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Reciprocal Transformations of Two Camassa–Holm Type Equations*

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Abstract *The relation between the Camassa–Holm equation and the Olver–Rosenau–Qiao equation is obtained, and we connect a new Camassa–Holm type equation proposed by Qiao etc. with the first negative flow of the KdV hierarchy by a reciprocal transformation.*

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1 Introduction

The Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0\,, \quad m = u - u_{xx} \tag{1}$$

is derived by Camassa and Holm as a model for the propagation of shallow water wave.[1] It is completely integrable with a Lax pair and can be written as a bi-Hamiltonian form[1−2] by rearranging the Hamiltonian operators of the KdV equation,[3]

$$m_t = -(\partial - \partial^3)\frac{\delta H_2}{\delta m} = -(m\partial + \partial m)\frac{\delta H_1}{\delta m}\,, \tag{2}$$

with Hamiltonian functionals

$$H_1 = \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x\,,$$

$$H_2 = \frac{1}{2}\int (u^3 + uu_x^2)\,\mathrm{d}x\,.$$

In fact the CH equation is linked to the first negative flow of the KdV hierarchy by a reciprocal type transformation.[4−5] It is solvable via inverse scattering transformation,[6−9] and a large number of works concern about the solutions of it such as multi-soliton solution, algebro-geometric solution.[10−12] Besides, nonlocal symmetries and a Darboux transformation of it are studied in Ref. [13]. It is worthwhile to remark that, unlike the KdV equation, the CH equation possesses peaked soliton solutions which are weak solutions with discontinuous first derivatives,[1−2,14−15] and the equation admitting peakon solution is called CH type equation.

The Olver–Rosenau–Qiao (ORQ) equation

$$n_t + [n(v^2 - v_x^2)]_x = 0, \quad n = v - v_{xx} \tag{3}$$

is proposed by rearranging the Hamiltonian operators of the mKdV equation,[16] and can be written as

$$n_t = (\partial - \partial^3)\frac{\delta \hat{H}_2}{\delta n} = \partial n \partial^{-1} n \partial \frac{\delta \hat{H}_1}{\delta n}\,, \tag{4}$$

where the Hamiltonian functionals are

$$\hat{H}_1 = \frac{1}{2}\int (u^2 + u_x^2)\,\mathrm{d}x\,,$$

$$\hat{H}_2 = \frac{1}{24}\int (3u^4 - u_x^4 + 6u^2u_x^2)\,\mathrm{d}x\,.$$

Furthermore, the ORQ equation is integrable with a Lax pair[17] and may be derived from the two dimension Euler equations.[18] It is interesting that this equation has a reciprocal link to the (modified) KdV hierarchy[19] and is also solvable by inverse scattering transformation.[20] Formation of singularities and the existence of peaked traveling-wave solutions, which have only constant amplitudes for the ORQ equation are discussed in Ref. [21].

Recently, a hybrid system that combines the CH and the ORQ equations is proposed[22]

$$\begin{aligned} &m_t = k_1[m(u^2 - u_x^2)]_x + k_2(2mu_x + m_x u)\,,\\ &m = u - u_{xx}\,, \end{aligned} \tag{5}$$

where k_1, k_2 are arbitrary constants. The above equation is a bi-Hamiltonian system using the Hamiltonian operators of the CH and the ORQ equations, i.e.,

$$\begin{aligned} m_t &= \Big(k_1\partial m \partial^{-1} m\partial + \frac{1}{2}k_2(\partial m + m\partial)\Big)\frac{\delta \widetilde{H}_1}{\delta m}\\ &= (\partial - \partial^3)\frac{\delta \widetilde{H}_2}{\delta m}\,, \end{aligned} \tag{6}$$

where

$$\widetilde{H}_1 = \int (u^2 + u_x^2)\,\mathrm{d}x\,,$$

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$$\widetilde{H}_2 = \frac{1}{4}\int\Big(k_1u^4 + 2k_1u^2u_x^2 - \frac{1}{3}k_1u_x^4 + 2k_2u^3 + 2k_2uu_x^2\Big)\,\mathrm{d}x\,.$$

Moreover, a Lax pair and infinite sequences of conserved densities as well as the peaked soliton (peakon) and multi-peakon solutions of the Eq. (5) are also studied.[22]

The aim of this paper is to give the detail relations between the CH equation, the ORQ equation and the first negative flow of the KdV hierarchy, and to apply a reciprocal transformation to the hybrid system (5) as well.

2 The Relation between CH Equation and ORQ Equation

It is well known that the CH equation is reciprocal linked to the first negative flow of the KdV hierarchy(See [5, 23] etc.), and the ORQ equation is also connected to the first negative flow of the (modified) KdV hierarchy by a reciprocal transformation.[19] Then a nature idea arises that what is the relation between the CH equation and the ORQ equation? These may be illustrated by the following graph

$$\begin{tikzcd} \mathrm{CH} \arrow[r, "\text{reciprocal}"] \arrow[d, "?"'] & \mathrm{KdV} \arrow[d, "\text{Miura}"'] \\ \mathrm{ORQ} \arrow[r, "\text{reciprocal}"'] & \mathrm{mKdV} \end{tikzcd}$$

In this section, we will present the reciprocal transformations between the CH equation, the ORQ equation and the first negative flow of the KdV hierarchy respectively, and then we will construct the transformation between the CH equation and the ORQ equation.

The Lax pair for the first negative flow of the KdV hierarchy is

$$\varphi_{yy} + Q\varphi - \lambda\varphi = 0\,,$$
$$\varphi_\tau - \frac{1}{2\lambda}\Big(p\varphi_y - \frac{1}{2}p_y\varphi\Big) = 0\,,$$

and the compatibility condition of the Lax pair yields

$$Q_\tau = p_y\,, \quad \Omega\cdot p = (\partial_y^2 + 4Q + 2Q_y\partial_y^{-1})p_y = 0\,. \tag{7}$$

Integrating the condition $\Omega\cdot p = 0$, we get

$$pp_{yy} - \frac{1}{2}p_y^2 + 2Qp^2 + f(\tau) = 0\,, \tag{8}$$

where f is an arbitrary function of τ. For non-zero f, scaling p as $p \to \sqrt{2f}p$, we get

$$Q = -\frac{p_{yy}}{2p} + \frac{p_y^2}{4p^2} - \frac{1}{4p^2}\,,$$

and therefore after a redefinition of the time τ, we may regard the first negative flow of the KdV hierarchy (7) as the following form

$$Q_\tau = p_y, \quad Q = -\frac{p_{yy}}{2p} + \frac{p_y^2}{4p^2} - \frac{1}{4p^2}\,. \tag{9}$$

Then, we will relate the flow (9) to the CH equation (1) and the ORQ equation (3) by two different reciprocal transformations. Because the two reciprocal transformations are the inverse of the transformations which transform the CH equation (1) and the ORQ equation (3) to the the first negative flow of KdV hierarchy (9) respectively, here we only give the results.

On the one hand, set $m = p^2,\ u = p^2 + p_{y\tau} - p^{-1}p_yp_\tau$, then we have a reciprocal transformation

$$\mathrm{d}x = -p^{-1}\,\mathrm{d}y + u\,\mathrm{d}\tau, \quad \mathrm{d}t = \mathrm{d}\tau\,. \tag{10}$$

Under the new variables x, t, the first negative flow of the KdV hierarchy (9) is transformed to the CH equation (1).

On the other hand, let $n = -2p/(p_y+1),\ v = -2p + (p_yp_\tau + p_\tau - pp_{y\tau})/p^2$, then we can define a reciprocal transformation

$$\mathrm{d}x = \frac{2}{n}\,\mathrm{d}y + \Big(p^2 - \frac{p_yp_\tau + p_\tau - pp_{y\tau}}{p^2}\Big)\,\mathrm{d}\tau\,,$$
$$\mathrm{d}t = \frac{1}{4}\,\mathrm{d}\tau\,. \tag{11}$$

Under the transformation (11), the first negative flow of the KdV hierarchy (9) is transformed to the ORQ equation (3).

Then the relations between the three equations are clear, and we have the following result.

Theorem 1 The Camassa–Holm equation is transformed to the Olver–Rosenau–Qiao equation by the transformation

$$x' = x - \frac{1}{2}\ln m\,, \quad t' = \frac{1}{4}t\,, \tag{12}$$

together with

$$n = \frac{4m^{3/2}}{2m - m_x}\,, \quad v = m^{-1/2}(m + u + u_x)\,. \tag{13}$$

Proof On the one hand, from the definitions (10)–(11), we have

$$\mathrm{d}x' = \Big(1 - \frac{m_x}{2m}\Big)\,\mathrm{d}x - \frac{m_t}{2m}\,\mathrm{d}t\,, \quad \mathrm{d}t' = \frac{1}{4}\,\mathrm{d}t\,,$$
$$\mathrm{d}x = \frac{n}{2m^{1/2}}\,\mathrm{d}x' + nm^{-3/2}m_t\,\mathrm{d}t'\,, \quad \mathrm{d}t = 4\,\mathrm{d}t'\,. \tag{14}$$

Then for any smooth function $g(x,t,m,n)$, we have $g_{x'} = (1/2)g_xnm^{-1/2}$, so direct calculation shows that

$$v - v_{x'} = 2m^{1/2}\,.$$

Besides, from the definition of n, we get

$$n - \frac{1}{2}m^{-1}nm_x = 2m^{1/2}\,,$$

therefore we have

$$n = 2m^{1/2} + m^{-1/2}m_{x'} = (1+\partial_{x'})2m^{1/2} = v - v_{x'x'}\,. \tag{15}$$

One the other hand,

$$\mathrm{d}n \ = n_{t'}\,\mathrm{d}t' + n_{x'}\,\mathrm{d}x' = \mathrm{d}n = \mathrm{d}\Big(\frac{4m^{3/2}}{2m - m_x}\Big)$$

$$= \Big(\frac{3m_x n}{2m} - \frac{1}{4}m^{-3/2}n^2(2m_x - m_{xx})\Big)\mathrm{d}x + \Big(\frac{3m_t n}{2m} - \frac{1}{4}m^{-3/2}n^2(2m_t - m_{xt})\Big)\mathrm{d}t\,. \quad (16)$$

Substituting (14) into (16) we get

$$n_{x'} = \Big[\frac{3nm_x}{2m} - \frac{1}{4}m^{-3/2}n^2(2m_x - m_{xx})\Big]\frac{n}{2m^{1/2}}\,, \quad (17)$$

$$n_{t'} = \Big[\frac{3nm_x}{2m} - \frac{1}{4}m^{-3/2}n^2(2m_x - m_{xx})\Big]nm^{-3/2}m_t + 6\frac{nm_t}{m} - m^{-3/2}n^2(2m_t - m_{xt})\,, \quad (18)$$

then solving m_{xx} from (17) and substituting it into (18), we obtain

$$\begin{aligned} n_{t'} &= -\,n_{x'}(4u + 4u_x) - 2m^{-1/2}n^2(u + u_x - m) \\ &= -\,4n_{x'}m^{1/2}(v - m^{1/2}) - 2n^2(v - 2m^{1/2}) \\ &= -\,\big(n(v^2 - v_{x'}^2)\big)_{x'}\,. \end{aligned}$$

Therefore the theorem is proven.

3 Reciprocal Transformation of the System (5)

In this section, we will consider a reciprocal transformation for the system (5). We have known that the CH equation and the ORQ equation are connected with the first negative flow of the KdV hierarchy by two different reciprocal transformations. Although the system (5) may be considered as the linear combination of the two equations, the spectral problem of it is changed, and we need a new reciprocal transformation.

Equation (5) has a Lax pair[22]

$$\varphi_x = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -k_1\lambda m - \lambda k_2 & 1 \end{pmatrix}\varphi\,, \quad (19)$$

$$\varphi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\varphi\,, \quad (20)$$

where $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ and

$$\begin{aligned} A &= -\frac{1}{\lambda^2} - \frac{1}{2}k_1(u^2 - u_x^2) - \frac{1}{2}k_2(u - u_x)\,, \\ B &= \frac{1}{\lambda}(u - u_x) + \frac{1}{2}\lambda m[k_1(u^2 - u_x^2) + k_2 u]\,, \\ C &= -\frac{1}{\lambda}(k_1(u + u_x) + k_2) - \frac{1}{2}\lambda[k_1^2 m(u^2 - u_x^2) \\ &\quad + k_1k_2(mu + u^2 - u_x^2) + k_2^2 u]\,. \end{aligned}$$

Moreover, setting $a = (1/2)\sqrt{k_1m^2 + k_2m}$, it is not hard to find from above Lax pair that the system (5) has a conservation law

$$a_t = (a(k_1(u^2 - u_x^2) + k_2u))_x\,,$$

which defines a reciprocal transformation

$$\mathrm{d}y = a\,\mathrm{d}x + a(k_1(u^2 - u_x^2) + k_2u))\mathrm{d}t\,, \quad \mathrm{d}\tau = 4\mathrm{d}t\,. \quad (21)$$

Under the transformation (21), the Lax pair (19)–(20) equals to the scalar form

$$\frac{1}{k_1m + k_2}(2\partial - 1)\frac{1}{m}(2\partial + 1)\varphi_1 = \mu\varphi_1\,, \quad (22)$$

$$\varphi_{1t} = \Big(k_1(u^2 - u_x^2) + k_2u - \frac{2(u - u_x)}{\mu m}\Big)\varphi_{1x} + \Big(\frac{1}{\mu} - \frac{u - u_x}{\mu m} + \frac{1}{2}k_2u_x\Big)\varphi_1\,, \quad (23)$$

where $\mu = -\lambda^2$.

Under the transformation (21), the Lax pair (22)–(23) is transformed to

$$\frac{1}{k_1m + k_2}\Bigg(\frac{4a^2}{m}\partial_y^2 + 4a\Big(\frac{a}{m}\Big)_y\partial_y - \frac{m + 2am_y}{m^2}\Bigg)\varphi_1 = \mu\varphi_1\,, \quad (24)$$

$$4\varphi_{1\tau} = \Big(\frac{1}{\mu} - \frac{u - au_y}{\mu m} + \frac{1}{2}k_2au_y\Big)\varphi_1 - \frac{2a(u - au_y)}{\mu m}\varphi_{1y}\,. \quad (25)$$

Then after the gauge transformation $\varphi_1 = \sqrt{m/a}\ \phi$, the Lax pair (24)–(25) is transformed to

$$\phi_{yy} + R\phi - \mu\phi = 0\,,$$
$$\phi_\tau = \frac{1}{2\mu}\Big(p\phi_y - \frac{1}{2}p_y\phi\Big)\,,$$

where

$$p = -a\frac{u - au_y}{m}\,,$$
$$R = -\frac{3m_y^2}{4m^2} + \frac{m_ya_y - m_y + am_{yy}}{2ma} + \frac{a_y^2 - 1 - 2aa_{yy}}{4a^2}\,.$$

Direct calculation shows that $R = -(p_{yy}/2p) + (p_y^2/4p^2) - (1/4p^2)$, so the CH type system (5) is transformed to the first negative flow for the KdV hierarchy

$$R_\tau = p_y\,, \quad R = -\frac{p_{yy}}{2p} + \frac{p_y^2}{4p^2} - \frac{1}{4p^2}\,. \quad (26)$$

In what follows we will discuss the reduction of the transformation (21). We only consider the case of $k_1 = 0$ and the case of $k_2 = 0$ is similarly to which of $k_1 = 0$. In this case, the hybrid CH-ORQ equation is reduced to

$$u_t = k_2(2u_xm + um_x)\,, \quad m = u - u_{xx}\,,$$

and the reciprocal transformation is reduced to

$$\mathrm{d}y = \Big(\frac{1}{2}\sqrt{k_2m}\Big)\mathrm{d}x + \Big(\frac{1}{2}\sqrt{k_2m}k_2u\Big)\mathrm{d}t, \quad \mathrm{d}\tau = 4\mathrm{d}t.$$

Therefore the equation and the transformation equal to which of the CH equation up a scaling for τ.[23]

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