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Nonlocal symmetries and negative hierarchies related to bilinear Bäcklund transformation*

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In this paper, nonlocal symmetries defined by bilinear Bäcklund transformation for bilinear potential KdV (pKdV) equation are obtained. By introducing an auxiliary variable which just satisfies the Schwartzian form of KdV (SKdV) equation, the nonlocal symmetry is localized and the Levi transformation is presented. Besides, based on three different types of nonlocal symmetries for potential KdV equation, three sets of negative pKdV hierarchies along with their bilinear forms are constructed. An impressive result is that the coefficients of the third type of (bilinear) negative pKdV hierarchy ($N > 0$) are variable, which are obtained via introducing an arbitrary parameter by considering the translation invariance of the pKdV equation.

Keywords: nonlocal symmetry, bilinear Bäcklund transformation, finite transformation, negative hierarchy

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1. Introduction

Lie group theory[1,2] is one of the most effective methods for seeking exact and analytic solutions of nonlinear partial differential equations (PDEs). Starting from Lie point symmetries, one can compute corresponding finite transformations and similar reductions to obtain explicit solutions directly. With the development of integrable systems and soliton theory, a variety of nonlocal symmetries have been intensely investigated in the literature. For example, a type of nonlocal symmetries which are related to Lax pair of integrable equations, that is so-called eigenfunction symmetries,[3–8] have played an important role in the topics of symmetry constraints, soliton equations with sources, positive and negative hierarchies, etc.

However, one usually cannot apply nonlocal symmetries directly to get explicit solutions via the classical Lie group method. One feasible way is to localize nonlocal symmetries by introducing another auxiliary variables into an extended system. In fact, this generalization of the concept of nonlocal symmetries by including pseudo-potentials was devised by Edelen,[9] Krasil'shchik and Vinogradov[10,11] in the 1980s by using their theory of coverings of differential equations. Galas[12] rederived one-soliton solutions for the KdV equation, Dym equation and AKNS system equations based on nonlocal symmetries of this sort. As an application of nonlocal symmetries for the bilinear KP equation and bilinear BKP equation, Hu *et al.*[13] derived two types of bilinear negative KP and bilinear negative BKP hierarchies, respectively. Recently the nonlocal symmetries are receiving great interest and much progress has been made on this topic. In our early two papers,[14,15] a class of nonlocal symmetries of the (potential) KdV equation in elegant and compact form are derived from Bäcklund transformation and Darboux transformation. Then we use these nonlocal symmetries to get abundant explicit solutions, especially the new interaction excitations between solitary waves and cnoidal waves, and to construct some integral models both in finite and infinite dimensions. One can also obtain the binary Darboux transformation starting from the nonlocal symmetries, by solving an initial value problem via introducing a suitable prolonged system.[16] In latter studies, based on this nonlocal symmetries approach, kinds of novel exact interaction solutions among solitons and other complicated waves have also been found for mKdV equation,[17] AKNS system,[18] KP equation,[19] the Hirota–Satsuma coupled Korteweg-de Vries system,[20] the (2+1) dimensional modified generalized long dispersive wave equation[21] and the nonlinear Schrödinger (NLS) equation.[22] In Ref. [23], Bluman and Yang introduced a new and complementary method for constructing nonlocally related PDE systems, which was on the basis of each point symmetry.

In this paper, our aim is to investigate nonlocal symmetries defined by bilinear Bäcklund transformation and their corresponding applications for bilinear potential KdV (pKdV) equation. In Section 2, the nonlocal symmetry is localized and the corresponding prolonged system for bilinear KdV equation is found. An impressive observation is that the process of localization also leads to the Schwartzian form of KdV equation, where the Schwartzian variable is realized by two solutions of

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the bilinear pKdV equation. Then via Lie's first theorem, the Levi transformation (the second Bäcklund transformation) is presented to give transformed solutions from trivial ones. In Section 3, combining different types of nonlocal symmetries with the bilinear Bäcklund transformation of pKdV equation, three sets of negative pKdV hierarchies along with their bilinear forms are constructed. Section 4 is a short summary and discussion.

2. Nonlocal symmetries related to bilinear Bäcklund transformation for bilinear KdV equation

It is known that for KdV equation

$$\omega_t + \omega_{xxx} - 6\omega\omega_x = 0, \tag{1}$$

we have the potential KdV equation given by

$$u_t + u_{xxx} - 3u_x^2 = 0, \tag{2}$$

with $\omega = u_x$.

Furthermore, for Eq. (2), there is the following Bäcklund transformation:[25]

$$u_x + u_{1,x} = -2\lambda + \frac{(u-u_1)^2}{2}, \tag{3}$$

$$u_t + u_{1,t} = 2u_x^2 + 2u_{1,x}^2 + 2u_x u_{1,x} - (u-u_1)(u_{xx} - u_{1,xx}), \tag{4}$$

with λ being an arbitrary parameter. The compatible condition $u_{1,xt} = u_{1,tx}$ of Eqs. (3) and (4) is just pKdV equation (2). Under the transformations $u = -2(\ln f)_x$ and $u_1 = -2(\ln g)_x$, the pKdV equation and its Bäcklund transformation can be expressed as bilinear forms, saying

$$(D_x^4 + D_x D_t) f f = 0, \tag{5}$$

$$(D_x^2 - \lambda) f g = 0, \tag{6}$$

$$(D_t + D_x^3 + 3\lambda D_x) f g = 0. \tag{7}$$

The well-known Hirota's bilinear operator $D_x^m D_t^n$ is defined by

$$D_x^m D_t^n ab = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\Big|_{x'=x,t'=t}.$$

In our early paper,[14] we obtained a nonlocal symmetry of u for pKdV equation given by

$$\sigma_u = \mathrm{e}^{\int u - u_1\,\mathrm{d}x}. \tag{8}$$

Here, the nonlocal symmetry of f for bilinear KdV equation can be derived from Eq. (8), i.e.,

$$\sigma_f = f\int \frac{g^2}{f^2}\mathrm{d}x, \tag{9}$$

where f and g satisfy bilinear Bäcklund transformation (6) and (7).

Clearly, the symmetry equation (9) for f is nonlocal. To localize Eq. (9), we introduce $g_1 \equiv g_1(x,t)$ by

$$\begin{cases} f^2 g_{1x} - g^2 = 0, \\ f^3 g_{1t} - 4f_{xx}g^2 + 8gf_x g_x - 4fg_x^2 + 8\lambda f g^2 = 0, \end{cases} \tag{10}$$

which leads nonlocal symmetry equation (9) to

$$\sigma_f = f g_1. \tag{11}$$

Then, by solving symmetry equations (6), (7), and (10) given by

$$\begin{cases} (D_x^2 - \lambda)(f\sigma_g + \sigma_f g) = 0, \\ (D_t + D_x^3 + 3\lambda D_x)(f\sigma_g + \sigma_f g) = 0, \end{cases} \tag{12}$$

and

$$\begin{cases} f^2\sigma_{g_1,x} + 2fg_{1x}\sigma_f - 2g\sigma_g = 0, \\ f^3\sigma_{g_1,t} + 8(f_x g - f g_x)\sigma_{g,x} \\ +8(f_x g_x - f_{xx} g + 2\lambda f g)\sigma_g \\ +(3f^2 g_{1t} - 4g_x^2 + 8\lambda g^2)\sigma_f - 4g^2\sigma_{f,xx} \\ +8gg_x\sigma_{f,x} = 0, \end{cases} \tag{13}$$

we obtain the corresponding symmetries σ_g and σ_{g_1} of g and g_1 respectively, saying

$$\sigma_g = ag, \quad \sigma_{g_1} = -g_1^2 + 2ag_1, \tag{14}$$

with a being an arbitrary constant.

From Eqs. (11) and (14), one can see that the prolongation of nonlocal symmetry equation (9) is closed after covering three dependent variables f, g, and g_1. Hence, the final prolonged system containing Eqs. (5), (6), (7), and (10) is found, which has the following Lie point symmetry

$$V = fg_1\frac{\partial}{\partial f} + ag\frac{\partial}{\partial g} + (-g_1^2 + 2ag_1)\frac{\partial}{\partial g_1}. \tag{15}$$

Due to Eq. (15) and Lie's first theorem, solving the following initial value problem:

$$\begin{aligned} &\frac{\mathrm{d}\bar{x}}{\mathrm{d}\varepsilon} = 0, \quad \frac{\mathrm{d}\bar{t}}{\mathrm{d}\varepsilon} = 0, \quad \frac{\mathrm{d}\bar{f}}{\mathrm{d}\varepsilon} = \bar{f}\bar{g}_1, \\ &\frac{\mathrm{d}\bar{g}}{\mathrm{d}\varepsilon} = a\bar{g}, \quad \frac{\mathrm{d}\bar{g}_1}{\mathrm{d}\varepsilon} = -\bar{g}_1^2 + 2a\bar{g}_1, \\ &\bar{x}|_{\varepsilon=0} = x, \quad \bar{t}|_{\varepsilon=0} = t, \quad \bar{f}|_{\varepsilon=0} = f, \\ &\bar{g}|_{\varepsilon=0} = g, \quad \bar{g}_1|_{\varepsilon=0} = g_1, \end{aligned} \tag{16}$$

yields the finite symmetry transformations

$$\begin{aligned} &\bar{x} = x, \quad \bar{t} = t, \quad \bar{f} = f\left(1 + \frac{\exp(2a\varepsilon) - 1}{2a}g_1\right), \\ &\bar{g} = \exp(a\varepsilon)g, \quad \bar{g}_1 = \frac{2a\exp(2a\varepsilon)g_1}{2a + (\exp(2a\varepsilon) - 1)g_1}. \end{aligned} \tag{17}$$

Remark 1 The bilinear Bäcklund transformation (6) and (7) in itself suggests a finite transformation from a solution f to another one g while the obtained finite transformation (17) arrives at a third solution $\bar{f}$. Actually, the transformation (17) is just the so-called Levi transformation or the second type of Bäcklund transformation.[24] This result shows the fact that two kinds of Bäcklund transformation possess the same infinitesimal form (9).

Remark 2 The form of σ_{g_1} in Eq. (14) embodies the Möbious (conformal) invariance property

$$g_1 \longrightarrow \frac{a+bg_1}{c+dg_1}, \quad (ad \neq cb).$$

Actually, one can check that the introduced invariant g_1 is nothing but just satisfy the Schwartzian form of KdV equation (SKdV equation)

$$g_{1,t} + \{g_1; x\} g_{1,x} + 6\lambda g_{1,x} = 0, \tag{18}$$

where $\{g_1; x\} = (g_{1,xxx}/g_{1,x}) - 3(g_{1,xx}/g_{1,x})^2/2$ is the Schwartzian derivative. Here, the Schwartzian variable g_1 shown in Eq. (10) is determined by the solutions $\{f, g\}$ of the biliear Bäcklund transformation.

Now by force of the finite symmetry transformation equation (17), one can get new solutions from any initial solutions. For example, we take the trivial solution $f = f_0$. From Eqs. (6), (7), and (10) with $\lambda = \lambda_0^2$, we obtain the following special solutions:

$$\begin{aligned} g &= \sinh(-\lambda_0 x + 4\lambda_0^3 t + x_0), \\ g_1 &= \frac{1}{4\lambda_0 f_0^2}[\sinh(2\lambda_0 x - 8\lambda_0^3 t - 2x_0) - 2\lambda_0 x + 24\lambda_0^3 t], \end{aligned} \tag{19}$$

where λ_0, x_0, and f_0 are three constants. Substituting Eq. (19) into Eq. (17) leads to the transformed solution of Eq. (5)

$$\begin{aligned} \bar{f} &= f_0 + \frac{\exp(2a\varepsilon)-1}{8a\lambda_0 f_0} \\ &\quad \times [\sinh(2\lambda_0 x - 8\lambda_0^3 t - 2x_0) - 2\lambda_0 x + 24\lambda_0^3 t], \end{aligned} \tag{20}$$

then to the solution of KdV equation (1), saying

$$\omega = 2(\ln\bar{f})_{xx} = 16\lambda_0^2(\exp(2a\varepsilon)-1)\frac{(\exp(2a\varepsilon)-1)(\cosh(Y)-1) - \lambda_0[(\exp(2a\varepsilon)-1)(12\lambda_0^2 t - x) + 4af_0^2]\sinh(Y)}{[(\exp(2a\varepsilon)-1)(\sinh(Y) + 2\lambda_0 x - 24\lambda_0^3 t) - 8a\lambda_0 f_0^2]^2}, \tag{21}$$

with $Y = -2\lambda_0 x + 8\lambda_0^3 t + 2x_0$.

Next, according to the classical Lie group method, the complete Lie point symmetries of the whole system (5)–(7), and (10) can be obtained. Then, abundant group invariant solutions related to the nonlocal symmetry of the KdV equation, which are expressed in terms of rational function, Bessel functions, period functions and their combinations, can be obtained by following the same procedure in Refs. [14] and [15]. What is different from four variables in the prolonged system of Refs. [14] and [15] is here three dependent variables are enough to ensure the localization of the nonlocal symmetry.

3. Three sets of negative pKdV hierarchies

The existence of infinitely many symmetries leads to the existence of integrable hierarchies and with the help of infinitely many nonlocal symmetries, one can extend the original system to its negative hierarchies.[13,26] In the following, starting from the nonlocal symmetry (8) related to Bäcklund transformation of Eq. (2), three sets of negative pKdV hierarchies are constructed and their corresponding bilinear forms are also presented.

3.1. The first type of negative pKdV hierarchy

Based on the nonlocal symmetry (8), a set of negative pKdV hierarchy is obtained, reading

$$\begin{cases} u_{t_{-N}} = -\sum_{i=1}^{N} \mathrm{e}^{\int u-u_i\,\mathrm{d}x}, \\ u_x + u_{i,x} = -2\lambda_i + \dfrac{(u-u_i)^2}{2}, \quad i = 1, 2, \ldots, N, \end{cases} \tag{22}$$

where λ_i is an arbitrary constant. In particular, when $N = 1$, one has the first equation of negative pKdV hierarchy, namely

$$2u_{xxt}u_t - 4u_x u_t^2 - u_{xt}^2 - 4\lambda_1 u_t^2 = 0. \tag{23}$$

Here, we have instead t_{-1} with t for simplicity. It is well known that the first negative flow in the KdV hierarchy is linked to Camassa–Holm equation via a hodograph transformation[27] or can be reduced to sinh-Gordon/sine-Gordon/Liouville equations.[28] Here, we can transform Eq. (23) into sine-Gordon and Liouville equations. In fact, by setting $\beta \equiv \beta(x,t) = -u_t$, we can rewrite Eq. (18) in the form

$$\beta_x = \left(-\frac{\beta_{xx}}{2\beta} + \frac{\beta_x^2}{4\beta^2}\right)_t, \tag{24}$$

which can be integrated once with respect to x to give

$$\beta(\ln\beta)_{xt} + \beta^2 = \beta_0(t), \tag{25}$$

where $\beta_0(t)$ is an arbitrary function of t.

Then, for non-zero $\beta_0(t)$, one can rescale β to $\sqrt{\beta_0(t)}\beta$, redefine t as $t/\sqrt{\beta_0(t)}$ and set $\beta = \exp(\mathrm{i}\eta)$ to give the sine-Gordon equation

$$\eta_{xt} = \sin\eta, \tag{26}$$

while for $\beta_0(t)=0$, by setting $\beta=-\exp\eta$, equation (25) becomes the Liouville equation

$$\eta_{xt}=\mathrm{e}^{\eta}. \tag{27}$$

Furthermore, by virtue of the dependent variable transformation

$$u=-2\frac{\psi_x}{\psi},\qquad u_i=-2\frac{\psi_{i,x}}{\psi_i},\quad (i=1,2,\ldots,N),$$

the negative pKdV hierarchy (22) is directly transformed into bilinear form

$$\begin{cases} D_xD_{t_{-N}}\psi\cdot\psi=\sum_{i=1}^{N}\psi_i^2, \\ (D_x^2-\lambda_i)\psi\cdot\psi_i=0,\quad i=1,2,\ldots,N. \end{cases} \tag{28}$$

3.2. The second type of negative pKdV hierarchy

For the nonlocal symmetry (8) being dependent with arbitrary parameter λ, we may derive a type of negative pKdV hierarchy by expanding the dependent variable in power series of λ. In this case, we have

$$\begin{cases} u_{t_{-N}}=-\frac{1}{N!}\left(\frac{\partial^N \mathrm{e}^{\int u-u_1\,\mathrm{d}x}}{\partial\lambda^N}\right)\Bigg|_{\lambda=0}, \\ u_x+u_{1,x}=-2\lambda+\frac{(u-u_1)^2}{2}. \end{cases} \tag{29}$$

Under transformations $u=-2\psi_x/\psi$ and $u_1=-2\psi_{1x}/\psi_1$, the negative pKdV hierarchy (29) becomes

$$\begin{cases} D_xD_{t_{-N}}\psi\cdot\psi=\frac{1}{N!}\left(\frac{\partial^N\psi_1^2}{\partial\lambda^N}\right)\Bigg|_{\lambda=0}, \\ (D_x^2-\lambda)\psi\cdot\psi_1=0. \end{cases} \tag{30}$$

Let $\psi_1=\psi_1(\lambda)$ has a formal series form

$$\psi_1=\sum_{i=0}^{\infty}\bar{\psi}_i\lambda^i, \tag{31}$$

where $\bar{\psi}_i$ is λ independent. Then, equation (30) can be rewritten as

$$\begin{cases} D_xD_{t_{-N}}\psi\cdot\psi=\sum_{k=0}^{N}\bar{\psi}_k\bar{\psi}_{N-k}, \\ D_x^2\psi\cdot\bar{\psi}_k=\psi\bar{\psi}_{k-1},\quad k=0,1,\ldots,N, \end{cases} \tag{32}$$

with $\bar{\psi}_{-1}=0$.

The negative KdV hierarchy in bilinear form (32) is just the special situation of the bilinear negative KP hierarchy for $y=0$ in Ref. [13].

3.3. The third type of negative pKdV hierarchy

By considering the translation invariance of pKdV equation (2) under the transformation $u\to u+c$, another parameter μ can be introduced in the Bäcklund transformation (3) and (4) with $u_1\to u_1+\mu$, saying

$$\begin{cases} u_x+u_{1,x}=-2\lambda+\frac{(u-u_1)^2}{2}+\mu(u_1-u)+\frac{1}{2}\mu^2, \\ u_t+u_{1,t}=2u_x^2+2u_{1,x}^2+2u_xu_{1,x} \\ \qquad -(u-u_1-\mu)(u_{xx}-u_{1,xx}), \end{cases} \tag{33}$$

and the corresponding symmetry of u becomes $\sigma'=\mathrm{e}^{\int u-u_1-\mu\,\mathrm{d}x}$. In this case, we make $\lambda=0$ and construct another negative pKdV hierarchy with the help of parameter μ. A novel set of negative pKdV hierarchy may be written down as follows:

$$\begin{cases} u_{t_{-N}}=-\frac{1}{N!}\left(\frac{\partial^N \mathrm{e}^{\int u-u_1-\mu\,\mathrm{d}x}}{\partial\mu^N}\right)\Bigg|_{\mu=0}, \\ u_x+u_{1,x}=\frac{(u-u_1)^2}{2}+\mu(u_1-u)+\frac{1}{2}\mu^2. \end{cases} \tag{34}$$

Due to the same transformation $u=-2\psi_x/\psi$ and $u_1=-2\psi_{1,x}/\psi_1$, we have

$$\begin{cases} D_xD_{t_{-N}}\psi\cdot\psi=\frac{1}{N!}\left(\frac{\partial^N \mathrm{e}^{-\mu x}\psi_1^2}{\partial\mu^N}\right)\Bigg|_{\mu=0}, \\ \left(D_x^2+\mu D_x+\frac{1}{4}\mu^2\right)\psi\cdot\psi_1=0. \end{cases} \tag{35}$$

Let $\psi_1=\psi_1(\mu)$ has a formal series form

$$\psi_1=\sum_{i=0}^{\infty}\bar{\phi}_i\mu^i, \tag{36}$$

where $\bar{\phi}_i$ is μ independent, and it leads Eq. (35) to

$$\begin{cases} D_xD_{t_{-N}}\psi\cdot\psi=\sum_{k=0}^{N}(-1)^{N-k}\frac{1}{(N-k)!}x^{N-k}\sum_{i=0}^{k}\bar{\phi}_i\bar{\phi}_{k-i}, \\ D_x^2\psi\cdot\bar{\phi}_k+D_x\psi\cdot\bar{\phi}_{k-1}+\frac{1}{4}\psi\cdot\bar{\phi}_{k-2}=0,\quad k=0,1,\ldots,N, \end{cases} \tag{37}$$

with $\bar{\phi}_{-2}=\bar{\phi}_{-1}=0$. For example, when $N=0$, there is

$$\begin{cases} D_xD_{t_0}\psi\cdot\psi=\bar{\phi}_0^2, \\ D_x^2\psi\cdot\bar{\phi}_0=0. \end{cases} \tag{38}$$

When $N=1$, we have

$$\begin{cases} D_xD_{t_{-1}}\psi\cdot\psi=-x\bar{\phi}_0^2+2\bar{\phi}_0\bar{\phi}_1, \\ D_x^2\psi\cdot\bar{\phi}_1+D_x\psi\cdot\bar{\phi}_0=0, \\ D_x^2\psi\cdot\bar{\phi}_0=0. \end{cases} \tag{39}$$

When $N=2$, we obtain

$$\begin{cases} D_xD_{t_{-2}}\psi\cdot\psi=\frac{x^2}{2}\bar{\phi}_0^2-2x\bar{\phi}_0\bar{\phi}_1+(2\bar{\phi}_0\bar{\phi}_2+\bar{\phi}_1^2), \\ D_x^2\psi\cdot\bar{\phi}_1+D_x\psi\cdot\bar{\phi}_0=0, \\ D_x^2\psi\cdot\bar{\phi}_0=0. \end{cases} \tag{40}$$

Remark 3 One can see that the coefficients of the third negative pKdV hierarchy (37) ($N\geq 1$) are variable, which are exactly different from the other two cases.

4. Summary and discussion

In this paper, the nonlocal symmetry depending on bilinear Bäcklund transformation for bilinear pKdV equation is localized by covering three dependent variables, which differs from that in Ref. [14]. Via Lie's first theorem, we obtain a Levi transformation which possesses the same infinitesimal form (9) with the original bilinear Bäcklund transformation. Furthermore, on the basis of the nonlocal symmetry (8) of pKdV equation, three different types of nonlocal symmetries are built and then applied to construct three sets of negative pKdV hierarchies. In the third case, by considering translation invariance of the pKdV equation, another arbitrary parameter is introduced to get new nonlocal symmetries, which are then expanded in power series to derive negative hierarchy. It is remarked that the third type of (bilinear) pKdV negative hierarchy with variable coefficients is fresh and worthy of our further investigation. To search for nonlocal symmetries of integrable systems and then to apply them to obtain new results are both of considerable interest. We know that it is convenient to construct N-soltions from the bilinear forms of integrable eqautions. Combining bilinear equations or their bilinear Bäcklund transformation with nonlocal symmetries may provide a direct way for seeking exact interaction solutions among solitons and other background waves, the rogue waves solutions, the new integrable models in bilinear forms, the nonlocal or local conservation laws, etc. We believe that this approach would also play an important role in supersymmetry systems and discrete equations.

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