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Groups Analysis and Localized Solutions of the (2+1)-Dimensional Ito Equation *

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By means of the modified Clarkson and Kruskal (CK) direct method and the variable separation approach, we investigate the (2+1)-dimensional Ito equation which was constructed by Ito in 1980. The full symmetry group with the Kac–Moody–Virasoro algebra structure and the variable separation solutions are obtained. By selecting appropriate arbitrary functions, some special soliton excitations are shown graphically. The results presented here would be beneficial for understanding the (2+1)-dimensional Ito equation better.

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At present, there are many powerful and efficient methods to investigate kinds of integrability, such as the inverse scattering method,[1] Darboux transformation (DT),[2] the Bäcklund transformation method (BT),[3] the Hirota bilinear method,[4] the tanh-expansion method,[5] the Lie symmetry method,[6] the algebra-geometrical approach,[7] and the variable separation approach.[8] Among these methods, the symmetry method and the variable separation approach have been applied to solve a wide range of problems and to explore many physically interesting solutions of nonlinear phenomena. By virtue of the symmetry group, on the one hand, one can arrive at new solutions from old ones; on the other hand, one can construct a special type of exact solutions called group invariant solutions which are invariant under some transformations. For (2+1)-dimensional differential equations, the variable separation approach established by Lou *et al.*[8] is another powerful method to provide some exact physically significant coherent solitons solutions, such as multiple solitoffs, dromions, lumps, ring solitons, breathers and instantons.

In 1980, in view of keeping the N-soliton solution, Ito[9] generalized the bilinear KdV equation

$$D_x(D_t + D_x^3)f \cdot f = 0$$

to a (2+1)-dimensional bilinear model

$$[D_t^2 + D_t D_x^3 + \mu D_t D_y + \nu D_t D_x] f \cdot f = 0, \tag{1}$$

where μ and ν are arbitrary constants. By the transformation $u = 2(\ln f)_{xx}$, one can rewrite Eq. (1) as

$$\begin{aligned} &u_{tt} + u_{xxxt} + 3(2u_x u_t + u u_{xt}) \\ &+ 3u_{xx}\int_{-\infty}^{x} u_t dx' + \mu u_{yt} + \nu u_{xt} = 0, \end{aligned} \tag{2}$$

which is called the (2+1)-dimensional Ito equation. For $\mu = 0$ and $\nu = 0$, Eq. (2) is reduced to the well-known (1+1)-dimensional Ito equation.[9−13] Compared with the (1+1)-dimensional Ito equation, slight attention is paid to the (2+1)-dimensional Ito equation. Recently, Li *et al.*[14] studied its soliton solutions, doubly-periodic wave solutions and periodic solitary wave solutions using the extended homoclinic test technique and the bilinear method, Wazwaz[15] studied its soliton solutions using the tanh-coth method and the Hirota bilinear method, and Zhao *et al.*[16] used the extend three-wave method to construct its multi-wave solutions. With the help of the Bell polynomials method, the bilinear forms, bilinear Bäklund transformations and Lax pairs of Eq. (2) were reobtained by Wang,[17] respectively.

In this Letter, we focus on the full symmetry group and localized solutions of the (2+1)-dimensional Ito equation, which have not been revealed up to now. For simplicity, introducing the potential transformation

$$u = w_x, \tag{3}$$

Eq. (2) is converted into

$$\begin{aligned} &w_{xtt} + w_{xxxxt} + 6w_{xx}w_{xt} + 3w_x w_{xxt} \\ &+ 3w_{xxx}w_t + \mu w_{xyt} + \nu w_{xxt} = 0. \end{aligned} \tag{4}$$

In Ref. [18], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. Then Lou *et al.*[19] were inspired to modify the CK direct method to find the full symmetry group for both

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integral and non-integral nonlinear DEs. Here we take this modified CK direct method to investigate Eq. (4).

For Eq. (4), one can take the simplified symmetry transformation ansatz as

$$w = \alpha + \beta W(\xi, \eta, \tau), \tag{5}$$

where α, β, ξ, η and τ are functions of $\{x, y, t\}$. It is required that $W \equiv W(\xi, \eta, \tau)$ also satisfies Eq. (4) with the new independent variables ξ, η and τ.

Substituting Eq. (5) into Eq. (4), eliminating $W_{\xi\xi\xi\tau}$ and their higher-order derivatives, then setting the coefficients of the polynomials of W and their derivatives to be zero, by solving these equations we can obtain

$$\begin{aligned}
\alpha =& -\frac{1}{18}\frac{\mu\eta_{0yy}x^2}{\eta_{0y}} + \frac{1}{3\delta\eta_{0y}^{4/3}}(\nu\eta_{0y}^2 \\
&- \delta\nu\eta_{0y}^{4/3} - \mu\eta_{0y}\xi_{0y})x + \alpha_0, \\
\beta =& \delta\eta_{0y}^{1/3}, \quad \xi = \delta\eta_{0y}^{1/3}x + \xi_0, \\
\eta =& \eta_0, \quad \tau = \frac{\eta_0}{\mu} + \tau_0,
\end{aligned} \tag{6}$$

where $\xi_0 \equiv \xi_0(y)$, $\eta_0 \equiv \eta_0(y)$, and $\alpha_0 \equiv \alpha_0(y)$ are arbitrary functions of y, and $\tau_0 \equiv \tau_0(t - \frac{y}{\mu})$ is an arbitrary function of $(t - \frac{y}{\mu})$. Here the constant δ possesses three discrete values determined by

$$\delta = 1, \quad -\frac{1}{2}(1 + i\sqrt{3}), \quad -\frac{1}{2}(1 - i\sqrt{3}), \tag{7}$$

with $i^2 = 1$.

Here we give the following final transformation group theorem of Eq. (4).

If $W(x, y, t)$ is a solution to Eq. (4), then it is $w(x, y, t)$ with

$$\begin{aligned}
w =& -\frac{1}{18}\frac{\mu\eta_{0yy}x^2}{\eta_{0y}} + \frac{1}{3\delta\eta_{0y}^{4/3}}(\nu\eta_{0y}^2 - \delta\nu\eta_{0y}^{4/3} \\
&- \mu\eta_{0y}\xi_{0y})x + \alpha_0 + \delta\eta_{0y}^{1/3}W(\xi, \eta, \tau),
\end{aligned} \tag{8}$$

where ξ, η, τ are given by Eq. (6) and the discrete values of δ are given by Eq. (7).

Hence starting from the equation itself, one can directly obtain the Lie symmetry group without the prior Lie algebra. From the above symmetry group theorem, one can also see that the full symmetry group $\mathcal{G}_{\rm Ito}$ of Eq. (4) is divided into three sectors which correspond to

$$\delta = 1, \quad -\frac{1 + i\sqrt{3}}{2}, \quad -\frac{1 - i\sqrt{3}}{2},$$

of the theorem respectively. That is, the full symmetry group $\mathcal{G}_{\rm Ito}$, expressed by the theorem for the complex (2+1)-dimensional Ito equation is the product of the usual Lie point symmetry group $\mathcal{S}$ (theorem with δ=1) and the discrete group $\mathcal{D}$,

$$\mathcal{G}_{Ito} = \mathcal{D} \otimes \mathcal{S}, \quad \mathcal{D} = \{I, R_1, R_2\}, \tag{9}$$

where I is the identity transformation, and

$$\begin{aligned}
& R_1 : w(x, y, t) \to -\frac{1 + i\sqrt{3}}{2}w \\
&\cdot\Big(-\frac{1 + i\sqrt{3}}{2}x, -\frac{1 + i\sqrt{3}}{2}y, t\Big), \\
& R_2 : w(x, y, t) \to -\frac{1 - i\sqrt{3}}{2}w \\
&\cdot\Big(-\frac{1 - i\sqrt{3}}{2}x, -\frac{1 - i\sqrt{3}}{2}y, t\Big).
\end{aligned}$$

Furthermore, via the full transformation group Eq. (8), the Lie point symmetries and the related Lie algebra can be recovered straightforwardly by a more simple limiting procedure. In fact, if we set

$$\begin{aligned}
\alpha_0 &= \epsilon g(y), \quad \eta_0(y) = y + \epsilon k(y), \\
\xi_0(y) &= \epsilon l(y), \quad \tau_0 = t - \frac{y}{\mu} + \epsilon s\Big(t - \frac{y}{\mu}\Big)
\end{aligned}$$

with an infinitesimal parameter ϵ, then the group Eq. (8) can be written as

$$\begin{aligned}
w =& w + \epsilon\sigma(W) \\
\sigma(W) =& \Big(\frac{1}{3}\dot{k}x + l\Big)w_x + kw_y + \Big(\frac{k}{\mu} + s\Big)w_t \\
&- \frac{1}{18}\mu\ddot{k}x^2 - \Big(\frac{1}{3}\mu\dot{l} - \frac{2}{9}\nu\dot{k}\Big)x - g + \frac{1}{3}\dot{k}w.
\end{aligned}$$

That is, the Lie point symmetries of Eq. (4) have the forms

$$\begin{aligned}
\sigma \equiv& \sigma_1(k) + \sigma_2(l) + \sigma_3(g) + \sigma_4(s) \\
\equiv& \Big[\frac{1}{3}\dot{k}\Big(xw_x + \frac{2}{3}\nu x + w\Big) \\
&+ k\Big(w_y + \frac{1}{\mu}w_t\Big) - \frac{1}{18}\mu\ddot{k}x^2\Big] \\
&+ \Big(lw_x - \frac{1}{3}\mu\dot{l}\Big) + (-g) + (sw_t).
\end{aligned} \tag{10}$$

The nonzero commutators among $\sigma_1(k)$, $\sigma_2(l)$, $\sigma_3(g)$ and $\sigma_4(s)$ are

$$\begin{aligned}
[\sigma_1(k_1), \sigma_1(k_2)] &= \sigma_1(k_1\dot{k}_2 - \dot{k}_1k_2), \\
[\sigma_2(l_1), \sigma_2(l_2)] &= \sigma_3(\frac{1}{3}\mu(l_1\dot{l}_2 - \dot{l}_1l_2)), \\
[\sigma_4(s_1), \sigma_4(s_2)] &= \sigma_4(s_1\dot{s}_2 - \dot{s}_1s_2), \\
[\sigma_1(k), \sigma_2(l)] &= \sigma_2(k\dot{l} - \frac{1}{3}\dot{k}l) + \frac{2}{9}\nu\dot{k}l\frac{\partial}{\partial w}, \\
[\sigma_1(k), \sigma_3(g)] &= \sigma_3(k\dot{g} + \frac{1}{3}\dot{k}g).
\end{aligned}$$

Hence, when $\nu = 0$, $\{\sigma_1(k), \sigma_2(l), \sigma_3(g), \sigma_4(s)\}$ constitutes an infinite-dimensional closed Kac–Moody–Virasoro type Lie algebra.

To seek exact solutions to Eq. (2) which may have some arbitrary functions, we take the variable separation approach. According to the standard procedure of the 'separation variable', make the following Bäcklund transformation

$$w = 2(\ln f)_x + w_0, \tag{11}$$

where $f \equiv f(x,y,t)$ is a function of the indicated variables and $w_0 = w_0(x,y)$ is a known seed solution to the model Eq. (4). Substituting the BT Eq. (11) into Eq. (4) leads to the bilinear form

$$\begin{aligned}&[D_t^2 + D_tD_x^3 + \mu D_tD_y\\ &+ (\nu + 3w_{0x})D_tD_x + h]f\cdot f = 0,\end{aligned} \tag{12}$$

where $h = h(y,t)$ is the integration function and the bilinear differential operators D_x, D_y, and D_t are defined by

$$\begin{aligned}&D_x^m D_y^n D_t^k f\cdot g = (\partial_x - \partial_x')^m(\partial_y - \partial_y')^n(\partial_t - \partial_t')^k\\ &\cdot f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t}.\end{aligned}$$

To find out some special solutions to Eq. (12), we look for the solutions in the form of

$$f = a_0 + a_1p + a_2q + a_3pq, \tag{13}$$

with $a_i \equiv a_i(y)(i = 0,1,2,3)$, $p \equiv p(x,y)$ and $q \equiv q(t,y)$ being functions of the indicated arguments. Since p is t-independent and q is x-independent, substituting Eq. (13) into Eq. (11) leads to

$$\begin{aligned}&(a_0a_3 - a_1a_2)(p_{xxx} + \mu p_y + 3w_{0x}p_x + \nu p_x)\\ &+ \mu(a_1a_{3y} - a_{1y}a_3)p^2 + \mu(a_0a_{2y} - a_{0y}a_2)\\ &+ \mu(a_0a_{3y} + a_1a_{2y} - a_3a_{0y} - a_2a_{1y})p\\ =&(a_0a_3 - a_1a_2)^2(c_0 + c_1p + c_2p^2),\end{aligned} \tag{14}$$

$$\begin{aligned}\mu q_y =& (a_0^2c_2 + a_1^2c_0 - a_0a_1c_1 + G_0F_2)\\ &- [(a_0a_3 + a_1a_2)c_1 - 2a_0a_2c_2 - 2a_1a_3c_0 + G_0]q\\ &+ (a_3^2c_0 - a_2a_3c_1 + a_2^2c_2)q^2,\end{aligned} \tag{15}$$

$$h = -2(a_3^2c_0 - a_2a_3c_1 + a_2^2c_2)q_t, \tag{16}$$

where $c_0 \equiv c_0(y), c_1 \equiv c_1(y), c_2 \equiv c_2(y)$ are all arbitrary functions of y, and G_0 is an arbitrary constant.

On the one hand, due to Eq. (16), w_{0x} can be determined as

$$\begin{aligned}w_{0x} =& 3^{-1}(a_0a_3 - a_1a_2)p_x^{-1}[(a_0a_3 - a_1a_2)^2\\ &\cdot(c_0 + c_1p + c_2p^2) - (a_0a_3 - a_1a_2)\\ &\cdot(p_{xxx} + \mu p_y + \nu p_x) + \mu(a_3a_{1y} - a_{3y}a_1)p^2\\ &+ \mu(a_{1y}a_2 - a_0a_{3y} - a_1a_{2y} + a_3a_{0y})p\\ &+ \mu(a_2a_{0y} - a_{2y}a_0)].\end{aligned} \tag{17}$$

On the other hand, we consider the solution to the Riccati Eq. (17) in the form of

$$q = F_1\exp(G_0t) + F_2, \tag{18}$$

where $F_1 \equiv F_1(y)$ and $F_2 \equiv F_2(y)$ can both be treated as arbitrary functions of y, while c_0, c_1 and c_2 are related to F_1 and F_2 by

$$\begin{aligned}c_0 =& \frac{1}{(a_0a_3 - a_1a_2)^2F_1}[-\mu(a_0 + a_2F_2)a_2F_{1y}\\ &+ \mu a_2^2F_1F_{2y} - (a_0 + a_2F_2)a_2F_1G_0],\end{aligned} \tag{19}$$

$$\begin{aligned}c_1 =& \frac{1}{(a_0a_3 - a_1a_2)^2F_1}[-\mu(a_0a_3 + a_1a_2\\ &+ 2a_2a_3F_2)F_{1y} + 2\mu a_2a_3F_1F_{2y}\\ &- (a_0a_3 + a_1a_2 + 2a_2a_3F_2)F_1G_0],\end{aligned} \tag{20}$$

$$\begin{aligned}c_2 =& \frac{1}{(a_0a_3 - a_1a_2)^2F_1}[-\mu(a_1 + a_3F_2)a_3F_{1y}\\ &+ \mu a_3^2F_1F_{2y} - (a_1 + a_3F_2)a_3F_1G_0].\end{aligned} \tag{21}$$

Finally, substituting Eq. (15) into Eqs. (13) and (3) yields the solutions to Eq. (2),

$$\begin{aligned}u = w_x =& -\frac{2(a_1 + a_3q)^2p_x^2}{(a_0 + a_1p + a_2q + a_3pq)^2}\\ &+ \frac{2(a_1 + a_3q)p_{xx}}{a_0 + a_1p + a_2q + a_3pq} + w_{0x},\end{aligned} \tag{22}$$

where w_{0x} and q are determined by Eqs. (19) and (20).

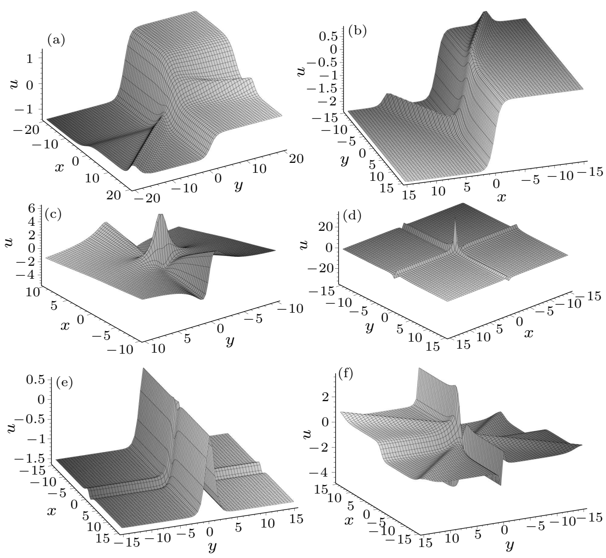

Fig. 1. Plots of different localized solutions expressed by Eqs. (22) and (23) at time $t = 0$, (a) $p = \exp(x) + \exp\left(\frac{1}{3}x\right)$; $F_1 = \exp(-y) + \frac{1}{3}\exp\left(\frac{1}{3}y\right)$; (b) $p = \exp(-x)$; $F_1 = \mathrm{sech}^2(y)$; (c) $p = \cosh(x)$; $F_1 = \cosh(y)$; (d) $p = \frac{1}{x^2}$; $F_1 = \frac{2}{1+y^2}$; (e) $p = 2 + \cosh(x)$; $F_1 = \tanh(y)$, and (f) $p = \exp(2x)$; $F_1 = \mathrm{sech}^2(y) + 2$.

One can see that since solution (22) has six arbitrary functions a_0, a_1, a_2, a_3, F_1 and F_2 of y, one arbitrary function p of $\{x,t\}$ and one arbitrary constant G_0, it gives quite abundant localized coherent structures of the (2+1)-dimensional Ito equation. To leave these interacted behaviors between solitary waves clear, some special localized excitations are illustrated in the following figure by selecting appropriate arbitrary functions. The parameters in the figure are chosen as

$$a_0 = a_3 = F_2 = 0, \quad a_1 = a_2 = F_3 = 1, \quad \mu = 2, \quad \nu = 1.$$

In summary, the symmetry study and the variable separation approaches are two useful methods in the study of nonlinear science, which are very valid for offering group invariant solutions and localized solutions. The Lie point symmetries of the (2+1)-dimensional Ito equation are clear in this work, while the nonlocal symmetries and corresponding nonlocal solutions are still unknown. Moreover, how to select appropriate arbitrary functions in variable separation solutions for obtaining abundant soliton structures, and more applications of localized solutions in physics and nature, need further study.

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