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Nonlocal symmetry and similarity reductions for a (2 + 1)-dimensional Korteweg–de Vries equation

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Abstract Based on the Lax pair, the nonlocal symmetries to (2 + 1)-dimensional Korteweg–de Vries equation are investigated, which are also constructed by the truncated Painlevé expansion method. Through introducing some internal spectrum parameters, infinitely many nonlocal symmetries are given. By choosing four suitable auxiliary variables, nonlocal symmetries are localized to a closed prolonged system. Via solving the initial-value problems, the finite symmetry transformations are obtained to generate new solutions. Moreover, rich explicit interaction solutions are presented by similarity reductions. In particular, bright soliton, dark soliton, bell-typed soliton and soliton interacting with elliptic solutions are found. Through computer numerical simulation, the dynamical phenomena of these interaction solutions are displayed in graphical way, which show meaningful structures.

Keywords Nonlocal symmetry · (2+1)-dimensional Korteweg–de Vries equation · Similarity reduction · Interaction solutions

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1 Introduction

Nowadays, many people pay attentions on integrable systems and soliton theory for investigating the real world. As the main part of nonlinear science, soliton theory [1–5] has been broadly investigated and widely applied in fields of mathematics, fluid physics, microphysics, solid state physics, condensed matter physics, hydrodynamics, fluid dynamics, cosmology, field theory and optics. With the research going on, symmetry theory [6] becomes increasingly important in nonlinear science. For a long time, the classical symmetry theory plays an important role in solving nonlinear systems, whether integrable or not. Constructing interaction solutions of nonlinear systems is much more difficult and tedious than general solitary wave solutions, but it is an important and meaningful research topic.

Recent studies [7–11] have shown that nonlocal symmetry method is one of the useful tools to derive nonlinear waves interacting with each other. One landmark in achieving further progress in solving nonlinear systems was the nonlocal symmetry method, which was first researched in 1980 [12]. Then, nonlocal symmetry approach gets further development in nonlinear science. Many mathematics [13–19] have used the nonlocal symmetry approach to investigate a myriad of important physical systems. In recent years, the nonlocal symmetries [20–22] relating to the DT and BT are investigated to construct interaction solutions. Lately, it comes to light that we can construct nonlocal sym-

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metry by virtue of Painlevé analysis. As relevant with singular manifold and consistent with the residual of the expansion, residual symmetry [23,24] can be called to represent the nonlocal symmetry.

In this paper, nonlocal symmetries, extended system, similarity reduction and explicit interaction solutions for (2 + 1)-dimensional KdV equation [25] are discussed,

$$4u_t - \alpha(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) \\ -\beta(u_{xxx} + 6uu_x) = 0, \tag{1}$$

with two arbitrary constants α and β. We can reduce Eq. (1) to some well-known equations with physical meanings.

If setting $\{\alpha \neq 0, \beta = 0\}$, Eq. (1) is reduced to Calogero–Bogoyavlenskii–Schiff equation [26] or Boiti–Leon–Pempinelli equation [27]

$$4u_t - \alpha(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) = 0, \tag{2}$$

which describes the propagation behaviour of the Riemann wave interacting with the long wave.

If setting $\{\alpha = 0, \beta \neq 0\}$, Eq. (1) is become famous KdV equation

$$4u_t - \beta(u_{xxx} + 6uu_x) = 0, \tag{3}$$

which can be extended to the Kadomtsev–Petviashvili (KP) equation [28].

Peng [25] first used Lax pair generating approach to construct Eq. (1). In [25], exact solutions, Bäcklund transformation and localized structures were studied from the method of singular manifold. In [29], Eq. (1) is examined to pass the Painlevé test and they derived some kinds of explicit solutions via applying Painlevé truncated extension approach. In [30], soliton solutions, quasiperiodic wave solutions and their links of Eq. (1) were investigated based on the Riemann–Bäcklund method. In [31], Wazwaz obtained multiple-soliton solutions and displayed that Eq. (1) does not have the resonance phenomenon. In [32], Wang et al. investigated the bilinear Bäklund transformation, infinite conservation laws and Lax pair with Darboux covariant property in Eq. (1) via binary Bell polynomials. In [33,34], Lü et al. constructed a direct bilinear Bäklund transformation on the basis of quadrilinear representation. In [35], interaction solutions of Eq. (1) were derived under consistent Riccati expansion approach.

Outline of the present paper is as follows: in Sect. 2, nonlocal symmetries for Eq. (1) are obtained by two ways, from the Lax pair and the truncated Painlevé extension. Then relations among the two ways are presented. By extending the original system, nonlocal symmetries are localized to a Lie symmetry. In Sect. 3, by obtaining the finite symmetry transformations, some new solutions are constructed. We derive some new explicit bright soliton, dark soliton and interaction solutions through similar reduction process. A short conclusion and discussion is included in the last section.

2 Nonlocal symmetry and localization

Setting $v_x = u_y$, Eq. (1) is converted into the following form

$$4u_t - \alpha(4uu_y + 2u_xv + u_{xxy}) - \beta(u_{xxx} + 6uu_x) = 0, \tag{4}$$

$$v_x - u_y = 0. \tag{5}$$

The Lax pair for Eqs. (4)–(5) meets

$$\psi_{xx} = (-u + \lambda)\psi, \tag{6}$$

$$\psi_t = \alpha\psi_{xxy} + \beta\psi_{xxx} + \gamma\psi_{xx} + \left(\frac{1}{2}\alpha v + \frac{3}{2}\beta u\right)\psi_x \\ + \alpha u\psi_y + \left(\frac{3}{4}\alpha u_y + \frac{3}{4}\beta u_x + \gamma u\right)\psi, \tag{7}$$

where $\{u, v\}$ is the solution to Eq. (4)–(5), with λ a spectral parameter, ϕ a spectral function and $\{\alpha, \beta, \gamma\}$ arbitrary constants.

Proposition 1 *If ψ denotes a solution to the Lax pair (6)–(7) with $\lambda = 0$, and then*

$$\sigma = (\sigma^u, \sigma^v) \equiv (\psi\psi_x, \psi\psi_y), \tag{8}$$

is a nonlocal symmetry for Eqs. (4)–(5).

Remark 1 In fact, even if $\lambda \neq 0$, the nonlocal symmetry for Eqs. (4)–(5) still can be expressed by (8). That is to say, when $\lambda = 0$, the nonlocal symmetry for Eqs. (4)–(5) is Eq. (8); when $\lambda \neq 0$, the nonlocal symmetry is $(\sigma^u, \sigma^v) = (F(\xi)\psi\psi_x, F(\xi)\psi\psi_y + (2\lambda\alpha)^{-1}F_\xi(\xi)\psi^2)$ with $\xi = t + (\lambda\alpha)^{-1}y$, which is also satisfied Eq. (8) by setting the arbitrary function $F(\xi) = 1$, which can be proved with direct computation. Existence of the arbitrary function will increase the complexity of the next similarity reduction process, so we take the case of $\lambda = 0$.

By using the similar method in Ref. [36], much more symmetries can be acquired through differentiating the given internal parameters. Then we can have the proposition as follows.

Proposition 2 *If $\sigma_0(\lambda)$ satisfies the above symmetry of Eqs. (4)–(5) with $\lambda \equiv \lambda_1, \lambda_2, \ldots, \lambda_r$, then*

$$\sigma_m \equiv \frac{\mathrm{d}^{\{m\}}}{\mathrm{d}\lambda^{\{m\}}}\sigma_0(\lambda) \equiv \frac{\mathrm{d}^{\{m_1\}}}{\mathrm{d}\lambda_1^{\{m_1\}}}\frac{\mathrm{d}^{\{m_2\}}}{\mathrm{d}\lambda_2^{\{m_2\}}}\cdots\frac{\mathrm{d}^{\{m_r\}}}{\mathrm{d}\lambda_r^{\{m_r\}}}\sigma_0(\lambda), \tag{9}$$

is also the symmetry for Eqs. (4)–(5) with $\{m\} \equiv \{m_1, m_2, \ldots, m_r\}$.

According to the above two propositions, many new nonlocal symmetries can be derived. Such as, if choosing ϕ and $\bar{\phi}$ satisfy the above Lax pair (6)–(7), then

$$\sigma(\lambda_1, \lambda_2) \equiv (\sigma^u(\lambda_1, \lambda_2), \sigma^v(\lambda_1, \lambda_2)), \tag{10}$$

with

$$\sigma^u(\lambda_1, \lambda_2) = (\lambda_1\psi + \lambda_2\bar{\psi})(\lambda_1\psi + \lambda_2\bar{\psi})_x,$$
$$\sigma^v(\lambda_1, \lambda_2) = (\lambda_1\psi + \lambda_2\bar{\psi})(\lambda_1\psi + \lambda_2\bar{\psi})_y,$$

and $\frac{\partial^{m_1+m_2}}{\partial\lambda_1^{m_1}\partial\lambda_2^{m_2}}\sigma(\lambda_1, \lambda_2)$ are also the symmetries for Eqs. (4)– (5).

It is known that exact solutions can be constructed directly through the nonlocal symmetry method, which need to transform nonlocal symmetries into local ones to obtain a prolonged system. Then this prolonged system is closed and satisfies a Lie point symmetry, namely equivalent with nonlocal symmetry (8).

Through introducing two new dependent variables ψ_1 and ψ_2, which are functions of $\{x, y, t\}$, with

$$\psi_1 = \psi_x, \quad \psi_2 = \psi_y, \tag{11}$$

so the above symmetry (8) can be converted into

$$\sigma^u = \psi\psi_1, \quad \sigma^v = \psi\psi_2. \tag{12}$$

In addition to the above variables, two additional potential variables $\phi \equiv \phi(x, y, t)$, $k \equiv k(x, y, t)$ should be introduced to calculate the symmetries for the three new variables ψ, ψ_1 and ψ_2. To be completely closed to the prolonged system, the new variable ϕ needs to satisfy the following compatibility conditions:

$$\phi_x = \psi^2, \quad \phi_y = k, \quad \phi_t = \frac{1}{2}\alpha v\psi^2 + \frac{3}{2}\beta(\lambda\psi^2 - \psi\psi_{xx}) - \beta\psi_1^2 - \alpha\psi_1\psi_2 + 2\beta\psi\psi_{1x} + \alpha\psi\psi_{1y} + \alpha\lambda k. \tag{13}$$

Then the symmetries can be generated

$$\sigma^{\psi} = -\frac{1}{4}\phi\psi, \quad \sigma^{\psi_1} = -\frac{1}{4}(\phi\psi_1 + \psi^3),$$
$$\sigma^{\psi_2} = -\frac{1}{4}(\phi\psi_2 + k\psi), \quad \sigma^k = -\frac{1}{2}k\phi, \quad \sigma^{\phi} = -\frac{1}{4}\phi^2, \tag{14}$$

where $\sigma^{\psi}, \sigma^{\psi_1}, \sigma^{\psi_2}, \sigma^k$ and σ^{ϕ} express the symmetries for ψ, ψ_1, ψ_2, k and ϕ, respectively.

Lastly, the prolongation of (8) can be successfully localized by introducing the variables ψ, ψ_1, ψ_2, k, and ϕ in the following vector expression

$$V = \psi\psi_1\frac{\partial}{\partial u} + \psi\psi_2\frac{\partial}{\partial v} - \frac{\phi\psi}{4}\frac{\partial}{\partial\psi} - \frac{1}{4}(\phi\psi_1 + \psi^3)\frac{\partial}{\partial\psi_1} - \frac{1}{4}(\phi\psi_2 + k\psi)\frac{\partial}{\partial\psi_2} - \frac{k\phi}{2}\frac{\partial}{\partial k} - \frac{\phi^2}{4}\frac{\partial}{\partial\phi}. \tag{15}$$

An meaningful and interesting result can be seen that the above potential variable ϕ must meet the following Schwartz expression for Eqs. (4)– (5)

$$\alpha S_y + \beta S_x + 4C_x - 4\alpha\lambda H_x = 0 \tag{16}$$

where $H = -\frac{\phi_y}{\phi_x}$, $C = -\frac{\phi_t}{\phi_x}$, and $S = \frac{\phi_{xxx}}{\phi_x} - \frac{3\phi_{xx}^2}{2\phi_x^2}$ are all invariable with respect to the following classical Möbius transformation

$$\phi \to \frac{a + b\phi}{c + d\phi} \quad (ad \neq bc). \tag{17}$$

By introducing a new transformation as follows,

$$\psi = \sqrt{\phi_x}, \tag{18}$$

then substituting it into the Lax pair (6)– (7), one can obtain

$$u = \lambda - \frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}, \tag{19}$$

$$v = -\frac{3\beta\lambda}{\alpha} + \frac{2}{\alpha}\frac{\phi_t}{\phi_x} - 2\lambda\frac{\phi_y}{\phi_x} + \frac{3\beta}{4\alpha}\frac{\phi_{xx}^2}{\phi_x^2} + \frac{\phi_{xx}\phi_{xy}}{\phi_x^2} - \frac{\phi_{xxy}}{\phi_x} - \frac{\beta}{2\alpha}\frac{\phi_{xxx}}{\phi_x}. \tag{20}$$

and the equivalent form of nonlocal symmetry for u and v

$$(\sigma^u, \sigma^v) \equiv \left(\frac{1}{2}\phi_{xx}, \frac{1}{2}\phi_{xy}\right). \tag{21}$$

Remark 2 Equation (21) is the residual symmetry of Eqs. (4)–(5). Under transformation (18), the nonlocal symmetry (8) can be connected with the residual symmetry (21). Equations (19)–(20) are nonauto-BT, which converts (4)–(5) into its Schwartz Eq. (16).

As Eqs. (4)–(5) pass the Painlevé test [29], they own the following truncated expansion form

$$u = u_0 + \frac{u_1}{\phi} + \frac{u_2}{\phi^2}, \quad v = v_0 + \frac{v_1}{\phi} + \frac{v_2}{\phi^2}. \tag{22}$$

with $u_0, u_1, u_2, v_0, v_1, v_2$, and ϕ being functions in $\{x, y, t\}$.

In order to construct nonlocal symmetry of (4)–(5) related to its truncated Painlevé expansion, we can substitute (22) into Eqs. (4)–(5) and eliminate the coefficients of different order powers for variable ϕ,

$$u_1 = 2\phi_{xx}, \quad v_1 = 2\phi_{xy}, \quad u_2 = -2\phi_x^2, \tag{23}$$

$$u_0 = \lambda - \frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}, \quad v_2 = -2\phi_x\phi_y, \tag{24}$$

$$v_0 = -\frac{3\beta\lambda}{\alpha} + \frac{2}{\alpha}\frac{\phi_t}{\phi_x} - 2\lambda\frac{\phi_y}{\phi_x} + \frac{3\beta}{4\alpha}\frac{\phi_{xx}^2}{\phi_x^2} + \frac{\phi_{xx}\phi_{xy}}{\phi_x^2} - \frac{\phi_{xxy}}{\phi_x} - \frac{\beta}{2\alpha}\frac{\phi_{xxx}}{\phi_x}. \tag{25}$$

Then the $(2+1)$-dimensional KdV Eqs. (4)–(5) can be successfully reduced to the Schwartz form (16), and both keep the Möbious invariance property (17). The variable ϕ has the point symmetry in the following form

$$\sigma_\phi = \kappa_0 + \kappa_1\phi + \kappa_2\phi^2. \tag{26}$$

where κ_0, κ_1, and κ_2 are arbitrary constants.

Remark 3 It is worthy to point out that the corresponding Schwartz expression for a differential equation can be obtained via localization of nonlocal symmetries from Lax pair and utilizing the method of singularity analysis. The Schwartz expression (16) and Möbious invariance property (17) both can be got through the above two ways.

By introducing another four new dependent variables f, g, h and p, nonlocal symmetries for Eqs. (4)–(5) are localized to the Lie point symmetry

$$\sigma^u = 2h, \quad \sigma^v = 2p, \quad \sigma^f = -2\phi f, \quad \sigma^g = -2\phi g,$$
$$\sigma^h = -2(f^2 + \phi h), \quad \sigma^p = -2(fg + \phi p), \quad \sigma^\phi = -\phi^2, \tag{27}$$

for the extended system

$$4u_t - \alpha(4uu_y + 2u_x v + u_{xxy}) - \beta(u_{xxx} + 6uu_x) = 0, \quad v_x - u_y = 0,$$
$$u = \lambda - \frac{1}{2}\frac{\phi_{xxx}}{\phi_x} + \frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}, \quad f = \phi_x, \quad g = \phi_y,$$
$$h = f_x, \quad p = f_y,$$
$$v = -\frac{3\beta\lambda}{\alpha} + \frac{2}{\alpha}\frac{\phi_t}{\phi_x} - 2\lambda\frac{\phi_y}{\phi_x} + \frac{3\beta}{4\alpha}\frac{\phi_{xx}^2}{\phi_x^2} + \frac{\phi_{xx}\phi_{xy}}{\phi_x^2} - \frac{\phi_{xxy}}{\phi_x} - \frac{\beta}{2\alpha}\frac{\phi_{xxx}}{\phi_x}. \tag{28}$$

In the last, the prolongation of nonlocal symmetries (21) localizes to an extended system by adding the above four variables with the pattern of prolonged symmetry vector field

$$V = 2h\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial v} - 2\phi f\frac{\partial}{\partial f} - 2\phi g\frac{\partial}{\partial g} - 2(f^2 + \phi h)\frac{\partial}{\partial h} - 2(fg + \phi p)\frac{\partial}{\partial p} - \phi^2\frac{\partial}{\partial \phi}. \tag{29}$$

3 Finite symmetry transformation and similarity reduction

Based on the theory in classical Lie group, the nonlocal symmetry (21) and Lie point symmetry (29) are equivalent with each other, and exact solutions can be constructed in two ways.

3.1 Finite symmetry transformation

Theorem 1 *When* $\{f, g, h, p, \phi, u, v\}$ *meets the extended system* (28)*, the following form of* $\{\bar f, \bar g, \bar h, \bar p, \bar\phi, \bar u, \bar v\}$ *also satisfies Eq.* (28)

$$\bar{u} = u + \frac{2\epsilon h}{1+\epsilon\phi} - \frac{2\epsilon^2 f^2}{(1+\epsilon\phi)^2}, \quad \bar{v} = v + \frac{2\epsilon p}{1+\epsilon\phi} - \frac{2\epsilon^2 fg}{(1+\epsilon\phi)^2}, \quad \bar{f} = \frac{f}{(1+\epsilon\phi)^2},$$
$$\bar{g} = \frac{g}{(1+\epsilon\phi)^2}, \quad \bar{h} = \frac{h}{(1+\epsilon\phi)^2} - \frac{2\epsilon f^2}{(1+\epsilon\phi)^3},$$
$$\bar{p} = \frac{p}{(1+\epsilon\phi)^2} - \frac{2\epsilon fg}{(1+\epsilon\phi)^3}, \quad \bar{\phi} = \frac{\phi}{1+\epsilon\phi}. \tag{30}$$

This symmetry group theorem can be proved through solving the initial-value problems as follows:

$$\frac{\mathrm{d}\bar{u}(\epsilon)}{\mathrm{d}\epsilon} = 2\bar{h}, \quad \frac{\mathrm{d}\bar{v}(\epsilon)}{\mathrm{d}\epsilon} = 2\bar{p}, \quad \frac{\mathrm{d}\bar{f}(\epsilon)}{\mathrm{d}\epsilon} = -2\bar{\phi}\bar{f}, \quad \frac{\mathrm{d}\bar{g}(\epsilon)}{\mathrm{d}\epsilon} = -2\bar{\phi}\bar{g},$$
$$\frac{\mathrm{d}\bar{h}(\epsilon)}{\mathrm{d}\epsilon} = -2(\bar{f}^2 + \bar{\phi}\bar{h}), \quad \frac{\mathrm{d}\bar{p}(\epsilon)}{\mathrm{d}\epsilon} = -2(\bar{f}\bar{g} + \bar{\phi}\bar{p}), \quad \frac{\mathrm{d}\bar{\phi}(\epsilon)}{\mathrm{d}\epsilon} = -\bar{\phi}^2,$$
$$\bar{f}(0) = f, \quad \bar{g}(0) = g, \quad \bar{h}(0) = h, \quad \bar{p}(0) = p, \quad \bar{\phi}(0) = \phi, \quad \bar{u}(0) = u, \quad \bar{v}(0) = v, \tag{31}$$

with ϵ the general group parameter.

Remark 4 From a known solution $\{u, v\}$ for Eqs. (4)–(5), a new solution $\{\bar{u}, \bar{v}\}$ can be derived by the above transformation in Theorem 1. It should be mentioned that the last expression in (30) is just the normal Möbious transformation.

3.2 Similarity reductions

For sake of similarity reductions of Eqs. (4)–(5), we research the extended system (28) through applying the approach of Lie point symmetry. Assuming the following infinitesimal transformations have no effect in Eq. (28),

$$\{x, y, t, f, g, h, p, \phi, u, v\} \to \{x+\epsilon X, y+\epsilon Y, t+\epsilon T, f+\epsilon F, g+\epsilon G, h+\epsilon H, p+\epsilon P, \phi+\epsilon\Phi, u+\epsilon U, v+\epsilon V\} \tag{32}$$

with

$$\sigma^u = Xu_x + Yu_y + Tu_t - U,$$
$$\sigma^g = Xg_x + Yg_y + Tg_t - G,$$
$$\sigma^f = Xf_x + Yf_y + fu_t - F,$$
$$\sigma^v = Xv_x + Yv_y + Tv_t - V,$$
$$\sigma^h = Xh_x + Yh_y + Th_t - H,$$
$$\sigma^p = Xp_x + Yp_y + Tp_t - P,$$
$$\sigma^\phi = X\phi_x + Y\phi_y + T\phi_t - \Phi, \tag{33}$$

Among them, $\{X, Y, T, F, G, H, P, \Phi, U, V\}$ are the functions in $\{x, y, t, f, g, h, p, \phi, u, v\}$ with ϵ a small parameter.

If we substitute Eq. (33) into the extended system (33)

$$4\sigma^u_t - \alpha(2\sigma^u_x v + 4u\sigma^u_y + 4\sigma^u u_y + 2u_x\sigma^v + \sigma^u_{xxy}) - \beta(6u\sigma^u_x + 6\sigma^u u_x + \sigma^u_{xxx}) = 0,$$
$$\sigma^v_x - \sigma^u_y = 0, \quad 4\sigma^u\phi_x^3 + 4u\sigma^\phi_x\phi_x^2 + 2\sigma^\phi_{xxx}\phi_x^2 - 2\sigma^\phi_{xx}\phi_{xx}\phi_x + \phi_{xx}^2\sigma^\phi_x = 0, \quad \sigma^f - \sigma^\phi_x = 0,$$
$$4\sigma^\phi_t\phi_x^2 - 2\alpha(v\sigma^\phi_x\phi_x^2 + \sigma^v\phi_x^3 - \sigma^\phi_{xx}\phi_x\phi_{xy} + \sigma^\phi_x\phi_{xx}\phi_{xy} - \sigma^\phi_{xy}\phi_x\phi_{xx} + \sigma^\phi_{xxy}\phi_x^2) + \beta(2\sigma^u\phi_x^3 + 2u\sigma^\phi_x\phi_x^2 + 2\sigma^\phi_{xx}\phi_x\phi_{xx} - \sigma^\phi_x\phi_{xx}^2) = 0,$$
$$\sigma^g - \sigma^\phi_y = 0, \quad \sigma^h - \sigma^f_x = 0, \quad \sigma^p - \sigma^f_y = 0, \tag{34}$$

an overdetermined system with infinitesimals $\{x, y, t, f, g, h, p, \phi, u, v\}$ can be derived via collecting and making all coefficients for potential variables and partial derivatives to be zero. By computing, these equations can yield

$$X = c_1 x + \frac{\beta}{\alpha}(c_2 - 3c_1)y + f_4,$$
$$Y = (c_2 - 2c_1)y + c_3, \quad T = c_2 t + c_4,$$
$$U = -2c_1 u - f_1 h, \quad V = (c_1 - c_2)v + \frac{\beta}{\alpha}(3c_1 - c_2)u - f_1 p - f_{1y} f - \frac{2}{\alpha} f_{4t}, \quad \Phi = \frac{1}{2} f_1\phi^2 + f_2\phi + f_3,$$
$$F = (-c_1 + f_1\phi + f_2)f,$$
$$G = (2c_1 - c_2 + f_1\phi + f_2)g + \frac{\beta}{\alpha}(3c_1 - c_2)f + \frac{1}{2} f_{1y}\phi^2 + f_{2y}\phi + f_{3y},$$
$$H = (-2c_1 + f_1\phi + f_2)h + f_1 f^2,$$
$$P = (c_1 - c_2 + f_1\phi + f_2)p + \frac{\beta}{\alpha}(3c_1 - c_2)h + (f_1 g + f_{1y}\phi + f_{2y})f, \tag{35}$$

with f_1, f_2 and f_3 being the arbitrary functions in y, f_4 being the arbitrary function in t, and $c_i (i = 1 \dots 4)$ four arbitrary constants. Specially, if setting $c_1 = c_2 = c_3 = c_4 = f_2 = f_3 = f_4 = 0$ and $f_1 = -2$, the derived symmetry is only Eq. (27); if setting $f_1 = 0$, the usual Lie point symmetry for Eqs. (4)–(5) can be obtained.

In order to construct more corresponding group invariant solutions, the symmetry constraint conditions $\sigma^i = 0$ $(i = f, g, h, p, \phi, u, v)$ defined by Eqs. (33) with Eqs. (35) should be solved. This is equal to solving the characteristic equations as follows:

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}f}{F} = \frac{\mathrm{d}g}{G} = \frac{\mathrm{d}h}{H} = \frac{\mathrm{d}p}{P} = \frac{\mathrm{d}\phi}{\Phi} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}v}{V}. \tag{36}$$

Below, we consider two nontrivial similarity reductions by solving (36) in detail.

Reduction 1 $c_3 \neq 0$. Generality, assuming $c_1 = c_2 = c_4 = 0, c_3 = 1, f_1 = c_5, f_2 = c_6, f_3 = c_7$ and $f_4 = \frac{1}{k}$, and redefining c by $c^2 = \frac{c_6^2 - 2c_5c_7}{4}$ $(c \neq 0)$. By the nonlocal symmetry approach, more general exact solutions can be obtained, which would be instructive to consider travelling waves of Eq. (1). Solving Eq. (36), we yield similarity solutions, which are expressed as

$$\begin{aligned}
u &= U - \frac{c_5}{c} H \tanh \Delta_1 - \frac{c_5^2}{2c^2} F^2 \tanh^2 \Delta_1, \\
v &= V - \frac{c_5}{c} P \tanh \Delta_1 - \frac{c_5^2}{2c^2} FG \tanh^2 \Delta_1, \\
f &= -F \mathrm{sech}^2 \Delta_1, \quad g = -G \mathrm{sech}^2 \Delta_1, \\
\phi &= -\frac{c_6}{c_5} - \frac{2c}{c_5} \tanh \Delta_1, \\
p &= \left(\frac{c_5}{c} FG \tanh \Delta_1 + P\right) \mathrm{sech}^2 \Delta_1, \\
h &= \left(\frac{c_5}{c} F^2 \tanh \Delta_1 + H\right) \mathrm{sech}^2 \Delta_1,
\end{aligned} \tag{37}$$

with $\Delta_1 = ck(\Phi + x)$. Here F, G, H, P, U, V and Φ are functions of $\{\xi, \eta\}$ and represent seven group invariants in (37), while $\xi = -kx + y$ and $\eta = t$ the corresponding similarity variables.

Substituting Eq. (37) into Eq. (34), we yield

$$\begin{aligned}
F &= \frac{2c^2 k}{c_5}(1 - k\Phi_\xi), \quad G = \frac{2c^2 k}{c_5}\Phi_\xi, \\
H &= \frac{-2c^2 k^3}{c_5}\Phi_{\xi\xi}, \\
U &= c^2 k^2 (k\Phi_\xi - 1)^2 - \frac{k^3 \Phi_{\xi\xi\xi}}{2(k\Phi_\xi - 1)} + \frac{k^4 \Phi_{\xi\xi}^2}{4(k\Phi_\xi - 1)^2}, \\
P &= \frac{2c^2 k^2}{c_5}\Phi_{\xi\xi}, \\
V &= \frac{\beta}{\alpha} c^2 k^2 (k\Phi_\xi - 1)^2 - 2c^2 k^2 (k\Phi_\xi - 1)\Phi_\xi \\
&\quad + \frac{(2\alpha - \beta k)k^2 \Phi_{\xi\xi\xi} - 4\Phi_\eta}{2\alpha(k\Phi_\xi - 1)} - \frac{(4\alpha - 3\beta k)k^3 \Phi_{\xi\xi}^2}{4\alpha(k\Phi_\xi - 1)^2},
\end{aligned} \tag{38}$$

where Φ meets the next reduced equation

$$\begin{aligned}
&(\alpha k^2 - \beta k^3)((8c^2 \Phi_{\xi\xi} - \Phi_{\xi\xi\xi\xi})(k\Phi_\xi - 1)^2 \\
&\quad + 4k(k\Phi_\xi - 1)\Phi_{\xi\xi}\Phi_{\xi\xi\xi} - (4c^2 + 3k^2\Phi_{\xi\xi}^2)) \\
&\quad + 4(k\Phi_\xi - 1)^2 \Phi_{\xi\eta} - 4k(k\Phi_\xi - 1)\Phi_{\xi\xi}\Phi_\eta = 0.
\end{aligned} \tag{39}$$

Then we can construct the interaction solutions by choosing

$$\Phi = r_0\xi + \omega_0\eta + mE_\pi(sn(r_1\xi + \omega_1\eta, n), \mu, n), \tag{40}$$

into the reduced Eq. (39), and $k = \frac{\alpha}{\beta}, r_0 = \frac{r_1\omega_1}{\omega_0} + \frac{\beta}{\alpha}$.

From (37) with (40), the solutions interacting between solitons and the cnoidal waves can be obtained. If the modulus $n \neq 1$ in Eq. (40), the interactional solutions for the potential u and v are exhibited in Fig. 1. In Fig. 1, clearly phenomena can be seen that a bright soliton and dark soliton travel on the cnoidal wave backgrounds rather than the usual backgrounds of plane wave, respectively. While setting $n = 1$, the interaction phenomena can be reduced back to bright soliton solutions and dark soliton solutions, which are shown in Fig. 2.

Reduction 2 Letting $c_1 = c_2 = 0, c_3 = \frac{1}{k}, f_1 = c_5, f_2 = c_6, f_3 = c_7$ and $f_4 = q_t$ with $q \equiv q(t)$, and setting $k_2 = \frac{c_4}{c_3}, c^2 = \frac{c_6^2 - 2c_5c_7}{4}$ $(c \neq 0)$. Solving (36), we yield similarity solutions as follows:

$$\begin{aligned}
u &= U - \frac{c_5}{c} H \tanh \Delta_2 - \frac{c_5^2}{2c^2} F^2 \tanh^2 \Delta_2, \\
v &= V - \frac{c_5}{c} P \tanh \Delta_2 - \frac{c_5^2}{2c^2} FG \tanh^2 \Delta_2, \\
f &= -F \mathrm{sech}^2 \Delta_2, \quad g = -G \mathrm{sech}^2 \Delta_2, \\
\phi &= -\frac{c_6}{c_5} - \frac{2c}{c_5} \tanh \Delta_2, \\
p &= \left(\frac{c_5}{c} FG \tanh \Delta_2 + P\right) \mathrm{sech}^2 \Delta_2, \\
h &= \left(\frac{c_5}{c} F^2 \tanh \Delta_2 + H\right) \mathrm{sech}^2 \Delta_2,
\end{aligned} \tag{41}$$

with $\Delta_2 = ck(\Phi + y)$. Here F, G, H, P, U, V, and Φ are functions of $\{\xi, \eta\}$ and represent seven group

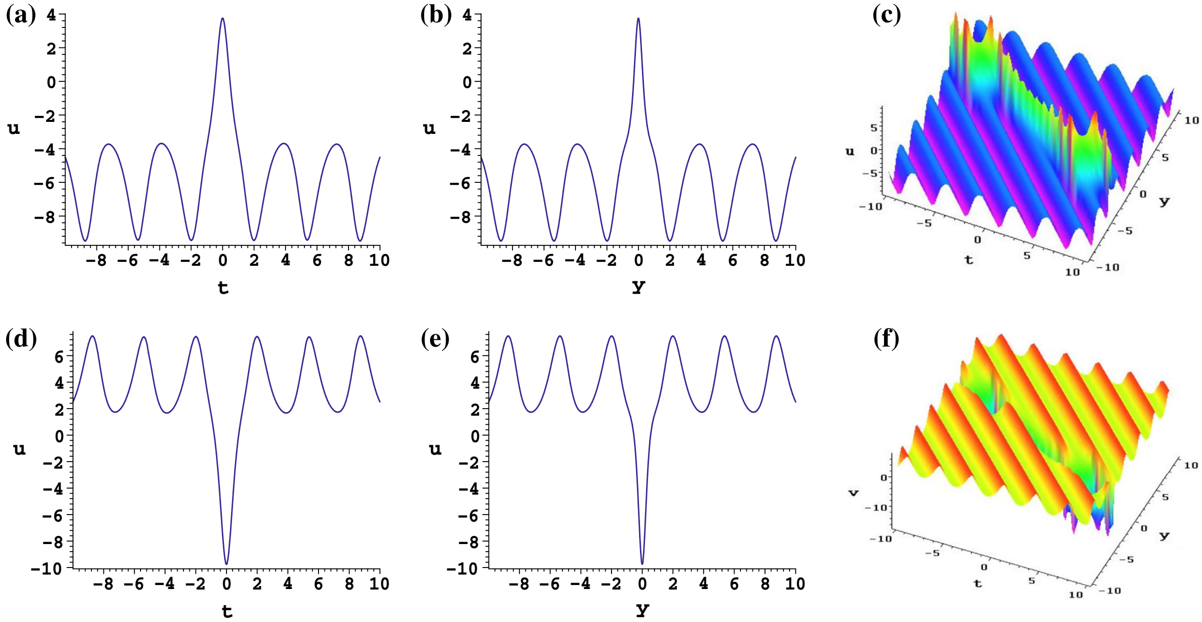

Fig. 1 Plots of bright soliton and dark soliton travelling on cnoidal waves for components u and v expressed by (37) at $x = 0$. The parameters are $\alpha = 1$, $\beta = 1$, $c = 1$, $\omega_0 = 1$, $\omega_1 = 1$, $r_1 = 1$, $m = 1$, $n = \frac{1}{2}$, $\mu = \frac{1}{2}$. **a**, **d** The wave travel modes along t axis; **b**, **e** the wave travel modes along y axis; **c**, **f** corresponding three-dimensional plots for components u and v

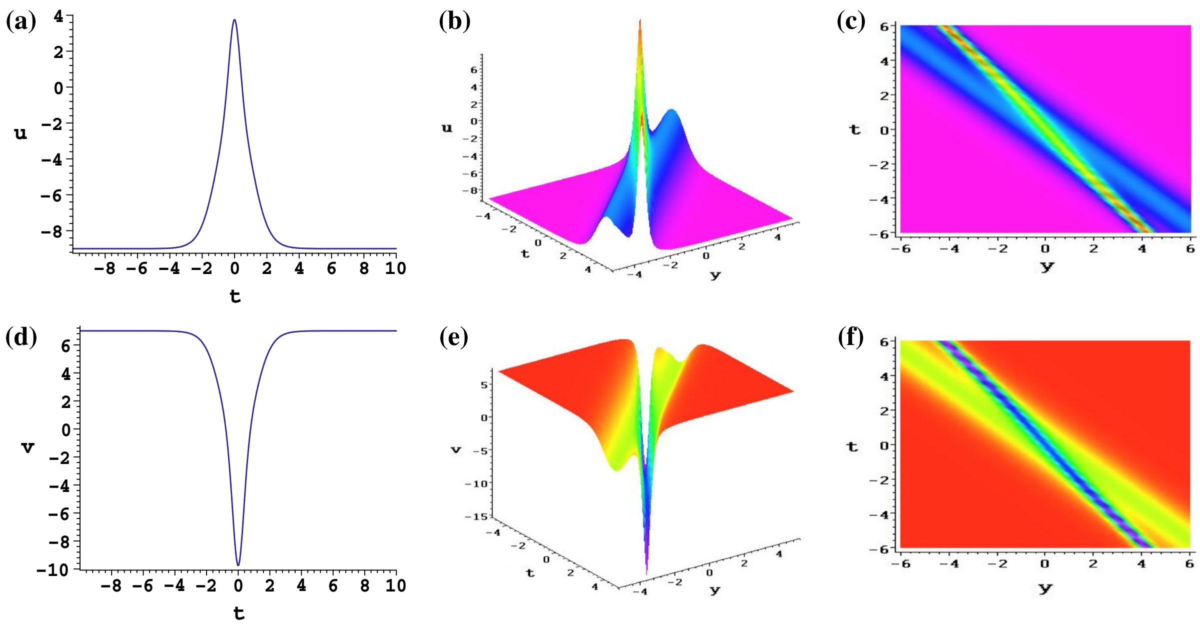

Fig. 2 Plots of bright soliton and dark soliton for components u and v expressed by (37) at $x = 0$. The parameters are $\alpha = 1$, $\beta = 1$, $c = 1$, $\omega_0 = 1$, $\omega_1 = 1$, $r_1 = 1$, $m = 1$, $n = 1$, $\mu = \frac{1}{2}$. **a**, **d** The wave travel patterns of the bright soliton and dark soliton along t axis; **b**, **e** the three-dimensional plots for components u and v; **c**, **f** the corresponding density plots

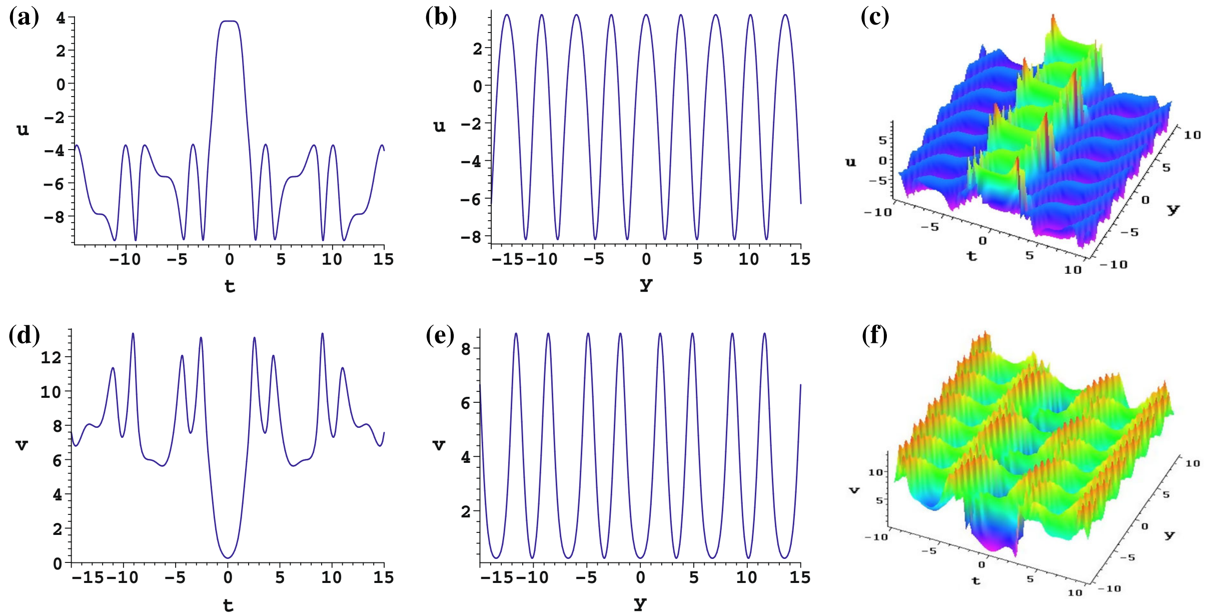

Fig. 3 Plots of first kind of special bright soliton and dark soliton travelling on cnoidal waves for the components u and v expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 1, m = 1, k_1 = 1, n = \frac{1}{2}, r_1 = 1, k_2 = 1, \mu = \frac{1}{2}, q = \sin(t)$. **a**, **d** The wave travel modes along t axis; **b**, **e** the wave travel modes along y axis; **c**, **f** corresponding three-dimensional plots for components u and v

invariants in (41), while $\xi = t - k_2 y$ and $\eta = x - \frac{k_1}{k_2} q$ the general similarity variables.

Substituting (41) into Eq. (34), we yield

$$
\begin{aligned}
F &= \frac{2c^2k_1}{c_5}(\Phi_\eta), \quad G = \frac{2c^2k_1}{c_5}(1 - k_2\Phi_\xi),\\
U &= c^2k_1^2\Phi_\eta^2 - \frac{\Phi_{\eta\eta\eta}}{2\Phi_\eta} + \frac{\Phi_{\eta\eta}^2}{4\Phi_\eta^2},\\
H &= \frac{-2c^2k_1}{c_5}\Phi_{\eta\eta}, \quad P = \frac{2c^2k_1k_2}{c_5}\Phi_{\xi\eta},\\
V &= c^2k_1(2 - 2k_1k_2\Phi_\xi + \frac{\beta k_1}{\alpha}\Phi_\eta)\Phi_\eta\\
&\quad + \left(\frac{2}{\alpha}\Phi_\xi + k_2\Phi_{\xi\eta\eta} - \frac{\beta}{2\alpha}\Phi_{\eta\eta\eta}\right)\frac{1}{\Phi_\eta}\\
&\quad + \left(\frac{3\beta}{4\alpha}\Phi_{\eta\eta} - k_2\Phi_{\xi\eta}\right)\frac{\Phi_{\eta\eta}}{\Phi_\eta^2},
\end{aligned} \tag{42}
$$

where Φ meets the reduced equation as follows

$$
\begin{aligned}
&(\alpha k_2\Phi_{\xi\eta\eta\eta} - \beta\Phi_{\eta\eta\eta\eta})\Phi_\eta^2 - 4(\Phi_\xi\Phi_{\eta\eta} - \Phi_{\xi\eta}\Phi_\eta)\Phi_\eta\\
&\quad - 3(\alpha k_2\Phi_{\xi\eta\eta} - \beta\Phi_{\eta\eta\eta})\Phi_\eta\Phi_{\eta\eta}\\
&\quad - (\alpha k_2\Phi_{\xi\eta} - \beta\Phi_{\eta\eta})(4c^2k_1^2\Phi_\eta^4 + \Phi_\eta\Phi_{\eta\eta\eta} - 3\Phi_{\eta\eta}^2) = 0.
\end{aligned} \tag{43}
$$

Then, another type of soliton–cnoidal wave solutions can be obtained via taking

$$
\Phi = r_0\xi + \omega_0\eta + mE_\pi(sn(r_1\xi + \omega_1\eta, n), \mu, n), \tag{44}
$$

into the reduced Eq. (43), and $\omega_0 = \frac{\alpha}{\beta}r_0k_2, \omega_1 = \frac{\alpha}{\beta}r_1k_2$.

Here, it should be pointed out that Φ meets different reduced equations in Eqs. (39) and (43). Although the expression of the chosen function (44) is the same with (40), the constraint conditions of the parameters are different. What is more important, the similarity solutions (41) contain an arbitrary function, which can enrich the solutions of Eq. (1). The exact solutions (41) with (44) for Eqs. (4)–(5) denote the interactional phenomena among solitons with cnoidal waves. Because of an arbitrary function q existing in Eq. (41), many new exact interaction solutions can be constructed. We illustrate three types of soliton–cnoidal wave interaction solutions in Figs. 3, 6 and 9 via choosing three different kinds of the arbitrary function q.

- The arbitrary function q is selected as $q = \sin(t)$. For Fig. 3, the solutions of Eqs. (4)–(5) for components

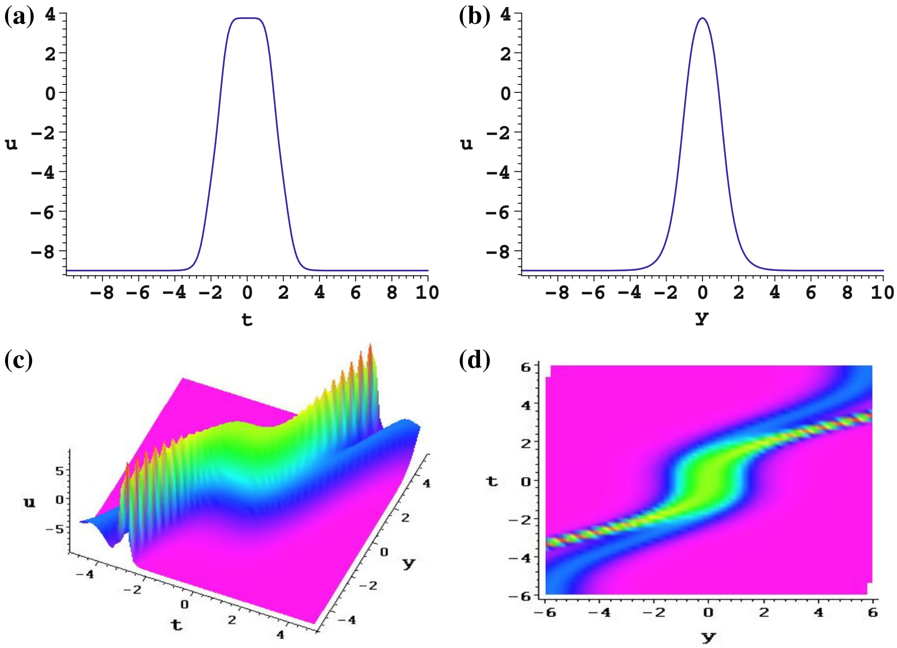

Fig. 4 Plots of first kind of bright soliton solution of u expressed by (37) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 1, m = 1, k_1 = 1, n = \frac{1}{2}, r_1 = 1, k_2 = 1, \mu = \frac{1}{2}, q = \sin(t)$. **a** The mode of wave structure along t axis; **b** the wave travel mode along y axis; **c** three-dimensional plot for component u; **d** corresponding density plot

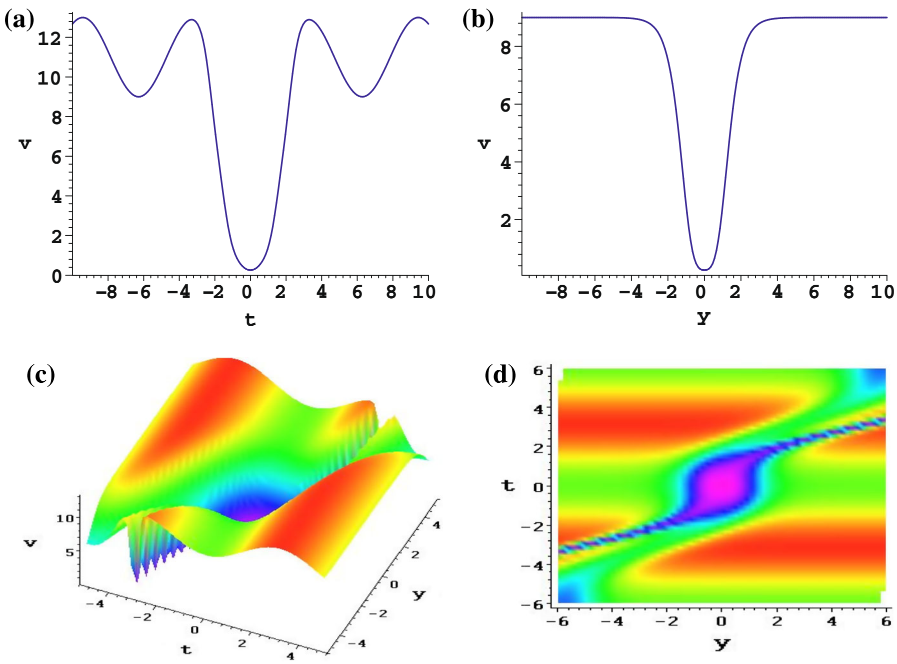

Fig. 5 Plots of first kind of dark soliton solution of v expressed by (37) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 1, m = 1, k_1 = 1, n = \frac{1}{2}, r_1 = 1, k_2 = 1, \mu = \frac{1}{2}, q = \sin(t)$. **a** The mode of wave structure along t axis; **b** the wave travel mode along y axis; **c** three-dimensional plot for component v; **d** corresponding density plot

u and v are bright soliton and dark soliton travelling on cnoidal waves with the parameters $\alpha = 1, \beta = 1, c = 1, r_0 = 1, m = 1, k_1 = 1, n = \frac{1}{2}, r_1 = 1, k_2 = 1, \mu = \frac{1}{2}$. In particular, when all other parameters are fixed but take the limit of the modulus $n = 1$, the periodic waves are reduced to bright soliton and dark soliton, which are shown in Figs. 4 and 5.

- The arbitrary function q is selected as $q = \cos(t)$. For Fig. 6, we plot the solutions of Eqs. (4)–(5) for components u and v. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, k_1 = 1, r_1 = 0.8, k_2 = 1, m = 1, n = 0.9, \mu = \frac{1}{2}$. For component u, a bright soliton can be seen travelling on the background of cnoidal wave. For component v, a dark soliton can be seen

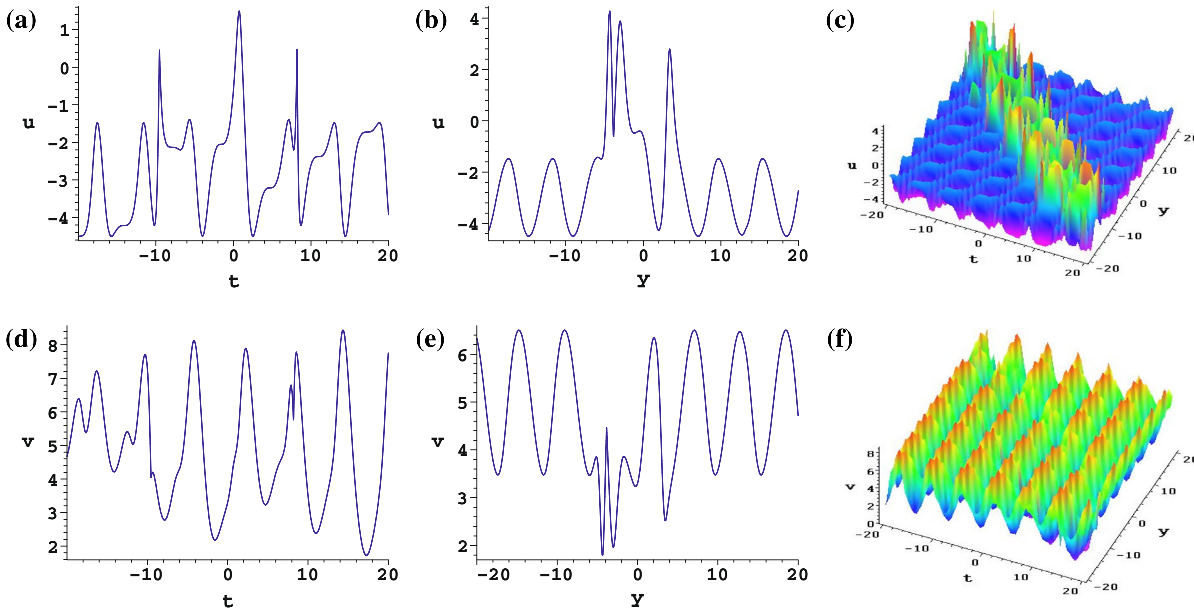

Fig. 6 Plots of second kind of soliton–cnoidal wave solutions for components u and v expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, m = 1, k_1 = 1, r_1 = 0.8, n = 0.9, k_2 = 1, \mu = \frac{1}{2}, q = \cos(t)$. **a**, **d** The wave travel mode along t axis; **b**, **e** the wave travel mode along y axis; **c**, **f** corresponding three-dimensional plots for components u and v

travelling on the background of cnoidal wave. Compared with Fig. 3, the bright soliton and dark soliton propagate along the diagonals in Fig. 6c, f, while the solitons propagate along the directions of y in Fig. 3c, f. In particular, when all other parameters are fixed but take the limit of the modulus $n = 1$, the periodic wave is reduced to bright solitary wave and the interaction solution becomes the two solitons for the component u in Fig. 7; the periodic wave is reduced to dark solitary wave for the component v in Fig. 8.

- The arbitrary function q is selected as $q = t$. For Fig. 9, we still graphically illustrate the interaction solutions of Eqs. (4)–(5) for components u and v, with selecting the same values of the parameters in Fig. 6. For component u, a bell-typed bright soliton can be seen travelling on the background of cnoidal wave. For component v, a bell-typed dark soliton can be seen travelling on the background of cnoidal wave. In particular, when all other parameters are fixed but take the limit of the modulus $n = 1$, the periodic waves are reduced to bright two solitons and dark two solitons, which are shown in Fig. 10. The waves are symmetric at $y = 0$.

By chosen the arbitrary function as $q = t$, $q = \cos(t)$ and $q = \sin(t)$, we construct different interaction behaviours, which have not yet been derived for Eqs. (4)–(5). The solutions describing solitons moving on the background of cnoidal waves instead of plane waves are very important in the real world. Shin [37,38] showed that solitons residing on cnoidal wave backgrounds travel faster than residing on backgrounds of plane wave. These solutions can be used to analyse some interesting physical phenomena, but there are few physical studies to report this kind of solutions. In fact, it is a common phenomenon to see solitons interacting with other waves, like the ocean waves and the optical waves. In the ocean, stokes waves are used to describe the finite amplitude wave with larger wave steepness in deep water. Solitary wave usually appears in the shallow water, which is also called shallow water wave. When the waves enter the shallow water, because of the significant influence of the submarine topography on the shape and height of the waves, it enables using cnoidal wave to analyse the propagation process of the finite amplitude wave in the shallow water. In the optics, localized states of the optically induced refractive index gratings and solitons in the optically induced lattices can be analysed by the soliton–cnoidal wave solutions [39–41]. Therefore, the theory of cnoidal waves and solitary waves has been widely used in shallow water and optical area.

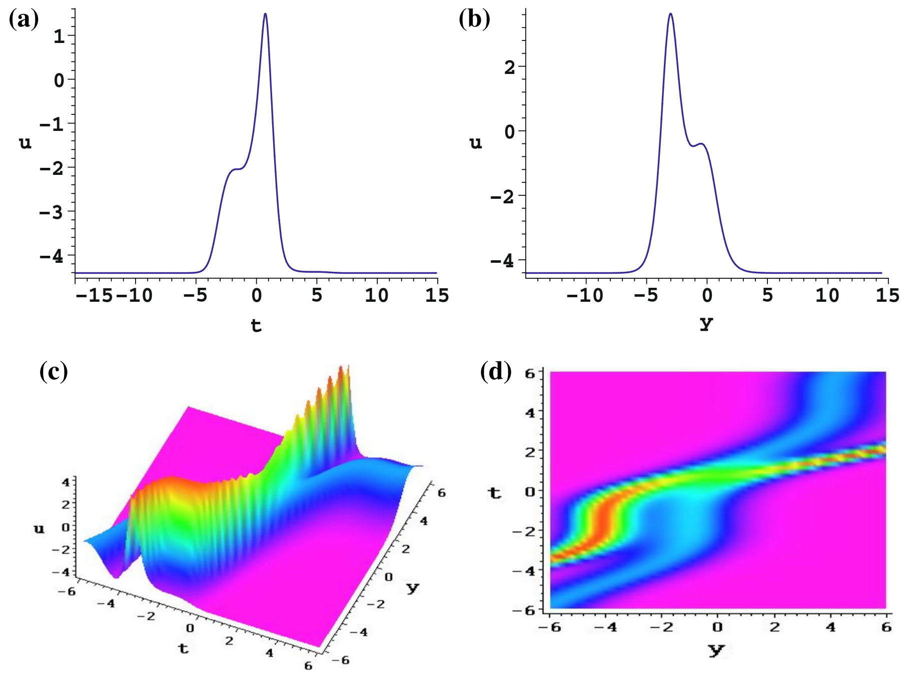

Fig. 7 Plots of second kind of bright soliton solution of u expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, k_1 = 1, r_1 = 0.8, k_2 = 1, m = 1, n = 0.9, \mu = \frac{1}{2}, q = \cos(t)$. **a** The mode of wave structure along t axis; **b** the wave travel mode along y axis; **c** three-dimensional plot for component u; **d** corresponding density plot

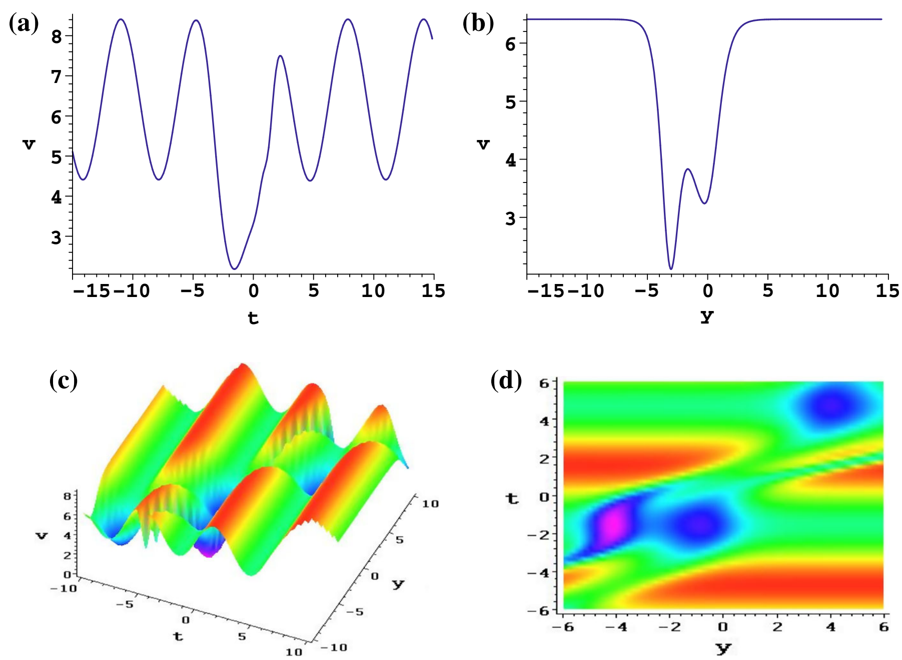

Fig. 8 Plots of second kind of dark soliton solution of v expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, k_1 = 1, r_1 = 0.8, k_2 = 1, m = 1, n = 0.9, \mu = \frac{1}{2}, q = \cos(t)$. **a** The mode of wave structure along t axis; **b** the wave travel mode along y axis; **c** three-dimensional plot for component v; **d** corresponding density plot

4 Summary and discussion

Nonlocal symmetry, extended system, similarity reductions, exact interaction solutions for Eqs. (4)–(5) are studied in this paper. Nonlocal symmetries for Eqs. (4)–(5) are obtained by two ways, from the Lax pair and the truncated Painlevé extension. Under transformation (18), the nonlocal symmetry (8) can be connected with the residual symmetry (21). With introducing of some internal parameters, much more symmetries are acquired through differentiating the given internal parameters. By extending the original system,

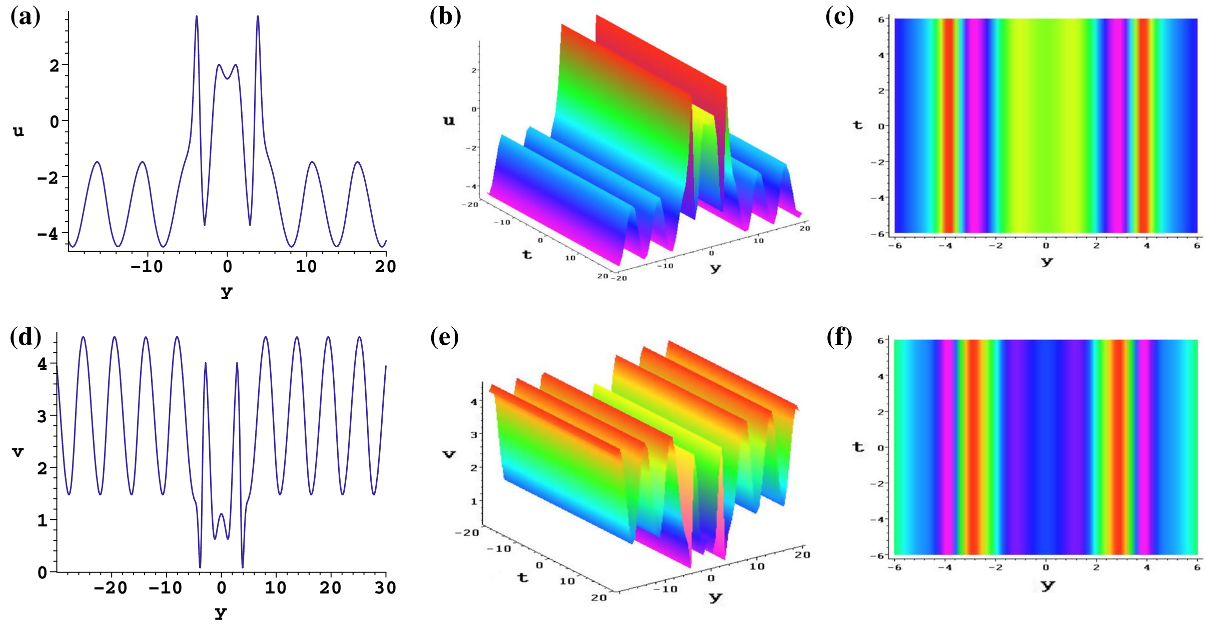

Fig. 9 Plots of third kind of interaction solutions for components u and v expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, k_1 = 1, r_1 = 0.8, k_2 = 1, m = 1, n = 0.9, \mu = \frac{1}{2}, q = t$. **a**, **d** The mode of wave structures along t axis; **b**, **e** the mode of wave structures along y axis; **c**, **f** corresponding three-dimensional plots for components u and v

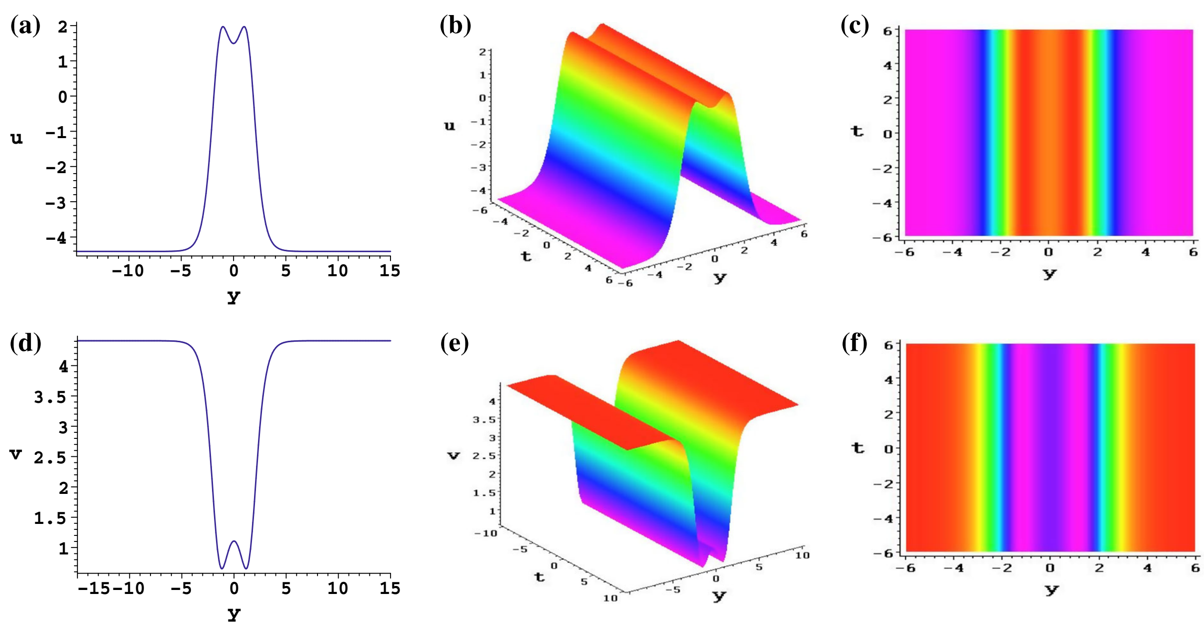

Fig. 10 Plots of third kind of bright two-soliton solution for component u and dark two-soliton solution for component v expressed by (41) at $x = 0$. The parameters are $\alpha = 1, \beta = 1, c = 1, r_0 = 0.5, k_1 = 1, r_1 = 0.8, k_2 = 1, m = 1, n = 0.9, \mu = \frac{1}{2}, q = t$. **a**, **d** The wave travel modes along y axis; **b**, **e** the three-dimensional plots for components u and v; **c**, **f** the corresponding density plots

nonlocal symmetries are successfully localized to an extended closed system. The corresponding Schwartz form of Eqs. (4)–(5) both reduced via localization of the nonlocal symmetry from Lax pair and utilizing the method of singularity analysis. Through solving the initial-value problems, the finite symmetry transformations are obtained to generate new solutions. Moreover, bright soliton, dark soliton, bell-typed soliton and soliton interacting with elliptic solutions are presented by similarity reductions. Because of the arbitrary function appeared in the interaction solutions, some interesting interaction behaviours are demonstrated by choosing the arbitrary function as $q = t$, $q = \cos(t)$ and $q = \sin(t)$, which can be used to illustrate some interesting physical phenomena and may be meaningful for studying waves in the ocean.

With the aid of the nonlocal symmetries, bright soliton, dark soliton and interacting with cnoidal wave solutions for Eqs. (4)–(5) are derived by similarity reductions. The method used here could be instructive to study other important physical systems. The relation between the nonlocal symmetry obtained by the Lax pair and other types of nonlocal symmetries, such as negative hierarchies and Darboux transformation, is also an meaningful subject. As localization is deemed to be the most important step to expand application of the nonlocal symmetries, there is no uniform method in localizing which type of the nonlocal symmetry to a Lie symmetry and constructing new interaction solutions by the nonlocal symmetry approach. It is well worthy of further exploration on Hamiltonian structure, conservation laws and generalized symmetry and some other important integrable properties of Eqs. (4)–(5) in the future work.

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