - \hat{\hat{L}}_{n,t}) - 2c = \sum_{\mu=1}^{M} c^{(\mu)}(S_{n+1}^{(\mu)} \bar{S}_n^{(\mu)} + S_n^{(\mu)} \bar{S}_{n+1}^{(\mu)}). \tag{20}$$

By using the relation $\frac{1}{2}(\hat{\hat{L}}_{n+1} - 2\hat{\hat{L}}_n + \hat{\hat{L}}_{n-1}) = \ln\left(1 + \frac{\varepsilon^2}{2}L_n\right)$, we propose the following discrete system

$$\mathrm{i}\frac{\mathrm{d}S_n^{(\mu)}}{\mathrm{d}t} - \frac{1}{\varepsilon^2}(S_{n+1}^{(\mu)} + S_{n-1}^{(\mu)} - 2S_n^{(\mu)}) - \frac{1}{2}L_n(S_{n+1}^{(\mu)} + S_{n-1}^{(\mu)}) = 0, \ \ \mu = 1, 2, \cdots, M, \tag{21}$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ln\left(1 + \frac{\varepsilon^2}{2}L_n\right)}{\varepsilon^2} = \sum_{\mu=1}^{M} \frac{c^{(\mu)}}{\varepsilon}[(S_{n+1}^{(\mu)} - S_{n-1}^{(\mu)})\bar{S}_n^{(\mu)} + S_n^{(\mu)}(\bar{S}_{n+1}^{(\mu)} - \bar{S}_{n-1}^{(\mu)})], \tag{22}$$

which converges to the coupled YO system (7) and (8) when $\varepsilon \to 0$.

For simplicity, by taking $\varepsilon = 1$ and applying the gauge transformation $S_n^{(\mu)} \to \exp(2\mathrm{i}t)S_n^{(\mu)}$, one can obtain the semi-discrete YO system

$$\mathrm{i}\frac{\mathrm{d}S_n^{(\mu)}}{\mathrm{d}t} = (S_{n+1}^{(\mu)} + S_{n-1}^{(\mu)})\left(1 + \frac{1}{2}L_n\right), \ \ \mu = 1, 2, \cdots, M, \tag{23}$$

$$\hat{\hat{L}}_{n+1,t} - \hat{\hat{L}}_{n,t} - 2c = \sum_{\mu=1}^{M} c^{(\mu)}(S_{n+1}^{(\mu)} \bar{S}_n^{(\mu)} + S_n^{(\mu)} \bar{S}_{n+1}^{(\mu)}), \tag{24}$$

$$\ln\left(1+\frac{L_n}{2}\right)=\frac{1}{2}(\hat{\hat{L}}_{n+1}-2\hat{\hat{L}}_n+\hat{\hat{L}}_{n-1}), \tag{25}$$

or

$$\mathrm{i}\frac{\mathrm{d}S_n^{(\mu)}}{\mathrm{d}t}=(S_{n+1}^{(\mu)}+S_{n-1}^{(\mu)})\left(1+\frac{1}{2}L_n\right),\ \ \mu=1,2,\cdots,M, \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(1+\frac{L_n}{2}\right)=\frac{1}{2}\sum_{\mu=1}^{M}c^{(\mu)}[(S_{n+1}^{(\mu)}-S_{n-1}^{(\mu)})\bar{S}_n^{(\mu)}+S_n^{(\mu)}(\bar{S}_{n+1}^{(\mu)}-\bar{S}_{n-1}^{(\mu)})]. \tag{27}$$

By setting $U_n=\ln\left(1+\frac{L_n}{2}\right)$, the semi-discrete YO system leads to

$$\mathrm{i}\frac{\mathrm{d}S_n^{(\mu)}}{\mathrm{d}t}=(S_{n+1}^{(\mu)}+S_{n-1}^{(\mu)})\exp(U_n),\ \ \mu=1,2,\cdots,M, \tag{28}$$

$$\frac{\mathrm{d}U_n}{\mathrm{d}t}=\frac{1}{2}\sum_{\mu=1}^{M}c^{(\mu)}[(S_{n+1}^{(\mu)}-S_{n-1}^{(\mu)})\bar{S}_n^{(\mu)}+S_n^{(\mu)}(\bar{S}_{n+1}^{(\mu)}-\bar{S}_{n-1}^{(\mu)})]. \tag{29}$$

In the subsequent two sections, we will consider general bright and dark soliton solutions for the semi-discrete coupled YO system (26) and (27) in details. For brevity, we call soliton solutions with bright or dark solitons for the SW components and solitons for the LW component by bright or dark soliton solutions.

3. Bright soliton solution for the semi-discrete coupled YO system

In this section, we construct bright soliton solutions for the semi-discrete coupled YO system (26) and (27). First, we briefly recall Bäcklund transformations of the semi-discrete BKP hierarchy by the following Lemma [43].

Lemma 3.1. *The following bilinear equations*

$$D_t g_n^{(\mu)}\cdot f_n=g_{n+1}^{(\mu)}f_{n-1}-g_{n-1}^{(\mu)}f_{n+1}, \tag{30}$$

$$D_t h_n^{(\mu)}\cdot f_n=h_{n+1}^{(\mu)}f_{n-1}-h_{n-1}^{(\mu)}f_{n+1}, \tag{31}$$

$$D_{y^{(\mu)}}f_{n+1}\cdot f_n=g_{n+1}^{(\mu)}h_n^{(\mu)}-g_n^{(\mu)}h_{n+1}^{(\mu)}, \tag{32}$$

for $\mu=1,\cdots,M$ *are satisfied by the pfaffians*

$$f_n=\mathrm{pf}(a_1,\cdots,a_{2N},c_{2N},\cdots,c_1), \tag{33}$$

$$g_n^{(\mu)}=\mathrm{pf}(d_0,a_1,\cdots,a_{2N},c_{2N},\cdots,c_1,\alpha^{(\mu)}), \tag{34}$$

$$h_n^{(\mu)} = \mathrm{pf}(d_0, a_1, \cdots, a_{2N}, c_{2N}, \cdots, c_1, \beta^{(\mu)}), \tag{35}$$

where the pfaffian elements are defined by

$$\begin{aligned}
&\mathrm{pf}(a_j, a_k) = \frac{p_j - p_k}{p_j p_k - 1}(p_j p_k)^n \exp(\xi_j + \xi_k),\\
&\mathrm{pf}(d_l, a_j) = p_j^{n+l} \exp(\xi_j), \ \ \mathrm{pf}(a_j, c_k) = \delta_{jk},\\
&\mathrm{pf}(d_l, c_j) = \mathrm{pf}(d_l, \alpha^{(\mu)}) = \mathrm{pf}(d_l, \beta^{(\mu)}) = \mathrm{pf}(a_j, \alpha^{(\mu)}) = \mathrm{pf}(a_j, \beta^{(\mu)}) = 0,\\
&\mathrm{pf}(c_j, c_k) = \begin{cases} \displaystyle\sum_{\mu=1}^{M} \frac{\exp(\zeta_j^{(\mu)} + \eta_k^{(\mu)})}{Q_j^{(\mu)} + P_k^{(\mu)}}, & N + 1 \leqslant j \leqslant 2N, \ 1 \leqslant k \leqslant N,\\ 0, & \text{otherwise}, \end{cases}\\
&\mathrm{pf}(c_j, \alpha^{(\mu)}) = \begin{cases} \exp(\eta_j^{(\mu)}), & 1 \leqslant j \leqslant N,\\ 0, & N + 1 \leqslant j \leqslant 2N, \end{cases}\\
&\mathrm{pf}(c_j, \beta^{(\mu)}) = \begin{cases} 0, & 1 \leqslant j \leqslant N,\\ \exp(\zeta_j^{(\mu)}), & N + 1 \leqslant j \leqslant 2N, \end{cases}
\end{aligned}$$

with

$$\xi_j = \left(p_j - \frac{1}{p_j}\right)t, \ \ \zeta_j^{(\mu)} = Q_j^{(\mu)} y^{(\mu)} + \zeta_{j0}^{(\mu)}, \ \ \eta_j^{(\mu)} = P_j^{(\mu)} y^{(\mu)} + \eta_{j0}^{(\mu)}.$$

Here p_j, $Q_j^{(\mu)}$, $P_j^{(\mu)}$, $\zeta_{j0}^{(\mu)}$ *and* $\eta_{j0}^{(\mu)}$ *are arbitrary constants.*

Proof. From the definition of the functions f_n, $g_n^{(\mu)}$ and $h_n^{(\mu)}$, we can derive the following pfaffian's rules:

$$\begin{aligned}
&f_{n+1} = \mathrm{pf}(d_0, d_1, \bullet), \ \ f_{n-1} = \mathrm{pf}(d_0, d_{-1}, \bullet), \ \ \partial_t f_n = \mathrm{pf}(d_{-1}, d_1, \bullet),\\
&\partial_{y^{(\mu)}} f_n = \mathrm{pf}(\bullet, \alpha^{(\mu)}, \beta^{(\mu)}), \ \ \partial_{y^{(\mu)}} f_{n+1} = \mathrm{pf}(d_0, d_1, \bullet, \alpha^{(\mu)}, \beta^{(\mu)}),\\
&g_{n+1}^{(\mu)} = \mathrm{pf}(d_1, \bullet, \alpha^{(\mu)}), \ \ g_{n-1}^{(\mu)} = \mathrm{pf}(d_{-1}, \bullet, \alpha^{(\mu)}), \ \ \partial_t g_n^{(\mu)} = \mathrm{pf}(d_0, d_{-1}, d_1, \bullet, \alpha^{(\mu)}),\\
&h_{n+1}^{(\mu)} = \mathrm{pf}(d_1, \bullet, \beta^{(\mu)}), \ \ h_{n-1}^{(\mu)} = \mathrm{pf}(d_{-1}, \bullet, \beta^{(\mu)}), \ \ \partial_t h_n^{(\mu)} = \mathrm{pf}(d_0, d_{-1}, d_1, \bullet, \beta^{(\mu)}),
\end{aligned}$$

where

$$\mathrm{pf}(d_0, d_1) = \mathrm{pf}(d_0, d_{-1}) = 1, \ \ \mathrm{pf}(d_{-1}, d_1) = \mathrm{pf}(\alpha^{(\mu)}, \beta^{(\mu)}) = 0,$$

and $(\bullet) = (a_1, \cdots, a_{2N}, c_{2N}, \cdots, c_1)$.

Now the algebraic identities of pfaffian

$$\begin{aligned}
&\mathrm{pf}(d_0, d_{-1}, d_1, \bullet, \alpha^{(\mu)})\mathrm{pf}(\bullet) = \mathrm{pf}(d_0, d_{-1}, \bullet)\mathrm{pf}(d_1, \bullet, \alpha^{(\mu)})\\
&\quad - \mathrm{pf}(d_0, d_1, \bullet)\mathrm{pf}(d_{-1}, \bullet, \alpha^{(\mu)}) + \mathrm{pf}(d_0, \bullet, \alpha^{(\mu)})\mathrm{pf}(d_{-1}, d_1, \bullet),\\
&\mathrm{pf}(d_0, d_{-1}, d_1, \bullet, \beta^{(\mu)})\mathrm{pf}(\bullet) = \mathrm{pf}(d_0, d_{-1}, \bullet)\mathrm{pf}(d_1, \bullet, \beta^{(\mu)})\\
&\quad - \mathrm{pf}(d_0, d_1, \bullet)\mathrm{pf}(d_{-1}, \bullet, \beta^{(\mu)}) + \mathrm{pf}(d_0, \bullet, \beta^{(\mu)})\mathrm{pf}(d_{-1}, d_1, \bullet),
\end{aligned}$$

and

$$\mathrm{pf}(d_0, d_1, \bullet, \alpha^{(\mu)}, \beta^{(\mu)})\mathrm{pf}(\bullet) = \mathrm{pf}(d_0, d_1, \bullet)\mathrm{pf}(\bullet, \alpha^{(\mu)}, \beta^{(\mu)}) - \mathrm{pf}(d_0, \bullet, \alpha^{(\mu)})\mathrm{pf}(d_1, \bullet, \beta^{(\mu)}) + \mathrm{pf}(d_1, \bullet, \alpha^{(\mu)})\mathrm{pf}(d_0, \bullet, \beta^{(\mu)}),$$

together with the above pfaffian expressions of τ-functions give the bilinear equations (30)–(32). □

Here we assume $c = 0$ in equation (24), which yields the bright-type soliton solution for the semi-discrete coupled YO system as shown below. Through the dependent variable transformation

$$S_n^{(\mu)} = \mathrm{i}^n \frac{g_n^{(\mu)}}{f_n}, \quad \bar{S}_n^{(\mu)} = (-\mathrm{i})^n \frac{\bar{g}_n^{(\mu)}}{f_n}, \quad L_n = 2\left(\frac{f_{n+1}f_{n-1}}{f_n^2} - 1\right), \quad \hat{\hat{L}} = 2\ln f_n, \tag{36}$$

equations (23)–(25) are cast into

$$D_t g_n^{(\mu)} \cdot f_n = g_{n+1}^{(\mu)} f_{n-1} - g_{n-1}^{(\mu)} f_{n+1}, \tag{37}$$

$$D_t \bar{g}_n^{(\mu)} \cdot f_n = \bar{g}_{n+1}^{(\mu)} f_{n-1} - \bar{g}_{n-1}^{(\mu)} f_{n+1}, \tag{38}$$

$$D_t f_{n+1} \cdot f_n = \sum_{\mu=1}^{M} \mathrm{i}\frac{c^{(\mu)}}{2}(g_{n+1}^{(\mu)}\bar{g}_n^{(\mu)} - g_n^{(\mu)}\bar{g}_{n+1}^{(\mu)}), \tag{39}$$

for $\mu = 1, \cdots, M$.

In order to carry out the reduction, we define

$$\mathrm{pf}(a_j', a_k') = \frac{p_j - p_k}{p_j p_k - 1}(p_j p_k)^n,$$
$$\mathrm{pf}(d_l, a_j') = p_j^{n+l}, \quad \mathrm{pf}(a_j', c_k') = \delta_{jk},$$
$$\mathrm{pf}(d_l, c_j') = \mathrm{pf}(d_l, \alpha^{(\mu)}) = \mathrm{pf}(d_l, \beta^{(\mu)}) = \mathrm{pf}(a_j', \alpha^{(\mu)}) = \mathrm{pf}(a_j', \beta^{(\mu)}) = 0,$$
$$\mathrm{pf}(c_j', c_k') = \begin{cases} \displaystyle\sum_{\mu=1}^{M} \frac{\exp(\zeta_j^{(\mu)} + \eta_k^{(\mu)} + \xi_j + \xi_k)}{Q_j^{(\mu)} + P_k^{(\mu)}}, & N+1 \leqslant j \leqslant 2N,\ 1 \leqslant k \leqslant N, \\ 0, & \text{otherwise,} \end{cases}$$
$$\mathrm{pf}(c_j', \alpha^{(\mu)}) = \begin{cases} \exp(\eta_j^{(\mu)} + \xi_j), & 1 \leqslant j \leqslant N, \\ 0, & N+1 \leqslant j \leqslant 2N, \end{cases}$$
$$\mathrm{pf}(c_j', \beta^{(\mu)}) = \begin{cases} 0, & 1 \leqslant j \leqslant N, \\ \exp(\zeta_j^{(\mu)} + \xi_j), & N+1 \leqslant j \leqslant 2N. \end{cases}$$

Then, the pfaffians f_n, $g_n^{(\mu)}$ and $h_n^{(\mu)}$ in equations (33)–(35) have alternative expressions in pfaffians

$$f_n = \mathrm{pf}(a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1'), \tag{40}$$

$$g_n^{(\mu)} = \mathrm{pf}(d_0, a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1', \alpha^{(\mu)}), \tag{41}$$

$$h_n^{(\mu)} = \mathrm{pf}(d_0, a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1', \beta^{(\mu)}). \tag{42}$$

Therefore, under the reduction conditions

$$P_j^{(\mu)} = \frac{1}{s^{(\mu)}}\left(p_j - \frac{1}{p_j}\right),\ \ 1 \leqslant j \leqslant N,\ \ Q_j^{(\mu)} = \frac{1}{s^{(\mu)}}\left(p_j - \frac{1}{p_j}\right),\ \ N+1 \leqslant j \leqslant 2N, \tag{43}$$

the following relation holds

$$\partial_t(f_n,\, g_n^{(\nu)},\, h_n^{(\nu)}) = \sum_{\mu=1}^{M} s^{(\mu)} \partial_{y^{(\mu)}}(f_n,\, g_n^{(\nu)},\, h_n^{(\nu)}), \tag{44}$$

and thus one can get

$$D_t f_{n+1} \cdot f_n = \sum_{\mu=1}^{M} s^{(\mu)} (g_{n+1}^{(\mu)} h_n^{(\mu)} - g_n^{(\mu)} h_{n+1}^{(\mu)}). \tag{45}$$

Lastly, by imposing the complex conjugate conditions

$$s^{(\mu)} = \mathrm{i}\frac{c^{(\mu)}}{2},\ \ p_{2N+1-j} = \bar{p}_j,\ \ \zeta^{(\mu)}_{p_{2N+1-j,0}} = \bar{\eta}^{(\mu)}_{j,0},\ \ \text{for}\ \ 1 \leqslant j \leqslant N, \tag{46}$$

and requiring $y^{(\mu)}$ being pure imaginary, one can realize $h_n^{(\mu)} = \bar{g}_n^{(\mu)}$. Thus equations (30), (31) and (45) become equations (37)–(39).

In summary, we arrive at the following theorem:

Theorem 3.1. *The bilinear equations (37)–(39) are satisfied by the pfaffians*

$$f_n = \mathrm{pf}(a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1'), \tag{47}$$

$$g_n^{(\mu)} = \mathrm{pf}(d_0, a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1', \alpha^{(\mu)}), \tag{48}$$

$$\bar{g}_n^{(\mu)} = \mathrm{pf}(d_0, a_1', \cdots, a_{2N}', c_{2N}', \cdots, c_1', \beta^{(\mu)}), \tag{49}$$

where the pfaffian elements are defined by

$$\mathrm{pf}(a_j', a_k') = \frac{p_j - p_k}{p_j p_k - 1}(p_j p_k)^n,$$
$$\mathrm{pf}(d_l, a_j') = p_j^{n+l},\ \ \mathrm{pf}(a_j', c_k') = \delta_{jk},$$
$$\mathrm{pf}(d_l, c_j') = \mathrm{pf}(d_l, \alpha^{(\mu)}) = \mathrm{pf}(d_l, \beta^{(\mu)}) = \mathrm{pf}(a_j', \alpha^{(\mu)}) = \mathrm{pf}(a_j', \beta^{(\mu)}) = 0,$$
$$\mathrm{pf}(c_j', c_k') = \begin{cases} \displaystyle\sum_{\mu=1}^{M} \frac{\exp(\zeta_j^{(\mu)} + \eta_k^{(\mu)} + \xi_j + \xi_k)}{Q_j^{(\mu)} + P_k^{(\mu)}}, & N+1 \leqslant j \leqslant 2N,\ 1 \leqslant k \leqslant N, \\ 0, & \text{otherwise}, \end{cases}$$
$$\mathrm{pf}(c_j', \alpha^{(\mu)}) = \begin{cases} \exp(\eta_j^{(\mu)} + \xi_j), & 1 \leqslant j \leqslant N, \\ 0, & N+1 \leqslant j \leqslant 2N, \end{cases}$$
$$\mathrm{pf}(c_j', \beta^{(\mu)}) = \begin{cases} 0, & 1 \leqslant j \leqslant N, \\ \exp(\zeta_j^{(\mu)} + \xi_j), & N+1 \leqslant j \leqslant 2N. \end{cases}$$

with

$$P_j^{(\mu)} = -\mathrm{i}\frac{2}{c^{(\mu)}}\left(p_j - \frac{1}{p_j}\right) \quad \text{for} \quad 1 \leqslant j \leqslant N,$$

$$Q_j^{(\mu)} = -\mathrm{i}\frac{2}{c^{(\mu)}}\left(p_j - \frac{1}{p_j}\right) \quad \text{for} \quad N+1 \leqslant j \leqslant 2N,$$

$$\xi_j = \left(p_j - \frac{1}{p_j}\right)t, \quad \zeta_j^{(\mu)} = \zeta_{j,0}^{(\mu)}, \quad \eta_j^{(\mu)} = \eta_{j,0}^{(\mu)},$$

and p_j, $\zeta_j^{(\mu)} \equiv \zeta_{j,0}^{(\mu)}$ *and* $\eta_j^{(\mu)} \equiv \eta_{j,0}^{(\mu)}$ *are constants satisfying* $p_{2N+1-j} = \bar{p}_j$, $\zeta_{2N+1-j,0}^{(\mu)} = \bar{\eta}_{j,0}^{(\mu)}$ *for* $1 \leqslant j \leqslant N$.

In what follows, we illustrate one and two bright soliton solutions for $M = 2$.

One-soliton solution: By taking $N = 1$ in (47)–(49), we get the τ-functions for the one-soliton solution

$$f_n = 1 + \frac{\mathrm{i}}{2}\frac{(c^{(1)}A_1\bar{A}_1 + c^{(2)}B_1\bar{B}_1)p_1\bar{p}_1}{(p_1\bar{p}_1 - 1)^2}\frac{p_1 - \bar{p}_1}{p_1 + \bar{p}_1}(p_1\bar{p}_1)^n \exp(\xi_1 + \bar{\xi}_1), \tag{50}$$

$$g_n^{(1)} = A_1 p_1^n \exp(\xi_1), \quad g_n^{(2)} = B_1 p_1^n \exp(\xi_1), \tag{51}$$

where $\xi_1 = \left(p_1 - \frac{1}{p_1}\right)t$, $A_1 = \exp[\eta_{1,0}^{(1)}]$ and $B_1 = \exp[\eta_{1,0}^{(2)}]$.

In order to avoid the singularity, the condition $\frac{\mathrm{i}(c^{(1)}A_1\bar{A}_1 + c^{(2)}B_1\bar{B}_1)(p_1 - \bar{p}_1)}{2(p_1 + \bar{p}_1)} > 0$ need to be satisfied. Further, the above τ-functions lead to the one-soliton solution as follows

$$S_n^{(1)} = \frac{A_1}{2}\exp(\mathrm{i}\xi'_{1I} - \theta_0)\mathrm{sech}(\xi'_{1R} + \theta_0), \tag{52}$$

$$S_n^{(2)} = \frac{B_1}{2}\exp(\mathrm{i}\xi'_{1I} - \theta_0)\mathrm{sech}(\xi'_{1R} + \theta_0), \tag{53}$$

$$L_n = \frac{(p_1\bar{p}_1 - 1)^2}{2p_1\bar{p}_1}\mathrm{sech}^2(\xi'_{1R} + \theta_0), \tag{54}$$

where $\xi'_1 = \xi'_{1R} + \mathrm{i}\xi'_{1I} = n\ln(\mathrm{i}p_1) + \xi_1$ and $\exp(2\theta_0) = \frac{\mathrm{i}}{2}\frac{(c^{(1)}A_1\bar{A}_1 + c^{(2)}B_1\bar{B}_1)p_1\bar{p}_1}{(p_1\bar{p}_1 - 1)^2}\frac{p_1 - \bar{p}_1}{p_1 + \bar{p}_1}$. The quantities $\frac{|A|}{2}\exp(-\theta_0)$ and $\frac{|B|}{2}\exp(-\theta_0)$ represent the amplitudes of the bright solitons in the SW components $S_n^{(1)}$ and $S_n^{(2)}$ respectively. The real quantity $\frac{(p_1\bar{p}_1 - 1)^2}{2p_1\bar{p}_1}$ denotes the amplitude of soliton in the LW component. As an example, we illustrate one-soliton in figure 1 for the nonlinearity coefficients $(c^{(1)}, c^{(2)}) = (1, -1)$. The parameters are chosen as $A_1 = 1$, $B_1 = 1 + \mathrm{i}$ and $p_1 = 1 + \mathrm{i}$.

Two-soliton solution: By taking $N = 2$ in (47)–(49), we get the τ-functions for the two-soliton solution

$$\begin{aligned} f_n = {} & 1 + C_{1\bar{1}}E_1\bar{E}_1 + C_{1\bar{2}}E_1\bar{E}_2 + C_{2\bar{1}}E_2\bar{E}_1 + C_{2\bar{2}}E_2\bar{E}_2 \\ & + |P_{12}|^2(P_{1\bar{1}}P_{2\bar{2}}C_{1\bar{2}}C_{2\bar{1}} - P_{1\bar{2}}P_{2\bar{1}}C_{1\bar{1}}C_{2\bar{2}})E_1E_2\bar{E}_1\bar{E}_2, \end{aligned} \tag{55}$$

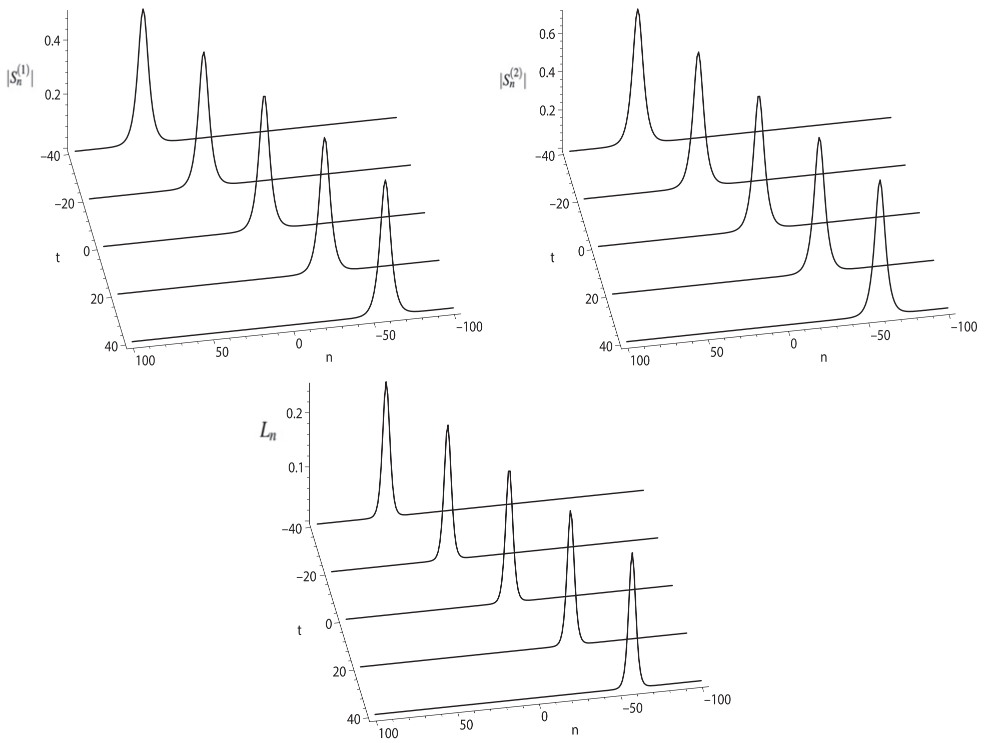

Figure 1. The profiles of evolutions of one-soliton solution (bright soliton for SW components).

$$g_n^{(1)} = A_1 E_1 + A_2 E_2 + P_{12}(A_1 P_{1\bar{1}} C_{2\bar{1}} - A_2 P_{2\bar{1}} C_{1\bar{1}}) E_1 E_2 \bar{E}_1 \\ + P_{12}(A_1 P_{1\bar{2}} C_{2\bar{2}} - A_2 P_{2\bar{2}} C_{1\bar{2}}) E_1 E_2 \bar{E}_2, \tag{56}$$

$$g_n^{(2)} = B_1 E_1 + B_2 E_2 + P_{12}(B_1 P_{1\bar{1}} C_{2\bar{1}} - B_2 P_{2\bar{1}} C_{1\bar{1}}) E_1 E_2 \bar{E}_1 \\ + P_{12}(B_1 P_{1\bar{2}} C_{2\bar{2}} - B_2 P_{2\bar{2}} C_{1\bar{2}}) E_1 E_2 \bar{E}_2, \tag{57}$$

with

$$P_{12} = \frac{p_1 - p_2}{p_1 p_2 - 1}, \quad P_{j\bar{k}} = \frac{p_j - \bar{p}_k}{p_j \bar{p}_k - 1},$$
$$E_j = p_j^n \exp(\xi_j), \quad C_{j\bar{k}} = \frac{\mathrm{i}}{2} \frac{(c^{(1)} A_j \bar{A}_k + c^{(2)} B_j \bar{B}_k) p_j \bar{p}_k}{(p_j \bar{p}_k - 1)^2} \frac{p_j - \bar{p}_k}{p_j + \bar{p}_k},$$

where $\xi_j = \left(p_j - \frac{1}{p_j}\right)t$, $A_j = \exp[\eta_{j,0}^{(1)}]$ and $B_j = \exp[\eta_{j,0}^{(2)}]$ for $j = 1, 2$. Figure 2 shows two-soliton interaction with the parameters as $(c^{(1)}, c^{(2)}) = (1, -1)$, $A_1 = 2$, $A_2 = 5$, $B_1 = 3$, $B_2 = 4$, $p_1 = 1 + \mathrm{i}$ and $p_2 = -\frac{1}{2} + \frac{2}{3}\mathrm{i}$.

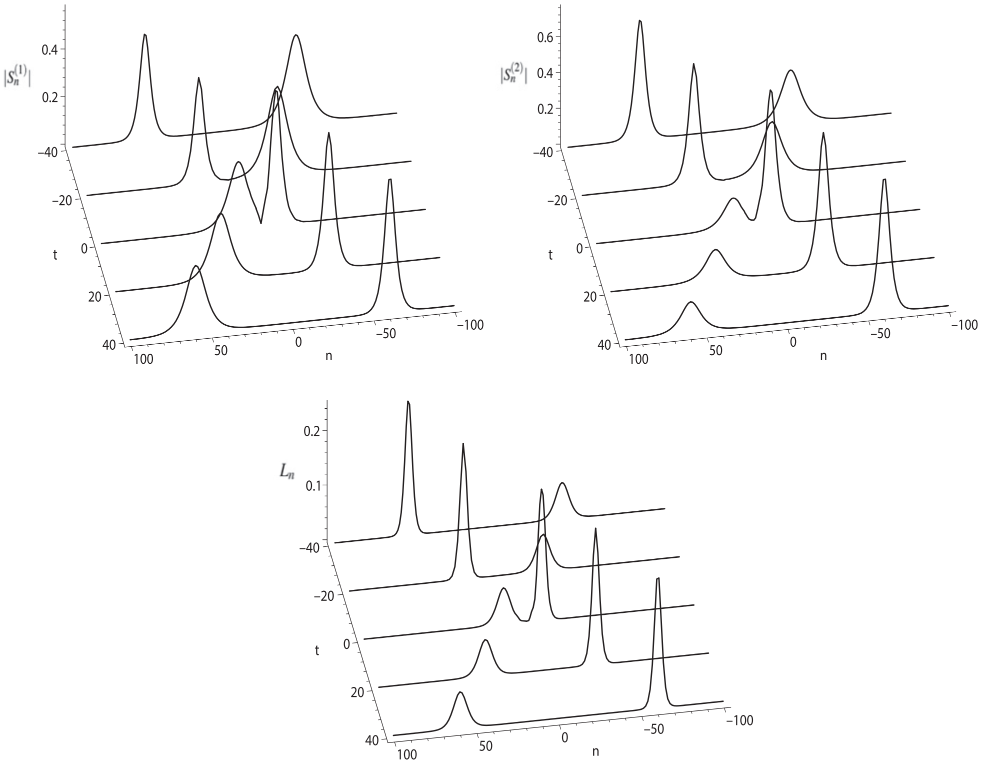

Figure 2. The profiles of evolutions of two-soliton solution (bright soliton for SW components).

4. Dark soliton solution for the semi-discrete coupled YO system

In this section, we consider dark soliton solutions for the semi-discrete coupled YO system (26) and (27). To this end, we need to introduce another set of Bäcklund transformations of the semi-discrete BKP hierarchy by the following lemma [43].

Lemma 4.1. *The following bilinear equations*

$$\left(D_t + \alpha^{(\mu)} - \frac{1}{\alpha^{(\mu)}}\right) g_n^{(\mu)} \cdot f_n = \alpha^{(\mu)} g_{n+1}^{(\mu)} f_{n-1} - \frac{1}{\alpha^{(\mu)}} g_{n-1}^{(\mu)} f_{n+1}, \tag{58}$$

$$\left(D_t + \frac{1}{\alpha^{(\mu)}} - \alpha^{(\mu)}\right) h_n^{(\mu)} \cdot f_n = \frac{1}{\alpha^{(\mu)}} h_{n+1}^{(\mu)} f_{n-1} - \alpha^{(\mu)} h_{n-1}^{(\mu)} f_{n+1}, \tag{59}$$

$$\left(D_{y^{(\mu)}} - \alpha^{(\mu)} + \frac{1}{\alpha^{(\mu)}}\right) f_{n+1} \cdot f_n = -\alpha^{(\mu)} g_{n+1}^{(\mu)} h_n^{(\mu)} + \frac{1}{\alpha^{(\mu)}} g_n^{(\mu)} h_{n+1}^{(\mu)}, \tag{60}$$

for $\mu = 1, \cdots, M$ *are satisfied by the pfaffians*

$$f_n = \tau_n^{0\cdots0}, \quad g_n^{(\mu)} = \tau_n^{0\ldots\overset{\mu}{\breve{1}}\ldots0}, \quad h_n^{(\mu)} = \tau_n^{0\ldots\overset{\mu}{\breve{-1}}\ldots0}, \tag{61}$$

where the τ-function $\tau_n^{l^{(1)}\cdots l^{(M)}}$ is the pfaffian $\tau_n^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(a_1, a_2, \cdots, a_{2N})$, whose elements are defined by

$$\mathrm{pf}(a_j, a_k) = c_{jk} + \frac{p_j - p_k}{p_j p_k - 1}\mathrm{pf}(d_0, a_j)\mathrm{pf}(d_0, a_k),$$

$$\mathrm{pf}(d_l, a_j) = p_j^{n+l}\prod_{\mu=1}^{M}\left(\frac{p_j - \alpha^{(\mu)}}{1 - \alpha^{(\mu)}p_j}\right)^{l^{(\mu)}}\exp(\xi_j),$$

$$\xi_j = \left(p_j - \frac{1}{p_j}\right)t + \sum_{\mu=1}^{M}\left(\frac{p_j - \alpha^{(\mu)}}{1 - \alpha^{(\mu)}p_j} - \frac{1 - \alpha^{(\mu)}p_j}{p_j - \alpha^{(\mu)}}\right)y^{(\mu)} + \xi_{j,0}.$$

Proof. From the definition of the τ-functions, we have the following formulae of pfaffians:

$$\tau_{n+1}^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(d_0, d_1, \bullet),\ \ \tau_{n-1}^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(d_0, d_{-1}, \bullet),$$
$$\partial_t\tau_n^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(d_{-1}, d_1, \bullet),\ \ \partial_{y^{(\mu)}}\tau_n^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(d_{-1}^{(\mu)}, d_1^{(\mu)}, \bullet),$$
$$\left(\partial_{y^{(\mu)}} - \alpha^{(\mu)} + \frac{1}{\alpha^{(\mu)}}\right)\tau_{n+1}^{l^{(1)}\cdots l^{(M)}} = \mathrm{pf}(d_0, d_1, d_{-1}^{(\mu)}, d_1^{(\mu)}, \bullet),$$
$$\tau_n^{l^{(1)}\cdots l^{(\mu)}+1\cdots l^{(M)}} = \mathrm{pf}(d_0, d_1^{(\mu)}, \bullet),$$
$$\alpha^{(\mu)}\tau_{n+1}^{l^{(1)}\cdots l^{(\mu)}+1\cdots l^{(M)}} = \mathrm{pf}(d_1, d_1^{(\mu)}, \bullet),$$
$$\frac{1}{\alpha^{(\mu)}}\tau_{n-1}^{l^{(1)}\cdots l^{(\mu)}+1\cdots l^{(M)}} = \mathrm{pf}(d_{-1}, d_1^{(\mu)}, \bullet),$$
$$\left(\partial_t + \alpha^{(\mu)} - \frac{1}{\alpha^{(\mu)}}\right)\tau_n^{l^{(1)}\cdots l^{(\mu)}+1\cdots l^{(M)}} = \mathrm{pf}(d_0, d_{-1}, d_1, d_1^{(\mu)}, \bullet),$$
$$\tau_n^{l^{(1)}\cdots l^{(\mu)}-1\cdots l^{(M)}} = \mathrm{pf}(d_0, d_{-1}^{(\mu)}, \bullet),$$
$$\frac{1}{\alpha^{(\mu)}}\tau_{n+1}^{l^{(1)}\cdots l^{(\mu)}-1\cdots l^{(M)}} = \mathrm{pf}(d_1, d_{-1}^{(\mu)}, \bullet),$$

where

$$\mathrm{pf}(d_0, d_1) = \mathrm{pf}(d_0, d_{-1}) = 1,\ \ \mathrm{pf}(d_{-1}, d_1) = 0,$$
$$\mathrm{pf}(d_{-1}^{(\mu)}, d_1^{(\mu)}) = 0,\ \ \mathrm{pf}(d_0, d_1^{(\mu)}) = \mathrm{pf}(d_0, d_{-1}^{(\mu)}) = 1,$$
$$\mathrm{pf}(d_1, d_1^{(\mu)}) = \alpha^{(\mu)},\ \ \mathrm{pf}(d_1, d_{-1}^{(\mu)}) = \mathrm{pf}(d_{-1}, d_1^{(\mu)}) = \frac{1}{\alpha^{(\mu)}},$$
$$\mathrm{pf}(d_l^{(\mu)}, a_j) = p_j^n\left(\frac{p_j - \alpha^{(\mu)}}{1 - \alpha^{(\mu)}p_j}\right)^l\prod_{\nu=1}^{M}\left(\frac{p_j - \alpha^{(\nu)}}{1 - \alpha^{(\nu)}p_j}\right)^{l^{(\nu)}}\exp(\xi_j),$$

and $(\bullet) = (a_1, \cdots, a_{2N})$.

Now the algebraic identity of pfaffians together with the previous rules for τ-functions in pfaffians

$$\mathrm{pf}(d_0, d_{-1}, d_1, d_1^{(\mu)}, \bullet)\mathrm{pf}(\bullet) = \mathrm{pf}(d_0, d_{-1}, \bullet)\mathrm{pf}(d_1, d_1^{(\mu)}, \bullet) - \mathrm{pf}(d_0, d_1, \bullet)\mathrm{pf}(d_{-1}, d_1^{(\mu)}, \bullet) + \mathrm{pf}(d_0, d_1^{(\mu)}, \bullet)\mathrm{pf}(d_{-1}, d_1, \bullet),$$

gives the bilinear equations (58) and (59) while the algebraic identities of pfaffians

$$\begin{aligned}&\mathrm{pf}(d_0, d_1, d_{-1}^{(\mu)}, d_1^{(\mu)}, \bullet)\mathrm{pf}(\bullet) = \mathrm{pf}(d_0, d_1, \bullet)\mathrm{pf}(d_{-1}^{(\mu)}, d_1^{(\mu)}, \bullet)\\ &\quad - \mathrm{pf}(d_0, d_{-1}^{(\mu)}, \bullet)\mathrm{pf}(d_1, d_1^{(\mu)}, \bullet) + \mathrm{pf}(d_0, d_1^{(\mu)}, \bullet)\mathrm{pf}(d_1, d_{-1}^{(\mu)}, \bullet),\end{aligned}$$

leads to the bilinear equation (60). □

Next, we consider $c \neq 0$ in (24), which implies the dark soliton solution of the semi-discrete YO system. By applying the dependent variable transformation

$$\begin{aligned}&S_n^{(\mu)} = (\mathrm{i}\alpha^{(\mu)})^n \exp[(\alpha^{(\mu)} - \bar{\alpha}^{(\mu)})t]\frac{g_n^{(\mu)}}{f_n}, \quad \bar{S}_n^{(\mu)} = (-\mathrm{i}\bar{\alpha}^{(\mu)})^n \exp[(\bar{\alpha}^{(\mu)} - \alpha^{(\mu)})t]\frac{\bar{g}_n^{(\mu)}}{f_n},\\ &L_n = 2\left(\frac{f_{n+1}f_{n-1}}{f_n^2} - 1\right), \quad \hat{\hat{L}} = 2\ln f_n\end{aligned} \tag{62}$$

with $|\alpha^{(\mu)}| = 1$, equations (23) and (25) are cast into

$$(D_t + \alpha^{(\mu)} - \bar{\alpha}^{(\mu)})g_n^{(\mu)} \cdot f_n = \alpha^{(\mu)} g_{n+1}^{(\mu)} f_{n-1} - \bar{\alpha}^{(\mu)} g_{n-1}^{(\mu)} f_{n+1}, \tag{63}$$

$$(D_t + \bar{\alpha}^{(\mu)} - \alpha^{(\mu)})\bar{g}_n^{(\mu)} \cdot f_n = \bar{\alpha}^{(\mu)} \bar{g}_{n+1}^{(\mu)} f_{n-1} - \alpha^{(\mu)} \bar{g}_{n-1}^{(\mu)} f_{n+1}, \tag{64}$$

$$D_t f_{n+1} \cdot f_n - c f_{n+1} \cdot f_n = \sum_{\mu=1}^{M} \mathrm{i}\frac{c^{(\mu)}}{2}(\alpha^{(\mu)} g_{n+1}^{(\mu)} \bar{g}_n^{(\mu)} - \bar{\alpha}^{(\mu)} g_n^{(\mu)} \bar{g}_{n+1}^{(\mu)}), \tag{65}$$

for $\mu = 1, \cdots, M$.

Now we carry out the reductions to obtain bilinear equations (63)–(65) from equations (58)–(60) in lemma 4.1. First, by taking

$$c_{jk} = \delta_{2N+1-j,k}, \quad j < k, \tag{66}$$

the τ-functions $\tau_n^{l^{(1)}\dots l^{(M)}}$ can be rewritten as

$$\tau_n^{l^{(1)}\dots l^{(M)}} = \mathrm{pf}(a_1', a_2', \cdots, a_{2N}')\prod_{j=1}^{2N} \mathrm{pf}(d_0, a_j) \tag{67}$$

where

$$\mathrm{pf}(a_j', a_k') = \delta_{2N+1-j,k}\frac{1}{\mathrm{pf}(d_0, a_j)\mathrm{pf}(d_0, a_{2N+1-j})} + \frac{p_j - p_k}{p_j p_k - 1}, \quad j < k. \tag{68}$$

Thus, if p_j satisfies the constraint condition:

$$\begin{aligned}&\sum_{\mu=1}^{M} s^{(\mu)}\left(\frac{p_j - \alpha^{(\mu)}}{1 - \alpha^{(\mu)}p_j} - \frac{1 - \alpha^{(\mu)}p_j}{p_j - \alpha^{(\mu)}} + \frac{p_{2N+1-j} - \alpha^{(\mu)}}{1 - \alpha^{(\mu)}p_{2N+1-j}} - \frac{1 - \alpha^{(\mu)}p_{2N+1-j}}{p_{2N+1-j} - \alpha^{(\mu)}}\right)\\ &\quad = p_j - \frac{1}{p_j} + p_{2N+1-j} - \frac{1}{p_{2N+1-j}},\end{aligned} \tag{69}$$

i.e.

$$\sum_{\mu=1}^{M}s^{(\mu)}\left(\alpha^{(\mu)}-\frac{1}{\alpha^{(\mu)}}\right)\left[\frac{1}{(1-\alpha^{(\mu)}p_j)\left(1-\frac{1}{\alpha^{(\mu)}}p_{2N+1-j}\right)}+\frac{1}{(1-\alpha^{(\mu)}p_{2N+1-j})\left(1-\frac{1}{\alpha^{(\mu)}}p_j\right)}\right]=\frac{1}{p_j}+\frac{1}{p_{2N+1-j}}, \tag{70}$$

then we have

$$\partial_t\tau_n^{l^{(1)}\cdots l^{(M)}}=\sum_{\mu=1}^{M}s^{(\mu)}\partial_{y^{(\mu)}}\tau_n^{l^{(1)}\cdots l^{(M)}}. \tag{71}$$

Applying the above relation (71) to equation (60), we have

$$\left[D_t-\sum_{\mu=1}^{M}s^{(\mu)}\left(\alpha^{(\mu)}-\frac{1}{\alpha^{(\mu)}}\right)\right]f_{n+1}\cdot f_n=-\sum_{\mu=1}^{M}s^{(\mu)}\left(\alpha^{(\mu)}g_{n+1}^{(\mu)}h_n^{(\mu)}-\frac{1}{\alpha^{(\mu)}}g_n^{(\mu)}h_{n+1}^{(\mu)}\right). \tag{72}$$

Furthermore, imposing the complex conjugate conditions

$$s^{(\mu)}=-\mathrm{i}\frac{c^{(\mu)}}{2},\quad p_{2N+1-j}=\bar{p}_j,\quad \xi^{(\mu)}_{p_{2N+1-j,0}}=\bar{\xi}^{(\mu)}_{j,0}+\mathrm{i}\frac{\pi}{2},\quad \text{for}\quad 1\leqslant j\leqslant N, \tag{73}$$

and requiring $y^{(\mu)}$ being pure imaginary, $|\alpha^{(\mu)}|=1$, one can realize $h_n^{(\mu)}=\bar{g}_n^{(\mu)}$.

Consequently, equations (58), (59) and (72) become equations (63)–(65). In summary, the following theorem holds:

Theorem 4.1. *The bilinear equations (63)–(65) with $|\alpha^{(\mu)}|=1$ are satisfied by*

$$f_n=\tau_n^{0\cdots 0},\quad g_n^{(\mu)}=\tau_n^{0\ldots\overset{\mu}{1}\ldots 0},\quad h_n^{(\mu)}=\tau_n^{0\ldots\overset{\mu}{-1}\ldots 0}, \tag{74}$$

where the τ-function $\tau_n^{l^{(1)}\cdots l^{(M)}}$ is the pfaffian $\tau_n^{l^{(1)}\cdots l^{(M)}}=\mathrm{pf}(a_1,a_2,\cdots,a_{2N})$, whose elements are defined by

$$\mathrm{pf}(a_j,a_k)=\delta_{2N+1-j,k}+\frac{p_j-p_k}{p_jp_k-1}\mathrm{pf}(d_0,a_j)\mathrm{pf}(d_0,a_k),$$

$$\mathrm{pf}(d_l,a_j)=p_j^{n+l}\prod_{\mu=1}^{M}\left(\frac{p_j-\alpha^{(\mu)}}{1-\alpha^{(\mu)}p_j}\right)^{l^{(\mu)}}\exp(\xi_j),$$

$$\xi_j=\left(p_j-\frac{1}{p_j}\right)t+\xi_{j,0}.$$

Here p_j, ξ_{j0} are constants satisfying the following constraints:

$$-\sum_{\mu=1}^{M} \mathrm{i}\frac{c^{(\mu)}}{2}(\alpha^{(\mu)} - \bar{\alpha}^{(\mu)})\, \Xi\, (\alpha^{(\mu)}, p_j, p_{2N+1-j}) = \frac{1}{p_j} + \frac{1}{p_{2N+1-j}}, \tag{75}$$

with

$$\Xi(\alpha^{(\mu)}, p_j, p_{2N+1-j}) = \frac{1}{(1 - \alpha^{(\mu)}p_j)(1 - \bar{\alpha}^{(\mu)}p_{2N+1-j})} + \frac{1}{(1 - \alpha^{(\mu)}p_{2N+1-j})(1 - \bar{\alpha}^{(\mu)}p_j)},$$

and $p_{2N+1-j} = \bar{p}_j$, $\xi^{(\mu)}_{2N+1-j,0} = \bar{\xi}^{(\mu)}_{j,0} + \mathrm{i}\frac{\pi}{2}$ *for* $1 \leqslant j \leqslant N$.

In the following, we will illustrate one and two dark soliton solutions for $M = 2$.

One-soliton solution: By taking $N = 1$ in (74), we get the τ-functions for the one-soliton solution:

$$f_n = 1 + \mathrm{i}\frac{p_1 - \bar{p}_1}{p_1\bar{p}_1 - 1}(p_1\bar{p}_1)^n \exp(\xi_1 + \bar{\xi}_1), \tag{76}$$

$$g_n^{(1)} = 1 + \mathrm{i}\frac{p_1 - \bar{p}_1}{p_1\bar{p}_1 - 1}\frac{(p_1 - \alpha^{(1)})(\bar{p}_1 - \alpha^{(1)})}{(1 - \alpha^{(1)}p_1)(1 - \alpha^{(1)}\bar{p}_1)}(p_1\bar{p}_1)^n \exp(\xi_1 + \bar{\xi}_1), \tag{77}$$

$$g_n^{(2)} = 1 + \mathrm{i}\frac{p_1 - \bar{p}_1}{p_1\bar{p}_1 - 1}\frac{(p_1 - \alpha^{(2)})(\bar{p}_1 - \alpha^{(2)})}{(1 - \alpha^{(2)}p_1)(1 - \alpha^{(2)}\bar{p}_1)}(p_1\bar{p}_1)^n \exp(\xi_1 + \bar{\xi}_1), \tag{78}$$

where $\xi_1 = \left(p_1 - \frac{1}{p_1}\right)t + \xi_{1,0}$ and p_1 is a complex constant satisfying

$$-\sum_{\mu=1}^{2} \mathrm{i}\frac{c^{(\mu)}}{2}(\alpha^{(\mu)} - \bar{\alpha}^{(\mu)})\left[\frac{1}{(1 - \alpha^{(\mu)}p_1)(1 - \bar{\alpha}^{(\mu)}\bar{p}_1)} + \frac{1}{(1 - \alpha^{(\mu)}\bar{p}_1)(1 - \bar{\alpha}^{(\mu)}p_1)}\right] = \frac{1}{p_1} + \frac{1}{\bar{p}_1}. \tag{79}$$

In order to avoid the singularity, the condition $\mathrm{i}\frac{p_1 - \bar{p}_1}{p_1\bar{p}_1 - 1} > 0$ need to be satisfied. Further, the above τ-functions lead to the one-soliton solution as follows

$$S_n^{(1)} = \frac{1}{2}\mathrm{e}^{\mathrm{i}(n\varphi_1 + \frac{n}{2}\pi + 2t\sin\varphi_1)}[1 + \mathrm{e}^{2\mathrm{i}\phi_1} - (1 - \mathrm{e}^{2\mathrm{i}\phi_1})\tanh(\xi'_{1R} + \theta_0)], \tag{80}$$

$$S_n^{(2)} = \frac{1}{2}\mathrm{e}^{\mathrm{i}(n\varphi_2 + \frac{n}{2}\pi + 2t\sin\varphi_2)}[1 + \mathrm{e}^{2\mathrm{i}\phi_2} - (1 - \mathrm{e}^{2\mathrm{i}\phi_2})\tanh(\xi'_{1R} + \theta_0)], \tag{81}$$

$$L_n = \frac{(p_1\bar{p}_1 - 1)^2}{2p_1\bar{p}_1}\mathrm{sech}^2(\xi'_{1R} + \theta_0), \tag{82}$$

where $\xi'_1 = \xi'_{1R} + \mathrm{i}\xi'_{1I} = n\ln(\mathrm{i}p_1) + \xi_1$, $\exp(\mathrm{i}\varphi_1) = \alpha^{(1)}$, $\exp(\mathrm{i}\varphi_2) = \alpha^{(2)}$, $\exp(2\theta_0) = \mathrm{i}\frac{p_1 - \bar{p}_1}{p_1\bar{p}_1 - 1}$, $\exp(2\mathrm{i}\phi_1) = \frac{(p_1 - \alpha^{(1)})(\bar{p}_1 - \alpha^{(1)})}{(1 - \alpha^{(1)}p_1)(1 - \alpha^{(1)}\bar{p}_1)}$, $\exp(2\mathrm{i}\phi_2) = \frac{(p_1 - \alpha^{(2)})(\bar{p}_1 - \alpha^{(2)})}{(1 - \alpha^{(2)}p_1)(1 - \alpha^{(2)}\bar{p}_1)}$ and the parameter p_1 is determined by equation (79). For the dark soliton in the SW components $S_n^{(1)}$ and $S_n^{(2)}$, their intensities approach 1 as $n \to \pm\infty$, and the intensities of the center of the solitons read $\cos\phi_1$ and $\cos\phi_2$. The real quantity $\frac{(p_1\bar{p}_1 - 1)^2}{2p_1\bar{p}_1}$ denotes the amplitude of the soliton in the LW component. We illustrate single dark soliton for the choice of the nonlinearity coefficients

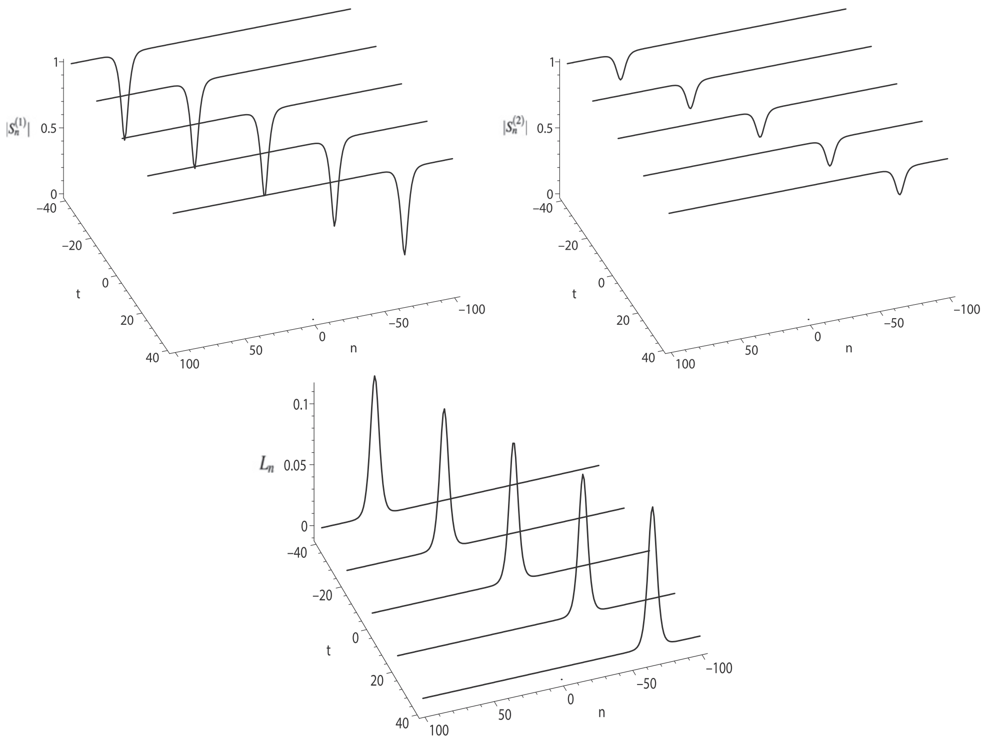

Figure 3. The profiles of evolutions of one-soliton solution (dark soliton for SW components).

$(c^{(1)}, c^{(2)}) = (1, -2)$ in figure 3. The parameters are chosen as $\alpha^{(1)} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $\alpha^{(2)} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$, $p_1 = 1 - 0.7796\mathrm{i}$ and $\xi_{1,0} = 0$.

Two-soliton solution: By taking $N = 2$ in (74), we get the τ-functions for the two-soliton solution:

$$f_n = 1 + E_1 + E_2 + \left| \frac{(p_1 - p_2)(p_1 - \bar{p}_2)}{(p_1 p_2 - 1)(p_1 \bar{p}_2 - 1)} \right|^2 E_1 E_2, \tag{83}$$

$$g_n^{(1)} = 1 + A_1 E_1 + A_2 E_2 + \left| \frac{(p_1 - p_2)(p_1 - \bar{p}_2)}{(p_1 p_2 - 1)(p_1 \bar{p}_2 - 1)} \right|^2 A_1 A_2 E_1 E_2, \tag{84}$$

$$g_n^{(2)} = 1 + B_1 E_1 + B_2 E_2 + \left| \frac{(p_1 - p_2)(p_1 - \bar{p}_2)}{(p_1 p_2 - 1)(p_1 \bar{p}_2 - 1)} \right|^2 B_1 B_2 E_1 E_2, \tag{85}$$

where

$$E_j = \mathrm{i}\frac{p_j - \bar{p}_j}{p_j \bar{p}_j - 1}(p_j \bar{p}_j)\exp(\xi_i + \bar{\xi}_j),$$
$$A_j = \frac{(p_j - \alpha^{(1)})(\bar{p}_j - \alpha^{(1)})}{(1 - \alpha^{(1)} p_j)(1 - \alpha^{(1)} \bar{p}_j)}, \quad B_j = \frac{(p_j - \alpha^{(2)})(\bar{p}_j - \alpha^{(2)})}{(1 - \alpha^{(2)} p_j)(1 - \alpha^{(2)} \bar{p}_j)},$$

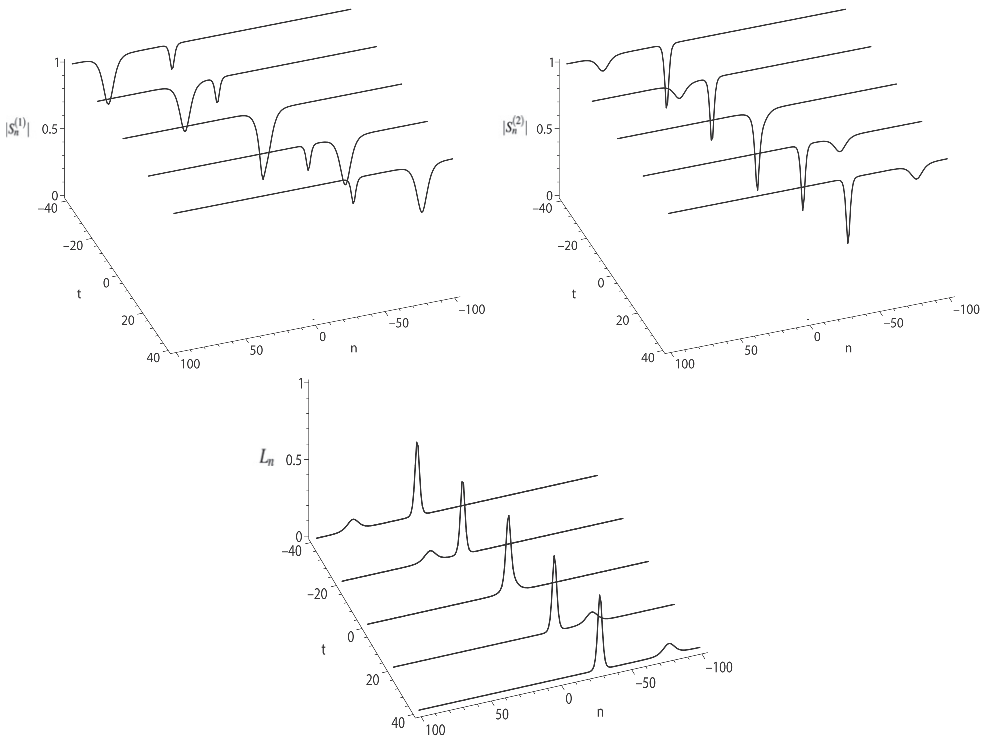

Figure 4. The profiles of evolutions of two-soliton solution (dark soliton for SW components).

and $\xi_j = \left(p_j - \frac{1}{p_j}\right)t + \xi_{j,0}$ and p_j are complex constants satisfying

$$-\sum_{\mu=1}^{2} \mathrm{i}\frac{c^{(\mu)}}{2}(\alpha^{(\mu)} - \bar{\alpha}^{(\mu)})\left[\frac{1}{(1-\alpha^{(\mu)}p_j)(1-\bar{\alpha}^{(\mu)}\bar{p}_j)} + \frac{1}{(1-\alpha^{(\mu)}\bar{p}_j)(1-\bar{\alpha}^{(\mu)}p_j)}\right] = \frac{1}{p_j} + \frac{1}{\bar{p}_j}, \tag{86}$$

for $j = 1,\ 2$. Figure 4 exhibits such a two-soliton interaction with the parameters as $(c^{(1)}, c^{(2)}) = \left(\frac{1}{9}, -\frac{1}{8}\right)$, $\alpha^{(1)} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$, $\alpha^{(2)} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$, $p_1 = 1 - 0.6714\mathrm{i}$, $p_2 = 0.5-1.5356\mathrm{i}$ and $\xi'_{1,0} = \xi'_{2,0} = 0$.

5. Conclusion and discussions

In the present paper, we have constructed an integrable semi-discrete analogue of the coupled YO system by using a bilinear approach. Moreover, both the bright and dark soliton solutions in terms of pfaffians have been derived based on the Bäcklund transformations of the semi-discrete BKP hierarchy. As far as we know, it is the first time to propose an integrable semi-discrete 1D YO system. It should be pointed out that fully discrete NLS and YO systems were constructed most recently [44], however, their semi-discrete limits cannot converge to known integrable models such as the AL lattice. We also remark that a (2+1)-dimensional differential-difference system in [45] can be reduced to our semi-discrete single-component YO system. It is expected that the proposed integrable semi-discrete YO system provides useful insights for the study of resonant interactions of LWs and SWs in discrete systems. Mixed-type soliton solutions with both bright and dark solitons and rogue wave solutions for the coupled semi-discrete YO system will be addressed in forthcoming articles.

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