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Bi-Hamiltonian Structure of Multi-Component Yajima-Oikawa Hierarchy

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Abstract: In this article, we construct the bi-Hamiltonian structure of the multi-component Yajima-Oikawa hierarchy.

Keywords: Bi-Hamiltonian Structure; Lax Pair; Yajima-Oikawa Hierarchy.

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is of fundamental importance in the theory of integrable systems and has been studied from various aspects [1–4]. It is well known that the Lax equation of the KP hierarchy is given by

$$L_{t_n}=[(L^n)_+,L],\ n\geq 1, \tag{1}$$

where $L=\partial+U_1\partial^{-1}+U_2\partial^{-2}+\cdots$ is a pseudo-differential operator and $(L^n)_+$ is defined as the differential part of L^n. The KP equation, which is derived from the propagation of two-dimensional dispersive waves on shallow water, can be obtained from the reduction of the KP hierarchy [5, 6]. Besides, many other well-known physical models, such as the KdV equation, the Boussinesq-Kaup equation, and the equation for a nonlinear string, are contained in the hierarchies obtained by the so-called k-reductions of the KP hierarchy [7, 8].

The k-constrained KP hierarchy is proposed in Sidorenko and Strampp and Konopelchenko et al. [9, 10] by the symmetry reduction of the KP hierarchy. It is not only mathematically important but also physically relevant. It is bi-Hamiltonian, has Darboux transformation, and is related to the W algebra theory [11–13]. Moreover, some physically applicable models, such as the Yajima-Oikawa model and the Melnikov model [14–17], which describe, respectively, the interacting waves appearing in plasma physics and hydrodynamics, can be found in the constrained KP hierarchy.

The pair of Yajima-Oikawa equations

$$iE_t+\frac{1}{2}E_{xx}-nE=0, \tag{2}$$

$$n_t+n_x+(|E|^2)_x=0, \tag{3}$$

describes the interaction of Langmuir and sound waves in plasmas [18]. Through a transformation $\phi=Ee^{i(t/2-x)}$, $\xi=x-t$, $\tau=t$ and a scaling for ξ, ϕ, the Yajima-Oikawa system is transformed to the long-wave-short-wave resonant interaction model (sometimes also called the Yajima-Oikawa system)

$$i\phi_\tau=-\phi_{\xi\xi}+n\phi, \tag{4}$$

$$n_\tau=-2(|\varphi|^2)_\xi, \tag{5}$$

which plays a central role in the transition to turbulence of boundary layers and other shear flows [19–22]. The Yajima-Oikawa system can be solved by the inverse scattering transform and thus has N-soliton solutions [18, 23]. Its rogue wave solutions are discussed by the Darboux dressing technique [24].

Moreover, the multi-component generalizations of the k-constrained KP hierarchy associated with the Lax operator

$$L_k=(L^k)_++\sum_{i=1}^{m}q_i\partial^{-1}r_i. \tag{6}$$

have been considered by Sidorenko and Strampp [25]. When $k=1$, $k=2$, and $k=3$, the multi-component generalizations of the AKNS hierarchy [26], Yajima-Oikawa hierarchy and Melnikov hierarchy are obtained. The recursion operators and bi-Hamiltonian structures have also been calculated for the multi-component generalizations of the k-constrained KP hierarchy under the constraint $r_i=q_i^*$, $(i=1,\ldots,m)$. However, the 'Hamiltonian operator' given by Sidorenko and Strampp for the coupled Yajima-Oikawa hierarchy is incorrect. Liu corrected the result and

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gave a bi-Hamiltonian structure for the coupled Yajima-Oikawa hierarchy [27]. Furthermore, the interaction between the multiple short waves and a long wave can be described by the following multi-component generalization of the Yajia-Oikawa system [28]:

$$i\phi_t^l+\phi_{xx}^l+n\phi^l=0,\ l=1,\ \dots \tag{7}$$

$$n_t=\sum_l c_l(|\phi^l|^2)_x. \tag{8}$$

The Painlevé analysis, soliton solutions, and energy-sharing collisions of the multi-component Yajima-Oikawa equation are also studied by them.

The purpose of this article is to construct the bi-Hamiltonian structure for the multi-component Yajima-Oikawa hierarchy using the trivector technique described in the work of Olver [29].

2 Bi-Hamiltonian Structure of Multi-Component Yajima-Oikawa Hierarchy

In this section, we derive the bi-Hamiltonian structure for the multi-component Yajima-Oikawa hierarchy.

The multi-component Yajima-Oikawa hierarchy defined by the Lax representation

$$L_{t_k}=[(L^{\frac{k}{2}})_+,L],\ L=\partial^2-u-Q^T\partial^{-1}R, \tag{9}$$

is the multi-component generalization of 2-constrained KP hierarchy, where Q, R are the n-component vector potentials defined as

$$Q=(q_1,q_2,\dots,q_n)^T,\ R=(r_1,r_2,\dots,r_n)^T. \tag{10}$$

It is not difficult to show that the aforementioned hierarchy is equal to the hierarchy of Yajima-Oikawa system (7) and (8) after a simple transformation, and we only consider the multi-component Yajima-Oijawa hierarchy (9).

In fact, the hierarchy (9) can also be obtained by the zero-curvature equation

$$M_t-N_x+[M,N]=0, \tag{11}$$

which is the compatibility condition of a $(n+2)\times(n+2)$ matrix spectral problem

$$\varphi_x=M\varphi,\ \varphi_t=N\varphi, \tag{12}$$

with

$$M=\begin{pmatrix}0&0&1\\ R&0_{n\times n}&0^T\\ \lambda+u&Q^T&0\end{pmatrix},\ N=\begin{pmatrix}N_{1,1}&A&N_{1,n+2}\\ B&S&C\\ N_{n+2,1}&D&N_{n+2,n+2}\end{pmatrix}, \tag{13}$$

where 0 and $0_{n\times n}$ are, respectively, n dimensional row vector and $n\times n$ zero matrix, and A, B, C, D, S and each entry $N_{i,j}$ are dependent on the potentials u, Q, R and the spectral parameter λ.

Substituting (13) into (11), we get

$$\begin{aligned}&D=A_x+N_{1,n+2}Q^T,\ S=\partial^{-1}(RA-CQ^T),\ B=-C_x+N_{1,n+2}R,\\ &N_{1,1}=\frac{1}{2}[\partial^{-1}(Q^TC-AR)-(N_{1,n+2})_x],\\ &N_{n+2,n+2}=\frac{1}{2}[\partial^{-1}(Q^TC-AR)+(N_{1,n+2})_x],\\ &N_{n+2,1}=\frac{1}{2}(Q^TC+AR)-\frac{1}{2}(N_{1,n+2})_{xx}+(\lambda+u)N_{1,n+2},\end{aligned}$$

and

$$\begin{pmatrix}u\\ Q\\ R\end{pmatrix}=(\lambda\mathcal{K}+\mathcal{J})\begin{pmatrix}N_{1,n+2}\\ C\\ A^T\end{pmatrix}, \tag{14}$$

where

$$\mathcal{K}=\begin{pmatrix}2\partial&0&0\\ 0^T&0_{n\times n}&-I_n\\ 0^T&I_n&0_{n\times n}\end{pmatrix}, \tag{15}$$

and

$$\mathcal{J}=\begin{pmatrix}-\frac{1}{2}\partial^3+u_x+2u\partial&\frac{1}{2}(Q_x^T+3Q^T\partial)&\frac{1}{2}(R_x^T+3R^T\partial)\\ Q_x+\frac{3}{2}Q\partial&(Q\partial^{-1}Q^T)^T+\frac{1}{2}Q\partial^{-1}Q^T&J_1\\ R_x+\frac{3}{2}R\partial&-J_1^*&(R\partial^{-1}R^T)^T+\frac{1}{2}R\partial^{-1}R^T\end{pmatrix}, \tag{16}$$

with

$$I_n=\mathrm{diag}(\underbrace{1,1,\cdots,1}_{n}),\ J_1=(\partial^2-u-Q^T\partial^{-1}R)I_n-\frac{1}{2}Q\partial^{-1}R^T.$$

It is easy to see that the operators $\mathcal{K}$ and $\mathcal{J}$ are skew-symmetric and that $\mathcal{K}$ is a Hamiltonian operator. In the following, we show $\mathcal{J}$ is a Hamiltonian operator and is compatible with $\mathcal{K}$.

Our main results are summarised as follows.

Theorem 1 The multi-component Yajima-Oikawa hierarchy (9) is a bi-Hamiltonian system

$$\begin{pmatrix} u \\ Q \\ R \end{pmatrix}_{t_k} = \mathcal{K} \begin{pmatrix} \frac{\delta H_{k+2}}{\delta u} \\ \frac{\delta H_{k+2}}{\delta Q} \\ \frac{\delta H_{k+2}}{\delta R} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta H_k}{\delta u} \\ \frac{\delta H_k}{\delta Q} \\ \frac{\delta H_k}{\delta R} \end{pmatrix}, \tag{17}$$

where the operators $\mathcal{K}$ and $\mathcal{J}$ are given by (15) and (16), respectively, and $H_k = \frac{2}{k}\mathrm{Res}(L^{\frac{k}{2}})$.

The proof of the theorem is given by the use of Olver's technique [29]. Let θ denote the basic uni-vector corresponding to the potential, and $\mathcal{D}$ is any skew-symmetric operator depending on the spatial variable x and the potential. In the proof, we will use mainly the properties listed following:

– the basic property of wedge product

$$\int \xi \wedge \eta \mathrm{d}x = (-1)^{mn} \int \eta \wedge \xi \mathrm{d}x, \tag{18}$$

for any m-form ξ and n-form η.

– the skew-symmetry of the operator $\mathcal{D}$

$$\int \xi \wedge \mathcal{D}\eta \mathrm{d}x = -\int (\mathcal{D}\xi) \wedge \eta \mathrm{d}x. \tag{19}$$

– the prolongation

$$-\mathrm{Pr}V_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta) = \theta \wedge \mathrm{Pr}V_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta. \tag{20}$$

The minus sign in (20) arises from the fact that we have interchanged a wedge product of θ using the formula (18).

Proof: Let θ_0, $\theta_1 = (\theta_{11}, \theta_{12}, \dots, \theta_{1n})^T$ and $\theta_2 = (\theta_{21}, \theta_{22}, \dots \theta_{2n})^T$ be the basic uni-vectors corresponding to u, Q, and R, respectively. We know that the operator $\mathcal{J}$ is the Hamiltonian if and only if (because the skew-symmetry is known)

$$\mathrm{Pr}V_{\mathcal{J}\theta}(\Theta_{\mathcal{J}}) = 0, \tag{21}$$

where $\theta = (\theta_0, \theta_1, \theta_2)$ and

$$\Theta_{\mathcal{J}} = \frac{1}{2}\int (\theta \wedge \mathcal{J}\theta)\mathrm{d}x,$$

is the associated bi-vector of $\mathcal{J}$.

To check whether $\mathcal{K}$ and $\mathcal{J}$ form a bi-Hamiltonian pair, we only need to prove

$$\mathrm{Pr}V_{K\theta}(\Theta_{\mathcal{J}}) = 0. \tag{22}$$

The proof of the theorem is rather technical and lengthy, and we give it in the Appendix.

Remark 1 When $n=2$, the operators $\mathcal{K}$ and $\mathcal{J}$ are just the Hamiltonian operators of the coupled Yajima-Oikawa hierarchy [27].

It is known that for a pseudo-differential symbol of the form $X = \sum_{i\le 0} X_i \partial^i + \sum_{i>0} X_i \partial^i$, Res$(X)$ is defined as [30]:

$$\mathrm{Res}(X) := \int \mathrm{res}(X)\mathrm{d}x, \ \mathrm{res}(X) = X_{-1},$$

Therefore, the Hamiltonian functions H_k, $(k=1, \dots)$ [11, 27] can be obtained by a direct computation:

$$H_1 = -\int u \mathrm{d}x$$
$$H_2 = -\int Q^T R \mathrm{d}x$$
$$\dots\dots,$$

Then, the second positive flow is obtained

$$\begin{pmatrix} u \\ Q \\ R \end{pmatrix}_{t_2} = J \begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta Q} \\ \frac{\delta H_2}{\delta R} \end{pmatrix} = \begin{pmatrix} -2(Q^T R)_x \\ -Q_{xx} + uQ \\ R_{xx} - uR \end{pmatrix} \tag{23}$$

with a Lax pair

$$\varphi_x = M\varphi, \ \varphi_t = \begin{pmatrix} -\lambda & -Q^T & 0 \\ R_x & 0_{n\times n} & -R \\ -Q^T R & -Q_x^T & -\lambda \end{pmatrix} \varphi.$$

If $n=1$, we get

$$q_{1t} = -q_{1xx} + uq_1, \ r_{1t} = r_{1xx} - ur_1, \ u_t = -2(q_1 r_1)_x. \tag{24}$$

In what follows, we connect the system (24) with the Yajima-Oikawa system [5]. Starting from the Yajima-Oikawa system (2) and (3), we can rewrite it as

$$iE_t + \frac{1}{2}E_{xx} - nE = 0,$$
$$-iE_t^* + \frac{1}{2}E_{xx}^* - nE^* = 0,$$
$$n_t + n_x + (|E|^2)_x = 0.$$

Then, after a transformation $\phi = Ee^{i(t/2-x)}$, $\xi = x - 1$, $\tau = t$, the aforementioned system is changed to

$$i\phi_\tau + \frac{1}{2}\phi_{\xi\xi} - n\phi = 0,$$
$$-i\phi_\tau^* + \frac{1}{2}\phi_{\xi\xi}^* - n\phi^* = 0,$$
$$n_\tau + (|\phi|^2)_\xi = 0.$$

After the transformation $t'=i\tau$, $x'=-i\sqrt{2}\xi$, $\phi=\sqrt[4]{2}\tilde{\phi}$ and the rewriting t', x' back to t, x, we get (24) as we set $q_1=\tilde{\phi}$, $r_1=\tilde{\phi}^*$, $n=-u$.

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Appendix

First, we prove that the operator $\mathcal{J}$ is Hamiltonian, namely to verify (21). To simplify the presentations and calculations, we define the function U and the n dimensional column vectors $\widetilde{Q}=(\widetilde{q_1}, \widetilde{q_2}, \dots, \widetilde{q_n})^T$, $\widetilde{R}=(\widetilde{r_1}, \widetilde{r_2}, \dots, \widetilde{r_n})^T$ as

$$U=-\frac{1}{2}\theta_{0xxx}+u_x\theta_0+2u\theta_{0x}+\frac{1}{2}\left(Q_x^T\theta_1+3Q^T\theta_{1x}\right)+\frac{1}{2}\left(R_x^T\theta_2+3R^T\theta_{2x}\right), \tag{25}$$

$$\widetilde{Q}=Q_x\theta_0+\frac{3}{2}Q\theta_{0x}+(Q\partial^{-1}Q^T)^T\theta_1+\frac{1}{2}Q\partial^{-1}(Q^T\theta_1-R^T\theta_2)+(\partial^2-u-Q^T\partial^{-1}R)\theta_2, \tag{26}$$

$$\widetilde{R}=R_x\theta_0+\frac{3}{2}R\theta_{0x}+(R\partial^{-1}R^T)^T\theta_2-\frac{1}{2}R\partial^{-1}(Q^T\theta_1-R^T\theta_2)+(-\partial^2+u-R^T\partial^{-1}Q)\theta_1. \tag{27}$$

It is easy to check that

$$U=-\frac{1}{2}\theta_{0xxx}+u_x\theta_0+2u\theta_{0x}+\frac{1}{2}\sum_{k=1}^{n}(q_{kx}\theta_{1k}+r_{kx}\theta_{2k}+3q_k\theta_{1kx}+3r_k\theta_{2kx}), \tag{28}$$

$$\widetilde{q_i}=q_{ix}\theta_0+\frac{3}{2}q_i\theta_{0x}+(\partial^2-u)\theta_{2i}+\sum_{k=1}^{n}q_k\partial^{-1}(q_i\theta_{1k}-r_k\theta_{2i})+\frac{1}{2}q_i\sum_{k=1}^{n}\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k}), \tag{29}$$

$$\widetilde{r_i}=r_{ix}\theta_0+\frac{3}{2}r_i\theta_{0x}+(-\partial^2+u)\theta_{1i}-\sum_{k=1}^{n}r_k\partial^{-1}(q_k\theta_{1i}-r_i\theta_{2k})-\frac{1}{2}r_i\sum_{k=1}^{n}\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k}). \tag{30}$$

From (16), we have

$$\mathcal{J}\theta=\begin{pmatrix}U\\ \widetilde{Q}\\ \widetilde{R}\end{pmatrix}=(U, \widetilde{q_1}, \dots, \widetilde{q_n}, \widetilde{r_1}, \dots, \widetilde{r_n})^T.$$

Then the associated bi-vector for $\mathcal{J}$ is

$$\begin{aligned}\Theta_{\mathcal{J}}&=\frac{1}{2}\int(\theta\wedge\mathcal{J}\theta)\mathrm{d}x\\ &=\frac{1}{2}\int\theta_0\wedge U+\theta_1\wedge\widetilde{Q}+\theta_2\wedge\widetilde{R}\mathrm{d}x\\ &=\frac{1}{2}\int\theta_0\wedge U\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{n}\int\theta_{1i}\wedge\widetilde{q_i}+\theta_{2i}\wedge\widetilde{r_i}\mathrm{d}x,\end{aligned}$$

substituting (28), (29), and (30) into the aforementioned equality and applying the properties (18), (19), we get

$$\begin{aligned}\Theta_{\mathcal{J}}=&\sum_{i=1}^{n}\int-\theta_{1ix}\wedge\theta_{2ix}-\theta_{1i}\wedge u\theta_{2i}+\theta_0\wedge(q_i\theta_{1ix}+r_i\theta_{2ix})\\ &-\frac{1}{2}\theta_{0x}\wedge(q_i\theta_{1i}+r_i\theta_{2i})\mathrm{d}x+\frac{1}{2}\sum_{i,j=1}^{n}\int(q_j\theta_{1i}-r_i\theta_{2j})\\ &\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\ &+\frac{1}{2}(q_i\theta_{1i}-r_i\theta_{2i})\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x+\int\theta_0\wedge u\theta_{0x}\mathrm{d}x.\end{aligned} \tag{31}$$

Calculation of (21) shows that

$$\begin{aligned}\mathrm{Pr}V_{\mathcal{J}\theta}(\Theta_{\mathcal{J}})=&\int\theta_0\wedge U\wedge\theta_{0x}\mathrm{d}x+\sum_{i=1}^{n}\int-\theta_{1i}\wedge U\wedge\theta_{2i}\\ &+\theta_0\wedge(\widetilde{q_i}\wedge\theta_{1ix}+\widetilde{r_i}\wedge\theta_{2ix})-\frac{1}{2}\theta_{0x}\wedge(\widetilde{q_i}\wedge\theta_{1i}+\widetilde{r_i}\wedge\theta_{2i})\mathrm{d}x\\ &+\sum_{i,j=1}^{n}\int(\theta_{1i}\wedge\widetilde{q_j}-\theta_{2j}\wedge\widetilde{r_i})\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\ &+\frac{1}{2}(\theta_{1i}\wedge\widetilde{q_i}-\theta_{2i}\wedge\widetilde{r_i})\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x.\end{aligned} \tag{32}$$

Substituting (28), (29), and (30) to the aforementioned expression and applying the properties (18), (19), the expression (32) can be divided into three parts: **I**, **II**, and **III**. Part **I** is about the terms for double summation without the variable θ_0. i.e.

$$\begin{aligned}\mathbf{I}=&\sum_{i,j=1}^{n}\int-\frac{1}{2}\theta_{1i}\wedge(q_{jx}\theta_{1j}+r_{jx}\theta_{2j}+3q_j\theta_{1jx}+3r_j\theta_{2jx})\wedge\theta_{2i}\\&+(\theta_{1i}\wedge\theta_{2jxx}+\theta_{2j}\wedge\theta_{1ixx})\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+\frac{1}{2}(\theta_{1i}\wedge\theta_{2ixx}+\theta_{2i}\wedge\theta_{1ixx})\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x\\=&\sum_{i,j=1}^{n}\int q_j(\theta_{1i}\wedge\theta_{1j}\wedge\theta_{2ix}-\theta_{1i}\wedge\theta_{1jx}\wedge\theta_{2i}-\theta_{1j}\wedge\theta_{2ix}\wedge\theta_{1i}\\&-\theta_{2i}\wedge\theta_{1jx}\wedge\theta_{1i})+r_j(\theta_{1ix}\wedge\theta_{2j}\wedge\theta_{2i}-\theta_{1i}\wedge\theta_{2jx}\wedge\theta_{2i}+\theta_{1i}\\&\wedge\theta_{2jx}\wedge\theta_{2i}+\theta_{2j}\wedge\theta_{1ix}\wedge\theta_{2i})\mathrm{d}x=0.\end{aligned}\tag{33}$$

Part **II** is the term of double summation contain θ_0, i.e.

$$\begin{aligned}\mathbf{II}=&\sum_{i,j=1}^{n}\int\theta_0\wedge\left\{\left[q_j\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})+\frac{1}{2}q_i\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\right]\wedge\theta_{1ix}\right.\\&\left.+\left[-r_j\partial^{-1}(q_j\theta_{1i}-r_i\theta_{2j})-\frac{1}{2}r_i\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\right]\wedge\theta_{2ix}\right\}\\&-\frac{1}{2}\theta_{0x}\wedge\left\{\left[q_j\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})+\frac{1}{2}q_i\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\right]\wedge\theta_{1i}\right.\\&\left.+\left[-r_j\partial^{-1}(q_j\theta_{1i}-r_i\theta_{2j})-\frac{1}{2}r_i\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\right]\wedge\theta_{2i}\right\}\\&+\left[\theta_{1i}\wedge\left(q_{jx}\theta_0+\frac{3}{2}q_j\theta_{0x}\right)-\theta_{2j}\wedge\left(r_{ix}\theta_0+\frac{3}{2}r_i\theta_{0x}\right)\right]\\&\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+\frac{1}{2}\left[\theta_{1i}\wedge\left(q_{ix}\theta_0+\frac{3}{2}q_i\theta_{0x}\right)-\theta_{2i}\wedge\left(r_{ix}\theta_0+\frac{3}{2}r_i\theta_{0x}\right)\right]\\&\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x\\=&\sum_{i,j=1}^{n}\int\left(\frac{1}{2}q_i\theta_{1ix}\wedge\theta_0+\frac{1}{2}q_{ix}\theta_{1i}\wedge\theta_0+\frac{1}{2}q_i\theta_{1i}\wedge\theta_{0x}\right)\\&\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\\&-\left(\frac{1}{2}r_i\theta_{2ix}\wedge\theta_0+\frac{1}{2}r_{ix}\theta_{2i}\wedge\theta_0+\frac{1}{2}r_i\theta_{2i}\wedge\theta_{0x}\right)\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\\&+\left(-q_j\theta_0\wedge\theta_{1ix}+\frac{1}{2}q_j\theta_{0x}\wedge\theta_{1i}+q_{jx}\theta_{1i}\wedge\theta_0+\frac{3}{2}q_j\theta_{1i}\wedge\theta_{0x}\right)\\&\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+\left(r_i\theta_0\wedge\theta_{2jx}-\frac{1}{2}r_i\theta_{0x}\wedge\theta_{2j}-r_{ix}\theta_{2j}\wedge\theta_0-\frac{3}{2}r_i\theta_{2j}\wedge\theta_{0x}\right)\\&\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\mathrm{d}x\\=&\sum_{i,j=1}^{n}\int-\frac{1}{2}q_i\theta_{1i}\wedge\theta_0\wedge(q_j\theta_{1j}-r_j\theta_{2j})+\frac{1}{2}r_i\theta_{2i}\wedge\theta_0\wedge(q_j\theta_{1j}-r_j\theta_{2j})\\&-q_j\theta_{1i}\wedge\theta_0\wedge(q_i\theta_{1j}-r_j\theta_{2i})+r_i\theta_{2j}\wedge\theta_0\wedge(q_i\theta_{1j}-r_j\theta_{2i})\mathrm{d}x\\=&\sum_{i,j=1}^{n}\int-\frac{1}{2}(q_i\theta_{1i}-r_i\theta_{2i})\wedge\theta_0\wedge(q_j\theta_{1j}-r_j\theta_{2j})\\&-(q_j\theta_{1i}-r_i\theta_{2j})\wedge\theta_0\wedge(q_i\theta_{1j}-r_j\theta_{2i})\mathrm{d}x\\=&0,\end{aligned}\tag{34}$$

The remainder terms of (32) are calculated as follows:

$$\begin{aligned}\mathbf{III}=&\sum_{i,j,k=1}^{n}\int\left\{\theta_{1i}\wedge\left[q_k\partial^{-1}(q_j\theta_{1k}-r_k\theta_{2j})+\frac{1}{2}q_j\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k})\right]\right.\\&\left.+\theta_{2j}\wedge\left[r_k\partial^{-1}(q_k\theta_{1i}-r_i\theta_{2k})+\frac{1}{2}r_i\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k})\right]\right\}\\&\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+\frac{1}{2}\left\{\theta_{1i}\wedge\left[q_k\partial^{-1}(q_i\theta_{1k}-r_k\theta_{2i})+\frac{1}{2}q_i\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k})\right]\right.\\&\left.+\theta_{2i}\wedge\left[r_k\partial^{-1}(q_k\theta_{1i}-r_i\theta_{2k})+\frac{1}{2}r_i\partial^{-1}(q_k\theta_{1k}-r_k\theta_{2k})\right]\right\}\\&\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x\\=&\sum_{i,j,k=1}^{n}\int q_k\theta_{1i}\wedge\partial^{-1}(q_j\theta_{1k}-r_k\theta_{2j})\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+r_k\theta_{2j}\wedge\partial^{-1}(q_k\theta_{1i}-r_i\theta_{2k})\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\mathrm{d}x\\=&\sum_{i,j,k=1}^{n}\int q_k\theta_{1i}\wedge\partial^{-1}q_j\theta_{1k}\wedge\partial^{-1}q_i\theta_{1j}+r_k\theta_{2j}\wedge\partial^{-1}r_i\theta_{2k}\wedge\partial^{-1}r_j\theta_{2i}\mathrm{d}x\\=&0.\end{aligned}\tag{35}$$

Then (33), (34), and (35) yields

$$\mathrm{Pr}V_{\mathcal{J}\theta}(\Theta_{\mathcal{J}})=0,\tag{36}$$

so the skew-symmetric operator $\mathcal{J}$ is Hamiltonian.

Next, we show the compatibility of the operators $\mathcal{K}$ and $\mathcal{J}$, i.e. the equality (22). Notice that

$$\mathcal{K}\theta=\begin{pmatrix}2\theta_{0x}\\-\theta_2\\\theta_1\end{pmatrix}.\tag{37}$$

Using the equality (31), we obtain

$$\begin{aligned}\mathrm{Pr}V_{\mathcal{K}\theta}(\Theta_{\mathcal{J}})=&\sum_{i=1}^{n}\int-\theta_{1i}\wedge2\theta_{0x}\wedge\theta_{2i}+\theta_0\wedge(-\theta_{2i}\wedge\theta_{1ix}+\theta_{1i}\wedge\theta_{2ix})\\&+\frac{1}{2}\theta_{0x}\wedge(\theta_{2i}\wedge\theta_{1i}-\theta_{1i}\wedge\theta_{2i})+\theta_0\wedge2\theta_{0x}\wedge\theta_{0x}\mathrm{d}x\\&+\sum_{i,j=1}^{n}\int[\theta_{1i}\wedge(-\theta_{2j})-\theta_{2j}\wedge\theta_{1i}]\wedge\partial^{-1}(q_i\theta_{1j}-r_j\theta_{2i})\\&+\frac{1}{2}[\theta_{1i}\wedge(-\theta_{2i})-\theta_{2i}\wedge\theta_{1i}]\wedge\partial^{-1}(q_j\theta_{1j}-r_j\theta_{2j})\mathrm{d}x\\=&\sum_{i=1}^{n}\int2\theta_{0x}\wedge\theta_{1i}\wedge\theta_{2i}-2\theta_{0x}\wedge\theta_{1i}\wedge\theta_{2i}\mathrm{d}x\\=&0.\end{aligned}\tag{38}$$

Equation (38) implies that the operators Q and P are compatible Hamiltonian operators and, therefore, the theorem is proved.

References

[1] Y. Cheng, Commun. Math. Phys. **171**, 661 (1995).
[2] Y. Y. Berkela and Y. M. Sidorendo, Math. Studii **17**, 47 (2002).
[3] H. X. Wu, Y. B. Zeng, and T. Y. Fan, J. Math. Phys. **49**, 093510 (2008).
[4] O. Chvartatskyi and Y. Sydorenko, J. Phys. Conf. Series **411**, 012010 (2013).
[5] Y. Cheng, J. Math. Phys. **33**, 3774 (1992).
[6] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970).
[7] A. M. Samoilenko, V. G. Samoilenko, and Y. M. Sidorenko, Ukr. Math. J. **51**, 86 (1999).
[8] P. M. Santini, Inverse Probl. **6**, 99 (1990).
[9] J. Sidorenko and W. Strampp, Inverse Probl. **7**, 37 (1991).
[10] B. G. Konopelchenko, J. Sidorenko, and W. Strampp, Phys. Lett. A **157**, 17 (1991).
[11] W. Oevel and W. Strampp, Commun. Math. Phys. **157**, 51 (1993).
[12] W. Oevel, Phys. A **195**, 533 (1993).
[13] Q. P. Liu and C. S. Xiong, Phys. Lett. B **327**, 257 (1994).
[14] V. K. Melnikov, Phys. Lett. A **118**, 22 (1986).
[15] V. K. Melnikov, Lett. Math. Phys. **7**, 129 (1983).
[16] V. K. Melnikov, Commun. Math. Phys. **112**, 639 (1987).
[17] V. K. Melnikov, Commun. Math. Phys. **120**, 481 (1989).
[18] N. Yajima and M. Oikawa, Prog. Theor. Phys. **56**, 1719 (1976).
[19] M. Funakoshi and M. Oikawa, J. Phys. Soc. Jpn. **52**, 1982 (1983).
[20] V. E. Zakharov, Sov. Phys. JETP **35**, 908 (1972).
[21] Y. C. Ma and L. G. Redekopp, Phys. Fluids **22**, 1872 (1979).
[22] M. T. Landahl, J. Fluid Mech. **56**, 775 (1972).
[23] A. A. Zabolotskii, Phys. Rev. A **80**, 063616 (2009).
[24] S. Chen, Phys. Lett. A **378**, 1095 (2014).
[25] J. Sidorenko and W. Strampp, J. Math. Phys. **34**, 1429 (1993).
[26] R. K. Bullough and P. J. Caudrey (Eds.), Soliton, Springer-Verlag, Berlin, 1980.
[27] Q. P. Liu, J. Math. Phys. **37**, 2307 (1996).
[28] T. Kanna, K. Sakkaravarthi, and K. Tamilselvan, Phys. Rev. E **88**, 062921 (2013).
[29] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer, Berlin, 1993.
[30] L. A. Dickey, Soliton Equations and Hamiltonian Systems, World Scientific, Singapore, 1991.