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Nonlocal symmetry and exact solutions of the (2+1)-dimensional modified Bogoyavlenskii–Schiff equation*

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In this paper, the truncated Painlevé analysis, nonlocal symmetry, Bäcklund transformation of the (2+1)-dimensional modified Bogoyavlenskii–Schiff equation are presented. Then the nonlocal symmetry is localized to the corresponding nonlocal group by the prolonged system. In addition, the (2+1)-dimensional modified Bogoyavlenskii–Schiff is proved consistent Riccati expansion (CRE) solvable. As a result, the soliton–cnoidal wave interaction solutions of the equation are explicitly given, which are difficult to find by other traditional methods. Moreover figures are given out to show the properties of the explicit analytic interaction solutions.

Keywords: (2+1)-dimensional modified Bogoyavlenskii–Schiff equation, nonlocal symmetry, consistent Riccati expansion, soliton–cnoidal wave solution

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1. Introduction

Soliton equations connect rich histories of exactly solvable systems constructed in mathematics, fluid physics, microphysics, cosmology, field theory, etc. To explain some physical phenomenon further, it becomes more and more important to seek exact solutions and interactions among solutions of nonlinear wave solutions. It is well known that there are many ways to obtain exact solutions of soliton equations, such as the Inverse Scattering transformation (IST),[1] Bäcklund transformation (BT),[2] Darboux transformation (DT),[3,4] Hirota bilinear method,[5] Painlevé method,[6,7] Lie symmetry method,[8–10] and so on. For a given nonlinear system, the Lie symmetry method proposed by Sophus Lie[11,12] during the nineteenth century is a standard method to find the corresponding Lie point symmetry algebras and groups.

The nonlocal symmetries are connected with integrable models and they enlarge the class of symmetries, therefore, to search for nonlocal symmetries of the nonlinear systems is an interesting work. Akhatov and Gazizov[13] provided a method for constructing nonlocal symmetries of differential equations based on the Lie–Bäcklund theory. Galas[14] obtained the nonlocal Lie–Bäcklund symmetries by introducing the pesudo-potentials as an auxiliary system. Guthrie[15] got nonlocal symmetries with the help of a recursion operator. Bluman[16] introduced the concept of potential symmetry for a differential system by writing the given system in a conserved form. Lou *et al.*[17–19] have made some efforts to obtain infinitely many nonlocal symmetries by inverse recursion operators, the conformal–invariant form (Schwartz form), and Darboux transformation. More recently, Lou, Hu, and Chen[20–22] obtained nonlocal symmetries that were related to the Darboux transformation with the Lax pair and Bäcklund transformation. Xin and Chen[23] gave a systemic method to find the nonlocal symmetry of nonlinear evolution equation and improved previous methods to avoid missing some important results such as integral terms or high-order derivative terms of nonlocal variables in the symmetries. In recent years, it was found that Painlevé analysis can be used to obtain nonlocal symmetries. The type of nonlocal symmetry related to the truncated Painlevé expansion is just the residual of the expansion with respect to singular manifold, and is also called residual symmetry.[24–28] The localization of this type of residual symmetry seems more easily performed than that coming from DT and BT. In order to develop some types of relatively simple and understandable methods to construct exact solutions, Lou proposed a consistent Riccati expansion (CRE) method to identify CRE solvable systems in Ref. [29]. A system is defined to be CRE solvable if it has a CRE. It has been revealed that many similar interaction solutions between a soliton and a cnoidal wave were found in various CRE solvable systems. By this method, recent studies[30–38] have found a lot of interaction solutions in many nonlinear equations.

In the present paper, we focus on investigating the nonlocal symmetry, prolonged system, Bäcklund transformation,

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CRE solvability, and the exact interaction solutions of the following (2+1)-dimensional modified Bogoyavlenskii–Schiff (mBS) equation[39–43]

$$u_t + u_{xxy} - 4u^2 u_y - 2u_x \partial_x^{-1}(u^2)_y = 0, \tag{1}$$

where subscript means a partial derivative such as $u_t = \partial u/\partial t$, $u_{xy} = \partial^2 u/\partial x \partial y$, and

$$\partial^{-1} u = \int_{-\infty}^{x} u(x,y,t)\,\mathrm{d}x.$$

This equation can be derived by the following CD-type $(2+1)$-dimensional breaking soliton equation[44–46]

$$u_{xt} + u_{xxxy} - 4u_x u_{xy} - 2u_{xx} u_y = 0, \tag{2}$$

with a Miura transformation,

$$u = -v_x - v^2. \tag{3}$$

It is obvious that if $y = x$ the equation becomes an mKdV equation,

$$u_t + 6u^2 u_x + u_{xxx} = 0, \tag{4}$$

which is widely researched by several authors. In order to treat the integral appearing in the equation, equation (1) is then rewritten as

$$\begin{cases} u_t + u_{xxy} - 4u^2 u_y - 2u_x v = 0, \\ v_x - 2uu_y = 0. \end{cases} \tag{5}$$

As equation (2) is a typical breaking soliton equation to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis and equation (4) may present the wave propagation of the bound particle, sound wave, and thermal pulse, equation (1) must have abundant physical phenomena. But little attention has been paid to this equation, except Refs. [39]–[43], where the Lax pair and soliton solutions are presented. Therefore, finding more types of solutions of Eq. (1) is of interest to understand the equation fully.

This paper is arranged as follows. In Section 2, the non-auto Bäcklund transformation and nonlocal symmetry of the (2+1)-dimensional modified mBS equation are obtained by making use of the truncated Painlevé expansion approach, then the nonlocal symmetry is localized by introducing another three dependent variables and the corresponding nonlocal transformation group is found. In Section 3, the (2+1)-dimensional modified mBS equation is verified to be consistent Riccati expansion solvable and the soliton–cnoidal wave solutions are constructed. The last section contains a summary and discussion.

2. The nonlocal symmetry from the truncated Painlevé expansion

For the (2+1)-dimensional mBS equation (5), there exists a truncated Painlevé expansion

$$u = \frac{u_0}{\phi} + u_1, \quad v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2, \tag{6}$$

with $u_0, u_1, v_0, v_1, v_2, \phi$ being the functions of x, y, and t, the function $\phi(x,y,t) = 0$ is the equation of singularity manifold.

Substituting Eq. (6) into Eq. (5) and balancing all the coefficients of different powers of ϕ, we can get

$$\begin{aligned} &u_0 = \phi_x, \quad v_0 = \phi_x \phi_y, \quad u_1 = -\frac{\phi_{xx}}{2\phi_x}, \\ &v_1 = -\phi_{xy}, \quad v_2 = \frac{1}{2}\left(\frac{\phi_t}{\phi_x} + (\frac{\phi_{xx}}{\phi_x})_y\right), \end{aligned} \tag{7}$$

and the (2+1)-dimensional mBS equation (5) is successfully satisfying the following Schwarzian form

$$P_x + S_y = 0. \tag{8}$$

Here, we denote

$$P = \frac{\phi_t}{\phi_x}, \quad S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{9}$$

where P is the usual Schwarzian variables, S is the Schwarzian derivative and both invariants under the Möbious transformation, i.e.,

$$\phi \to \frac{a + b\phi}{c + d\phi} \quad (ad \neq bc). \tag{10}$$

If we take a special case $a = 0, b = c = 1, d = \varepsilon$, then equation (10) can be rewritten as

$$\phi \to \phi - \varepsilon\phi^2 + o(\varepsilon^2),$$

which means equation (8) possesses the point symmetry[24]

$$\sigma^{\phi} = -\phi^2. \tag{11}$$

From the standard truncated Painlevé expansion (6), we have the following non-auto-Bäcklund transformation theorem of Eq. (5).

Theorem 1 (non-auto-BT theorem) If the function ϕ satisfies Eq. (8), then,

$$\begin{aligned} u &= \frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}, \\ v &= \frac{\phi_x \phi_y}{\phi^2} - \frac{\phi_{xy}}{\phi} + \frac{1}{2}\left(\frac{\phi_t}{\phi_x} + \left(\frac{\phi_{xx}}{\phi_x}\right)_y\right), \end{aligned} \tag{12}$$

is a non-auto BT between ϕ and the solution u, v of the (2+1)-dimensional mBS equation (5).

One knows that the symmetry equations for Eq. (5) read

$$\begin{aligned} &\sigma^u_t + \sigma^u_{xxy} - 8\sigma^u u u_y - 4u^2 \sigma^u_y - 2\sigma^u_x v - 2u_x \sigma^v = 0, \\ &\sigma^v_x - 2\sigma^u u_y - 2u\sigma^u_y = 0, \end{aligned} \tag{13}$$

where σ^u and σ^v denote the symmetries of u and v, respectively. From the truncated Painlevé expansion (6) and the Theorem 1, a new nonlocal symmetry of Eq. (5) is presented and studied as follows.

Theorem 2 The equation (5) has the nonlocal symmetry given by

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} = \begin{pmatrix} \phi_x \\ -\phi_{xy} \end{pmatrix}, \tag{14}$$

where u, v and ϕ satisfy the non-auto BT (12).

Proof The nonlocal symmetries (14) are residual of the singularity manifold ϕ. The nonlocal symmetries (14) will also be obtained with substituting the Möbious transformation symmetry σ^ϕ into the linearized equation (7).

To find out the group of the nonlocal symmetry (14)

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \bar{u}(\varepsilon) \\ \bar{v}(\varepsilon) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \varepsilon \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix},$$

we have to solve the following initial value problem

$$\begin{aligned} &\frac{\mathrm{d}\bar{u}(\varepsilon)}{\mathrm{d}\varepsilon} = \bar{\phi}_x, \quad \frac{\mathrm{d}\bar{v}(\varepsilon)}{\mathrm{d}\varepsilon} = -\bar{\phi}_{xy}, \\ &\bar{u}(\varepsilon)\mid_{\varepsilon=0} = u, \quad \bar{v}(\varepsilon)\mid_{\varepsilon=0} = v, \end{aligned} \tag{15}$$

with ε being the infinitesimal parameter.

However, since it is difficult to solve Eqs. (15) for $\bar{u}(\varepsilon)$ and $\bar{v}(\varepsilon)$ due to the intrusion of the function $\bar{\phi}(\varepsilon)$ and its differentiations, we introduce new variables to eliminate the space derivatives of $\bar{\phi}(\varepsilon)$

$$f = \phi_x, \quad g = \phi_y, \quad h = f_y. \tag{16}$$

Now the nonlocal symmetry (14) of the original equation (5) becomes a Lie point symmetry of the prolonged system (5), (12), and (16), saying

$$\begin{pmatrix} \sigma^u \\ \sigma^v \\ \sigma^\phi \\ \sigma^f \\ \sigma^g \\ \sigma^h \end{pmatrix} = \begin{pmatrix} f \\ -h \\ -\phi^2 \\ -2\phi f \\ -2\phi g \\ -2fg - 2\phi h \end{pmatrix}. \tag{17}$$

The result (17) indicates that the nonlocal symmetries (14) are localized in the properly prolonged systems (5), (12), and (16) with the Lie point symmetry vector

$$V = f\partial_u + h\partial_v - \phi^2\partial_\phi - 2\phi f\partial_f - 2\phi g\partial_g - 2(fg + \phi h)\partial_h. \tag{18}$$

In other words, the symmetries related to the truncated Painlevé expansion are just a special Lie point symmetry of the prolonged system.

Now we have obtained the localized nonlocal symmetries, an interesting question is what kind of finite transformation would correspond to the Lie point symmetry (18). We have the following theorem.

Theorem 3 If $\{u, v, \phi, f, g, h\}$ is a solution of the prolonged systems (5), (12), and (16), then $\{\bar{u}, \bar{v}, \bar{\phi}, \bar{f}, \bar{g}, \bar{h}\}$ is given by

$$\begin{aligned} \bar{u} &= u + \frac{\varepsilon f}{\varepsilon\phi + 1}, \\ \bar{v} &= v - \frac{\varepsilon h}{\varepsilon\phi + 1} + \frac{\varepsilon^2 fg}{(\varepsilon\phi + 1)^2}, \\ \bar{\phi} &= \frac{\phi}{\varepsilon\phi + 1}, \\ \bar{f} &= \frac{f}{(\varepsilon\phi + 1)^2}, \\ \bar{g} &= \frac{g}{(\varepsilon\phi + 1)^2}, \\ \bar{h} &= -\frac{h}{(\varepsilon\phi + 1)^2} - \frac{2\varepsilon fg}{(\varepsilon\phi + 1)^3} \end{aligned}$$

with arbitrary group parameter ε.

Proof Using Lie's first theorem on vector (18) with the corresponding initial condition

$$\begin{aligned} &\frac{\mathrm{d}\bar{u}(\varepsilon)}{\mathrm{d}\varepsilon} = \bar{f}(\varepsilon), \quad \bar{u}(0) = u, \\ &\frac{\mathrm{d}\bar{v}(\varepsilon)}{\mathrm{d}\varepsilon} = -\bar{h}(\varepsilon), \quad \bar{v}(0) = v, \\ &\frac{\mathrm{d}\bar{\phi}(\varepsilon)}{\mathrm{d}\varepsilon} = -\bar{\phi}^2(\varepsilon), \quad \bar{\phi}(0) = \phi, \\ &\frac{\mathrm{d}\bar{f}(\varepsilon)}{\mathrm{d}\varepsilon} = -2\bar{\phi}(\varepsilon)\bar{f}(\varepsilon), \quad \bar{f}(0) = f, \\ &\frac{\mathrm{d}\bar{g}(\varepsilon)}{\mathrm{d}\varepsilon} = -2\bar{\phi}(\varepsilon)\bar{g}(\varepsilon), \quad \bar{g}(0) = g, \\ &\frac{\mathrm{d}\bar{h}(\varepsilon)}{\mathrm{d}\varepsilon} = -2(\bar{f}(\varepsilon)\bar{g}(\varepsilon) + \bar{\phi}(\varepsilon)\bar{h}(\varepsilon)), \quad \bar{h}(0) = h. \end{aligned}$$

One can easily obtain the solutions of the above equations given in Theorem 3, thus the theorem is proved.

Actually, the above group transformation is equivalent to the truncated Painlevé expansion (6) since the singularity manifold equations (5), (12), and (16) are form-invariant under the transformation

$$1 + \varepsilon\phi \to \phi$$

with $\varepsilon f \to \phi_x$, $\varepsilon g \to \phi_y$, $\varepsilon h \to \phi_{xy}$.

Next let us study Lie point symmetries of the prolonged systems instead of the single Eq. (5). According to the classical Lie point symmetry method, the Lie point symmetries for the whole prolonged systems possess the form

$$\begin{aligned} \sigma^u &= Xu_x + Yu_y + Tu_t - U, \\ \sigma^v &= Xv_x + Yv_y + Tv_t - V, \\ \sigma^\phi &= X\phi_x + Y\phi_y + T\phi_t - \Phi, \end{aligned}$$

$$\sigma^f = Xf_x + Yf_y + Tf_t - F,$$
$$\sigma^g = Xg_x + Yg_y + Tg_t - G,$$
$$\sigma^h = Xh_x + Yh_y + Th_t - H, \tag{19}$$

where $X, Y, T, U, V, \Phi, F, G, H$ are functions of x, y, t, u, v, ϕ, f, g, h, which means that the prolonged systems (5), (12), and (16) are invariant under the transformations

$$u \to u + \varepsilon\sigma^u, \quad v \to v + \varepsilon\sigma^v,$$
$$\phi \to \phi + \varepsilon\sigma^\phi, \quad f \to f + \varepsilon\sigma^f,$$
$$g \to g + \varepsilon\sigma^g, \quad h \to h + \varepsilon\sigma^h, \tag{20}$$

with the infinitesimal parameter ε.

The symmetries σ^k ($k = u, v, \phi, f, g, h$) are defined as the solution of the linearized equations of the prolonged systems (5), (12), and (16).

$$\begin{aligned}
&\sigma^u_t + \sigma^u_{xxy} - 8\sigma^u u u_y - 4u^2\sigma^u_y - 2\sigma^u_x v - 2u_x\sigma^v = 0,\\
&\sigma^v_x - 2\sigma^u u_y - 2u\sigma^u_y = 0,\\
&(\sigma^\phi_{xxxy} + \sigma^\phi_{xt})\phi_x^3 - (3\sigma^\phi_{xxy}\phi_{xx} + \sigma^\phi_t\phi_{xx} + 3\sigma^\phi_{xx}\phi_{xxy}\\
&\qquad + \sigma^\phi_{xx}\phi_t + \sigma^\phi_{xxx}\phi_{xy} + \sigma^\phi_{xy}\phi_{xxx})\phi_x^2 + (3\sigma^\phi_{xy}\phi_{xx}^2 + 3\sigma^\phi_x\phi_{xx}\phi_{xxy}\\
&\qquad + \sigma^\phi_x\phi_t\phi_{xx} + 6\sigma^\phi_{xx}\phi_{xx}\phi_{xy} + \sigma^\phi_x\phi_{xxx}\phi_{xy})\phi_x\\
&\qquad - 6\sigma^\phi_x\phi_{xx}^2\phi_{xy} = 0,\\
&\sigma^f - \sigma^\phi_x = 0,\\
&\sigma^g - \sigma^\phi_y = 0,\\
&\sigma^h - \sigma^f_y = 0.
\end{aligned} \tag{21}$$

We substitute the expressions (19) into the symmetry equations (21) and collect the coefficients of the independent partial derivatives of dependent variables u, v, ϕ, f, g, h. Then we obtain a system of overdetermined linear equations for the infinitesimals $X, Y, T, U, V, \Phi, F, G, H$, which can be easily given by solving the determining equations

$$\begin{aligned}
&X = c_1x + f_1, \quad Y = c_2y + c_3,\\
&T = (2c_1 + c_2)t + c_4,\\
&U = -c_1u - \frac{1}{2}f_2 f,\\
&V = -(c_1 + c_2)v - \frac{1}{2}(f_{1t} + f_2 h),\\
&\Phi = \frac{1}{2}f_2\phi^2 + f_3\phi + f_4,\\
&F = f_2\phi f + f_3 f,\\
&G = \frac{1}{2}f_{2y}\phi^2 + f_2\phi g + f_{3y}\phi + f_3 g + f_{4y},\\
&H = f_{2y}\phi f + f_2 g f + f_2\phi h + f_{3y}f + f_3 h,
\end{aligned} \tag{22}$$

where $f_1 \equiv f_1(t)$ is an arbitrary function of t,

$$f_2 \equiv f_2(y), \quad f_3 \equiv f_3(y), \quad \text{and} \quad f_4 \equiv f_4(y)$$

are arbitrary functions of y and c_1, c_2, c_3, and c_4 are arbitrary constants. When

$$c_1 = c_2 = c_3 = c_4 = f_1 = f_3 = f_4 = 0$$

and $f_2 = -2$, the obtained symmetry is just Eq. (17), and when $f_2 = 0$, the related symmetry is only the general Lie point symmetry of the original equation (5). To obtain more group invariant solutions, we would like to solve the symmetry constraint condition $\sigma^k = 0$ defined by Eq. (19) with Eq. (22), which is equivalent to solving the following characteristic equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}v}{V} = \frac{\mathrm{d}\phi}{\Phi} = \frac{\mathrm{d}f}{F} = \frac{\mathrm{d}g}{G} = \frac{\mathrm{d}h}{H}. \tag{23}$$

To solve the characteristic equations, one special case is listed in the following.

Without loss of generality, we assume $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, $f_1 = 1/c_5$, $f_2 = c_6$, $f_3 = c_7$, and $f_4 = c_8$. For simplicity, we introduce $\Delta^2 = c_7^2 - 2c_6c_8$. We find the similarity solutions after solving out the characteristic equations (23)

$$\begin{aligned}
&\phi = -\frac{c_7}{c_6} - \frac{\Delta}{c_6}\tanh\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],\\
&f = -F_2\,\mathrm{sech}^2\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],\\
&g = -F_3\,\mathrm{sech}^2\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],\\
&h = F_4\,\mathrm{sech}^2\left[\frac{1}{2}c_5\Delta(F_1 + x)\right]\\
&\qquad - \frac{2c_6}{\Delta}F_2F_3\,\mathrm{sech}^2\left[\frac{1}{2}c_5\Delta(F_1 + x)\right]\tanh\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],\\
&u = F_5 + \frac{c_6}{\Delta}F_2\tanh\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],\\
&v = F_6 + \left[\left(\frac{c_6}{\Delta}\right)^2F_2F_3 - \frac{c_6}{\Delta}F_4\right]\tanh\left[\frac{1}{2}c_5\Delta(F_1 + x)\right],
\end{aligned} \tag{24}$$

where

$$F_1 = F_1(\xi,\eta), \quad F_2 = F_2(\xi,\eta),$$
$$F_3 = F_3(\xi,\eta), \quad F_4 = F_4(\xi,\eta),$$
$$F_5 = F_5(\xi,\eta), \quad F_6 = F_6(\xi,\eta)$$

are the group-invariant functions while $\xi = y$ and $\eta = t - c_5x$ are the similarity variables. Substituting Eqs. (24) into the prolonged systems (5), (12), and (16), the invariant functions F_1, F_2, F_3, F_4, F_5, and F_6 satisfy the reduction systems

$$F_2 = -\frac{c_5\Delta^2}{2c_6}(c_5F_{1\eta}),$$
$$F_3 = -\frac{c_5\Delta^2}{2c_6}F_{1\xi},$$
$$F_4 = \frac{c_5\Delta^2}{2c_6}F_{1\xi\eta},$$

$$F_5 = \frac{c_5 F_{1\eta\eta}}{2(F_{1\eta}-c_5)},$$
$$F_6 = \frac{2}{(c_5F_{1\eta}-1)^2}((F_{1\xi\eta}F_{1\eta\eta}-F_{1\xi\eta\eta}F_{1\eta})c_5^3 + F_{1\xi\eta\eta}c_5^2 - F_{1\eta}^2 c_5 + F_{1\eta}), \tag{25}$$

in the above equations, F_1 satisfies the following reduction equation

$$\Delta^2 F_{1\xi\eta}F_{1\eta}^4 c_5^6 - 4\Delta^2 F_{1\xi\eta}F_{1\eta}^3 c_5^5 + (6\Delta^2 F_{1\xi\eta}F_{1\eta}^2 + F_{1\eta}F_{1\xi\eta}F_{1\eta\eta\eta} - 3F_{1\xi\eta}F_{1\eta\eta}^2 + 3F_{1\eta}F_{1\eta\eta}F_{1\xi\eta\eta} - F_{1\xi\eta\eta\eta}F_{1\eta}^2)c_5^4$$
$$+ (2F_{1\eta}F_{1\xi\eta\eta\eta} - 4\Delta^2 F_{1\eta}F_{1\xi\eta} - F_{1\xi\eta}F_{1\eta\eta\eta} - 3F_{1\eta\eta}F_{1\xi\eta\eta})c_5^3 + (\Delta^2 F_{1\xi\eta} - F_{1\xi\eta\eta\eta})c_5^2 + F_{1\eta}F_{1\eta\eta}c_5 - F_{1\eta\eta} = 0. \tag{26}$$

It is obvious that once the solutions F_1 are solved out with Eq. (26), the solutions for F_2, F_3, F_4, F_5, and F_6 can be solved out directly from Eq. (25). So the explicit solutions for the (2+1)-dimensional modified mBS equation (5) are immediately obtained by substituting F_1, F_2, F_3, F_4, F_5, and F_6 into Eq. (24).

3. CRE solvable and soliton–cnoidal waves solution

3.1. CRE solvable

For the (2+1)-dimensional mBS equation (5), we aim to look for its truncated Painlevé expansion solution in the following possible form

$$u = u_0 + u_1 R(w),$$
$$v = v_0 + v_1 R(w) + v_2 R(w)^2, \quad (w = w(x,y,t)), \tag{27}$$

where $R(w)$ is a solution of the Riccati equation

$$R_w = b_0 + b_1 R + b_2 R^2, \tag{28}$$

with b_0, b_1, b_2 being arbitrary constants. By vanishing all the coefficients of the power of $R(w)$ after substituting Eq. (28) with Eq. (27) into Eq. (5), we have nine over-determined equations for only six undetermined functions u_0, u_1, v_0, v_1, v_2, and w, it is fortunate that the overdetermined system may be consistent, thus we obtain

$$u_1 = b_2 w_x, \quad u_0 = \frac{1}{2}(b_1 w_x + \frac{w_{xx}}{w_x}),$$
$$v_2 = b_2^2 w_x w_y, \quad v_1 = b_2\left(b_1 w_x w_y + w_{xy}\right),$$
$$v_0 = b_0 b_2 w_x w_y + \frac{1}{2} b_1 w_{xy} + \frac{w_t}{2w_x} + \left(\frac{w_{xy}}{2w_x^2}\right)_x, \tag{29}$$

and the function w must satisfy

$$\delta w_x (C_1 w_x)_x + P_{1x} + S_{1y} = 0,$$
$$(\delta = 4b_0 b_2 - b_1^2), \tag{30}$$

where

$$P_1 = \frac{w_t}{w_x},$$
$$S_1 = \frac{w_{xxx}}{w_x} - \frac{3}{2}\left(\frac{w_{xx}}{w_x}\right)^2. \tag{31}$$

From above discussion, it is shown that equation (5) really has the truncated Painlevé expansion solution related to the Riccati equation (28). At this point, we call the expansion (27) a consistent Riccati expansion (CRE) and the (2+1)-dimensional mBS equation is CRE solvable.[29]

In summary, we have the following theorem:

Theorem 4 If w is a solution of

$$\delta w_x (C_1 w_x)_x + P_{1x} + S_{1y} = 0, \tag{32}$$

then,

$$u = \frac{1}{2}\left(b_1 w_x + \frac{w_{xx}}{w_x}\right) + b_2 w_x R,$$
$$v = v_0 + v_1 R + v_2 R^2, \tag{33}$$

is a solution of Eq. (5), with $R \equiv R(w)$ being a solution of the Riccati equation (28).

3.2. Soliton–cnoidal wave interaction solutions

Obviously, the Riccati equation (28) has a special solution $R(w) = \tanh(w)$, while the truncated Painlevé expansion solution (27) becomes

$$u = u_0 + u_1 \tanh(w),$$
$$v = v_0 + v_1 \tanh(w) + v_2 \tanh^2(w), \tag{34}$$

where u_0, u_1, v_0, v_1, v_2, and w are determined by Eqs. (28), (29), and (30).

We know the solution (34) is just consistent with Theorem 4. As consistent tanh-function expansion (CTE) (34) is a special case of CRE, it is quite clear that a CRE solvable system must be CTE solvable, and *vice versa*. If a system is CTE solvable, some important interaction solitary wave solutions can be constructed directly. In order to say the relation clearly, we give out the following Bäcklund transformation.

Theorem 5 (BT) If w is a solution of Eq. (30) with $\delta = 4$, then

$$u = u_0 - w_x \tanh(w),$$
$$v = v_0 - w_{xy}\tanh(w) + w_x w_y \tanh^2(w), \tag{35}$$

is a solution of Eq. (5), where $\{u_0, v_0\}$ is determined by Eq. (29) with $b_0 = 1$, $b_1 = 0$, $b_2 = -1$, $\delta = 4$.

In order to obtain the solution of Eq. (5), we consider w in the form

$$w = k_1 x + l_1 y + d_1 t + g, \tag{36}$$

where g is a function of x, y and t. It will lead to the interaction solutions between a soliton and other waves. By means of Theorem 5, some nontrivial solutions of (2+1)-dimensional mBS equation can be obtained from some quite trivial solutions of Eq. (23), which are listed as follows.

Case 1 In Eq. (30), we take a trivial solution for w, saying

$$w = kx + ly + dt + c, \tag{37}$$

with k, l, d, and c being arbitrary constants. Then substituting Eq. (36) into Theorem 5 yields the following kink soliton and ring soliton solution of the (2+1)-dimensional mBS equation (5)

$$\begin{aligned} u &= -k\tanh(kx + ly + dt + c), \\ v &= -\frac{d+l}{2k} - kl\,\mathrm{sech}^2(kx + ly + dt + c). \end{aligned} \tag{38}$$

Case 2 To find out the interaction solutions between soliton and cnoidal periodic wave, let

$$\begin{aligned} &w = k_1 x + l_1 y + d_1 t + W(X), \\ &(X \equiv k_2 x + l_2 y + d_2 t), \end{aligned} \tag{39}$$

where $W_1 \equiv W_1(X) = W_X$ satisfies

$$W_{1X}^2 = a_0 + a_1 W_1 + a_2 W_1^2 + a_3 W_1^3 + a_4 W_1^4, \tag{40}$$

with a_0, a_1, a_2, a_3, and a_4 being constants. Substituting Eq. (39) with Eq. (40) into Theorem 4, we have the following relations

$$\begin{aligned} a_0 &= a_1\frac{k_1}{k_2} - a_2\left(\frac{k_1}{k_2}\right)^2 + a_3\left(\frac{k_1}{k_2}\right)^3 - 4\left(\frac{k_1}{k_2}\right)^4, \\ a_4 &= 4, \\ d_2 &= -a_1 l_2\frac{k_2^3}{k_1} + 2a_2k_2^2l_2 - 3a_3k_1k_2l_2 \\ &\quad + 16k_1^2l_2 + \frac{d_1k_2}{k_1}, \end{aligned} \tag{41}$$

which leads to the following explicit solutions of Eq. (5) in the form of

$$\begin{aligned} u &= \frac{1}{2}\frac{k_2^2 W_{1X}}{k_1 + k_2 W_1} - (k_1 + k_2 W_1)\tanh(k_1 x + l_1 y + d_1 t + W), \\ v &= -(k_1 + k_2 W_1)(l_1 + l_2 W_1) + \frac{1}{2}\frac{d_1 + d_2 W_1 + k_2^2 l_2 W_{1XX}}{k_1 + k_2 W_1} - \frac{1}{2}\frac{k_2^3 l_2 W_{1X}^2}{(k_1 + k_2 W_1)^2} + d_1 t + W)^2. \end{aligned} \tag{42}$$

It is known that an equation by the definition of the elliptic functions can be written out in terms of Jacobi elliptic functions. The formula (42) exhibits the interactions between soliton and abundant cnoidal periodic waves. To show these soliton–cnoidal waves more intuitively, we just take a simple solution of Eq. (40) as

$$W_1 = \mu_0 + \mu_1\,\mathrm{sn}(mX, n), \tag{43}$$

where $\mathrm{sn}(mX, n)$ is the usual Jacobi elliptic sine function. The modulus n of the Jacobi elliptic function satisfies: $0 \le n \le 1$. When $n \to 1$, $\mathrm{sn}(\xi)$ degenerate as hyperbolic function $\tanh(\xi)$, when $n \to 0$, $\mathrm{sn}(\xi)$ degenerates as a trigonometric function $\sin(\xi)$. Substituting Eq. (43) with Eq. (41) into Eq. (40) and setting the coefficients of $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$, $\mathrm{sn}(\xi)$ equal zero, yields

$$\begin{aligned} a_1 &= (1-n^2)m^3 - (2n^2 - 10)m^2\frac{k_1}{k_2} + 24m\left(\frac{k_1}{k_2}\right)^2 \\ &\quad + 16\left(\frac{k_1}{k_2}\right)^3, \\ a_3 &= 8m + 16\frac{k_1}{k_2}, \\ a_2 &= (5-n^2)m^2 + 24m\frac{k_1}{k_2} + 24\left(\frac{k_1}{k_2}\right)^2, \\ \mu_0 &= -\frac{1}{2}m - \frac{k_1}{k_2}, \\ \mu_1 &= \frac{1}{2}mn. \end{aligned} \tag{44}$$

Hence, one kind of soliton–cnoidal wave solution is obtained by taking Eq. (43) and

$$W = \mu_0 X + \mu_1 \int_{X_0}^{X} sn(mY, n)dY, \tag{45}$$

with the parameter requirement (44) into the general solution (42).

The solution given in Eq. (42) with Eq. (41) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave. In Fig. 1, we plot the interaction solution of the potential u when the value of the Jacobi elliptic function modulus $n \neq 1$. In Fig .2, we plot the interaction solution of the potential v when the value of the Jacobi elliptic function modulus $n \neq 1$. This kind of solution can be easily applicable to the analysis of interesting physical phenomenon. In fact, there are plentiful solitary waves and cnoidal periodic waves in the real physics world.

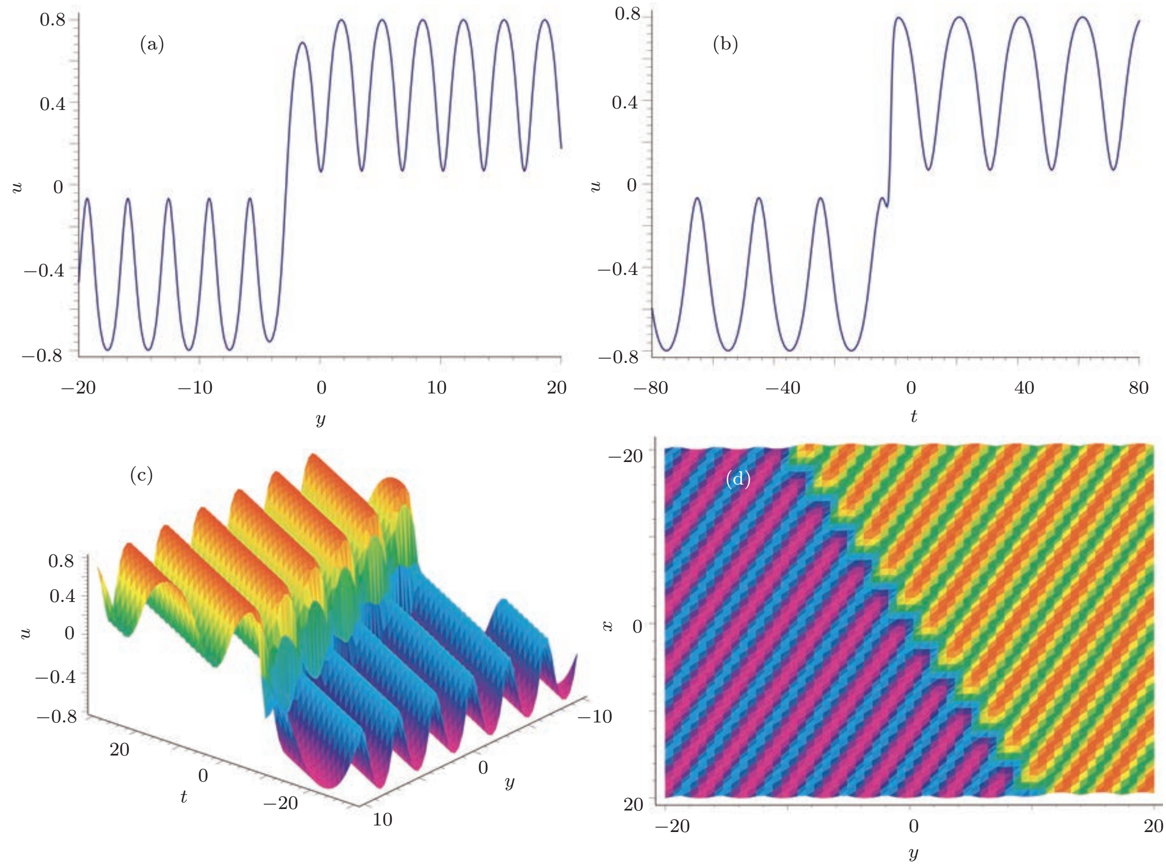

Fig. 1. (color online) The first type of soliton–cnoidal wave interaction solution for u with the parameters $m = 1$, $n = 1/2$, $k_2 = 1$, $\mu_0 = 1$, and $\mu_1 = 1/4$: (a) one-dimensional image at $x = 0$, $t = 2$; (b) one-dimensional image at $x = 0$, $y = 2$; (c) the three-dimensional plot; (d) overhead view for u at $t = 0$.

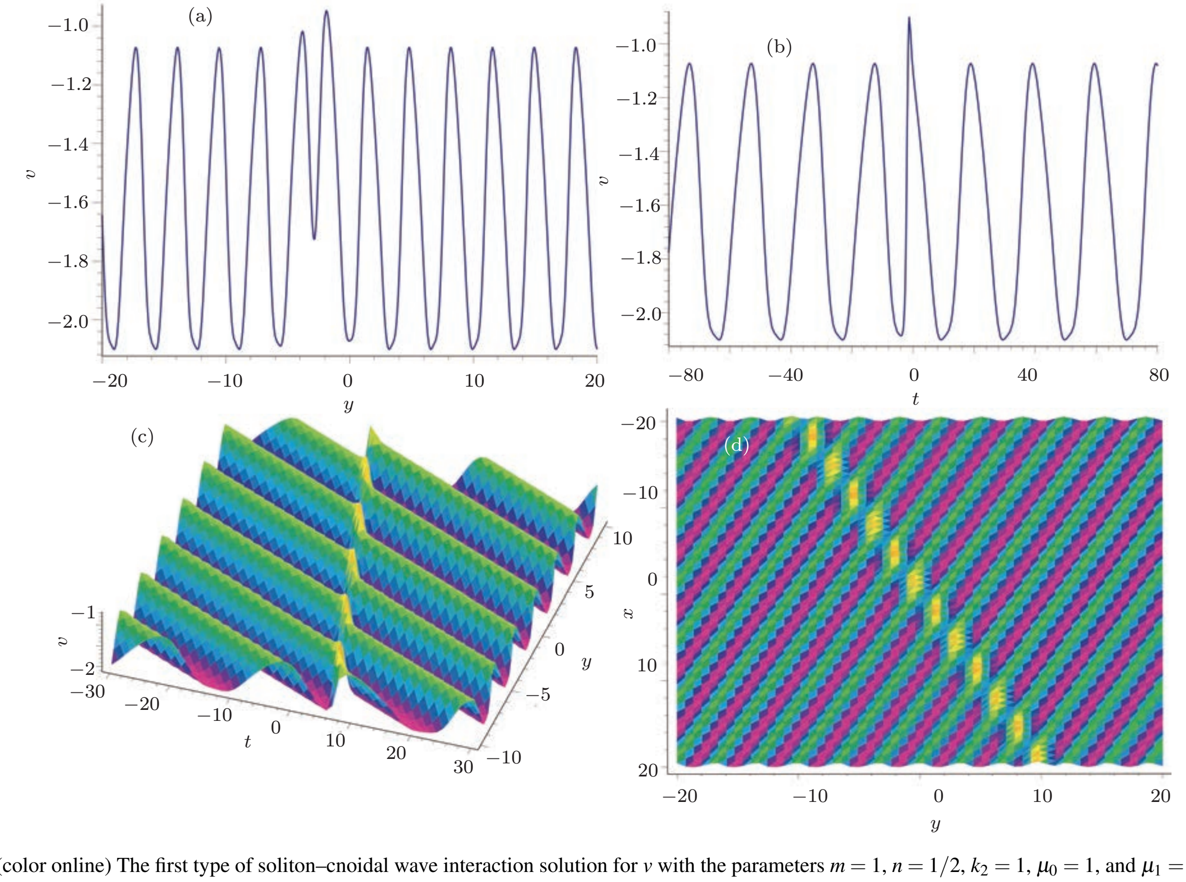

Fig. 2. (color online) The first type of soliton–cnoidal wave interaction solution for v with the parameters $m = 1$, $n = 1/2$, $k_2 = 1$, $\mu_0 = 1$, and $\mu_1 = 1/4$: (a) one-dimensional image at $x = 0$, $t = 2$; (b) one-dimensional image at $x = 0$, $y = 2$; (c) the three-dimensional plot; (d) overhead view for v at $t = 0$.

4. Summary and discussion

In summary, the (2+1)-dimensional mBS equation is investigated by using the truncated Painlevé analysis. The nonlocal symmetries, Bäcklund transformations and CRE solvable of the equation are found. Then by means of the CRE method, the soliton–cnoidal wave solutions of the (2+1)-dimensional mBS equation are obtained. By a special form of CRE, i.e., the consistent tanh-function expansion (CTE), kink soliton+cnoidal periodic wave solution and ring soliton+cnoidal periodic wave solution are explicitly expressed by the Jacobi elliptic functions and the corresponding elliptic integral. The interactions between solitons and cnoidal periodic waves display some interesting and physical phenomena. The CRE method used here can be developed to find other kinds of solutions and integrable models. It can also be used to find interaction solutions among different kinds of nonlinear waves. The CRE method did provide us with the result which is quite nontrivial and difficult to be obtained by other traditional approaches.

In addition, the generalized mKdV equation has been investigated in many aspects, but numerical methods relevant to the (2+1)-dimensional mBS equation have been reported little in the current articles. So uncovering more integrable properties of the equation, such as the Darboux transformation, Hamiltonian structure and the conservation, are interesting and meaningful work. The details on the CRE method and other methods to solve interaction solutions among different kinds of nonlinear waves and the investigation of other integrability properties such as Hamiltonian structure and generalized nonlocal symmetry of the (2+1)-dimensional mBS equation deserves further study.

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